Bistable traveling wave passing an obstacle: perturbation results

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Abstract

We study the existence of generalized transition fronts for a bistable reaction diffusion equation, $u_t - \Delta u = f(u)$, in a heterogeneous medium, $\Omega = \mathbb{R}^N \backslash K$ where K is a compact set of $\mathbb{R}^N$, with Neumann boundary condition and $t \in \mathbb{R}$. In the paper, *Bistable traveling waves around an obstacle* (2009), H. Berestycki, F. Hamel and H. Matano prove the existence of a generalized transition front when K is smooth enough and satisfies some geometric properties. We are interested in an extension of this result when $\Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon$ and K_ε is a small perturbation of K. We prove that as soon as K_ε is close to K in the $C^{2,\alpha}$ topology generalized transition front still exist while it does not if the perturbation is not smooth enough.

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1 Introduction and main results

1.1 Problem and motivations

This work is concerned with the following parabolic semilinear problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u) & \text{in } \Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon, \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega_\varepsilon = \partial K_\varepsilon, \end{cases} \tag{1.1}$$

considering a family of obstacles $(K_\varepsilon)_{0<\varepsilon\leq 1}$ which are $C^{2,\alpha}$-compact sets in $\mathbb{R}^N$, for $0 < \alpha < 1$, such that for all $0 < \varepsilon \leq 1$, $K_\varepsilon \subset B_{R_0}$, for some given $R_0 > 0$, and $K_\varepsilon \to K$ for the $C^{2,\alpha}$ topology, where K is a $C^{2,\alpha}$-compact subset of $\mathbb{R}^N$ and is either star-shaped or directionally convex (see the definitions at the end of this section).

Remark 1.1 *When we write $K_\varepsilon \to K$ for the $C^{2,\alpha}$ topology we mean that for each x_0 in ∂K, and for some $r>0$ such that $\partial K_\varepsilon \cap B_r(x_0) \neq \emptyset$ there exists a couple of parametrization of K_ε and K, ψ_ε and ψ, $C^{2,\alpha}(B_r(x_0))$ functions such that $\|\psi_\varepsilon - \psi\|_{C^{2,\alpha}(B_r(x_0))} \to 0$ as $\varepsilon \to 0$. For more details about the $C^{2,\alpha}$ topology one can look at [1], chapter 6.*

We assume that f is of bistable type, meaning that it is a $C^{1,1}([0,1])$ function such that,

$$\exists\, \theta \in (0,1) |\ f(0) = f(\theta) = f(1) = 0, \quad f'(0) < 0, \quad f'(1) < 0, \\ f(s) < 0 \ \ \forall\, s \in (0,\theta),\ f(s) > 0 \ \ \forall\, s \in (\theta,1), \tag{1.2}$$

with positive mass:

$$\forall\, 0 \leq s < 1, \ \int_s^1 f(\tau)d\tau > 0. \tag{1.3}$$

Our goal is to study how does the shape of K influence the behavior of the solutions of (1.1) and as a consequence how a propagating planar traveling front interacts with our obstacles $(K_\varepsilon)_\varepsilon$.

Existence of traveling fronts for the homogeneous equation $u_t = \Delta u + f(u)$ in $\mathbb{R}^N$ has been studied since the article of Kolmogorov, Petrovsky and Piskunov, [2] in 1937. Recently more and more interest has been observed in the study of traveling front solutions in spatially dependent media because of its importance in several scientific fields. For more descriptions and references about those fronts one can look at the introduction of [3], and references therein. As in H. Berestycki, F. Hamel and H. Matano's paper we study here the existence of a particular type of generalized transition wave, a notion introduced by H. Berestycki and F. Hamel in [4, 5, 6] (see at the end of Section 1.2, Remark 1.7 for a precise definition).

Coming back to our framework, it follows from (1.2) and (1.3) that there exists a unique solution (up to translation) (c, ϕ) of

$$\begin{cases} \phi''(z) - c\phi'(z) + f(\phi(z)) = 0, & z \in \mathbb{R}, \\ \phi(-\infty) = 0, \ \ \phi(+\infty) = 1, & \\ 0 < \phi(z) < 1, & z \in \mathbb{R}, \end{cases} \tag{1.4}$$

with $c > 0$. For further details see [7] for example.
As stated in [3] (section 2 and 3), as soon as $K_\varepsilon \subset \{x, x_1 \leq L\}$, $L \in \mathbb{R}$, there exists a unique time global solution u_ε of (1.1) such that $0 < u_\varepsilon < 1$, $(u_\varepsilon)_t > 0$ in $\mathbb{R} \times \overline{\Omega_\varepsilon}$ and that behaves like a planar traveling front for very small times, i.e

$$u_\varepsilon(t,x) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \overline{\Omega_\varepsilon}. \tag{1.5}$$

To prove these results the authors constructed some sub and super solutions ω^+ and ω^- depending only on x_1 and t. Then using (1.5) and some extension of the comparison principle they proved the uniqueness of the entire (time-global) solution u_ε.
Moreover, they proved that the behavior of u_ε for large time is determined by the solution of

the associated stationary problem. We will then start with the study of the following elliptic problem:

$$\begin{cases} \Delta u_{\varepsilon,\infty} + f(u_{\varepsilon,\infty}) = 0 & \text{in } \Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon, \\ \nu \cdot \nabla u_{\varepsilon,\infty} = 0 & \text{on } \partial\Omega_\varepsilon = \partial K_\varepsilon, \\ 0 < u_{\varepsilon,\infty} \leq 1 & \text{in } \overline{\Omega_\varepsilon}, \\ \lim\limits_{|x|\to+\infty} u_{\varepsilon,\infty}(x) = 1. \end{cases} \tag{1.6}$$

Remark 1.2 *We add the extra consition* $\lim\limits_{|x|\to+\infty} u_{\varepsilon,\infty}(x) = 1$*, because from [3], section 5, as soon as* K_ε *is a compact set,* $f \in C^1([0,1])$ *satisfies (1.2) and (1.3),* (ϕ, c) *solution of (1.4) with* $c > 0$ *and under some asymptotic and monotonicity conditions on* u_ε*, which are satisfied in our framework, our entire solution* u_ε *converges toward* $u_{\varepsilon,\infty}$ *as* $t \to +\infty$ *locally uniformly in* $x \in \overline{\Omega}_\varepsilon$*, and* $u_{\varepsilon,\infty}$ *is a classical solution of*

$$\begin{cases} \Delta u + f(u) = 0 & \textit{in } \Omega_\varepsilon, \\ \nu \cdot \nabla u = 0 & \textit{on } \partial\Omega_\varepsilon, \\ 0 < u \| eq1 & \textit{in } \overline{\Omega}_\varepsilon, \end{cases} \tag{1.7}$$

such that $u_{\varepsilon,\infty}(x) \to 1$ *as* $|x| \to +\infty$*.*

Before stating the main results, let explain what we mean by star-shaped or directionally convex obstacles.

