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INVARIANCE PRINCIPLES FOR SELF-SIMILAR SET-INDEXED RANDOM FIELDS

HERMINE BIERMÉ [1,2] & OLIVIER DURIEU [2]

ABSTRACT. For a stationary random field $(X_j)_{j\in\mathbb{Z}^d}$ and some measure μ on $\mathbb{R}^d$, we consider the set-indexed weighted sum process $S_n(A)=\sum_{j\in\mathbb{Z}^d}\mu(nA\cap R_j)^{\frac{1}{2}}X_j$, where R_j is the unit cube with lower corner j. We establish a general invariance principle under a p-stability assumption on the X_j's and an entropy condition on the class of sets A. The limit processes are self-similar set-indexed Gaussian processes with continuous sample paths. Using Chentsov's type representations to choose appropriate measures μ and particular sets A, we show that these limits can be Lévy (fractional) Brownian fields or (fractional) Brownian sheets.

1. Introduction

Let $(X_j)_{j\in\mathbb{Z}^d}$ be a centered stationary random field with $X_0\in L^2$. One can naturally associate to $(X_j)_{j\in\mathbb{Z}^d}$ a set-indexed process by considering, for $A\in\mathcal{B}(\mathbb{R}^d)$,

$$S(A)=\sum_{j\in A}X_j.$$

When the $X_j's$ are independent and identically distributed with $\mathrm{Var}(X_0)=1$, the Central Limit Theorem ensures that, for convenient sets A, the normalized sequence $\left(n^{-d/2}S(nA)\right)_{n\geq 1}$ converges in distribution to a centered Gaussian variable with variance given by $\lambda(A)$, where λ is the Lebesgue measure on $\mathbb{R}^d$. In order to establish invariance principles Alexander and Pyke [1] consider a "smoothed" version defined as

$$S_n(A)=\sum_{j\in\mathbb{Z}^d}b_{n,j}(A)X_j, \tag{1}$$

where $b_{n,j}(A)=\lambda(nA\cap R_j)$, with R_j the unit cube with lower corner j, and restrict the process to a sub-class $\mathcal{A}$ of Borel sets satisfying a metric entropy condition. The limit process $(W(A))_{A\in\mathcal{A}}$ is a Brownian process indexed by $\mathcal{A}$ which is a centered Gaussian process with

$$\mathrm{Cov}(W(A),W(B))=\lambda(A\cap B),\ \ A,B\in\mathcal{A}.$$

When $d=1$ one recovers the classical Donsker Theorem, with Brownian motion at the limit, by considering the class $\mathcal{R}=\{[0,t];t\in[0,1]\}$. For dimension $d>2$, it is usual to consider the class $\mathcal{R}=\{[0,t];t\in[0,1]^d\}$ with $[0,t]=[0,t_1]\times\ldots\times[0,t_d]$ for $t=(t_1,\ldots,t_d)\in[0,1]^d$ such that $(W([0,t]))_{t\in[0,1]^d}=(\mathcal{B}(t))_{t\in[0,1]^d}$ where $(\mathcal{B}(t))_{t\in\mathbb{R}^d}$ is the Brownian sheet characterized by

$$\mathrm{Cov}(\mathcal{B}(t),\mathcal{B}(s))=\frac{1}{2^d}\prod_{i=1}^{d}(|t_i|+|s_i|-|t_i-s_i|),\ \text{for all } t,s\in\mathbb{R}^d. \tag{2}$$

Contrarily to the Brownian motion, the Brownian sheet does not have stationary increments. The independence property is also lost. A second generalization for d-dimensionally indexed Brownian field was introduced by Lévy [22] as a centered Gaussian random field $(W(t))_{t\in\mathbb{R}^d}$ with

$$\mathrm{Cov}(W(t),W(s))=\frac{1}{2}(|t|+|s|-|t-s|),\ \text{for all } t,s\in\mathbb{R}^d,$$

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where $|\cdot|$ denotes the Euclidean norm. Such a field has stationary increments and is linearly additive [23], which means that $(W(a+rs))_{r\in\mathbb{R}}$ has independent increments for any $a, s \in \mathbb{R}^d$. Then, a natural question is to find a class $\mathcal{A}$ and weights $(b_{n,j}(A))_{A\in\mathcal{A}}$ to get the Lévy Brownian field W as the limit of an invariance principle. To our knowledge this question was only raised by Ossiander and Pyke [25] but their construction does not fit the setting of (1). The main ingredient is the geometric Chentsov construction of the Lévy Brownian field [7], which allows to identify $(W(t))_{t\in\mathbb{R}^d}$ as $(M(A_t))_{t\in\mathbb{R}^d}$, for some convenient Borel sets A_t and M a Gaussian random measure with control measure μ given in a specific way (see Section 8.3 of [27]).

In this paper, we also deal with dependent data $(X_j)_{j\in\mathbb{Z}^d}$. Central limit theorems for stationary random sequences have been extensively studied under several dependence assumptions, see Hall and Heyde [16], Dedecker et al. [10], Bradley [5]. Extensions to stationary random fields are often more difficult due to the lack of order of $\mathbb{R}^d$. Nevertheless, limit theorems for dependent random fields can be found in the literature. Bolthausen [4] proved a central limit theorem for α-mixing random fields. Goldie and Greenwood [15] gave an invariance principle for the set-indexed process (1) with $b_{n,j}(A) = \lambda(nA \cap R_j)$ in case of uniform ϕ-mixing random fields (see also Chen [6]). Later, Dedecker [9] (see also [8]) obtained the invariance principle under a projective criterion.

Here we adopt the setting of physical dependence measure (p-stability) introduced in Wu [33] in dimension 1 and extended by El Machkouri, Volný and Wu [14] to general dimension $d \geq 1$. Several limit theorems are proved under p-stability, see [34] and references therein. An invariance principle for the set-indexed process (1) with $b_{n,j}(A) = \lambda(nA \cap R_j)$ is obtained in [14].

In Section 2 we recall the definition of p-stability and its main properties. We also give several examples of random fields satisfying p-stability. In Section 3, we consider Central Limit Theorem for weighted sums $\sum_{j\in\mathbb{Z}^d} b_{n,j}X_j$ of 2-stable random fields, under appropriate assumptions on weights. A similar result was already obtained by Wang [32], but its proof, based on an m-dependent approximation and on a coefficient averaging procedure, requires asymptotic properties for averaged coefficients (named regular property in Definition 1 of [32]). Following [14] we use an m_n-dependent approximation and a result of Heinrich [18] which enable us to state the Central Limit Theorem under conditions that only depend on the coefficients. In Section 4, we prescribe the coefficients to be of the form

$$b_{n,j}(A) = \sqrt{\mu(nA \cap R_j)},$$

for some measure μ defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ absolutely continuous with respect to the Lebesgue measure and depending on a Borel set A in a particular class $\mathcal{A}$. We provide general assumptions on both μ and $\mathcal{A}$ which ensure the convergence for finite dimensional distributions of the normalized set-indexed sequence $(S_n(A))_{A\in\mathcal{A}}$ defined in (1) to a centered set-indexed Gaussian process $(\sigma W(A))_{A\in\mathcal{A}}$ with $\sigma > 0$ and

$$\mathrm{Cov}(W(A), W(B)) = \mu(A \cap B) = \frac{1}{2}\left(\mu(A) + \mu(B) - \mu(A \triangle B)\right).$$

An invariance principle is obtained under an additional entropy assumption on the class $\mathcal{A}$, which may be obtained using Vapnik-Cheronenkis dimension of $\mathcal{A}$. Section 5 illustrates previous results with particular examples of self-similar processes. Our setting allows on the one hand to recover classical results for Brownian sheet considering $\mu = \lambda$ the Lebesgue measure and $\mathcal{A} = \mathcal{R}$. On the other hand, it allows more flexibility and we can use Chentsov random fields representations. The Lévy Brownian field may be obtained as a particular case of field defined for a Chentsov measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

$$\mu(dx) = \frac{dx}{|x|^{d-2H}} \text{ for some } H \in (0, d/2]$$

and considering $(D_t)_{t\in[0,1]^d}$ with D_t the Euclidean ball of diameter $[\![0, t]\!]$. The limit random field is H-self-similar but does not have stationary increments except when $H = \frac{1}{2}$, which corresponds to the Lévy Brownian field. Chentsov's type construction for self-similar fields with stationary increments has been considered by Takenaka in [29] and generalized to α-stable fields in [30, 28].

This construction involves Borel sets V_t of $\mathbb{R}^d \times \mathbb{R}$, identified in our setting with $\mathbb{R}^{d+1}$, indexed by $t \in \mathbb{R}^d$, and a Takenaka measure on $(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}))$ defined by

$$\mu(dx, dr) = r^{2H-d-1} 1_{r>0} dx dr \text{ for } H \in (0, 1/2).$$

The limit field $(B_H(t))_{t\in\mathbb{R}^d}$ is the well-known Lévy fractional Brownian field of Hurst parameter H, satisfying

$$\mathrm{Cov}(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \text{ for all } t, s \in \mathbb{R}^d.$$

Note that such a field is well defined for any $H \in (0, 1)$. However Chentsov's construction is only possible when $H \leq \frac{1}{2}$. The case $H = \frac{1}{2}$ corresponds to the Lévy Brownian field, whose construction was previously considered. Generalizations to others self-similar Brownian sheets are studied using products of Lévy Chentsov or Takenaka measures. Invariance principles are obtained using a tightness criterion which involves entropy conditions on the considered class of sets. These entropy conditions may be obtained using Vapnik-Cheronenkis dimension of the class, which are computed for the different class of sets considered for theses examples in the last section of this paper. Some technical proofs are postponed to Appendix.

2. Measure of dependence

Let $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ be a sequence of i.i.d. random variables, g be a measurable function from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ and consider the stationary random field $(X_j)_{j\in\mathbb{Z}^d}$ defined by

$$X_j = g(\varepsilon_{j-k} : k \in \mathbb{Z}^d), \qquad j \in \mathbb{Z}^d. \tag{3}$$

Such a process is called Bernoulli shift in [12]. The following measure of dependence has been introduced by Wu [33] (see also [34]). Let ε_0' be a copy of ε_0 independent of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ and define $(\varepsilon_j^*)_{j\in\mathbb{Z}^d}$ by $\varepsilon_0^* = \varepsilon_0'$ and $\varepsilon_j^* = \varepsilon_j$ for $j \neq 0$. For every $p \in (0, +\infty]$, if $X_0 \in L^p$, the *physical dependence measure* $\delta_{p,j}$ of X_j is given by

$$\delta_{p,j} = \|X_j - X_j^*\|_p$$

where $(X_j^*)_{j\in\mathbb{Z}^d}$ denotes the stationary random field defined by (3) with $(\varepsilon_j^*)_{j\in\mathbb{Z}^d}$ instead of $(\varepsilon_j)_{j\in\mathbb{Z}^d}$. Then, a random field $(X_j)_{j\in\mathbb{Z}^d}$ is called *p-stable* if it is defined as above and

$$\Delta_p = \sum_{j\in\mathbb{Z}^d} \delta_{p,j} < +\infty.$$

Note that p-stability implies p'-stability for any $p' \in (0, p]$. Let us also remark that when X_0 is only assumed to be in L^1 and $X_j - X_j^* \in L^p$ for all $j \in \mathbb{Z}^d$ with $\Delta_p < +\infty$, Lemma 1 of [14] implies in fact that $X_0 \in L^p$ and thus $(X_j)_{j\in\mathbb{Z}^d}$ is *p-stable*. The following proposition is proved in El Machkouri et al. [14] for a finite set $\Gamma \subset \mathbb{Z}^d$. Its generalization to infinite set is straightforward, as remarked in [32], using the convention that $\displaystyle\sum_{j\in\Gamma} = \lim_{M\to+\infty} \sum_{j\in\Gamma;|j|\leq M}$, with $|j|$ the Euclidean norm of j.

Proposition 2.1. *Let $(X_j)_{j\in\mathbb{Z}^d}$ be a centered p-stable random field with $p \geq 2$.*

(i) For all $\Gamma \subset \mathbb{Z}^d$ and for all real numbers $(a_j)_{j\in\Gamma}$ such that $\displaystyle\sum_{j\in\Gamma} a_j^2 < +\infty$,

$$\left\| \sum_{j\in\Gamma} a_j X_j \right\|_p \leq \sqrt{2p} \left(\sum_{j\in\Gamma} a_j^2 \right)^{\frac{1}{2}} \Delta_p.$$

(ii) The random field satisfies the short range property

$$\sum_{k\in\mathbb{Z}^d} |\mathrm{Cov}(X_0, X_k)| < +\infty.$$

Let us give some examples of fields satisfying such an assumption.

