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On closed subgroups of the group of homeomorphisms of a manifold

Frédéric Le Roux

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Abstract

Let $M$ be a triangulable compact manifold. We prove that, among closed subgroups of $\text{Homeo}_0(M)$ (the identity component of the group of homeomorphisms of $M$), the subgroup consisting of volume preserving elements is maximal.

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1 Introduction

The theory of groups acting on the circle is very rich (see in particular the monographs [Ghy01, Nav07]). The theory is far less developed in higher dimension, where it seems difficult to discover more than some isolated islands in a sea of chaos. In this note, we are interested in the closed subgroups of the group $\text{Homeo}_0(M)$, the identity component of the group of homeomorphisms of some compact topological $n$-dimensional manifold $M$. We will show that, when $n \geq 2$, for any good (nonatomic and with total support) probability measure $\mu$, the subgroup of elements that preserve $\mu$ is maximal among closed subgroups.

Let us recall some related results in the case when $M$ is the circle. De La Harpe conjectured that $\text{PSL}(2, \mathbb{R})$ is a maximal closed subgroup ([Bes]). Ghys proposed a list of closed groups acting transitively, asking whether, up to conjugacy, the list was complete ([Ghy01]); the list consists in the whole group, $\text{SO}(2)$, $\text{PSL}(2, \mathbb{R})$, the group $\text{Homeo}_{k,0}(\mathbb{S}^1)$ of elements that commutes with some rotation of order $k$, and the group $\text{PSL}_k(2, \mathbb{R})$ which is defined analogously. The first conjecture was solved by Giblin and Markovic in [GM06]. These authors also answered Ghys’s question affirmatively, under the additional hypothesis that the group contains some non trivial arcwise connected component. Thinking of the two-sphere with these results in mind, one is naturally led to the following questions.

Question 1. Let $G$ be a proper closed subgroup of $\text{Homeo}_0(\mathbb{S}^2)$ acting transitively. Assume that $G$ is not a (finite dimensional) Lie group. Is $G$ conjugate to one of the two subgroups: (1) the centralizer of the antipodal map $x \mapsto -x$, (2) the subgroup of area-preserving elements?

Note that the centralizer of the antipodal map is the group of lifts of homeomorphisms of the projective plane; it is the spherical analog of the groups $\text{Homeo}_{k,0}(\mathbb{S}^1)$. 

Question 2. Is $\text{PSL}(2, \mathbb{C})$ maximal among closed subgroups of $\text{Homeo}_0(S^2)$?

On the circle the group of measure-preserving elements coincides with $\text{SO}(2)$. It is not a maximal closed subgroup since it is included in $\text{PSL}(2, \mathbb{R})$. In contrast, we propose to prove that the closed subgroup of area-preserving homeomorphisms of the two-sphere is maximal. To put this into a general context, let $M$ be a compact topological manifold whose dimension is greater or equal to 2. We assume that $M$ is triangulable and (for simplicity) without boundary. Let us equip $M$ with a probability measure $\mu$ which is assumed to be good: this means that every finite set has measure zero, and every non-empty open set has positive measure. We consider the group $\text{Homeo}_0(M)$ of homeomorphisms of $M$ that are isotopic to the identity, and the subgroup $\text{Homeo}_0(M, \mu)$ of elements that preserve the measure $\mu$. According to the famous Oxtoby-Ulam theorem ([OU41, GP75], see also [Fat80]), if $\mu'$ is another good probability measure on $M$ then it is homeomorphic to $\mu$, meaning that there exists an element $h \in \text{Homeo}_0(M)$ such that $h_*\mu = \mu'$. In particular the subgroup $\text{Homeo}_0(M, \mu')$ is isomorphic to $\text{Homeo}_0(M, \mu)$. We equip these transformation groups with the topology of uniform convergence, which turns them into topological groups. The subgroup $\text{Homeo}_0(M, \mu)$ is easily seen to be closed in $\text{Homeo}_0(M)$. Note that according to Fathi’s theorem (first theorem in [Fat80]), $\text{Homeo}_0(M, \mu)$ coincides with the identity component in the group of measure preserving homeomorphisms. The aim of the present note is to prove the following.