Definition 1.3 K is called star-shaped, if either $K = \emptyset$, or there is $x \in \overset{\circ}{K}$ such that, for all $y \in \partial K$ and $t \in [0,1)$, the point $x + t(y-x)$ lies in $\overset{\circ}{K}$ and $\nu_K(y) \cdot (y-x) \geq 0$, where $\nu_K(y)$ denotes the outward unit normal to K at y.

Definition 1.4 K is called directionally convex with respect to a hyperplane P if there exists a hyperplane $P = \{x \in \mathbb{R}^N, x \cdot e = a\}$ where e is a unit vector and a is some real number, such that

- for every line Σ parallel to e the set $K \cap \Sigma$ is either a single line or empty,
- $K \cap P = \pi(K)$ where $\pi(K)$ is the orthogonal projection of K onto P.

1.2 Main results

Our main result is the following theorem

Theorem 1.5 *Assume* $f \in C^{1,1}([0,1])$ *satisfies (1.2) and (1.3). Consider* $(K_\varepsilon)_{0<\varepsilon\leq 1}$ *a family of* $C^{2,\alpha}$*-compact sets of* $\mathbb{R}^N$ *(i.e for all* $\varepsilon \in (0,1]$*,* $K_\varepsilon \subset B_{R_0}$ *for some* $R_0 > 0$*) and let* $\Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon$ *a smooth, open, connected subset of* $\mathbb{R}^N$ *(with* $N \geq 2$*). Assume that* $K_\varepsilon \to K$ *for the* $C^{2,\alpha}$ *topology as* $\varepsilon \to 0$*, where* K *is a* $C^{2,\alpha}$*-compact subset of* $\mathbb{R}^N$ *that is either star-shaped or directionally convex with respect to some hyperplane* P*. Then there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$*, the unique solution of (1.6) is* $u_{\varepsilon,\infty} \equiv 1$

This means that for obstacles that are compact sets in $\mathbb{R}^N$ and close enough (in the $C^{2,\alpha}$ sense) to some star-shaped or directionally convex domains then the unique solution of (1.6) is the constant 1.
This Theorem yields some properties about the solution u_ε of the parabolic problem (1.1).

Corollary 1.6 *Assume that f satisfies (1.2) and that there exists a solution ϕ to (1.4) with $c > 0$ (if f satisfies (1.2) and (1.3), ϕ exists and $c > 0$). Let $(K_\varepsilon)_{0<\varepsilon\leq 1}$ be a family of compact domains in $\mathbb{R}^N$ such that for all $0 < \varepsilon \leq 1$, $K_\varepsilon \subset B_{R_0}$, for some given $R_0 > 0$, and $K_\varepsilon \to K$ for the $C^{2,\alpha}$ topology, with $0 < \alpha < 1$ and where $K \subset \mathbb{R}^N$ is either star-shaped or directionally convex with respect to some hyperplane P. Then for all $0 < \varepsilon \leq 1$, there exists an entire solution $u_\varepsilon(t, x)$ of (1.1) such that $0 < u_\varepsilon < 1$ and $\partial_t u_\varepsilon > 0$ over $\mathbb{R} \times \overline{\Omega_\varepsilon}$ and there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*

$$u_\varepsilon(t, x) - \phi(x_1 + ct) \to 0$$

as $t \to \pm\infty$ uniformly in $x \in \overline{\Omega_\varepsilon}$, and as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$.

We will prove Theorem 1.5 in section 2 below, and Corollary 1.6 in section 3.

Remark 1.7 *For all $0 < \varepsilon < \varepsilon_0$ the solution $u_\varepsilon(t, x)$ given in Corollary 1.6 is a generalized, almost planar, invasion front between 0 and 1 with global mean speed c, in the sense that*

$$\sup_{(t,x)\in\mathbb{R}\times\overline{\Omega_\varepsilon}, x_1+ct\geq A} |u_\varepsilon(t, x) - 1| \xrightarrow[A\to+\infty]{} 0$$

$$\sup_{(t,x)\in\mathbb{R}\times\overline{\Omega_\varepsilon}, x_1+ct\leq -A} |u_\varepsilon(t, x)| \xrightarrow[A\to+\infty]{} 0$$

Before proving the previous statements, let give some examples of domains $(K_\varepsilon)_\varepsilon$ and K to illustrate our results.

1.3 Examples

We assume that $N = 2$ and we construct two families of obstacles; one which converges to a star shaped domain and the other which converges to a directionally convex domain. The black plain line represents the limit K and the dashed parts represent the small perturbations (of order ε).

Star-shaped domain:

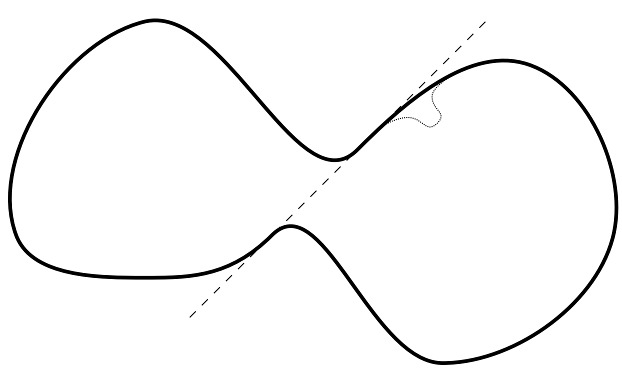

Figure 1: Obstacles converging toward a star-shaped domain

The long-dashed line is used during the construction of K and it is on this line that we could find the center(s) of the domain (i.e the point x in Definition 1.3). We can clearly see that for all $\varepsilon > 0$, K_ε is not star-shaped, because the points just behind the dashed area cannot be linked to the center x by a straight line.