Example 1. (linear random fields) This example is the first example given in [14] or [33] in dimension 1. Let $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ be a sequence of i.i.d. centered random variables with $\varepsilon_0 \in L^p$ for some $p \geq 1$ and consider a real filter $a = (a_j)_{j\in\mathbb{Z}^d}$. When $\sum_{j\in\mathbb{Z}^d} |a_j| < +\infty$, one can define in L^p the centered linear random field

$$X_j = \sum_{k\in\mathbb{Z}^d} a_k \varepsilon_{j-k}, \qquad j \in \mathbb{Z}^d.$$

When $p \geq 2$, from Rosenthal's inequality (Theorem 2.12 of [16]), this random field may be defined in L^p under the weaker assumption that $\sum_{j\in\mathbb{Z}^d} a_j^2 < +\infty$. Note that, since $p \geq 1$, $X = (X_j)_{j\in\mathbb{Z}^d}$ is a stationary centered random field. When $p \geq 2$, X is a second order random field with

$$\mathrm{Cov}(X_{j+l}, X_l) = \sum_{k\in\mathbb{Z}^d} a_k a_{k-j}.$$

Moreover, one has clearly $\delta_{p,j} = |a_j| \|\varepsilon_0 - \varepsilon_0'\|_p$ so X is p-stable if and only if

$$\sum_{j\in\mathbb{Z}^d} |a_j| < +\infty.$$

Example 2. (functional of a linear random field) One can consider more general fields obtained as functionals of linear random fields. More precisely, let $Y = (Y_j)_{j\in\mathbb{Z}^d}$ be a centered L^p linear random field with filter $a = (a_j)_{j\in\mathbb{Z}^d}$ and $p \geq 1$, as introduced in the previous example and consider

$$X_j = f(Y_j), \qquad j \in \mathbb{Z}^d,$$

for some real function $f : \mathbb{R} \to \mathbb{R}$. As remarked in [14], when f is a Lipschitz continuous function, the p-stability of Y implies the p-stability of X. More generally, when f is H-Hölder continuous with $H \in (0, 1]$, X is p-stable if $f(Y_0) \in L^p$ and

$$\mathbb{E}\left(|\varepsilon_0 - \varepsilon_0'|^{pH}\right)^{1/p} \sum_{j\in\mathbb{Z}^d} |a_j|^H < +\infty.$$

In particular, this allows the process Y to be heavy-tailed.

Example 3. (functional of a Gaussian linear random field) Another interesting example is given by considering for Y a Gaussian stationary process, assuming $\varepsilon_0 \sim \mathcal{N}(0, 1)$. Normalizing Y in such a way that $\mathrm{Var}(Y_j) = \sum_{k\in\mathbb{Z}^d} a_k^2 = 1$, one can consider the first Wiener chaos (or Gaussian space) $\mathcal{H}_1$, defined as the closed linear subspace of L^2 generated by the family $\{Y_j, Y_j^*; j \in \mathbb{Z}^d\}$. Then, for every $q \geq 2$, the symbol $\mathcal{H}_q$ stands for the qth *Wiener chaos*, defined as the closed linear subspace of L^2 generated by the family $\{H_q(Y_j), H_q(Y_j^*); j \in \mathbb{Z}^d\}$, where H_q is the qth Hermite polynomial given by

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right). \tag{4}$$

We write by convention $\mathcal{H}_0 = \mathbb{R}$. In this framework, we have for all $q, q' \geq 1$ and $j, j' \in \mathbb{Z}^d$, (see Lemma 1.1.1 of [24]),

$$\mathrm{Cov}\left(H_q(\widetilde{Y_j}), H_{q'}(\widetilde{Y_{j'}})\right) = q! \delta_{q,q'} \mathrm{Cov}(\widetilde{Y_j}, \widetilde{Y_{j'}})^q,$$

where $\delta_{q,q'}$ is the Kronecker symbol and $\widetilde{Y}$ is either Y or Y^*. It follows straightforwardly that

$$\|H_q(Y_j) - H_q(Y_j^*)\|_2^2 = 2q!\left(\mathrm{Var}(Y_j)^q - \mathrm{Cov}(Y_j, Y_j^*)^q\right) \leq q! q \|Y_j - Y_j^*\|_2^2,$$

since $\mathrm{Var}(Y_j) = \mathrm{Var}(Y_j^*) = 1$. Moreover using Hypercontractivity properties (see [21] p.65) one also has

$$\begin{aligned}\|H_q(Y_j) - H_q(Y_j^*)\|_p &\leq (p-1)^{q/2}\|H_q(Y_j) - H_q(Y_j^*)\|_2 \\ &\leq (p-1)^{q/2}\sqrt{q!q}\|Y_j - Y_j^*\|_2.\end{aligned}$$

It follows that 2-stability of Y implies p-stability of $H_q(Y) = (H_q(Y_j))_{j\in\mathbb{Z}^d}$, for $q \geq 1$. More generally, considering $f(Y) = (f(Y_j))_{j\in\mathbb{Z}^d}$ such that $f(Y_0) \in L^2$ and $\mathbb{E}(f(Y_0)) = 0$, according to the chaos expansion (see Theorem 1.1.1 of [24]) one has for all $j \in \mathbb{Z}^d$ the following equality in L^2

$$f(\widetilde{Y_j}) = \sum_{q=1}^{+\infty} c_q(f) H_q(\widetilde{Y_j}), \tag{5}$$

with $c_q(f) = \frac{1}{q!}\mathbb{E}(f(Y_0)H_q(Y_0))$ satisfying $\sum_{q=1}^{+\infty} q! c_q(f)^2 < +\infty$. Assuming that f is such that

$$\sum_{q=1}^{+\infty} \sqrt{qq!}(p-1)^{q/2}|c_q(f)| < +\infty, \tag{6}$$

we obtain that, on the one hand (5) holds also in L^p, and on the other hand that

$$\|f(Y_j) - f(Y_j^*)\|_p \leq \left(\sum_{q=1}^{+\infty} \sqrt{qq!}(p-1)^{q/2}|c_q(f)|\right)\|Y_j - Y_j^*\|_2.$$

Therefore, under Assumption (6), the 2-stability of Y implies also the p-stability of $f(Y)$.

Example 4. (Volterra field) Another example, quoted in [14], is given by Volterra fields defined by

$$X_j = \sum_{k,l\in\mathbb{Z}^d} a_{k,l}\varepsilon_{j-k}\varepsilon_{j-l},$$

where $(\varepsilon_j)_{j\in\mathbb{Z}^d}$ is a sequence of i.i.d. centered random variables with $\varepsilon_0 \in L^p$ for some $p \geq 2$ and $a = (a_{k,l})_{k,l\in\mathbb{Z}^d}$ is a sequence vanishing on the diagonal ($a_{k,k} = 0$ for all $k \in \mathbb{Z}^d$) and such that $\sum_{k,l\in\mathbb{Z}^d} a_{k,l}^2 < +\infty$. These assumptions ensure that X_j is well defined on L^2 and $(X_j)_{j\in\mathbb{Z}^d}$ is a stationary second order centered random field with

$$\mathrm{Cov}(X_{j+j'}, X_{j'}) = \sum_{k,l\in\mathbb{Z}^d} a_{k,l}\left(a_{k-j,l-j} + a_{l-j,k-j}\right).$$

One can easily compute $X_j - X_j^* = (\varepsilon_0 - \varepsilon_0')\left(\sum_{l\in\mathbb{Z}^d}(a_{j,l} + a_{l,j})\varepsilon_{j-l}\right)$. Then, by Rosenthal inequality (see Theorem 2.12 of [16]), one can find $C_p > 0$ such that

$$\mathbb{E}(|X_j - X_j^*|^p) \leq C_p\mathbb{E}(|\varepsilon_0 - \varepsilon_0'|^p)\left(\|\varepsilon_0\|_p^p\sum_{l\in\mathbb{Z}^d}|a_{j,l} + a_{l,j}|^p + \|\varepsilon_0\|_2^p\left(\sum_{l\in\mathbb{Z}^d}|a_{j,l} + a_{l,j}|^2\right)^{p/2}\right).$$

Hence, $X_j - X_j^* \in L^p$ and

$$\|X_j - X_j^*\|_p \leq C_p^{1/p}\|\varepsilon_0 - \varepsilon_0'\|_p\left(\|\varepsilon_0\|_p + \|\varepsilon_0\|_2\right)\left(\sum_{l\in\mathbb{Z}^d}|a_{j,l} + a_{l,j}|^2\right)^{1/2},$$

using $\left(\sum_{l\in\mathbb{Z}^d}|a_{j,l}+a_{l,j}|^p\right)^{1/p} \leq \left(\sum_{l\in\mathbb{Z}^d}|a_{j,l}+a_{l,j}|^2\right)^{1/2}$, since $p\geq 2$. Then under the additional assumption that

$$\sum_{j\in\mathbb{Z}^d}\left(\sum_{l\in\mathbb{Z}^d}|a_{j,l}+a_{l,j}|^2\right)^{1/2} < +\infty, \tag{7}$$

we obtain that $\Delta_p < +\infty$ which implies that $X_0 \in L^p$ and $(X_j)_{j\in\mathbb{Z}^d}$ is p-stable. Finally, let us remark that, in particular, (7) is implied by the assumption that $\sum_{k,l\in\mathbb{Z}^d}|a_{k,l}| < +\infty$.

3. **A central limit theorem for weighted sums**

In this section we establish a central limit theorem for normalized weighted sums of 2-stable random fields. This situation arises for example when one works with linear random fields $Y_i = \sum_{j\in\mathbb{Z}^d} a_{i-j}X_j$ for which the partial sum $\sum_{i\in\Gamma_n} Y_i$ on $\Gamma_n = \{0,\dots,n\}^d$ can be written as $\sum_{j\in\mathbb{Z}^d} b_{n,j}X_j$ where $b_{n,j} = \sum_{i\in\Gamma_n} a_{i-j}$. It is the case in Wang [32] or in Peligrad and Utev [26] for linear random sequences.
It also arises when one considers the smoothed partial sum process indexed by sets $S_n(A) = \sum_{j\in\Gamma_n}\lambda(nA\cap R_j)X_j$, where λ is the Lebesgue measure on $\mathbb{R}^d$ and R_j is the unit cube with lower corner j, as in [1], [9] or [14]. In this paper we are interested in the second situation but replacing the Lebesgue measure by different ones. Let us start with the following general result.

Let $\ell^2(\mathbb{Z}^d)$ be the space of all sequences $a=(a_j)_{j\in\mathbb{Z}^d}$ indexed by $\mathbb{Z}^d$ such that $\|a\|^2 := \sum_{j\in\mathbb{Z}^d} a_j^2 < +\infty$. For $k\in\mathbb{Z}^d$, we denote by τ_k the shift of vector k on elements of $\ell^2(\mathbb{Z}^d)$ defined by $(\tau_k a)_j := a_{j+k}$, $j\in\mathbb{Z}^d$.

Theorem 3.1. *Let $(X_j)_{j\in\mathbb{Z}^d}$ be a centered 2-stable random field (i.e. $\Delta_2 < +\infty$) and $b_n = (b_{n,j})_{j\in\mathbb{Z}^d}$, $n\in\mathbb{N}$, be sequences of real numbers in $\ell^2(\mathbb{Z}^d)$. Then $S_n = \sum_{j\in\mathbb{Z}^d} b_{n,j}X_j$ is well defined in L^2. If further*

(i) $\sup_{j\in\mathbb{Z}^d}\frac{|b_{n,j}|}{\|b_n\|} \xrightarrow[n\to\infty]{} 0$,

(ii) for all $e\in\mathbb{Z}^d$ with $|e|=1$, $\frac{\|\tau_e b_n - b_n\|}{\|b_n\|} \xrightarrow[n\to\infty]{} 0$,

then $\|b_n\|^{-2}\operatorname{Var}(S_n)\to\sigma^2$ as $n\to+\infty$ and

$$\frac{S_n}{\|b_n\|} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2),$$

where $\sigma^2 = \sum_{k\in\mathbb{Z}^d}\operatorname{Cov}(X_0,X_k)$ and $\xrightarrow{\mathcal{D}}$ means convergence in distribution.