Theorem. The group $\text{Homeo}_0(M, \mu)$ is maximal among closed subgroups of the group $\text{Homeo}_0(M)$.

In what follows we consider some element $f \in \text{Homeo}_0(M)$ that does not preserves the measure $\mu$, and we denote by $G_f$ the subgroup of $\text{Homeo}_0(M)$ generated by

$$\{f\} \cup \text{Homeo}_0(M, \mu).$$

Our aim is to show that the group $G_f$ is dense in $\text{Homeo}_0(M)$.

2 Localization

In this section we show how to find some element in $G_f$ that has small support and contracts the volume of some given ball.

Good balls A ball is any subset of $M$ which is homeomorphic to a euclidean ball in $\mathbb{R}^n$, where $n$ is the dimension of $M$. We will need to consider balls which are locally flat and whose boundary has measure zero. More precisely, let us denote by $B_r(0)$ the euclidean ball with radius $r$ and center 0 in $\mathbb{R}^n$. A ball $B$ will be called good if $\mu(\partial B) = 0$ and if there exists a topological embedding (continuous one-to-one map) $\gamma : B_2(0) \to M$ such that $\gamma(B_1(0)) = B$. Note that, due to countable additivity, if $\gamma : B_1(0) \to M$ is any topological embedding, then for almost every $r \in (0, 1)$ the ball $\gamma(B_r(0))$ is good.
Oxtoby-Ulam theorem We will need the following consequence of the Oxtoby-Ulam theorem. Let $B_1, B_2$ be two good balls in the interior of some manifold $M'$, with or without boundary (what we have in mind is either $M' = M$ or $M'$ is a euclidean ball). Let $\mu'$ be a good probability measure on $M'$ which assigns measure zero to the boundary $\partial M'$. Denote by $\text{Homeo}_0(M', \mu')$ the identity component of the group of homeomorphisms of $M'$ which are supported in the interior of $M'$ and preserve $\mu'$. Assume $\mu'(B_1) = \mu'(B_2)$. Then there exists $\phi \in \text{Homeo}_0(M', \mu')$ such that $\phi(B_1) = B_2$. To construct $\phi$, we first choose a good ball $B$ in the interior of $M'$ that contains $B_1, B_2$ in its interior. According to the annulus theorem ([Kir69, Qui82]), we may find a homeomorphism $\phi'$ supported in the ball $B$ that sends $B_1$ onto $B_2$. A first use of the Oxtoby-Ulam theorem provides a homeomorphism $\phi_1$ supported in $B_2$ and sending the measure $(\phi'_s, \mu'_1)|_{B_2}$ to the measure $\mu'_1|_{B_2}$. A second use of the same theorem gives a homeomorphism $\phi_2$ supported in $B \setminus B_2$ and sending the measure $(\phi'_s, \mu'_1)|_{B \setminus B_2}$ to the measure $\mu'_1|_{B \setminus B_2}$. Then $\phi$ is obtained as $\phi_2 \phi_1 \phi'$. Note that, since $\phi$ is supported in the ball $B$, Alexander’s trick ([Ale23]) provides an isotopy from the identity to $\phi$ within the homeomorphisms of $B$ that preserves the measure $\mu'$, which shows that $\phi$ belongs to $\text{Homeo}_0(M', \mu')$.

Triangulations We will also need triangulations which have good properties with respect to the measure $\mu$. We begin with any triangulation $\mathcal{T}$ of $M$. We would like the $(n - 1)$-skeleton of $\mathcal{T}$ to have measure zero, but some $(n - 1)$-dimensional simplices may have positive measure. We fix this as follows. Each $n$-dimensional simplex $s$ of $\mathcal{T}$ is homeomorphic to the standard $n$-dimensional simplex; let $\mu_s$ be a probability measure on $s$ which is the homeomorphic image of the Lebesgue measure on the standard simplex. The measure