Directionally convex domain:

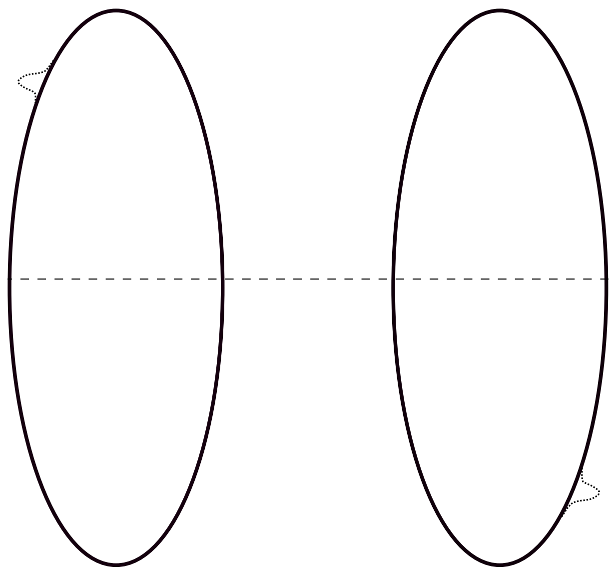

Figure 2: Obstacles converging toward a directionally convex domain

In this second figure, the hyperplanes P, for which the plain black domain could be directionally convex are necessarily horizontal (i.e $\{(x, y) \in \mathbb{R}^2 | (x, y) \cdot (0, 1) = a\}$) and a has to be 0 (assuming the center of the ellipses are on the x-axis), else the second property in Definition 1.4 is not satisfied. Indeed the first property of Definition 1.4 eliminates every vertical hyperplanes P and the second property eliminates the diagonal ones. Adding small perturbations (of order ε) on each side of each ellipse, one gets that for all $\varepsilon > 0$ K_ε does not satisfy the second property of Definition 1.4.

One need to be careful on the shape of the perturbations. Indeed considering an ellipse, which is star-shaped and directionally convex and adding on each side of the vertical axis some well chosen perturbations (keeping the domain smooth), the obstacle is not star-shaped, neither directionally convex anymore (see figure below),

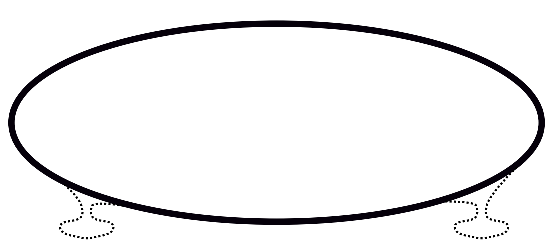

Figure 3: Obstacles converging only in C^0

and $K_\varepsilon \to K$, an ellipse, as $\varepsilon \to 0$, but the convergence of K_ε cannot be $C^{2,\alpha}$ (see section 4 for more details) but only C^0 which is not enough to get Theorem 1.5, as we will see in section 4.

2 Proof of the main Theorem

In this section, to simplify the notation we will write u_ε in stead of $u_{\varepsilon,\infty}$. As we are only working on the stationary problem, there is no confusion with the solution of the parabolic system. To prove Theorem 1.5, we will use the following Proposition:

Proposition 2.1 *For all $0 < \delta < 1$, if u_ε is a solution of (1.6), then there exists $R = R(\delta) > R_0$, such that $u_\varepsilon(x) \geq 1 - \delta$ for all $|x| \geq R$ and for all $0 < \varepsilon < 1$.*

This proposition means that u_ε converges toward 1 as $|x| \to +\infty$ uniformly in ε.
Let first admit this result and prove Theorem 1.5.

2.1 Proof of Theorem 1.5

Using Schauder Estimates (see [1], chapter 6.7), we obtain that

$$\|u_\varepsilon\|_{C^{2,\alpha}(\Omega)} \leq C.$$

As $K_\varepsilon \to K$ for the $C^{2,\alpha}$ topology when $\varepsilon \to 0$, C is independent of ε. So there exists a sequence $(\varepsilon_n)_n$ in $]0, \eta \wedge 1[$ such that $\varepsilon_n \to 0$ and $u_{\varepsilon_n} \to u$ in C^2_{loc} as $n \to +\infty$ and u satisfies:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega = \mathbb{R}^N \backslash K, \\ \nu \cdot \nabla u = 0 & \text{on } \partial\Omega = \partial K, \end{cases} \tag{2.1}$$

because of the compact injection $C^{2,\alpha}_{\text{loc}} \hookrightarrow C^2_{\text{loc}}$. Using Proposition 2.1 we get $\lim\limits_{|x| \to +\infty} u(x) = 1$. And K is either star-shaped or directionally convex. We now recall the following results from [3]:

Theorem 2.2 *Let f be a Lipschitz-continuous function in $[0,1]$ such that $f(0) = f(1) = 0$ and f is nonincreasing in $[1-\delta, 1]$ for some $\delta > 0$. Assume that*

$$\forall\, 0 \leq s < 1, \ \int_s^1 f(\tau) d\tau > 0. \tag{2.2}$$

Let Ω be a smooth, open, connected subset of $\mathbb{R}^N$ (with $N \geq 2$) with outward unit normal ν, and assume that $K = \mathbb{R}^N \backslash \Omega$ is compact. Let $0 \leq u \leq 1$ be a classical solution of

$$\begin{cases} \Delta u = f(u) & \textit{in } \Omega, \\ \nu \cdot \nabla u = 0 & \textit{on } \partial\Omega, \\ u(x) \to 1 & \textit{as } |x| \to +\infty. \end{cases} \tag{2.3}$$

If K is star shaped or directionally convex, then

$$u \equiv 1 \text{ in } \overline{\Omega}. \tag{2.4}$$

Remark 2.3 *The main difficulty here was that, for every $\varepsilon \in (0,1)$, K_ε is not necessarily star-shaped nor directionally convex and thus we could not use directly Theorem 2.2. The purpose of the article is to find an equivalent to Theorem 2.2 but in the case of a family of obstacles $(K_\varepsilon)_\varepsilon$ that converges toward a star-shaped or directionally convex domain.*

It follows from Theorem 2.2 that $u \equiv 1$. It also proves that the limit u is unique and thus $u_\varepsilon \to u$ as $\varepsilon \to 0$ in C^2 (and not only along a subsequence).

Now we need to prove that there exists $\varepsilon_0 > 0$ such that $u_\varepsilon \equiv 1$ for all $0 < \varepsilon < \varepsilon_0$. Let assume that for all $\varepsilon > 0$, $u_\varepsilon \not\equiv 1$. Then there exists $x_0 \in \overline{\Omega_\varepsilon}$ such that $u_\varepsilon(x_0) = \min_{x \in \Omega_\varepsilon} u_\varepsilon(x) < 1$. As u_ε is a solution of (1.6), the Hopf lemma yields that,

$$\text{if } x_0 \in \partial K_\varepsilon \text{ then } \frac{\delta u_\varepsilon}{\delta \nu}(x_0) < 0,$$

which is impossible due to Neuman boundary conditions. Hence $x_0 \in \Omega_\varepsilon$.