Remark. Note that Wang [32] in Theorem 2 proved a similar result. The dependence assumption is the same but our assumptions *(i)* and *(ii)* on the coefficients do not involve averaged coefficients as in Definition 1 of [32]. However when $\|b_n\|\to\infty$, our conditions *(i)* and *(ii)* are equivalent to Wang's ones. Its proof is based on an approximation by m-dependent random fields, a coefficient-averaging procedure inspired by Peligrad and Utev [26], and a big/small blocking summation in order to apply a central limit theorem for triangular array of weighted i.i.d. random variables. Our proof is inspired by El Machkouri et al. [14] and based on a theorem of Heinrich [18] for m_n-dependent random fields.

Proof. 1) By stationarity, we have

$$\begin{aligned}\mathrm{Var}(S_n) &= \sum_{k\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d} b_{n,j}b_{n,k}\,\mathrm{Cov}(X_j,X_k)\\ &= \sum_{k\in\mathbb{Z}^d}\left(\sum_{j\in\mathbb{Z}^d} b_{n,j}b_{n,j+k}\right)\mathrm{Cov}(X_0,X_k). \qquad (8)\end{aligned}$$

Thus, using Cauchy-Schwarz inequality, $\mathrm{Var}(S_n) \le \|b_n\|^2 \sum_{k\in\mathbb{Z}^d} |\,\mathrm{Cov}(X_0,X_k)|$ which is finite by Proposition 2.1. By assumption *(ii)* and triangular inequalities, we obtain that for all $k\in\mathbb{Z}^d$,

$$\frac{\|\tau_k b_n - b_n\|}{\|b_n\|} \xrightarrow[n\to\infty]{} 0.$$

Writing $\|\tau_k b_n - b_n\|^2 = \|\tau_k b_n\|^2 + \|b_n\|^2 - 2\sum_{j\in\mathbb{Z}^d} b_{n,j}b_{n,j+k}$, we infer

$$\frac{1}{\|b_n\|^2}\sum_{j\in\mathbb{Z}^d} b_{n,j}b_{n,j+k} \xrightarrow[n\to\infty]{} 1.$$

Therefore, by (8) and the dominated convergence theorem,

$$\mathrm{Var}\left(\frac{S_n}{\|b_n\|}\right) \xrightarrow[n\to\infty]{} \sum_{k\in\mathbb{Z}^d}\mathrm{Cov}(X_0,X_k) = \sigma^2. \qquad (9)$$

2) We will apply a theorem of Heinrich [18] for m_n-dependent random field.

Theorem (Heinrich, 1988). *Let $(V_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^d$ be a sequence of finite sets such that $|V_n| \to +\infty$ and $(m_n)_{n\in\mathbb{N}}$ be a sequence of positive integers such that $m_n \to +\infty$. Assume for each $n\in\mathbb{N}$, $(U_{n,j})_{j\in\mathbb{Z}^d}$ is an m_n-dependent random field with $\mathbb{E}(U_{n,j}) = 0$ for all $j\in\mathbb{Z}^d$ which satisfies:*

- $\mathrm{Var}\left(\sum_{j\in V_n} U_{n,j}\right) \xrightarrow[n\to\infty]{} \sigma^2 < +\infty$,
- *there exists a constant $C>0$, such that $\sum_{j\in V_n}\mathbb{E}(U_{n,j}^2) \le C$, for all $n\in\mathbb{N}$,*
- *for all $\varepsilon>0$, $L_n(\varepsilon) = m_n^{2d}\sum_{j\in V_n}\mathbb{E}\left(U_{n,j}^2 1_{\{|U_{n,j}|\ge\varepsilon m_n^{-2d}\}}\right) \xrightarrow[n\to\infty]{} 0$.*

Then $\sum_{j\in V_n} U_{n,j}$ converges in distribution to $\mathcal{N}(0,\sigma^2)$.

We will have two steps of approximations of S_n, first by a sum on a finite set V_n and then by a sum of m_n-dependent variables.

3) Let $V_n = \left\{j\in\mathbb{Z}^d : |j| \le \varphi(n)\right\}$ with $\varphi(n)\to+\infty$ such that

$$\frac{\sum_{j\in V_n} b_{n,j}^2}{\|b_n\|^2} \xrightarrow[n\to\infty]{} 1$$

and set $S_{V_n} = \sum_{j\in V_n} b_{n,j}X_j$. Using Proposition 2.1, we obtain

$$\frac{\|S_n - S_{V_n}\|_2}{\|b_n\|} \le \frac{2}{\|b_n\|}\left(\sum_{j\in\mathbb{Z}^d\setminus V_n} b_{n,j}^2\right)^{\frac{1}{2}}\Delta_2 \xrightarrow[n\to\infty]{} 0. \qquad (10)$$

4) For $m\in\mathbb{N}$ and $j\in\mathbb{Z}^d$, we consider the σ-field $\mathcal{F}_{m,j} = \sigma\{\varepsilon_{j+i} : |i|\le m\}$ and we define the random variable $X_{m,j} = \mathbb{E}(X_j|\mathcal{F}_{m,j})$. We denote $S_{m,V_n} = \sum_{j\in V_n} b_{n,j}X_{m,j}$. In the same way, we

introduce $\mathcal{F}^*_{m,j} = \sigma\left\{\varepsilon^*_{j+i} : |i| \le m\right\}$, $X^*_{m,j} = \mathbb{E}(X^*_j|\mathcal{F}^*_{m,j})$, $j \in \mathbb{Z}^d$, and we define the coefficients

$$\delta_{m,p,j} = \|X_j - X_{m,j} - (X^*_j - X^*_{m,j})\|_p \text{ and } \Delta_{m,p} = \sum_{j\in\mathbb{Z}^d} \delta_{m,p,j},$$

for all $p \in (0, +\infty]$. By Lemma 2 of El Machkouri et al. [14], $\Delta_p < +\infty$ implies $\Delta_{m,p} \to 0$ as $m \to +\infty$. Thus, by Proposition 2.1 for $p = 2$, we get

$$\sup_n \frac{\|S_{V_n} - S_{m,V_n}\|_2}{\|b_n\|} \le 2\Delta_{m,2} \xrightarrow[m\to\infty]{} 0 \tag{11}$$

5) Fix a sequence $(m_n)_{n\in\mathbb{N}}$ and define $U_{n,j} = \frac{b_{n,j}}{\|b_n\|} X_{m_n,j}$. We will show that assumptions of Heinrich's theorem are satisfied.
We have $\sum_{j\in V_n} U_{n,j} = \|b_n\|^{-1} S_{m_n,V_n}$ and equations (9), (10), (11) show that

$$\mathrm{Var}\left(\frac{S_{m_n,V_n}}{\|b_n\|}\right) \xrightarrow[n\to\infty]{} \sigma^2.$$

We also have, using $\mathbb{E}(X^2_{m_n,0}) \le \mathbb{E}(X^2_0)$,

$$\sum_{j\in V_n} \mathbb{E}(U^2_{n,j}) \le \frac{1}{\|b_n\|^2} \sum_{j\in V_n} b^2_{n,j}\mathbb{E}(X^2_0) \le \mathbb{E}(X^2_0) < +\infty.$$

On the other hand,

$$L_n(\varepsilon) = \frac{m_n^{2d}}{\|b_n\|^2} \sum_{j\in V_n} b^2_{n,j}\mathbb{E}\left(X^2_{m_n,j} 1_{\{b_{n,j}m_n^{2d}|X_{m_n,j}|\ge\varepsilon\|b_n\|\}}\right).$$

Thus by Jensen inequality and for a fixed sequence $(a_n)_{n\in\mathbb{N}}$, we get

$$\begin{aligned}\mathbb{E}\left(X^2_{m_n,j} 1_{\{b_{n,j}m_n^{2d}|X_{m_n,j}|\ge\varepsilon\|b_n\|\}}\right) &\le \mathbb{E}\left(X^2_j 1_{\{b_{n,j}m_n^{2d}|X_{m_n,j}|\ge\varepsilon\|b_n\|\}}\right)\\ &\le \mathbb{E}\left(X^2_j 1_{\{|X_j|\ge a_n\}}\right) + a_n^2 P\left(b_{n,j}m_n^{2d}|X_{m_n,j}| \ge \varepsilon\|b_n\|\right)\end{aligned}$$

Therefore,

$$\begin{aligned}L_n(\varepsilon) &\le m_n^{2d}\frac{\sum_{j\in V_n} b^2_{n,j}}{\|b_n\|^2}\left(\mathbb{E}\left(X^2_0 1_{\{|X_0|\ge a_n\}}\right) + a_n^2\frac{b^2_{n,j}}{\|b_n\|^2}\frac{m_n^{4d}}{\varepsilon^2}\mathbb{E}(X^2_0)\right)\\ &\le m_n^{2d}\mathbb{E}\left(X^2_0 1_{\{|X_0|\ge a_n\}}\right) + \frac{m_n^{6d}}{\varepsilon^2}\sup_{j\in\mathbb{Z}^d}\frac{b^2_{n,j}}{\|b_n\|^2}a_n^2\mathbb{E}(X^2_0).\end{aligned}$$

By assumption *(i)*, we can choose the sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \to +\infty$ and $\sup_{j\in\mathbb{Z}^d}\frac{|b_{n,j}|}{\|b_n\|}a_n \to 0$ as $n \to +\infty$. Then we can choose $m_n \to +\infty$ such that $L_n(\varepsilon) \to 0$ as $n \to +\infty$.

Thus by Heinrich's theorem, we have $\frac{S_{m_n,V_n}}{\|b_n\|} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2)$, and by (10) and (11) we obtain $\frac{S_n}{\|b_n\|} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,\sigma^2)$. □

Theorem 3.1 will be used in the next section to derive limit theorems for set-indexed sums of random fields.

4. **Invariance principle for some set-indexed sums**

Let $d \ge 1$ be a fixed integer. Let $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product and $|\cdot|$ the associated Euclidean norm on $\mathbb{R}^d$. For each $j \in \mathbb{Z}^d$, we denote by R_j the rectangle of $\mathbb{R}^d$ given by $[j_1, j_1+1) \times \cdots \times [j_d, j_d+1)$ for $j = (j_1, \ldots, j_d)$. We consider a σ-finite measure μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ absolutely continuous with respect to the Lebesgue measure. In this section we focus on weighted sums where the weights are indexed by a Borelian set A as

$$b_{n,j}(A) = \sqrt{\mu(nA \cap R_j)}. \tag{12}$$

We are mainly interested in two kind of measures related to Chentsov random fields (see [27] Chapter 8). The first one is defined on $\mathbb{R}^d$ by $\mu(dx) = \frac{dx}{|x|^{d-2H}}$ for some $H \in (0, d/2]$. The Lévy Chentsov random field is defined by considering a random Gaussian measure with μ as control measure for $H = 1/2$. The second one is used as control measure for the construction of Takenaka random fields on $\mathbb{R}^d \times \mathbb{R}$ (identified with $\mathbb{R}^{d+1}$ in the sequel) and defined by $\mu(dx, dr) = r^{2H-d-1}1_{r>0}dxdr$ for $H \in (0, 1/2)$. Both of these measures satisfy the following general assumptions (see Section 5).

Assumption 1. *μ is a σ-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, absolutely continuous with respect to the Lebesgue measure, and such that:*

(1) There exists $\beta > 0$ such that $\mu(nA) = n^\beta \mu(A)$, for all $n \in \mathbb{N}$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

(2) Let us denote $\pi(x) = \min_{1\le i\le d} |\langle x, e_i\rangle|$, for $x \in \mathbb{R}^d$ and $(e_i)_{1\le i\le d}$ the canonical basis of $\mathbb{R}^d$,

(2a) $\limsup_{\pi(j)\to+\infty} \mu(R_j) < +\infty$.

(2b) For all $e \in \mathbb{Z}^d$ with $|e| = 1$ one has

$$\mu(R_{j+e}) = \mu(R_j) + \underset{\pi(j)\to+\infty}{o}(\mu(R_j)).$$

Under Assumption 1, we will establish limit theorems for the set-indexed process defined in (1) with respect to (12), that is

$$S_n(A) = \sum_{j\in\mathbb{Z}^d} \sqrt{\mu(nA \cap R_j)} X_j. \tag{13}$$

Condition *(1)* will guarantee the self-similarity of the limit random field. One can notice that if *(1)* holds then $\mu(cA) = c^\beta \mu(A)$, for all $c \ge 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$.

In the following, we need the notion of regular Borel set. We call A a *regular Borel set* if $\lambda(\partial A) = 0$, where ∂A denotes the boundary of A and λ is the Lebesgue measure on $\mathbb{R}^d$. Since μ is absolutely continuous with respect to the Lebesgue measure, if A is regular, then $\mu(\partial A) = 0$.