$$\mu' = \frac{1}{N} \sum \mu_s$$

(where $N$ denotes the number of $n$-dimensional simplices of $\mathcal{T}$) is a good probability measure on $M$ for which the $n - 1$-dimensional simplices have measure zero. We apply the Oxtoby-Ulam theorem to get a homeomorphism $h$ of $M$ sending $\mu'$ to $\mu$. Then we consider the image triangulation $\mathcal{T}_0 = h_*(\mathcal{T})$, whose $(n - 1)$-skeleton has measure zero. In addition to this, all the simplices of $\mathcal{T}_0$ have the same mass. Using successive barycentric subdivisions we get a sequence $(\mathcal{T}_p)_{p \geq 0}$ of nested triangulations with both properties: the $(n - 1)$-skeleton have no mass and all the simplices have the same mass. Denote by $m_p$ the common mass of the simplices of $\mathcal{T}_p$, and by $d_p$ the supremum of the diameters of the simplices of $\mathcal{T}_p$ (for some metric which is compatible with the topology on $M$). Then the sequences $(m_p)$ and $(d_p)$ tend to zero.

Here is a useful consequence. Let $O$ be any open subset of $M$. We define inductively $\mathcal{O}_p$ as the set of all the $n$-dimensional open simplices of $\mathcal{T}_p$ that are included in $O$ but not in some $s \in \mathcal{O}_{p-1}$. The elements of $\mathcal{O} := \cup \mathcal{O}_p$ are pairwise

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1One may probably avoid the use of the annulus theorem here, since the ball $B$ may be constructed explicitly by gluing the two good balls $B_1$ and $B_2$ to a piecewise linear tube connecting them.
disjoint and their closures cover $O$. Since the $(n-1)$-skeleton of our triangulations have no mass, we have the equality

$$\mu(O) = \sum_{U \in \mathcal{O}} \mu(U) \quad (1).$$

We call a (closed) simplex of some $T_p$ good if it is a good ball in $M$. We notice that for every $p > 0$, all the $n$-dimensional simplices that are disjoint from the $(n-1)$-skeleton of $T_0$ are good. Thus equality (1) still holds if, in the definition of the $O_p$'s, we replace the simplices by the simplices whose closure is good. As a consequence, if two probability measures $\mu, \mu'$ give the same mass to all the good simplices of $T_p$ for every $p$, then they are equal.

In the first Lemma we look for elements of the group $G_f$ that do not preserve the measure and have small support.

**Lemma 2.1.** For every positive $\varepsilon$ there exists a good ball $B$ of measure less than $\varepsilon$ and an element $g \in G_f$ which is supported in $B$ and does not preserve the measure $\mu$.

**Proof.** By hypothesis the probability measures $\mu$ and $f_* \mu$ are not equal. According to the discussion preceding the Lemma, there exists some $p > 0$ and some simplex of the triangulation $T_p$ whose closure $B_1$ is a good ball, and such that $\mu(B_1) \neq \mu(f^{-1}(B_1))$. To fix ideas let us assume that

$$\mu(f^{-1}(B_1)) > \mu(B_1).$$

This implies the same inequality for at least one of the simplices of $T_{p+1}$ that are included in $B_1$; thus, by induction, we see that we may choose $p$ to be arbitrarily large. Note that we have $\mu(f^{-1}(M \setminus B_1)) < \mu(M \setminus B_1)$. Thus the same reasoning, applied to $M \setminus B_1$, provides a (closed) simplex $B_2$ of some $T_{p'}$, disjoint from $B_1$, such that

$$\mu(f^{-1}(B_2)) < \mu(B_2).$$

Again, by induction, we may assume that $p' = p$ and this is an arbitrarily large integer. In particular $B_1$ and $B_2$ are good balls with the same mass. Let $B'$ be a ball whose interior contains $B_1$ and $B_2$. Since $B_1$ and $B_2$ have the same measure, by the above mentioned version of the Oxtoby-Ulam theorem there exists $\phi \in \text{Homeo}_0(M, \mu)$ supported in $B'$ and sending $B_1$ onto $B_2$. Now we consider the element

$$g = f^{-1} \phi f$$

of the group $G_f$. It has support in the ball $B = f^{-1}(B')$. It sends the ball $f^{-1}(B_1)$ to the ball $f^{-1}(B_2)$, and we have

$$\mu(f^{-1}(B_1)) > \mu(B_1) = \mu(B_2) > \mu(f^{-1}(B_2))$$

so that $g$ does not preserve the measure $\mu$, as required by the Lemma.