If $u_\varepsilon(x_0) > \theta$,

$$-\Delta u_\varepsilon(x_0) = f(u_\varepsilon(x_0)) > 0,$$

which is impossible since x_0 is a minimizer. So, for all $0 < \varepsilon < 1$,

$$0 \leq \min_{x \in \Omega_\varepsilon} u_\varepsilon(x) \leq \theta,$$

which contradicts $u_\varepsilon \to u \equiv 1$. Thus there exists ε_0 such that for all $\varepsilon < \varepsilon_0$, $u_\varepsilon \equiv 1$. □

2.2 Proof of Proposition 2.1

We will now prove Proposition 2.1, starting with the following lemma:

Lemma 2.4 *There exists $\omega = \omega(r)$ with $r \in \mathbb{R}^+$ such that*

$$\begin{cases} -\omega''(r) = f(\omega(r)), & \forall r \in \mathbb{R}^+_*, \\ \omega(0) = 0, \ \ \omega'(0) > 0, & \\ \omega' > 0, \ \ 0 < \omega < 1 & \text{in } \mathbb{R}^*_+, \\ \lim_{r \to +\infty} \omega(r) = 1. & \end{cases} \tag{2.5}$$

We can prove this lemma using some results about traveling fronts in the multistable case (a) or using an ODE approach (b).

Proof of Lemma 2.4

- (a) Proof using traveling waves

We know from [7] that there exists a unique solution $(\omega_1, c_1) \in C^2(\mathbb{R}) \times \mathbb{R}$ of

$$\begin{cases} -\omega_1''(x) + c_1\omega_1'(x) = f(\omega_1(x)) & \text{in } \mathbb{R}, \\ \omega_1(-\infty) = 0, \ \ \omega_1(+\infty) = 1, & \\ \omega_1' > 0, & \text{in } \mathbb{R}, \\ 0 < \omega_1 < 1, & \text{in } \mathbb{R}. \end{cases} \tag{2.6}$$

Because of (1.3), $c_1 > 0$. One can prove the existence and uniqueness of $v \in C^2(\mathbb{R}^+)$

$$\begin{cases} v''(z) - cv'(z) + f(v(z)) = 0, & \forall z \in (0, +\infty), \\ v(0) = 0, \ \ v(+\infty) = 1, & \\ v'(z) > 0, & \forall z \in (0, +\infty), \\ 0 < v(z) < 1, & \forall z \in (0, +\infty), \end{cases} \tag{2.7}$$

for $c \leq c_1$ (see [7, 8]). Then taking $c = 0 \leq c_1$ there exists $\omega \in C^2(\mathbb{R}^+)$ such that

$$\begin{cases} \omega''(z) + f(\omega(z)) = 0, & \forall z \in (0, +\infty), \\ \omega(0) = 0, \ \ \omega(+\infty) = 1, & \\ \omega'(z) > 0, & \forall z \in (0, +\infty), \\ 0 < \omega(z) < 1, & \forall z \in (0, +\infty). \end{cases}$$

The Hopf Lemma yields that $\omega'(0) > 0$. $\square$

- (b) Proof using an ODE approach.

  We want to prove the global existence and uniqueness of the following ODE:

$$\begin{cases} -\omega'' = f(\omega) & \text{in } (0, +\infty), \quad (1) \\ \omega(0) = 0, & \\ \omega'(0) = \sqrt{2F(1)}, & \end{cases} \tag{2.8}$$

where $F(z) = \int_0^z f(s)ds$. Using (1.3), $F(1) > 0$.
From Cauchy-Lipschitz theorem we know that there exists a unique maximal solution ω of (2.8) in $I \subset (0, +\infty)$. To prove the global existence, i.e $I = (0, +\infty)$, let prove that $\omega' > 0$ in I and $0 < \omega < 1$ in I.

We start by proving that $\omega' > 0$. We know that $\omega'(0) > 0$. Suppose that there exists $r_0 \in I$ such that $\omega'(r_0) = 0$. Then multiplying (2.8) (1) by ω' and integrating between 0 and r_0, one gets:

$$-F(1) + F(w(r_0)) = 0 = \int_1^{\omega(r_0)} f(z)dz. \tag{2.9}$$

Without loss of generality, we extend f linearly (as a C^1 function) outside $[0, 1]$. Note that by the Maximum Principle any solution with such a f will take values in $[0, 1]$, hence is a solution of the original problem. The last equation (2.9) is impossible, which implies that $\omega > 0$ in I.

Next assume by contradiction that there exists $r_1 \in I$ such that $\omega(r_1) = 1$. Using the same method as above (multiplying by ω' and integrating between 0 and r_1) one gets:

$$\frac{(\omega'(r_1))^2}{2} - F(1) + F(1) = 0,$$

which is impossible. Hence $0 < w < 1$ in I.

If we assume that $I \subsetneq (0, +\infty)$, i.e there exists $r_\infty \in (0, +\infty)$ such that $I = (0, r_\infty)$, it means that $\lim \omega(r) = +\infty$ as $r \to r_\infty$. This is impossible because $0 < \omega < 1$ in I. Thus $I = (0, +\infty)$.

We have proved that there exists a unique global solution ω of (2.8) and that $\omega'(r) > 0$ and $0 < \omega(r) < 1$ for all $r \in (0, +\infty)$. As ω is increasing and bounded from above, it has a limit when $r \to +\infty$ such that $0 < \omega(+\infty) \leq 1$. Moreover $\omega(+\infty) > \theta$. Indeed if we assume that $\omega \leq \theta$ in $\mathbb{R}_+$ then one gets that ω is convex (since $\omega'' = -f(\omega) \geq 0$) and increasing, it then goes to $+\infty$ when $r \to +\infty$, which is impossible. It immediately follows from elliptic regularity estimates that $f(\omega(+\infty)) = 0$. Hence $\omega(+\infty) = 1$.
One has proved Lemma 2.4. $\square$

Proof of Proposition 2.1. Now we introduce a function f_δ with the same hypothesis as f but such that $f_\delta \leq f$, $f_\delta = f$ in $[0, 1-\delta]$ and $f_\delta(1 - \frac{\delta}{2}) = 0$ (see figure below). Notice that $\int_0^{1-\frac{\delta}{2}} f_\delta(z)dz > 0$ for δ small.