4.1. **Central limit theorem.** We have the following central limit theorem for weighted set-indexed sums of 2-stable random fields.

Theorem 4.1. *Let $(X_j)_{j\in\mathbb{Z}^d}$ be a centered 2-stable random field (i.e. $\Delta_2 < +\infty$) and μ be a measure on $\mathbb{R}^d$ satisfying Assumption 1. For all regular Borel set A in $\mathbb{R}^d$, with $\mu(A) < +\infty$, the sequence $S_n(A)$ defined in (13) verifies*

$$\frac{1}{n^{\frac{\beta}{2}}} S_n(A) \xrightarrow[n\to\infty]{\mathcal{D}} \sigma\mathcal{N}(0, \mu(A)),$$

where $\sigma^2 = \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(X_0, X_k)$.

It is a direct consequence of Theorem 3.1 and of the following proposition.

Proposition 4.2. *Let μ be a measure on $\mathbb{R}^d$ satisfying Assumption 1. For all regular Borel set A in $\mathbb{R}^d$ with $0 < \mu(A) < +\infty$, set $b_{n,j}(A) = \mu(nA \cap R_j)^{\frac{1}{2}}$, $j \in \mathbb{Z}^d$, and $b_n(A) = (b_{n,j}(A))_{j\in\mathbb{Z}^d}$. Then $b_n(A) \in \ell^2(\mathbb{Z}^d)$ for all $n \in \mathbb{N}$ and*

(i) $\|b_n(A)\|^2 = n^\beta \mu(A)$,

(ii) $\sup_{j\in\mathbb{Z}^d} \frac{b_{n,j}(A)}{\|b_n(A)\|} \xrightarrow[n\to\infty]{} 0$,

(iii) for all $e \in \mathbb{Z}^d$ with $|e| = 1$, $\frac{\|\tau_e b_n(A) - b_n(A)\|}{\|b_n(A)\|} \xrightarrow[n\to\infty]{} 0$.

Proof. Fix a regular Borel set A in $\mathbb{R}^d$ with $0 < \mu(A) < +\infty$. First, observe that by *(1)* of Assumption 1,

$$\|b_n(A)\|^2 = \sum_{j\in\mathbb{Z}^d} \mu(nA \cap R_j) = \mu(nA) = n^\beta \mu(A).$$

This shows that $b_n \in \ell^2(\mathbb{Z}^d)$ and *(i)*.
Further, by condition *(2a)* of Assumption 1, one can find $M > 0$ such that $\sup_{\pi(j)>M} \mu(R_j) < +\infty$ and thus $\sup_{\pi(j)>M} \dfrac{b_{n,j}(A)}{\|b_n(A)\|} \to 0$ as $n \to +\infty$. Moreover,

$$\begin{aligned}\sup_{\pi(j)\leq M} \mu(nA \cap R_j) &\leq \mu\left(nA \cap \left(\bigcup_{\pi(j)\leq M} R_j\right)\right) \\ &\leq n^\beta \mu\left(A \cap \left\{x \in \mathbb{R}^d; \pi(x) \leq n^{-1}(M+1)\right\}\right),\end{aligned}$$

using *(1)* of Assumption 1. Since μ is absolutely continuous with respect to the Lebesgue measure, by the dominated convergence theorem, we obtain $\sup_{\pi(j)\leq M} \dfrac{b_{n,j}(A)}{\|b_n(A)\|} \to 0$ as $n \to +\infty$, which ends to prove *(ii)*.
Let us now check condition *(iii)*. Let $e \in \mathbb{Z}^d$ with $|e| = 1$ be fixed and define the sets:

$$\begin{aligned}V_{1,n}(A) &= \left\{j \in \mathbb{Z}^d : R_j \cup R_{j+e} \subset nA\right\}, \\ V_{2,n}(A) &= \left\{j \in \mathbb{Z}^d : (R_j \cup R_{j+e}) \cap nA \neq \emptyset \text{ and } (R_j \cup R_{j+e}) \cap (nA)^c \neq \emptyset\right\},\end{aligned}$$

where A^c denotes the complement of A in $\mathbb{R}^d$. We have

$$\begin{aligned}\|\tau_e b_n(A) - b_n(A)\|^2 &= \sum_{j\in V_{1,n}(A)} (b_{n,j+e}(A) - b_{n,j}(A))^2 + \sum_{j\in V_{2,n}(A)} (b_{n,j+e}(A) - b_{n,j}(A))^2 \\ &= \qquad (I_n) \qquad + \qquad (II_n) \qquad .\end{aligned}$$

Let us start to deal with the term (II_n). For all $\varepsilon > 0$, we denote the *ε-inside-neighborhood* of ∂A by

$$\tilde{\partial}_\varepsilon A = \left\{x \in \mathbb{R}^d : \inf_{y\in\partial A} |x - y| \leq \varepsilon\sqrt{d}\right\} \cap \overline{A}, \tag{14}$$

where $\sqrt{d}$ comes from the euclidean norm and is here to simplify the notations. We have

$$\begin{aligned}(II_n) &\leq \sum_{j\in V_{2,n}(A)} \mu(nA \cap (R_j \cup R_{j+e})) \\ &= \mu\left(nA \cap \bigcup_{j\in V_{2,n}(A)} R_j\right) + \mu\left(nA \cap \bigcup_{j\in V_{2,n}(A)} R_{j+e}\right) \\ &\leq 2\mu(\tilde{\partial}_2(nA)) \\ &= 2n^\beta \mu(\tilde{\partial}_{\frac{2}{n}} A).\end{aligned}$$

By regularity of A and absolute continuity of μ, we have $\mu(\tilde{\partial}_\varepsilon A) \to 0$ as $\varepsilon \to 0$, using dominated convergence theorem. Thus, $(II_n) = o(n^\beta)$.
On the other hand, using $(\sqrt{a} - \sqrt{b})^2 \leq |a - b|$, we have

$$(I_n) = \sum_{j\in V_{1,n}(A)} (\mu(R_j)^{\frac{1}{2}} - \mu(R_{j+e})^{\frac{1}{2}})^2 \leq \sum_{j\in V_{1,n}(A)} |\mu(R_j) - \mu(R_{j+e})| \tag{15}$$

Let $\varepsilon > 0$. Remark that by *(2b)* of Assumption 1, there exists $M > 0$ such that for $\pi(j) \geq M$,

$$|\mu(R_j) - \mu(R_{j+e})| \leq \varepsilon\mu(R_j)$$

Therefore

$$\sum_{j\in V_{1,n}(A);\pi(j)\geq M} |\mu(R_j) - \mu(R_{j+e})| \leq \varepsilon\mu(nA).$$

Moreover

$$\sum_{j\in V_{1,n}(A);\pi(j)\le M} |\mu(R_j)-\mu(R_{j+e})| \le 2\mu\left(nA\cap\left(\bigcup_{\pi(j)\le M+1} R_j\right)\right)$$
$$\le 2n^\beta \mu\left(A\cap\left\{x\in\mathbb{R}^d;\pi(x)\le n^{-1}(M+2)\right\}\right),$$

using *(1)* of Assumption 1. Since μ is absolutely continuous with respect to the Lebesgue measure, by dominated convergence theorem, one can find n_M such that for all $n\ge n_M$

$$\mu\left(A\cap\left\{x\in\mathbb{R}^d;\pi(x)\le n^{-1}(M+2)\right\}\right)\le\varepsilon.$$

Hence, it is now clear that $(I_n)=o(n^\beta)$ and condition *(iii)* follows. □

In the following we consider the set-indexed process and extend the central limit theorem to a functional central limit theorem (or invariance principle). We first discuss the finite dimensional convergence.

4.2. **Finite dimensional convergence.** Again, let μ be a σ-finite measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ and $(X_j)_{j\in\mathbb{Z}^d}$ be a stationary $\mathbb{R}$-valued random field with $\mathbb{E}(X_0)=0$. For a class $\mathcal{A}$ of regular Borel sets of $\mathbb{R}^d$, we define the $\mathcal{A}$-indexed process

$$S_n(A)=\sum_{j\in\mathbb{Z}^d}\mu(nA\cap R_j)^{\frac12}X_j,\qquad A\in\mathcal{A}.$$

As before, we will use the notation $b_n(A)$ to designate the $\mathbb{Z}^d$-indexed sequence $b_{n,j}(A)=\mu(nA\cap R_j)^{\frac12}, j\in\mathbb{Z}^d$.

We have the following finite dimensional convergence for the normalized set-indexed process $(S_n(A))_{A\in\mathcal{A}}$.

Theorem 4.3. *Let $(X_j)_{j\in\mathbb{Z}^d}$ be a centered 2-stable random field (i.e. $\Delta_2<+\infty$), let μ be a measure on $\mathbb{R}^d$ satisfying Assumption 1 and let $\mathcal{A}$ be a class of regular Borel sets of $\mathbb{R}^d$ with $\mu(A)<+\infty$ for any $A\in\mathcal{A}$. Then*

$$\left(\frac{1}{n^{\beta/2}}S_n(A)\right)_{A\in\mathcal{A}}\xrightarrow[n\to\infty]{fdd}(\sigma W(A))_{A\in\mathcal{A}},$$

where $\xrightarrow{fdd}$ means convergence for finite dimensional distributions, $(W(A))_{A\in\mathcal{A}}$ is a centered Gaussian process with covariances

$$\mathrm{Cov}(W(A),W(B))=\mu(A\cap B),$$

and $\sigma^2=\sum_{k\in\mathbb{Z}^d}\mathrm{Cov}(X_0,X_k)$.

Remark. If the class $\mathcal{A}$ is stable by scalar multiplications, *(1)* of Assumption 1 guarantees that the limit random field $(W(A))_{A\in\mathcal{A}}$ is a self-similar process of order $\beta/2$, namely $(cW(A))_{A\in\mathcal{A}}\overset{f.d.d.}{=}c^{\frac\beta2}(W(A))_{A\in\mathcal{A}}$ for all $c\ge0$.

To prove the theorem we will use the following lemma which is a complement to Proposition 4.2.

Lemma 4.4. *For all regular Borel sets A and B with $\mu(A)<+\infty$ and $\mu(B)<+\infty$, as $n\to\infty$,*

$$\langle b_n(A),b_n(B)\rangle=n^\beta\mu(A\cap B)+o(n^\beta).$$

Proof. We define the sets

$$V_{n,1}=\left\{j\in\mathbb{Z}^d:R_j\subset n(A\cap B)\right\},$$
$$V_{n,2}=\left\{j\in\mathbb{Z}^d:R_j\cap n(A\cap B)\ne\emptyset\right\}\setminus V_{n,1},$$
$$V_{n,3}=\left\{j\in\mathbb{Z}^d:R_j\cap nA\ne\emptyset\text{ and }R_j\cap nB\ne\emptyset\right\}\setminus V_{n,1}.$$

We can write

$$\begin{aligned}\langle b_n(A), b_n(B)\rangle &= \sum_{j\in V_{n,1}} \mu(n(A\cap B)\cap R_j) + \sum_{j\in V_{n,3}} \mu(nA\cap R_j)^{\frac12}\mu(nB\cap R_j)^{\frac12}\\ &= \mu(n(A\cap B)) - \sum_{j\in V_{n,2}} \mu(n(A\cap B)\cap R_j) + \sum_{j\in V_{n,3}} \mu(nA\cap R_j)^{\frac12}\mu(nB\cap R_j)^{\frac12}.\end{aligned}$$

Now, if $j\in V_{n,2}$, the set $n(A\cap B)\cap R_j$ is contained in the 1-inside-neighborhood of $\partial(n(A\cap B))$ as defined in (14). Remark also that $\tilde\partial_1(n(A\cap B)) = n\tilde\partial_{\frac1n}(A\cap B)$. Therefore,

$$\begin{aligned}\sum_{j\in V_{n,2}} \mu(n(A\cap B)\cap R_j) &\le \mu\left(n(A\cap B)\cap \bigcup_{j\in V_{n,2}} R_j\right)\\ &\le \mu(n\tilde\partial_{\frac1n}(A\cap B))\\ &= n^\beta \mu(\tilde\partial_{\frac1n}(A\cap B)) = o(n^\beta).\end{aligned}$$

In the same way, if $j\in V_{n,3}$, $n(A\cup B)\cap R_j$ is contained in the 1-inside-neighborhood of $\partial(nA)$ or in the 1-inside-neighborhood of $\partial(nB)$, and

$$\begin{aligned}\sum_{j\in V_{n,3}} \mu(nA\cap R_j)^{\frac12}\mu(nB\cap R_j)^{\frac12} &\le \mu\left(n(A\cup B)\cap \bigcup_{j\in V_{n,3}} R_j\right)\\ &\le n^\beta\mu(\tilde\partial_{\frac1n}A) + n^\beta\mu(\tilde\partial_{\frac1n}B) = o(n^\beta).\end{aligned}$$