It remains to see that in the above construction we may have chosen $B$ to be a good ball of arbitrarily small measure. Since $\mu$ has no atom, for every $\varepsilon > 0$
there exists some $\eta > 0$ such that every subset of $M$ of diameter less than $\eta$ has measure less than $\varepsilon$. Thus by choosing $p = p'$ large enough we may require that

$$
\mu(f^{-1}(B_1)) + \mu(f^{-1}(B_2)) < \varepsilon.
$$

Then we choose $B$ as a ball whose interior contains $f^{-1}(B_1)$ and $f^{-1}(B_2)$ and which still has measure less than $\varepsilon$. Finally we shrink $B$ a little bit to turn it into a good ball. This completes the proof of the Lemma.

We subdivide the euclidean unit ball $B_1(0)$ of $\mathbb{R}^n$ into the half-balls $B_1^- = B_1(0) \cap \{x \leq 0\}$ and $B_1^+ = B_1(0) \cap \{x \geq 0\}$. Let $\Sigma$ be the disk $B_1^- \cap B_1^+$ that separates the half-balls. We consider a given ball $B$ and some homeomorphism $g$ supported in $B$. For every homeomorphism $\gamma : B_1(0) \to B$ we let $\gamma = \gamma(B_1^\pm)$; we say that $\gamma$ is thin if $\gamma(\Sigma)$ has measure zero. We now consider the set $\mathcal{I}(\gamma, g)$ of all the numbers of the type

$$
\mu(g(\gamma^+)) - \mu(\gamma^+)
$$

where $\gamma$ is thin.

**Lemma 2.2.** If $g$ does not preserve the measure $\mu$ then $\mathcal{I}(\gamma, g)$ contains an interval $[a^-, a^+]$ with $a^- < 0 < a^+$.

**Proof.** First we want to prove that there exists some $\gamma : B_1(0) \to B$ which is thin and such that $\mu(g(\gamma^+)) \neq \mu(\gamma^+)$. Since $g$ does not preserve the measure $\mu$, we may find some good ball $b$ in the interior of $B$ such that $\mu(b) \neq \mu(f^{-1}(b))$. To fix ideas we assume that $\mu(b) < \mu(f^{-1}(b))$. Thanks to the Oxtoby-Ulam theorem we may identify $B$ with a euclidean ball in $\mathbb{R}^n$, $b$ with another euclidean ball inside $B$, and $\mu$ with the restriction of the Lebesgue measure on $\mathbb{R}^n$. All our balls are centered at the origin. Let $b'$ be a ball slightly greater than $b$, and $T$ be a thin tube in $B \setminus b'$ connecting the boundary of $B$ and that of $b'$. There exists a homeomorphism $\gamma : B_1(0) \to B$ such that $\gamma^+ = T \cup b'$. The construction may be done so that the (Lebesgue) measure of $\gamma^+$ is arbitrarily close to that of $b$, and then we have $\mu(\gamma^+) < \mu(g^{-1}(\gamma^+))$, as wanted.

We can find a continuous family $(R_t)_{t \in [0,1]}$ of rotations of $B_1(0)$ such that $R_0$ is the identity and $R_1$ is a rotation that exchanges $B_1^-$ and $B_1^+$. Setting $\gamma_t := \gamma \circ R_t$, we have $\gamma_t^+ = \gamma_0^+ = \gamma^-$. Note that it may happen that $\gamma_t(\Sigma)$ has positive measure for some $t$. To remedy for this we consider $\gamma' = \phi \circ \gamma$, where $\phi : B \to B$ is a homeomorphism that fixes $\gamma(\Sigma)$, such that the image under $\gamma'$ of the Lebesgue measure on $B_1(0)$ is equivalent to the restriction of $\mu$ to the ball $B$, in the sense that both measures share the same measure zero sets; such a $\phi$ is provided by the Oxtoby-Ulam theorem. This ensures that $\gamma_t' := \gamma' \circ R_t$ is thin for every $t$. Note that $\gamma_t^\pm = \gamma_0^\pm$ and $\gamma_1^\pm = \gamma_1^\pm$. We have

$$
\mu(g(\gamma_1^+)) - \mu(\gamma_1^+) = \mu(g(\gamma_0^-)) - \mu(\gamma_0^-) = (1 - \mu(g(\gamma_0^+))) - (1 - \mu(\gamma_0^+)) = -(\mu(g(\gamma_0^+)) - \mu(\gamma_0^+)) \neq 0.
$$

Thus the set $\mathcal{I}(\gamma, g)$ contains the interval

$$
\{\mu(g(\gamma_t^+)) - \mu(\gamma_t^+), t \in [0,1]\}
$$
which contains both a positive and a negative number, as required by the lemma.