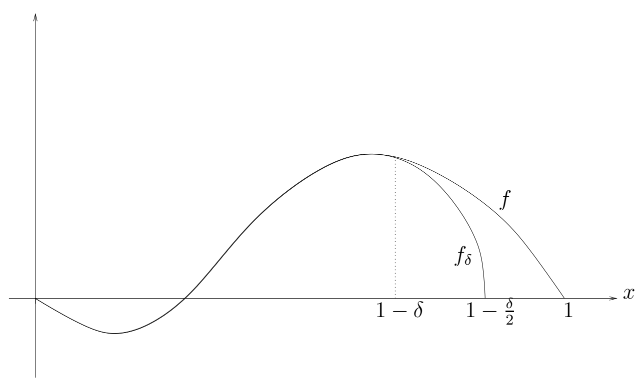

Figure 4: $f_\delta(x)$

For the same reason than in Lemma 2.4 there exists $\omega = \omega_\delta$ such that

$$\begin{cases} -\omega_\delta''(x) = f_\delta(\omega_\delta(x)) & \text{in } (0, +\infty), \\ \omega_\delta(0) = 0, \ \ \omega_\delta(+\infty) = 1 - \frac{\delta}{2}, & \\ 0 < \omega_\delta < 1 - \frac{\delta}{2} & \text{in } (0, +\infty), \\ \omega_\delta' > 0 & \text{in } (0, +\infty). \end{cases} \tag{2.10}$$

Next, for any $R > R_0$ consider $\omega_\delta(|x| - R)$ for every $|x| \geq R$ and let $z(x) = \omega_\delta(|x| - R)$, one gets:

$$\Delta z + f(z) = \omega_\delta'' + \frac{N-1}{|x|}\omega_\delta' + f(\omega_\delta) = \frac{N-1}{|x|}\omega_\delta' + f(\omega_\delta) - f_\delta(\omega_\delta) > 0 \text{ in } \{|x| > R\}.$$

So

$$-\Delta z < f(z) \text{ in } \{|x| > R\}. \tag{2.11}$$

We want to prove that

$$\omega_\delta(|x| - R_0) < u_\varepsilon(x), \qquad \forall x \in \mathbb{R}^N, |x| \geq R_0.$$

We know from (1.6) that $u_\varepsilon(x) \to 1$ as $|x| \to +\infty$. Hence there exists $A = A(\varepsilon) > 0$ such that $u_\varepsilon(x) \geq 1 - \frac{\delta}{3}$, for all $|x| \geq A$.
One gets $u_\varepsilon(x) \geq \omega(|x| - A)$, for all $|x| \geq A$.

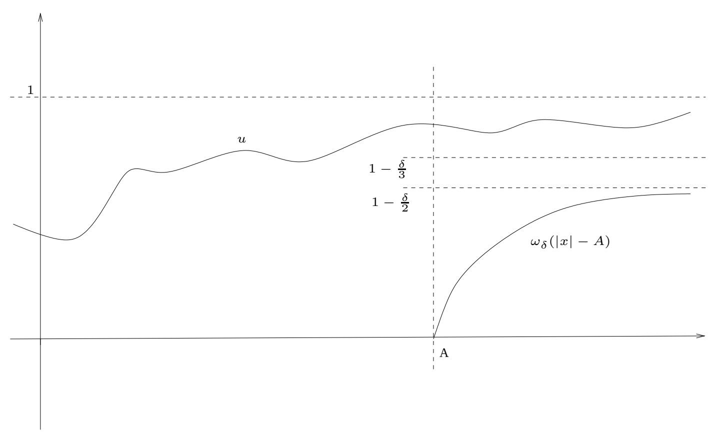

Figure 5: u_ε and $\omega_\delta(|\cdot| - A)$

Consider

$$\overline{R} = \inf\Big\{R \geq R_0; u_\varepsilon(x) > \omega_\delta(|x| - R), \text{ for all } \{|x| \geq R\}\Big\}. \tag{2.12}$$

As $\overline{R} \geq R_0$ and $K_\varepsilon \subset B_{R_0}$, u_ε is always defined on $\{|x| > \overline{R}\}$. One will prove that $\overline{R} = R_0$. As ω_δ is increasing, we know that

$$\forall R \geq A \quad u_\varepsilon(x) \geq \omega(|x| - R), \quad \forall |x| \geq R.$$

Hence $\overline{R} \leq A$.
Assume that $\overline{R} > R_0$. Then there are two cases to study:

- either $\inf\Big\{u_\varepsilon(x) - \omega_\delta(|x| - \overline{R}), \quad \forall |x| > \overline{R}\Big\} > 0$, (1)
- or $\inf\Big\{u_\varepsilon(x) - \omega_\delta(|x| - \overline{R}), \quad \forall |x| > \overline{R}\Big\} = 0$. (2)

In the first case (1), one gets $u_\varepsilon(x) > \omega_\delta(|x| - \overline{R})$ for all $|x| > \overline{R}$. As ∇u_ε and ω'_δ are bounded one can translate ω_δ to the left such that both graphs touch at one point, i.e there exists $R^* < \overline{R}$ such that $u_\varepsilon(x) \geq \omega_\delta(|x| - R^*)$ for all $|x| > R^*$, and $u_\varepsilon(x_0) = \omega_\delta(|x_0| - R^*)$ for some $|x_0| > R^*$. This contradicts the optimality of $\overline{R}$.

In the second case (2), there necessarily exists x_0 with $|x_0| > \overline{R}$ such that $u_\varepsilon(x_0) = \omega_\delta(|x_0| - \overline{R})$. Let $v(x) = u_\varepsilon(x) - \omega_\delta(|x| - \overline{R})$, for all $|x| > \overline{R}$. As u_ε is a solution of (1.6) and using (2.11), v satisfies:

$$\begin{cases} -\Delta v > f(v) & \text{in } \{|x| > \overline{R}\}, \\ v > 0 & \text{on } \{|x| = \overline{R}\}. \end{cases} \tag{2.13}$$

From the maximum principle $v(x) \geq 0$, for all $|x| \geq \overline{R}$. But there exists x_0 such that $|x_0| > \overline{R}$ (x_0 is an interior point) and $v(x_0) = 0$, i.e $v(\cdot)$ reaches its minimum 0 inside the domain. This implies that $v(\cdot) \equiv 0$, which is impossible because $v(\cdot) > 0$, for all $|x| = \overline{R}$.