□

Proof of Theorem 4.3. We will use the Cramér-Wold device and for simplicity only consider the case $k=2$. Let A and B be two regular Borel sets of $\mathbb{R}^d$ with $\mu(A)<+\infty$ and $\mu(B)<+\infty$, and fix $\lambda_1,\lambda_2\in\mathbb{R}$. We have

$$\lambda_1 S_n(A) + \lambda_2 S_n(B) = \sum_{j\in\mathbb{Z}^d} c_{n,j} X_j$$

with $c_{n,j} = \lambda_1 b_{n,j}(A) + \lambda_2 b_{n,j}(B)$, $j\in\mathbb{Z}^d$. We can assume $\mu(A)>0$ and $\mu(B)>0$, otherwise the result follows from Theorem 4.1. Denoting $c_n = (c_{n,j})_{j\in\mathbb{Z}^d}$, we can write

$$\|c_n\|^2 = \lambda_1^2\|b_n(A)\|^2 + \lambda_2^2\|b_n(B)\|^2 + 2\lambda_1\lambda_2\langle b_n(A), b_n(B)\rangle.$$

Thus by Proposition 4.2 and Lemma 4.4, we get

$$\frac{\|c_n\|^2}{n^\beta} \xrightarrow[n\to\infty]{} c := \lambda_1^2\mu(A) + \lambda_2^2\mu(B) + 2\lambda_1\lambda_2\mu(A\cap B) \ge 0.$$

When $c=0$, since $\mathrm{Var}(\lambda_1 S_n(A) + \lambda_2 S_n(B)) \le \|c_n\|^2\sigma^2$, the random variables $n^{-\frac\beta2}(\lambda_1 S_n(A) + \lambda_2 S_n(B))$ converge to 0 in L^2.
Now assume $c>0$. We will show that the assumptions of Theorem 3.1 are satisfied for the sequence $c_n = (c_{n,j})_{j\in\mathbb{Z}^d}$. For all $j\in\mathbb{Z}^d$, we have

$$\frac{|c_{n,j}|}{\|c_n\|} \le |\lambda_1|\frac{\|b_n(A)\|}{\|c_n\|}\frac{|b_{n,j}(A)|}{\|b_n(A)\|} + |\lambda_2|\frac{\|b_n(B)\|}{\|c_n\|}\frac{|b_{n,j}(B)|}{\|b_n(B)\|}.$$

Since the sequences $\dfrac{\|b_n(A)\|}{\|c_n\|}$ and $\dfrac{\|b_n(B)\|}{\|c_n\|}$ converge respectively to $\mu(A)^{\frac12}c^{-\frac12}$ and $\mu(B)^{\frac12}c^{-\frac12}$, and so are bounded, Proposition 4.2 shows that condition *(i)* of Theorem 3.1 holds.
Further, since for all $e\in\mathbb{Z}^d$,

$$\frac{\|\tau_e c_n - c_n\|}{\|c_n\|} \le |\lambda_1|\frac{\|\tau_e b_n(A) - b_n(A)\|}{\|c_n\|} + |\lambda_2|\frac{\|\tau_e b_n(B) - b_n(B)\|}{\|c_n\|},$$

the same argument shows that condition *(ii)* also holds. Then Theorem 3.1 shows that

$$n^{-\beta}\,\mathrm{Var}(\lambda_1 S_n(A) + \lambda_2 S_n(B)) \to \sigma^2 c$$

as $n \to +\infty$ and

$$\frac{1}{n^{\frac{\beta}{2}}}(\lambda_1 S_n(A) + \lambda_2 S_n(B)) \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0, \sigma^2 c).$$

We derive the convergence

$$\frac{1}{n^{\frac{\beta}{2}}}(S_n(A), S_n(B)) \xrightarrow[n\to\infty]{\mathcal{D}} \sigma(W(A), W(B)),$$

where the Gaussian process W verifies

$$\begin{aligned}\mathrm{Cov}(W(A), W(B)) &= \lim_{n\to+\infty} \frac{1}{2n^\beta\sigma^2}[\mathrm{Var}(S_n(A) + S_n(B)) - \mathrm{Var}(S_n(A)) - \mathrm{Var}(S_n(B))] \\ &= \frac{1}{2}[\mu(A) + \mu(B) + 2\mu(A \cap B) - \mu(A) - \mu(B)] \\ &= \mu(A \cap B).\end{aligned}$$

□

4.3. **Invariance principle.** We keep the notations of the preceding section. We will establish an invariance principle for the set-indexed process $(S_n(A))_{A\in\mathcal{A}}$ under an entropy condition on the class $\mathcal{A}$.

We equip the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$ with the pseudo-metric ρ defined by $\rho(A, B) = \mu(A \triangle B)^{\frac{1}{2}}$. For a class $\mathcal{A}$ of Borel sets, we define the *covering number* $N(\mathcal{A}, \rho, \varepsilon)$ as the smallest number of ρ-balls of radius ε needed to cover $\mathcal{A}$ and $N(\mathcal{A}, \rho, \varepsilon) = +\infty$ if there is no such finite covering. The entropy $H(\mathcal{A}, \rho, \varepsilon)$ is the logarithm of $N(\mathcal{A}, \rho, \varepsilon)$. We also denote by $\mathcal{C}(\mathcal{A})$ the space of continuous functions on $\mathcal{A}$ equipped with the supremum norm.

Theorem 4.5. *Let μ be a measure on $\mathbb{R}^d$ satisfying Assumption 1 and let $\mathcal{A}$ be a class of regular Borel sets of $\mathbb{R}^d$ such that $\mu(A) < +\infty$ for any $A \in \mathcal{A}$. Assume that one of the two following conditions holds:*

(i) there exists $p \geq 2$ such that the centered random field $(X_j)_{j\in\mathbb{Z}^d}$ is p-stable (i.e. $\Delta_p < +\infty$) and

$$\int_0^1 N(\mathcal{A}, \rho, \varepsilon)^{\frac{1}{p}} d\varepsilon < +\infty; \tag{16}$$

(ii) for all $p \geq 2$, the centered random field $(X_j)_{j\in\mathbb{Z}^d}$ is p-stable with $\sup_{p\geq 2} \Delta_p < +\infty$ and

$$\int_0^1 H(\mathcal{A}, \rho, \varepsilon)^{\frac{1}{2}} d\varepsilon < +\infty; \tag{17}$$

then $\left(n^{-\frac{\beta}{2}} S_n(A)\right)_{A\in\mathcal{A}}$ converges in distribution in $\mathcal{C}(\mathcal{A})$ to $(\sigma W(A))_{A\in\mathcal{A}}$, where $(W(A))_{A\in\mathcal{A}}$ is the centered Gaussian process with covariances

$$\mathrm{Cov}(W(A), W(B)) = \mu(A \cap B),$$

and $\sigma^2 = \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(X_0, X_k)$.

Remark. In Section 5, we present some applications to obtain particular limit processes. To this aim, it is possible to only use condition *(i)* of the theorem. Nevertheless, the entropy condition (17) is weaker and is the standard condition that we can expect in such an invariance principle.

Proof. The finite dimensional convergence of the process $(n^{-\frac{\beta}{2}} S_n(A))_{A\in\mathcal{A}}$ is a direct consequence of Theorem 4.3. To prove the tightness, we will use the following lemma.

Lemma 4.6. *If $(X_j)_{j\in\mathbb{Z}^d}$ is a p-stable random field for some $p \geq 2$, then for all Borel sets A and B,*

$$n^{-\frac{\beta}{2}} \|S_n(A) - S_n(B)\|_p \leq \sqrt{2p}\Delta_p \rho(A, B).$$

Proof. We have $S_n(A) - S_n(B) = \sum_{j\in\mathbb{Z}^d} (b_{n,j}(A) - b_{n,j}(B))X_j$ and by Proposition 2.1,

$$n^{-\frac{\beta}{2}}\|S_n(A) - S_n(B)\|_p \leq \sqrt{2p}\Delta_p n^{-\frac{\beta}{2}}\|b_n(A) - b_n(B)\|.$$

Now,

$$n^{-\beta}\|b_n(A) - b_n(B)\|^2 = \mu(A) + \mu(B) - 2n^{-\beta}\langle b_n(A), b_n(B)\rangle,$$

while

$$\begin{aligned}\langle b_n(A), b_n(B)\rangle &= \sum_{j\in\mathbb{Z}^d} \mu(nA\cap R_j)^{\frac{1}{2}}\mu(nB\cap R_j)^{\frac{1}{2}} \\ &\geq \sum_{j\in\mathbb{Z}^d} \mu(n(A\cap B)\cap R_j) = \mu(n(A\cap B)).\end{aligned}$$

Thus, $n^{-\beta}\|b_n(A) - b_n(B)\|^2 \leq \mu(A) + \mu(B) - 2\mu(A\cap B) = \rho(A,B)^2$ and the proof of the lemma is complete. □

First, assume condition *(i)* holds. Note that (16) implies that $N(\mathcal{A},\rho,\varepsilon) < +\infty$ for all $\varepsilon > 0$, i.e. $(\mathcal{A},\rho)$ is totally bounded. By Theorem 11.6 in Ledoux and Talagrand [21], as a consequence of Lemma 4.6 and condition (16), we obtain that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \geq 1$,

$$\mathbb{E}\left(\sup_{A,B\in\mathcal{A},\,\rho(A,B)<\delta} n^{-\frac{\beta}{2}}|S_n(A) - S_n(B)|\right) < \varepsilon.$$

This last property implies that the process $(n^{-\frac{\beta}{2}}S_n(A))_{A\in\mathcal{A}}$ is tight in $\mathcal{C}(\mathcal{A})$ and completes the proof of the theorem under *(i)*.

Now, assume condition *(ii)* holds. Let us introduce the Young function ψ_2, defined on $\mathbb{R}_+$ by $\psi_2(x) = \exp(x^2) - 1$. Its associated Orlicz space is composed by random variables X such that $\mathbb{E}\psi_2(|X|/a) < +\infty$ for some $a > 0$ and is equipped with the norm $\|X\|_{\psi_2} = \inf\{a > 0 : \mathbb{E}\psi_2(|X|/a) \leq 1\}$. It is well known (see Lemma 4 in [14]) that there exists a positive constant C such that for every random variable X,

$$\|X\|_{\psi_2} \leq C\sup_{p\geq 2}\frac{\|X\|_p}{\sqrt{p}}.$$

Thus, using Lemma 4.6, we obtain

$$n^{-\frac{\beta}{2}}\|S_n(A) - S_n(B)\|_{\psi_2} \leq C\sqrt{2}\sup_{p\geq 2}\Delta_p\rho(A,B). \tag{18}$$

Now, applying Theorem 11.6 in [21], the inequality (18) and the condition (17) show that the process $(n^{-\frac{\beta}{2}}S_n(A))_{A\in\mathcal{A}}$ is tight in $\mathcal{C}(\mathcal{A})$ and completes the proof of the theorem under *(ii)*.

□

To apply Theorem 4.5 to particular classes of Borel sets, we will need an upper bound of the associated covering number (in order to establish condition (16)). In the next section, we introduce the so-called Vapnik-Chervonenkis dimension which provides a useful tool to derive such bounds.

4.4. **Vapnik-Chervonenkis dimension and covering numbers.** Let E be a set and $\mathcal{A}$ a class of subsets of E. If C is a finite subset of E of cardinality k, we denote

$$\mathcal{A}\cap C = \{A\cap C :\ A\in\mathcal{A}\}$$

and we say that $\mathcal{A}$ shatters C if $\mathrm{Card}(\mathcal{A}\cap C) = 2^k$. The class $\mathcal{A}$ has a finite *Vapnik-Chervonenkis dimension* if there exists $k\in\mathbb{N}$ such that no set of cardinality k can be shattered by $\mathcal{A}$. The VC-dimension $V(\mathcal{A})$ of $\mathcal{A}$ is the smallest k with this property.

For example the VC-dimension of the collection of all intervals of the form $(-\infty,t]$ in $\mathbb{R}$ is 2. More generally, the VC-dimension of the collection of all rectangles $(-\infty,t_1]\times\ldots\times(-\infty,t_d]$ in $\mathbb{R}^d$ is $d+1$.