**Corollary 2.3.** Let \( \gamma_0 : B_1(0) \to M \) be a topological embedding in \( M \) with \( \mu(\gamma_0(\Sigma)) = 0 \), let \( B_0 = \gamma_0(B_1(0)) \), and let \( \varepsilon > 0 \) be less than the measure of \( \gamma_0^+ \). Then there exists some element \( g \in G_f \), supported in \( B_0 \), such that

\[
\mu(g(\gamma_0^+)) = \mu(\gamma_0^+) - \varepsilon.
\]

In the situation of the corollary we will say that \( g \) transfers a mass \( \varepsilon \) from \( \gamma_0^+ \) to \( \gamma_0^- \).

**Proof.** Lemma 2.1 provides some element \( g' \in G_f \) that does not preserve the measure \( \mu \), and which is supported on a good ball \( B \) whose measure is less than the minimum of \( \mu(\gamma_0^+) - \varepsilon \) and \( \mu(\gamma_0^-) \). Then Lemma 2.2 provides some homeomorphism \( \gamma : B_1(0) \to B \) which is thin and such that \( g' \) transfers some mass \( a \) from \( \gamma^+ \) to \( \gamma^- \):

\[
\mu(g'(\gamma^+)) - \mu(\gamma^+) = a.
\]

Since such a number \( a \) may be chosen freely in an open interval containing 0, we may assume that \( a = \frac{\varepsilon}{N} \) for some positive integer \( N \). Choose some homeomorphism \( \Phi_1 \in \text{Homeo}_0(M, \mu) \) that sends \( B \) inside \( B_0 \), \( \gamma^+ \) inside \( \gamma_0^+ \) and \( \gamma^- \) inside \( \gamma_0^- \). Such a \( \Phi_1 \) is provided by Oxtoby-Ulam theorem, thanks to the fact that we have chosen the measure of \( B \) to be small enough and that \( \mu(\gamma(\Sigma)) = \mu(\gamma_0(\Sigma)) = 0 \).

Now the conjugate \( g_1 = \Phi_1 g' \Phi_1^{-1} \) transfers a mass \( a \) from \( \gamma_0^+ \) to \( \gamma_0^- \):

\[
\mu(g_1(\gamma_0^+)) = \mu(\gamma_0^+) - a.
\]

We repeat the process with \( \gamma_1 = g_1 \circ \gamma_0 \) instead of \( \gamma_0 \), getting an element \( g_2 \in G_f \) that transfers a mass \( a \) from \( \gamma_1^+ \) to \( \gamma_1^- \):

\[
\mu(g_2 g_1(\gamma_0^+)) = \mu(g_2(\gamma_1^+)) = \mu(\gamma_1^+) - a = \mu(g_1(\gamma_0^+)) - a = \mu(\gamma_0^+) - 2a.
\]

We repeat the process \( N \) times, and get the final homeomorphism \( g \) as a composition of the \( N \) homeomorphisms \( g_N, \ldots, g_1 \).

**3 Proof of the theorem**

We consider as before some element \( f \in \text{Homeo}_0(M) \setminus \text{Homeo}_0(M, \mu) \). Let \( g \) be some other element in \( \text{Homeo}_0(M) \). In order to prove the theorem we want to approximate \( g \) with some element in the group \( G_f \) generated by \( f \) and \( \text{Homeo}_0(M, \mu) \). We fix a triangulation \( T_0 \) for which the \((n-1)\)-skeleton has zero measure. The first step of the proof consists in finding an element \( g' \in G_f \) satisfying the following property: for every simplex \( s \) of \( T_0 \), the measure of \( g'(s) \) coincides with the measure of \( g^{-1}(s) \). To achieve this, the (very natural) idea
is to use corollary 2.3 to progressively transfer some mass from the simplices $s$ whose mass is larger than the mass of their image under $g^{-1}$, to those for which the opposite holds.