Then $\overline{R} = R_0$ which does not depend on ε and

$$\forall |x| \geq R_0 \quad u_\varepsilon(x) \geq \omega(|x| - R_0).$$

But $\omega_\delta(x) \to 1 - \frac{\delta}{2}$ as $|x| \to +\infty$ implies that there exists $\hat{R}$, independent of ε, such that for all $|x| > \hat{R} + R_0$, $u_\varepsilon(x) > \omega_\delta(|x| - R_0) \geq 1 - \delta$. One has proved Proposition 2.1. □

3 The associated parabolic problem and its properties

In this section we will use Theorem 1.5, to apply some results of [3] and derive Corollary 1.6. We investigate the following semilinear parabolic problem:

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon + f(u_\varepsilon) & \text{in } \Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon, \\ \nu \cdot \nabla u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon = \partial K_\varepsilon, \end{cases} \tag{3.1}$$

where K_ε is a compact set in $\mathbb{R}^N$. Notice that in this section, u_ε is the solution of the parabolic problem. **Proof of Corollary 1.6** We know from H. Berestycki, F. Hamel and H. Matano's paper [3] that their exists an entire solution u_ε of (3.1) in Ω_ε such that $0 < u_\varepsilon(t,x) < 1$, $\partial_t u_\varepsilon(t,x) > 0$ for all $(t,x) \in \mathbb{R} \times \overline{\Omega_\varepsilon}$ and

$$u_\varepsilon(t,x) - \phi(x_1 + ct) \to 0 \text{ as } t \to -\infty \text{ uniformly in } x \in \overline{\Omega_\varepsilon},$$

and as $|x| \to +\infty$ uniformly in $t \in \mathbb{R}$. Furthermore, there exists a classical solution $u_{\varepsilon,\infty}$ of

$$\begin{cases} \Delta u_{\varepsilon,\infty} + f(u_{\varepsilon,\infty}) = 0 & \text{in } \Omega_\varepsilon = \mathbb{R}^N \backslash K_\varepsilon, \\ \nu \cdot \nabla u_{\varepsilon,\infty} = 0 & \text{on } \partial\Omega_\varepsilon = \partial K_\varepsilon, \\ 0 < u_{\varepsilon,\infty} \leq 1 & \text{in } \overline{\Omega_\varepsilon}, \\ \lim\limits_{|x| \to +\infty} u_{\varepsilon,\infty}(x) = 1, \end{cases} \tag{3.2}$$

such that

$$u_\varepsilon(t,x) - \phi(x_1 + ct)u_{\varepsilon,\infty}(x) \to 0 \text{ as } t \to +\infty \text{ uniformly in } x \in \overline{\Omega_\varepsilon}.$$

Then using Theorem 1.5, there exists $\varepsilon_0 < 1$ such that for all $0 < \varepsilon < \varepsilon_0$, the only solution $u_{\varepsilon,\infty}$ of (3.2) is identically equal to 1. We have proved Corollary 1.6. □

4 Discussion about the convergence of $(K_\varepsilon)_{0<\varepsilon<1}$

In this section we discuss the hypothesis of convergence of $(K_\varepsilon)_{0<\varepsilon<1}$. Until now we assumed that $K_\varepsilon \to K$ in $C^{2,\alpha}$, with $0 < \alpha < 1$ in order to use the Schauder estimates and ensure the convergence of u_ε as $\varepsilon \to 0$. One can wonder if we can weaken this hypothesis: would the C^0 or C^1 convergence be enough?

We will prove that the C^0 convergence is not enough.

4.1 Example of a family of obstacles that converges only in C^0

In this subsection we construct a family of obstacles that are neither star-shaped nor directionally convex but converges uniformly to K which is convex. We want to prove that for all $\varepsilon \in]0,1]$ there exists a solution of (3.2) which is not identically equal to 1. To do so we will use the counterexample of section 6.3 in [3].

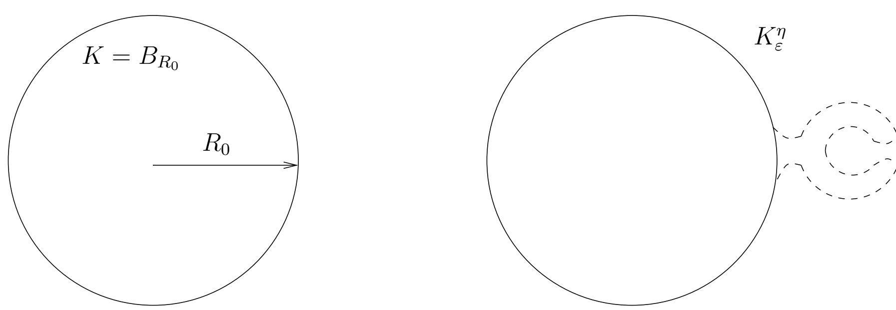

Figure 6: Liouville counterexample

Zooming on the dashed part:

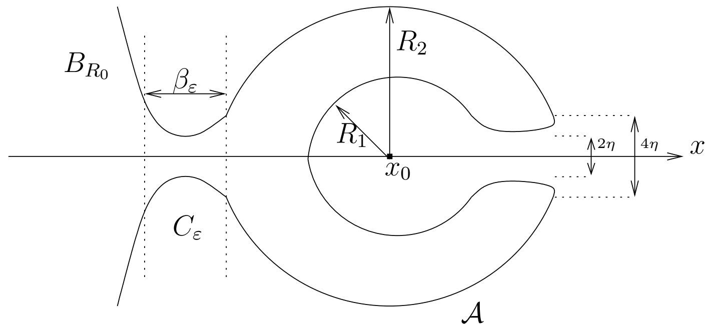

Figure 7: Zoom on the perturbation V_ε

We consider an obstacle $K_\varepsilon = K_\varepsilon^\eta$ (see figure 6 and 7), for $\varepsilon > 0$, such that:

$$\begin{cases} \big(\mathcal{A} \cap \{x; x_1 \leq x_1^0\}\big) \cup B_{R_0} \cup C_\varepsilon \subset K_\varepsilon^\eta, \\ \mathcal{A} \cap \big\{x; x_1 > x_1^0, |x'| > 2\eta\big\} \subset K_\varepsilon^\eta, \\ K_\varepsilon^\eta \subset \Big(\mathcal{A} \cap \big\{x; x_1 > x_1^0, |x'| > \eta\big\}\Big) \cup B_{R_0} \cup \Big(\mathcal{A} \cap \big\{x; x_1 \leq x_1^0\big\}\Big) \cup C_\varepsilon. \end{cases} \tag{4.1}$$

where $x' = (x_2, ..., x_N)$ and $\mathcal{A} = \{x : R_1 \leq |x - x^0| \leq R_2\}$, R_0, $R_1 < R_2$, are three positive constants, $x^0 = (x_1^0, 0, 0, ..., 0)$ is the center of the annular region $\mathcal{A}$ with $x_1^0 = R_0 + R_2 + \beta_\varepsilon$, C_ε is some corridor that links smoothly $\mathcal{A}$ and B_{R_0} which length is β_ε and $\eta > 0$, small enough.
Now let explain why the convergence of K_ε is only true for the C^0 topology.