In some situations, a bound of the covering number can be derive from the Vapnik-Chervonenkis dimension:

If $(E, \mathcal{E})$ is a measurable space, μ is a probability measure on $(E, \mathcal{E})$ and $\mathcal{A} \subset \mathcal{E}$ is a class of finite VC-dimension, then for all $0 < \varepsilon < 1$, we have:

$$N(\mathcal{A}, \rho, \varepsilon) \leq KV(\mathcal{A})(4e)^{V(\mathcal{A})} \left(\frac{1}{\varepsilon}\right)^{2(V(\mathcal{A})-1)}, \tag{19}$$

where $\rho(A, B) = \mu(A \triangle B)^{\frac{1}{2}}$ and K is a universal constant (see van der Vaart and Wellner [31] Theorem 2.6.4).

The computation of the VC-dimension of the classes we consider in application of Theorem 4.5 is the object of Section 6.

5. Particular examples

5.1. **Brownian sheet.** Here we consider the case $\mu = \lambda$ the Lebesgue measure on $\mathbb{R}^d$. It is clear that for all $A \in \mathcal{B}(\mathbb{R}^d)$, for all $n \geq 0$, $\lambda(nA) = n^d \lambda(A)$ and since $\lambda(R_j) = 1$ for all $j \in \mathbb{Z}^d$, Assumption 1 holds. Thus by Theorem 4.5, if a class $\mathcal{A}$ of regular Borel sets satisfies (16) for some $p \geq 2$ and if $(X_i)_{i \in \mathbb{Z}^d}$ is a centered p-stable random field then the process

$$\frac{1}{n^{\frac{d}{2}}} S_n(A) = \frac{1}{n^{\frac{d}{2}}} \sum_{j \in \mathbb{Z}^d} \lambda(nA \cap R_j)^{\frac{1}{2}} X_j, \quad A \in \mathcal{A},$$

converges in distribution in $\mathcal{C}(\mathcal{A})$ to the process $(\sigma\mathcal{B}(A))_{A \in \mathcal{A}}$ where $(\mathcal{B}(A))_{A \in \mathcal{A}}$ is the $\mathcal{A}$-indexed standard Brownian motion, i.e. the centered Gaussian process with covariances

$$\mathrm{Cov}(\mathcal{B}(A), \mathcal{B}(B)) = \lambda(A \cap B), \qquad A, B \in \mathcal{A}.$$

The same result has already been proved by El Machkouri et al. [14] with the difference that they used the weights $\lambda(nA \cap R_j)$ instead of $\lambda(nA \cap R_j)^{\frac{1}{2}}$. In fact, in this situation, the number of coefficients $\lambda(nA \cap R_j)$ which are not 0 or 1 is asymptotically negligible compared to $n^{\frac{d}{2}}$. This explains that the limit process remains the same in both cases.

Let us recall that the standard Brownian sheet indexed by $\mathbb{R}^d$ is the centered Gaussian process $(\mathcal{B}(t))_{t \in \mathbb{R}^d}$ with covariances given in (2). Since $(\mathcal{B}(t))_{t \in \mathbb{R}^d} = (\mathcal{B}([0, t]))_{t \in \mathbb{R}^d}$, the Brownian sheet may be obtained as the limit in the invariance principle. Indeed, it is well known that the class $\mathcal{R} = \left\{[0, t],\, t \in [0, 1]^d\right\}$ has the finite VC-dimension $V(\mathcal{R}) = d + 1$. Then, since λ is a probability measure on $[0, 1]^d$, by (19) we get the bound

$$N(\mathcal{R}, \rho, \varepsilon) \leq K\varepsilon^{-2d},$$

where $\rho(A, B) = \lambda(A \cap B)^{\frac{1}{2}}$ and K only depends on d. Thus (16) holds for $p > 2d$, and if $(X_i)_{i \in \mathbb{Z}^d}$ is a centered p-stable random field for such a p, the process $\left(n^{-\frac{d}{2}} S_n([0, t])\right)_{t \in [0,1]^d}$ converges in distribution to $(\sigma\mathcal{B}(t))_{t \in [0,1]^d}$, where $(\mathcal{B}(t))_{t \in [0,1]^d}$ is the standard Brownian sheet.

In dimension $d = 1$, as stated in Theorem 3 of Wu [33], the result remains true under the weaker condition $p = 2$. It is a consequence of Corollary 3 in [11] (see also [17]). It seems that our approach (as in [14]) only allows to get the condition $p > 2$ and thus does not give the optimal result concerning the required moments for the X_i. One can also expect that in higher dimension the condition $p > 2d$ could be weakened. Nevertheless, in the present paper, our main interest is not to get optimal condition on p but to see how one can obtain different limit processes by choosing different measures μ and classes of sets. This is the aim of the following subsections.

5.2. **Lévy Chentsov random fields.** Another generalization of the Brownian motion to multi-dimensionally indexed Brownian motion, introduced by Lévy [22], is the so-called Lévy Brownian field $(W(t))_{t \in \mathbb{R}^d}$ defined as the mean zero Gaussian field with covariances

$$\mathrm{Cov}(W(t), W(s)) = \frac{1}{2}\left(|t| + |s| - |t - s|\right), \quad t, s \in \mathbb{R}^d,$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. Note that this field is self-similar of order $\frac{1}{2}$ with stationary increments in the strong sense, see [27] Section 8.1. Chentsov [7] (see also [27]) gave a construction of the Lévy Brownian field as a set-indexed random process. Let $\mathcal{D}$ be the class of all balls of

diameter $[\![0,t]\!]$, $t\in\mathbb{R}^d$, the Lévy Brownian field can be seen (up to a multiplicative constant) as the set-indexed Gaussian process $(W(A))_{A\in\mathcal{D}}$ with mean zero and covariances $\mathrm{Cov}(W(A),W(B)) = \mu(A\cap B)$, where the control measure is $\mu(dx)=|x|^{1-d}dx$. In this section we show that using weights depending on μ, Theorem 4.5 leads to a Lévy Brownian field as limit.

More generally, we define the Lévy Chentsov measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ as

$$\mu(dx)=\frac{dx}{|x|^{d-2H}} \text{ for some } H\in(0,d/2].$$

Lemma 5.1. *The Lévy Chentsov measure μ satisfies Assumption 1 for $\beta=2H$.*

The proof is postponed to Appendix.

Here we are interested in the class of subsets of $\mathbb{R}^d$ given by

$$D_t=B\left(\frac{t}{2},\frac{|t|}{2}\right)=\left\{x\in\mathbb{R}^d\,:\,\left|x-\frac{t}{2}\right|<\frac{|t|}{2}\right\},\qquad t\in\mathbb{R}^d,$$

i.e. the class of balls of diameter $[\![0,t]\!]$, $t\in\mathbb{R}^d$. This class has the same VC-dimension than the classical class of rectangles (the proof is postponed to Section 6).

Proposition 5.2. *The VC-dimension of the class $\mathcal{D}=\left\{D_t,\,t\in\mathbb{R}^d\right\}$ is $d+1$.*

The set-indexed Gaussian process $(W(D_t))_{t\in\mathbb{R}^d}$ obtained as a limit in Theorem 4.3 has covariances given by

$$\mathrm{Cov}(W(D_t),W(D_s))=\mu(D_t\cap D_s)=\frac{1}{2}(\mu(D_t)+\mu(D_s)-\mu(D_t\triangle D_s)).$$

Remark that in polar coordinates, $D_t=\left\{(r,\theta)\in\mathbb{R}_+\times S^{d-1}\,:\,0<r<\langle\theta,t\rangle\right\}$, where S^{d-1} is the sphere of $\mathbb{R}^d$ ($S^0=\{-1,1\}$), and that $\mu(dx)=r^{2H-1}drd\theta$. So,

$$\mu(D_t)=\int_{S^{d-1}}\int_{\mathbb{R}_+}1_{\{r<\langle\theta,t\rangle\}}r^{2H-1}drd\theta=\frac{1}{4H}\int_{S^{d-1}}|\langle\theta,t\rangle|^{2H}d\theta.$$

Therefore, using rotation invariance, one has $\mu(D_t)=C_{H,d}^2|t|^{2H}$ with $C_{H,d}^2=\frac{1}{4H}\int_{S^{d-1}}|\langle\theta,e\rangle|^{2H}d\theta$ for any fixed $e\in S^{d-1}$. Note that, when $d=1$, we have $C_{H,1}^2=\frac{1}{2H}$. One can define the standard centered Gaussian random field $(W_H(t))_{t\in\mathbb{R}^d}$ with covariances

$$\mathrm{Cov}(W_H(t),W_H(s))=\frac{1}{2}\left(|t|^{2H}+|s|^{2H}-C_{H,d}^{-2}\mu(D_t\triangle D_s)\right),$$

such that $(W(D_t))_{t\in\mathbb{R}^d}=(C_{H,d}W_H(t))_{t\in\mathbb{R}^d}$ in distribution.

Note that, when $d=1$, one simply computes

$$\mathrm{Cov}(W_H(t),W_H(s))=C_{H,1}^{-2}\mu(D_t\cap D_s)=2^{-2H}\left(|t|+|s|-|t-s|\right)^{2H}.$$

When $d\geq 2$, for $s,t\in\mathbb{R}^d$, we have

$$\begin{aligned}\mu(D_t\cap D_s^c) &= \int_{S^{d-1}}\int_0^{+\infty}1_{\langle\theta,s\rangle\leq r<\langle\theta,t\rangle}r^{-1+2H}drd\theta\\ &= \frac{1}{2H}\left(\int_{0<\langle\theta,s\rangle<\langle\theta,t\rangle}|\langle\theta,t\rangle|^{2H}-|\langle\theta,s\rangle|^{2H}d\theta+\int_{\langle\theta,s\rangle<0<\langle\theta,t\rangle}|\langle\theta,t\rangle|^{2H}d\theta\right).\end{aligned}$$

Similarly, by a change of variables,

$$\mu(D_s\cap D_t^c)=\frac{1}{2H}\left(\int_{\langle\theta,s\rangle<\langle\theta,t\rangle<0}|\langle\theta,s\rangle|^{2H}-|\langle\theta,t\rangle|^{2H}d\theta+\int_{\langle\theta,s\rangle<0<\langle\theta,t\rangle}|\langle\theta,s\rangle|^{2H}d\theta\right).$$

As a consequence, when $H=\frac{1}{2}$ we have the explicit formula, for t, $s\in\mathbb{R}^d$,

$$\rho(D_t,D_s)^2=\mu(D_t\triangle D_s)=\mu(D_{t-s})=C_{\frac{1}{2},d}^2|t-s|$$

Thus, we see that $(W_{\frac{1}{2}}(t))_{t\in\mathbb{R}^d}$ is the Lévy Brownian field $(W(t))_{t\in\mathbb{R}^d}$.

Now, let us restrict the class to the sets indexed by the unit ball $B^d = \left\{t \in \mathbb{R}^d : |t| \leq 1\right\}$ and set

$$\mathcal{D} = \left\{D_t,\, t \in B^d\right\}. \tag{20}$$

In this case one can find a constant $C > 0$ such that for all $t,\, s \in B^d$,

$$\rho(D_t, D_s)^2 = \mu(D_t \triangle D_s) \leq C^2 |t-s|^{\min(2H,1)}.$$

We infer that the ε-covering number of the class $\mathcal{D}$ with respect to the pseudo-metric ρ is bounded by the $(C^{-1}\varepsilon)^{-1/\min\{H,\frac{1}{2}\}}$-covering number of B^d with respect to the Euclidean norm. Thus,

$$N(\mathcal{D}, \rho, \varepsilon) \leq N\left(B^d, |\cdot|, (C^{-1}\varepsilon)^{-1/\min\{H,\frac{1}{2}\}}\right) = \mathcal{O}\left(\varepsilon^{-d/\min\{H,\frac{1}{2}\}}\right)$$

which leads to the condition $p > \max\{\frac{d}{H}, 2d\}$ in application of Theorem 4.5.

As mentioned in Section 4.4, another way to bound the covering number is to use the VC-dimension of the class $\mathcal{D}$. It is clear that the VC-dimension of the class $\mathcal{D} = \left\{D_t,\, t \in B^d\right\}$ remains the same than the one of $\mathcal{D} = \left\{D_t,\, t \in \mathbb{R}^d\right\}$. Thus $V(\mathcal{D}) = d+1$. Since $\mu(B^d)$ is finite we can normalize μ to get a probability measure. Using (19) and $V(\mathcal{D}) = d+1$, we obtain

$$N(\mathcal{D}, \rho, \varepsilon) \leq K\left(\frac{1}{\varepsilon}\right)^{2d},$$

where K is a constant which depends only on d and H. We thus obtain the condition $p > 2d$ which is independent of H and is better than the previous one in the case $H < 1/2$.