Here are some details. Given a triangulation $T$ for which the $(n - 1)$-skeleton has zero measure, we choose two $n$-dimensional simplices $s, s'$ of $T$, and some positive $\varepsilon$ less than $\mu(s)$; let us explain how to transfer a mass $\varepsilon$ from $s$ to $s'$. First assume that $s$ and $s'$ are adjacent. Then we may choose an embedding $\gamma : B_1(0) \to s \cup s'$ with $\gamma(\Sigma) \subset s \cap s'$ and $\gamma^- \subset s$ and $\gamma^+ \subset s'$, and we apply corollary 2.3. Thus we get an element $h \in G_{f}$, supported in $s \cup s'$, such that $\mu(h(s)) = \mu(s) - \varepsilon$, and consequently $\mu(h(s')) = \mu(s') + \varepsilon$. Now consider the general case, when $s$ and $s'$ are not adjacent. Since $M$ is connected, there exists a sequence $s_0 = s, \ldots, s_\ell = s'$ of simplices of $T$ in which two successive elements are adjacent. As described before we may transfer mass $\varepsilon$ from $s_0$ to $s_1$, then from $s_1$ to $s_2$, and so on. Thus by successive adjacent transfers of mass we get some element in $h \in G_{f}$ that transfers mass $\varepsilon$ from $s$ to $s'$. Note that the masses of all the other elements do not change, that is, $\mu(h(\sigma)) = \mu(\sigma)$ for every simplex $\sigma$ of $T$ different from $s$ and $s'$.

Now we go back to our triangulation $T_0$, and we construct $g'$ the following way. If each simplex $s$ has the same measure as its inverse image $g^{-1}(s)$ then there is nothing to do. In the opposite case there exists some simplex $s$ of $T_0$ such that $\mu(s) > \mu(g^{-1}(s))$. We also select some other simplex $s'$ such that $\mu(s') \neq \mu(g^{-1}(s'))$, and we use the previously described construction of a homeomorphism $g_1 \in G_{f}$ that transfers the mass $\mu(s) - \mu(g^{-1}(s))$ from the simplex $s$ to the simplex $s'$. After doing so the number of simplices $g_1(s) \in g_1(T_0)$ whose mass differs from the mass of $g^{-1}(s)$ has decreased by at least one compared to $T_0$. We proceed recursively until we get an element $g' \in G_{f}$ such that $\mu(g'(s)) = \mu(g^{-1}(s))$ for every simplex $s$ in $T_0$, as wanted for this first step.

For the second and last step we consider the triangulations $(g^{-1})_*(T_0)$ and $g'_*(T_0)$. The homeomorphism $g'g$ sends the first one to the second one, and each simplex $g^{-1}(s) \in (g^{-1})_*(T_0)$ has the same measure as its image $g'(s) \in g'_*(T_0)$. We apply Oxtoby-Ulam theorem independently on each $g'(s)$ to get a homeomorphism $\Phi_s : g'(s) \to g'(s)$, which is the identity on $\partial g'(s)$, and which sends the measure $(g'g)_*(\mu|_{g^{-1}(s)})$ to the measure $\mu|_{g'(s)}$. The homeomorphism

$$\Phi := \left( \prod_s \Phi_s \right) g'g$$

preserves the measure $\mu$. Furthermore by Alexander’s trick each $\Phi_s$ is isotopic to the identity, hence $\Phi$ is isotopic to the identity, and belongs to the group $\Homeo_0(M, \mu)$. Now the homeomorphism $g'' = g'^{-1}\Phi$ belongs to the group $G_f$ and for each simplex $s$ of the triangulation $T_0$ we have $g''(s) = g^{-1}(s)$. We may have chosen the triangulation $T_0$ so that each simplex has diameter less than some given $\varepsilon$. Every point $x$ in $M$ belongs to some $n$-dimensional closed simplex $g^{-1}(s)$ of the triangulation $(g^{-1})_*T_0$, and since both $g(x)$ and $g''(x)$ belong to $s$ they are a distance less than $\varepsilon$ apart. In other words the uniform distance from $g$ to $g''$ is less than $\varepsilon$. This proves that $g$ belongs to the closure of $G_f$, and completes the proof of the theorem.
References