We want that for all $\varepsilon > 0$, the annular region $\mathcal{A}$ stays an annular region (see arguments in the next section, Corollary 4.1 and 4.2). To do so we need to reduce vertically and horizontally the perturbation in order to stay with an annular region.
To simplify the proof, assume that $N = 2$. Let g be the parametrization of $K = B_{R_0}$, i.e

$$K = \Big\{(x, y) \in \mathbb{R}^2 | (x, y) = g(t) = \big(R_0 \cos(t), R_0 \sin(t)\big) \;\; \forall t \in [0, 2\pi[\Big\}. \tag{4.2}$$

Let f_ε be the parametrization of K_ε for all $0 < \varepsilon \leq 1$. To define f_ε, we start with the case when $\varepsilon = 1$:

$$K_1 = \left\{(x, y) \in \mathbb{R}^2 | (x, y) = f_1(t) = \left\{ \begin{array}{ll} g(t) & \forall t \in]\theta, 2\pi - \theta[, \\ h(t) & \forall t \in [0, \theta] \cup [2\pi - \theta, 2\pi[\end{array} \right. \right\}, \tag{4.3}$$

where θ is some small positive number and h is such that f_1 is a $C^{2,\alpha}$ function. Now one can define f_ε and K_ε:

$$K_\varepsilon = \left\{(x, y) \in \mathbb{R}^2 | (x, y) = f_\varepsilon(t) = \left\{ \begin{array}{ll} g(t) & \forall t \in]\varepsilon\theta, 2\pi - \varepsilon\theta[, \\ h_\varepsilon(t) & \forall t \in [0, \varepsilon\theta] \cup [2\pi - \varepsilon\theta, 2\pi[\end{array} \right. \right\}, \tag{4.4}$$

where h_ε is such that f_ε is a $C^{2,\alpha}$ function and such that for every $(x,y) \in \mathcal{A} \cap K_\varepsilon$, $(x,y) = h_\varepsilon(t) = \varepsilon h(t)$. This last condition ensures that $\mathcal{A}$ stays an annular region.
Then one can easily see that $K_\varepsilon \to K$ as $\varepsilon \to 0$ for the C^0 topology, i.e $\|f - f_\varepsilon\|_{C^0([0,2\pi[)} \to 0$ as $\varepsilon \to 0$
Now assume that $K_\varepsilon \to K$ as $\varepsilon \to 0$ for the C^1 topology. One can notice (see figure 8 below) that for $\varepsilon = 1$ there exists a $t_m \in [0,\theta]$ such that $f_1'(t_m) = e_1 = (1,0)$ or $\nu(x_m,y_m) = e_2 = (0,1)$, where ν is the outward unit normal and $(x_m,y_m) = f_1(t_m) \in \mathcal{A} \cap K_\varepsilon$.

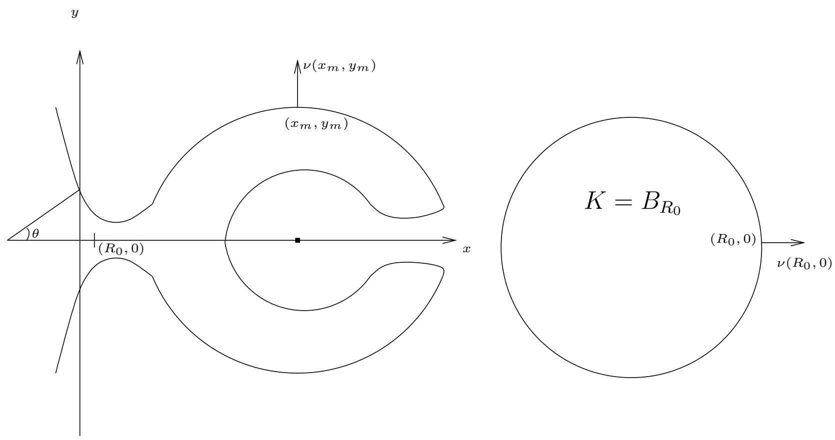

Figure 8: Outward unit normal

Let consider $t_\varepsilon \in [0, \varepsilon\theta]$ such that $\nu(x_\varepsilon, y_\varepsilon) = e_2$, where $(x_\varepsilon, y_\varepsilon) = f_\varepsilon(t_\varepsilon)$ the point at the top of the big sphere in $\mathcal{A}$. This point always exists because we parametrize K_ε such that $\mathcal{A}$ stays an annular region. As the convergence is C^1 one should have that $\nu(x_\varepsilon, y_\varepsilon) \to \nu(R_0, 0)$ as $\varepsilon \to 0$, because $(x_\varepsilon, y_\varepsilon) \to (R_0, 0)$ as $\varepsilon \to 0$. But $\nu(x_\varepsilon, y_\varepsilon) = e_2$ for every $\varepsilon \in (0,1]$ and $\nu(R_0, 0) = e_1$, which is impossible. The convergence can not be C^1.