To summarize, in this setting, Theorem 4.5 becomes:

Corollary 5.3. *Let $H \in (0, \frac{d}{2}]$, $\mu(dx) = \frac{dx}{|x|^{d-2H}}$ and $\mathcal{D}$ be the class defined in (20). Assume that $(X_j)_{j\in\mathbb{Z}^d}$ is a centered p-stable random field for some $p > 2d$ and set $S_n(t) = S_n(D_t) = \sum_{j\in\mathbb{Z}^d} \mu(nD_t \cap R_j)^{\frac{1}{2}} X_j$. Then the random field $\left(n^{-H} S_n(t)\right)_{t\in B^d}$ converges in distribution in $\mathcal{C}(B^d)$ to the random field $(\sigma W_H(t))_{t\in\mathcal{B}^d}$ where $\sigma^2 = C_{H,d}^2 \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}(X_0, X_k)$. In particular, for $H = \frac{1}{2}$, $(W_{\frac{1}{2}}(t))_{t\in\mathcal{B}^d}$ is the Lévy Brownian field.*

Let us remark that the random field $(W_H(t))_{t\in\mathbb{R}^d}$ is H-self-similar, but for $H \neq 1/2$ it does not have stationary increments.

5.3. **Takenaka random fields.** For $H \in (0,1)$, the only H-self-similar Gaussian random field with stationary increments in the strong sense is the Lévy fractional Brownian field $(B_H(t))_{t\in\mathbb{R}^d}$ defined as the centered Gaussian field with covariances

$$\mathrm{Cov}(B_H(t), B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),$$

see [27] Section 8. The Chentsov representation for such field is only possible when $H \leq 1/2$ and was given by Takenaka [29] in the case $H < 1/2$.

We consider the Takenaka measure defined on $\mathbb{R}^d \times \mathbb{R}$ (identified with $\mathbb{R}^{d+1}$) by

$$\mu(dx, dr) = r^{2H-d-1} 1_{r>0} dx dr \text{ for } H \in (0, 1/2).$$

Lemma 5.4. *The Takenaka measure satisfies Assumption 1 on $\mathbb{R}^{d+1}$ with $\beta = 2H$.*

The proof is postponed to Appendix.

We now define a class of cones in $\mathbb{R}^d \times \mathbb{R}_+$. For $t \in \mathbb{R}^d$, set

$$\mathcal{C}_t = \left\{(x,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |x-t| \leq r\right\}$$

and $\mathcal{C} = \left\{\mathcal{C}_t,\, t \in \mathbb{R}^d\right\}$. Note that for each $t \in \mathbb{R}^d$, we have $\mu(\mathcal{C}_t) = +\infty$ but $\mu(\mathcal{C}_t \triangle \mathcal{C}_0) < +\infty$. We set $V_t = \mathcal{C}_t \triangle \mathcal{C}_0$ and define the class

$$\mathcal{V} = \left\{V_t,\, t \in \mathbb{R}^d\right\}.$$

In Section 6 we show:

Proposition 5.5. *The VC-dimension of $\mathcal{C}$ is $d+2$ and the VC-dimension of $\mathcal{V}$ is also $d+2$.*

Computations (see [27] Lemma 8.4.2) show that there exists a positive constant $c_{H,d}$ such that $\mu(V_t) = c_{H,d}^2 |t|^{2H}$ and

$$\mu(V_t \triangle V_s) = \mu(\mathcal{C}_t \triangle \mathcal{C}_s) = \mu(V_{t-s}) = c_{H,d}^2 |t-s|^{2H}.$$

Therefore the set-indexed Gaussian process $(W(V_t))_{t\in\mathbb{R}^d}$ obtained as a limit in Theorem 4.3 is equal in distribution to $(c_{H,d} B_H(t))_{t\in\mathbb{R}^d}$.

Now, let us restrict the class $\mathcal{V}$ to

$$\mathcal{V} = \left\{V_t,\, t \in B^d\right\} \tag{21}$$

where B^d is the unit ball of $\mathbb{R}^d$. Since $\rho(V_t, V_s) = c_{H,d}|t-s|^H$, we get

$$N(\mathcal{V}, \rho, \varepsilon) \le N\left(B^d, |\cdot|, (c_{H,d}^{-1}\varepsilon)^{-1/H}\right) = \mathcal{O}\left(\varepsilon^{-d/H}\right),$$

which gives $p > d/H$ in Theorem 4.5. Contrarily to the preceding section, we are not able to improve this bound by using the VC-dimension of the class. Actually, we have $\mu\left(\cup_{t\in B^d} V_t\right) = +\infty$ and we cannot use (19).

To summarize, in this setting, applying Theorem 4.5 with $d+1$ instead of d we get:

Corollary 5.6. *Let $H \in (0, \frac{1}{2})$, $\mu(dx, dr) = r^{2H-d-1} 1_{r>0} dx dr$ and $\mathcal{V}$ be the class defined in* (21)*. Assume that $(X_j)_{j\in\mathbb{Z}^{d+1}}$ is a centered p-stable random field for some $p > d/H$ and set $S_n(t) = S_n(V_t) = \sum_{j\in\mathbb{Z}^{d+1}} \mu(nV_t \cap R_j)^{\frac{1}{2}} X_j$, $t \in B^d$. Then the random field $\left(n^{-H} S_n(t)\right)_{t\in B^d}$ converges in distribution in $\mathcal{C}(B^d)$ to the random field $(\sigma B_H(t))_{t\in\mathcal{B}^d}$ where $(B_H(t))_{t\in\mathcal{B}^d}$ is the H-Lévy fractional Brownian field and $\sigma^2 = c_{H,d}^2 \sum_{k\in\mathbb{Z}^{d+1}} \mathrm{Cov}(X_0, X_k)$.*

5.4. **Fractional Brownian sheets.** In [32], Wang obtain an invariance principle for fractional Brownian sheet, considering sums indexed by rectangle and a stationary random field obtained as the convolution of a filter and a p-stable centered random field. Let us recall (see [2, 20]) that for $\mathbf{H} = (H_1, \ldots, H_d) \in (0,1)^d$ the fractional Brownian sheet $(\mathcal{B}_{\mathbf{H}}(t))_{t\in\mathbb{R}^d}$ is defined as a centered Gaussian random field with covariances

$$\mathrm{Cov}(\mathcal{B}_{\mathbf{H}}(t), \mathcal{B}_{\mathbf{H}}(s)) = \frac{1}{2^d} \prod_{i=1}^d \left(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}\right).$$

Now, for $i = 1, \ldots, d$, let us consider the Takenaka measure defined on $\mathbb{R}\times\mathbb{R}$ (identified with $\mathbb{R}^2$) by

$$\mu_{H_i}(dx_i, dr_i) = \frac{1}{2} r_i^{2H_i - 2} 1_{r_i>0} dx_i dr_i \text{ for } H_i \in (0, 1/2),$$

and $V_{t_i} = \mathcal{C}_{t_i} \triangle \mathcal{C}_0$, where $\mathcal{C}_{t_i} = \{(x,r) \in \mathbb{R}\times\mathbb{R}_+ : |x-t| \le r\}$, for $t_i \in \mathbb{R}$, and remark that for all $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$,

$$\mathrm{Cov}(\mathcal{B}_{\mathbf{H}}(t), \mathcal{B}_{\mathbf{H}}(s)) = \mu(V_t \cap V_s),$$

with $\mu = \mu_{H_1} \otimes \ldots \otimes \mu_{H_d}$ a measure on $\mathbb{R}^{2d}$ and $V_t = V_{t_1} \times \ldots \times V_{t_d} \subset \mathbb{R}^{2d}$. Now it is clear that μ satisfies Assumption 1 on $\mathbb{R}^{2d}$ with $\beta = 2\sum_{i=1}^d H_i$. Then, restricting to the class of set $\mathcal{V} = \left\{V_t,\, t \in [0,1]^d\right\}$, according to Lemma 8 of [3], for any $t, s \in [0,1]^d$ one can find $c > 0$ such that

$$\rho(V_t, V_s)^2 = \mu(V_t \triangle V_s) = \mathrm{Var}(\mathcal{B}_{\mathbf{H}}(t) - \mathcal{B}_{\mathbf{H}}(s)) \le c \sum_{i=1}^d |t_i - s_i|^{2H_i}.$$

It follows that with $Q = \sum_{i=1}^d \frac{1}{H_i}$,

$$N(\mathcal{V}, \rho, \varepsilon) = \mathcal{O}(\varepsilon^{-Q}),$$

which gives the condition $p > Q$ in Theorem 4.5 to get the invariance principle.

Another self-similar sheet may be obtained by considering, for the index $\mathbf{H} = (H_1, \ldots, H_d) \in (0, 1/2]^d$, the measure $\mu_{\mathbf{H}}$ defined on $\mathbb{R}^d$ as the product of Lévy Chentsov measures

$$\mu_{\mathbf{H}}(dx) = \bigotimes_{i=1}^{d} |x_i|^{2H_i - 1} dx_i,$$

which still satisfies Assumption 1 on $\mathbb{R}^d$ with $\beta = 2\sum_{i=1}^{d} H_i$. Theorem 4.3 shows that, if $(X_i)_{i \in \mathbb{Z}^d}$ is a centered 2-stable random field, then the finite dimensional convergence holds

$$\left(n^{-\frac{\beta}{2}} S_n([0,t])\right)_{t \in \mathbb{R}^d} \xrightarrow[n \to \infty]{fdd} (\sigma \mathcal{W}_{\mathbf{H}}(t))_{t \in \mathbb{R}^d},$$

where $\sigma^2 = \prod_{i=1}^{d} 2H_i \sum_{k \in \mathbb{Z}^d} \mathrm{Cov}(X_0, X_k)$ and $(\mathcal{W}_{\mathbf{H}}(t))_{t \in \mathbb{R}^d}$ is a self-similar Brownian sheet with covariances

$$\begin{aligned} \mathrm{Cov}(\mathcal{W}_{\mathbf{H}}(t), \mathcal{W}_{\mathbf{H}}(s)) &= \prod_{i=1}^{d} \frac{1}{2^{H_i}} (|t_i| + |s_i| - |t_i - s_i|)^{2H_i} \\ &= \prod_{i=1}^{d} \min(|t_i|, |s_i|)^{2H_i} 1_{\{t_i s_i \geq 0\}}, \end{aligned}$$

for all $t = (t_1, \ldots, t_d)$ and $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$. In particular, this random field also generalizes Brownian sheet obtained for $H_1 = \ldots = H_d = 1/2$ since in this case the measure $\mu_{\mathbf{H}}$ is the Lebesgue measure. This generalization is clearly different from the fractional Brownian sheet defined above. It is also different from the set-indexed fractional Brownian motion introduced by Herbin and Merzbach [19].

Further, restricting to the class of sets to $\mathcal{R} = \left\{[0,t],\, t \in [0,1]^d\right\}$, if $(X_i)_{i \in \mathbb{Z}^d}$ is a centered p-stable random field with $p > Q$, where $Q = \sum_{i=1}^{d} \frac{1}{H_i}$ as previously, Theorem 4.5 gives that the random field $\left(n^{-\frac{\beta}{2}} S_n([0,t])\right)_{t \in [0,1]^d}$ converges in distribution in $\mathcal{C}([0,1]^d)$ to the random field $(\sigma \mathcal{W}_{\mathbf{H}}(t))_{t \in [0,1]^d}$.

6. Vapnik-Chervonenkis dimension computations

This section is devoted to the computation of the VC-dimension of the class of balls introduced in Section 5.2 and of the class of cones introduced in Section 5.3. We thus give the proofs of Proposition 5.2 and Proposition 5.5. To do these computations, we will use the following general result, see Dudley [13] Theorem 7.2.