4.2 Existence of a non constant solution u_ε of (1.6)

We want to prove that for all $0 < \varepsilon < 1$ their exists a solution $0 < u_\varepsilon < 1$ of

$$\begin{cases} -\Delta u_\varepsilon = f(u_\varepsilon) & \text{in } \mathbb{R}^N \backslash K_\varepsilon^\eta = \Omega_\varepsilon^\eta, \\ \nu \cdot \nabla u_\varepsilon = 0 & \text{on } \partial K_\varepsilon^\eta = \partial \Omega_\varepsilon^\eta, \\ u_\varepsilon(x) \to 1 \text{ as } |x| \to +\infty. \end{cases} \tag{4.5}$$

Here as in section 2, u_ε denotes the solution of the elliptic problem. We will follow the same steps as in [3], section 6.
First, let notice that it is enough to find $\omega \not\equiv 1$ solution of

$$\begin{cases} -\Delta \omega = f(\omega) & \text{in } B_R \backslash K_\varepsilon^\eta, \\ \nu \cdot \nabla \omega = 0 & \text{on } \partial K_\varepsilon^\eta, \\ \omega = 1 & \text{on } \partial B_R, \end{cases} \tag{4.6}$$

for some $R > 0$ large enough such that $K_\varepsilon^\eta \subset B_R$.
Indeed then ω extended by 1 outside B_R is a supersolution of (4.5) and one can define:

$$\psi(x) = \begin{cases} 0 & \text{if } \{|x| < R\}\backslash K_\varepsilon^\eta, \\ U(|x| - R) & \text{if } |x| \geq R, \end{cases} \tag{4.7}$$

where $U : \mathbb{R}^+ \to (0,1)$ satisfies $U'' + f(U) = 0$ in $\mathbb{R}_+^*$, $U(0) = 0$, $U'(\xi) > 0 \ \forall\ \xi \geq 0$, $U(+\infty) = 1$. It exists as soon as (1.3) is satisfied (see proof of Lemma 2.4). As $U(|\cdot| - R)$ is a subsolution, ψ is a subsolution of (4.5).
Hence there exists a solution $\psi < u_\varepsilon < \omega$ of (4.5). If we prove that $\omega \not\equiv 1$ then $0 < u_\varepsilon < 1$ (with the maximum principle).

Now let consider our problem (4.6) and replace ω by $v = 1 - \omega$. The problem becomes

$$\begin{cases} -\Delta v = -f(1-v) & \text{in } B_R\backslash K_\varepsilon^\eta, \\ \nu \cdot \nabla v = 0 & \text{on } \partial K_\varepsilon^\eta, \\ v = 0 & \text{on } \partial B_R. \end{cases} \tag{4.8}$$

Using exactly the same arguments as in [3] one proves that, if we considere:

$$v_0(x) = \begin{cases} 1 & \text{if } x \in B_{R_2}(x^0)\backslash K_\varepsilon^\eta \cap \Big\{x; x_1 - x_1^0 \leq \frac{2R_1+R_2}{3}\Big\}, \\ \frac{3}{R_2-R_1}\left(\frac{R_1+2R_2}{3} - (x_1 - x_1^0)\right) & \begin{array}{l}\text{if } x \in B_{R_2}(x^0)\backslash K_\varepsilon^\eta \\ \cap\Big\{x; \frac{2R_1+R_2}{3} \leq x_1 - x_1^0 \leq \frac{R_1+2R_2}{3}\Big\},\end{array} \\ 0 & \begin{array}{l}\text{if } x \in \Big[B_R\backslash\big(B_{R_2}(x^0) \cup C_\varepsilon \cup B_{R_0}(0)\big)\Big] \\ \cup\Big[B_{R_2}(x^0)\backslash K_\varepsilon^\eta \cap \{x, x_1 - x_1^0 \geq \frac{R_1+2R_2}{3}\}\Big],\end{array} \end{cases} \tag{4.9}$$

then there exists $v \in H^1(B_R\backslash K_\varepsilon^\eta) \cap \{v = 0 \text{ on } \partial B_R\} = H_0^1$ such that $\|v - v_0\|_{H_0^1} < \delta$ for some $\delta > 0$ small enough, that is a local minimizer of the associated energy functional in H_0^1 when the width η of the channel is small enough. For more clarity we will give the main step of the proof but for details see [3], section 6.3.

We introduce the energy functional in a domain D:

$$J_D(\omega) = \int_D \Big\{\frac{1}{2}|\nabla\omega|^2 - G(\omega)\Big\}dx, \tag{4.10}$$

defined for functions of $H^1(D)$, where

$$G(t) = \int_0^t g(s)ds. \tag{4.11}$$

Using Proposition 6.6 in [3] one gets the following Corollary

Corollary 4.1 *In $B_{R_1}(x^0)$, $v_0 \equiv 1$ is a strict local minimum of $J_{B_{R_1}(x^0)}$ in the space $H^1(B_{R_1}(x^0))$. More precisely, their exist $\alpha > 0$ and $\delta > 0$ for which*

$$J_{B_{R_1}(x^0)}(v) \geq J_{B_{R_1}(x^0)}(v_0) + \alpha\|v - v_0\|^2_{H^1(B_{R_1}(x^0))}, \tag{4.12}$$

for all $v \in H^1(B_{R_1}(x^0))$ such that $\|v - v_0\|^2 \leq \delta$.

And then using Proposition 6.8 of [3] and Corollary 4.1 one gets the following Corollary

Corollary 4.2 *There exist* $\gamma > 0$ *and* $\eta_0 > 0$ *(which depend on* ε*) such that for all* $0 < \eta < \eta_0$ *and* $v \in H_0^1$ *such that* $\|v - v_0\|^2 = \delta$*, then*

$$J_{B_R \backslash K_\varepsilon^\eta}(v_0) < J_{B_R \backslash K_\varepsilon^\eta}(v) - \gamma.$$

The functional $J_{B_R \backslash K_\varepsilon^\eta}$ admits a local minimum in the ball of radius δ around v_0 in $H^1(B_R \backslash K_\varepsilon^\eta) \cap \{v = 0 \text{ on } \partial B_R\}$. This yields a (stable) solution v of (4.8) for small enough $\eta > 0$. Furthermore, provided that δ is chosen small enough, this solution does not coincide either with 1 or with 0 in $B_R \backslash K_\varepsilon^\eta$.
We have proved that for all $\varepsilon \in]0, 1]$, $u_\varepsilon \not\equiv 1$.

One has proved that C^0 convergence of the domain is not sufficient. One can now wonder whether this C^1 convergence would be sufficient. The main difficulties here would be the following

- We cannot apply Theorem 1.5 directly to prove the sufficiency of the C^1 convergence. In deed without the $C^{2,\alpha}$ convergence of the obstacles we cannot use Schauder theory. One can try to relax some assumptions on the regularity of the domain in the Schauder theory, as it has already been done in the literature (L^∞ assumption for coefficient in an elliptic or parabolic equation is enough to get the maximum principle). This remark refers to some technical arguments that will not be further explored in this article.

- One can try to construct a counterexample, as for the C^0 convergence. The problem here would be that we could not use the energy arguments anymore because if we want the convergence to be C^1, the perturbations cannot draw any holes (most important argument in the construction of v_0).

Thus the optimal space of convergence for the obstacles K_ε is still an open problem.

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