Theorem (Dudley, 1978). *If G is an m-dimensional vector space of real functions on a set E and $\mathcal{G} = \{\{x \in E : g(x) \geq 0\}, g \in G\}$, then $V(\mathcal{G}) = m + 1$.*

Proof of Proposition 5.2.
1) First consider the class $\mathcal{H}$ of closed half-spaces delimited by hyperplanes which contain zero. For each $x \in \mathbb{R}^d$, we define

$$H_x = \left\{y \in \mathbb{R}^d : \langle x, y \rangle \geq 0\right\}$$

and $\mathcal{H} = \left\{H_x,\, x \in \mathbb{R}^d\right\}$. By Dudley's theorem, it is easy to see that $V(\mathcal{H}) = d + 1$.
2) Recall that here, $\mathcal{D}$ is the class of balls D_t of diameter $[\![0,t]\!]$. Let S be a finite set of points in $\mathbb{R}^d$. We shall see that S is shattered by $\mathcal{D}$ if and only if it is shattered by $\mathcal{H}$. Together with the preceding result, this proves the proposition.
2.1) Let assume that $\mathcal{H}$ shatters S, fix an arbitrary subset B of S and write $C = S \setminus B$. First, remark that 0 does not belong to S, otherwise the empty set cannot be obtained as the trace of an element of $\mathcal{H}$. Now, since $\mathcal{H}$ shatters S, there exists $x \in \mathbb{R}^d$ such that for all $y \in B$, $\langle x, y \rangle < 0$ and for all $z \in C$, $\langle x, z \rangle \geq 0$. It is then clear that there exists a ball D_t which contains B and such that $D_t \setminus \{0\}$ is contained in the open half-space $\{y : \langle x, y \rangle < 0\}$ (take $t = -ax$ with $a > 0$ big

enough). Thus points of C are not in D_t (recall that C does not contain 0). Therefore $\mathcal{D}$ shatters S.
2.2) Let assume that $\mathcal{D}$ shatters S and fix an arbitrary subset B of S. Again write $C = S \setminus B$. There exists t and s such that $D_t \cap S = B$ and $D_s \cap S = C$. As a consequence $B \subset D_t \setminus D_s$ and $C \subset D_s \setminus D_t$. But there exists a hyperplane containing 0 that separates $D_t \setminus D_s$ and $D_s \setminus D_t$ and so separates B and C. The set B being arbitrary, we showed that $\mathcal{H}$ shatters S. □

Proof of Proposition 5.5.
1) In $\mathbb{R}^d \times \mathbb{R}$ we consider the class $\mathcal{H}$ of closed half-spaces delimited by hyperplanes containing the last direction,

$$H_{x,a} = \left\{(y,r) \in \mathbb{R}^d \times \mathbb{R} : \langle x, y\rangle - a \geq 0\right\}, \qquad x \in \mathbb{R}^d, a \in \mathbb{R}.$$

Writing $g_{x,a}(y) = \langle x, y\rangle - a$, since $\left\{g_{x,a} : x \in \mathbb{R}^d, a \in \mathbb{R}\right\}$ is a vector space of dimension $d+1$, Dudley's theorem shows that $V(\mathcal{H}) = d+2$.
2) Recall that $\mathcal{C}$ is the class of cones $\mathcal{C}_t = \left\{(x,r) \in \mathbb{R}^d \times \mathbb{R}_+ : |x-t| \leq r\right\}$. Assume that there exist a set S of $d+2$ points which is shattered by $\mathcal{C}$. Then for all subsets B and C of S such that $B \cap C = \emptyset$ and $B \cup C = S$, there exist t and s such that $\mathcal{C}_t \cap S = B$ and $\mathcal{C}_s \cap S = C$. Thus $B \subset \mathcal{C}_t \setminus \mathcal{C}_s$ and $C \subset \mathcal{C}_s \setminus \mathcal{C}_t$. But $\mathcal{C}_t \setminus \mathcal{C}_s$ and $\mathcal{C}_s \setminus \mathcal{C}_t$ can be separated by a hyperplane containing the last direction. This shows that $\mathcal{H}$ shatters S which is a contradiction because $V(\mathcal{H}) = d+2$. Hence $V(\mathcal{C}) \leq d+2$.
3) To show that $V(\mathcal{C}) \geq d+2$ we have to find a set S of cardinality $d+1$ which is shattered by $\mathcal{C}$. We denote by e_i the i-th vector of the canonical basis of $\mathbb{R}^d$, $i = 1, \ldots, d$, and by $e_0 = 0$ the null vector in $\mathbb{R}^d$. We will show that the subset $S = \left\{(e_0, d^{-\frac{1}{2}}), (e_1, 1), \ldots, (e_d, 1)\right\}$ of $\mathbb{R}^d \times \mathbb{R}_+$ is shattered by $\mathcal{C}$. First, remark that S is contained in $\mathcal{C}_0$ and that the empty set can be obtained as the trace of $\mathcal{C}_{(-1,\ldots,-1)}$. For $t = -(\frac{1}{d}, \ldots, \frac{1}{d})$, since $|e_0 - t| = |t| = \frac{1}{\sqrt{d}}$ and $|e_i - t|^2 = 1 + \frac{3}{d}$, we have $\mathcal{C}_t \cap S = \{(e_0, d^{-\frac{1}{2}})\}$. Now for each $k \in \{1, \ldots, d\}$ and each choice of distinct $i_1, \ldots, i_k \in \{1, \ldots, d\}$ set $s = s(\alpha, i_1, \ldots, i_k) = \alpha \sum_{j=1}^{k} e_{i_j}$, with $\alpha > 0$. Then $|s|^2 = k\alpha^2$, if $i \in \{i_1, \ldots, i_k\}$, $|e_i - s|^2 = k\alpha^2 - 2\alpha + 1$, and if $i \notin \{0, i_1, \ldots, i_k\}$, $|e_i - s|^2 = k\alpha^2 + 1 > 1$. So, we can see that for $\alpha = \frac{2}{k}$, $\mathcal{C}_s \cap S = \{(e_{i_1}, 1), \ldots, (e_{i_k}, 1)\}$ and for $\alpha = \frac{1}{\sqrt{d}\sqrt{k}}$, $\mathcal{C}_s \cap S = \left\{(e_0, d^{-\frac{1}{2}}), (e_{i_1}, 1), \ldots, (e_{i_k}, 1)\right\}$.
4) We get the VC-dimension of the class $\left\{\mathcal{C}_t \triangle \mathcal{C}_0,\, t \in \mathbb{R}^d\right\}$ as a consequence of Proposition 6.1 which follows. □

Proposition 6.1. *Let E be a set and $\mathcal{A}$ a class of subset of E with finite VC-dimension. Let A_0 be a subset of E and set*

$$\mathcal{A}_0 = \{A \triangle A_0,\, A \in \mathcal{A}\}.$$

Then $V(\mathcal{A}_0) = V(\mathcal{A})$.

Proof. 1) We will show $V(\mathcal{A}_0) \geq V(\mathcal{A})$. Let S be a set of cardinality $V(\mathcal{A}) - 1$ which is shattered by $\mathcal{A}$. Fix $B \subset S$ and $C = S \setminus B$. Denote $B_0 = B \cap A_0$ and $C_0 = C \cap A_0$. Since $\mathcal{A}$ shatters S, there exist A such that

$$D := (B \setminus B_0) \cup C_0 \subset A \text{ and } A \cap (S \setminus D) = \emptyset.$$

Then $B \setminus B_0 \subset A \setminus A_0$ and $B_0 \subset A_0 \setminus A$. Thus $B \subset A \triangle A_0$. On the other hand, $C \setminus C_0 \subset X \setminus (A \cup A_0)$ and $C_0 \subset A \cap A_0$. Thus $C \cap (A \triangle A_0) = \emptyset$. Therefore $\mathcal{A}_0$ shatters S.
2) We will show $V(\mathcal{A}_0) \leq V(\mathcal{A})$. Let S be a set of cardinality $V(\mathcal{A})$, so it cannot be shattered by $\mathcal{A}$. Assume that $\mathcal{A}_0$ shatters S and fix arbitrary subsets $B \subset S$ and $C = S \setminus B$. Again set $B_0 = B \cap A_0$ and $C_0 = C \cap A_0$. By assumption, there exist $A \in \mathcal{A}$ such that

$$D := (B \setminus B_0) \cup C_0 \subset A \triangle A_0 \text{ and } (A \triangle A_0) \cap (S \setminus D) = \emptyset$$

and we can see that $B \subset A$ and $A \cap C = \emptyset$. The set B being arbitrary, we infer that $\mathcal{A}$ shatters S, a contradiction. □

Appendix

Proof of Lemma 5.1.

Consider the Lévy Chentsov measure $\mu(dx) = \frac{dx}{|x|^{d-2H}}$ for some $H \in (0, d/2]$.

1. For each Borel set A in $\mathbb{R}^d$ and for all $n \geq 0$, we have $\mu(nA) = n^{2H}\mu(A)$. Indeed,

$$\mu(nA) = \int 1_{nA}(x)\frac{dx}{|x|^{d-2H}} = \int 1_A(y)n^d\frac{dy}{|ny|^{d-2H}} = n^{2H}\mu(A).$$

Hence *(1)* of Assumption 1 is satisfied with $\beta = 2H$.

2. (a) Note that for any $j \in \mathbb{Z}^d$ with $|j| > 2$ we have $\mu(R_j) < \lambda(R_j) = 1$ since $H < d/2$ which proves *(2a)* of Assumption 1.

(b) Let $e \in \mathbb{Z}^d$ with $|e| = 1$.

$$\begin{aligned}\mu(R_j) - \mu(R_{j+e}) &= \int_{R_j}\left(\frac{1}{|x|^{d-2H}} - \frac{1}{|x+e|^{d-2H}}\right)dx \\ &= \int_{R_j}\frac{1}{|x|^{d-2H}}\left(1 - \left(\frac{|x|}{|x+e|}\right)^{d-2H}\right)dx\end{aligned}$$

Now, as $|x|$ goes to infinity, we have

$$1 - \left(\frac{|x|}{|x+e|}\right)^{d-2H} = \frac{d-2H}{|x+e|}\left\langle\frac{x+e}{|x+e|}, e\right\rangle + o\left(\frac{1}{|x+e|}\right).$$

Since by Cauchy-Schwarz inequality, $|\langle\frac{x+e}{|x+e|}, e\rangle| \leq 1$, for all $\varepsilon > 0$ there exists $M > 0$ such that for $|j| \geq M$ and $x \in R_j$, we have

$$\left|1 - \left(\frac{|x|}{|x+e|}\right)^{d-2H}\right| \leq \frac{d-2H+\varepsilon}{|x+e|}.$$

Fix such an $\varepsilon > 0$ and so $M > 0$. For all $j \in \mathbb{Z}^d$, with $|j| \geq M$, we obtain

$$|\mu(R_j) - \mu(R_{j+e})| \leq (d-2H+\varepsilon)\int_{R_j}\frac{1}{|x+e||x|^{d-2H}}dx$$

It follows that μ satisfies *(2b)* of Assumption 1 since $\mu(R_j) = \int_{R_j}\frac{1}{|x|^{d-2H}}dx$. □

Proof of Lemma 5.4.

Consider the Takenaka measure $\mu(dx, dr) = r^{2H-d-1}1_{r>0}dxdr$ for $H \in (0, 1/2)$.

1. For each Borel set A in $\mathbb{R}^{d+1}$ and for all $n \geq 0$, we have $\mu(nA) = n^{2H}\mu(A)$. Indeed,

$$\begin{aligned}\mu(nA) &= \int 1_{nA}(x, r)r^{2H-d-1}1_{r>0}dxdr \\ &= \int 1_A(y, s)n^{d+1}(ns)^{2H-d-1}dydr = n^{2H}\mu(A).\end{aligned}$$

2. Note that we may have $\mu(R_j) = +\infty$ for some $j \in \mathbb{Z}^{d+1}$. However, let $e_{d+1} = (0, \ldots, 0, 1)$.

(a) If $j \in \mathbb{Z}^{d+1}$ is such that $|\langle j, e_{d+1}\rangle| > 1$, one has $\mu(R_j) \leq \lambda(R_j) = 1$, which proves *(2a)* of Assumption 1.

(b) Let $j \in \mathbb{Z}^{d+1}$ with $|\langle j, e_{d+1}\rangle| > 2$. Then, for all $e \in \mathbb{Z}^{d+1}$ with $\langle e, e_{d+1}\rangle = 0$, one has $\mu(R_{j+e}) = \mu(R_j)$. Therefore it only remains to consider the case $e = \pm e_{d+1}$. As previously we can write

$$\mu(R_{j\pm e_{d+1}}) - \mu(R_j) = \int_{R_j}\left(1 - (1 \pm r^{-1})^{d+1-2H}\right)r^{2H-d-1}1_{r>0}dxdr.$$

For $\varepsilon > 0$ one can find $M > 0$ such that $|1 - (1 \pm r^{-1})^{d+1-2H}| \leq \varepsilon$ for any $(x, r) \in R_j$ with $|\langle j, e_{d+1}\rangle| > M$ and therefore

$$\left|\mu(R_{j\pm e_{d+1}}) - \mu(R_j)\right| \leq \varepsilon\mu(R_j).$$

□

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