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ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA

JEAN-YVES CHARBONNEL

Abstract. The commuting variety of a reductive Lie algebra \( g \) is the underlying variety of a well defined subscheme of \( g \times g \). In this note, it is proved that this scheme is normal. In particular, its ideal of definition is a prime ideal.

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1. Introduction

In this note, the base field \( k \) is algebraically closed of characteristic 0, \( g \) is a reductive Lie algebra of finite dimension, \( \ell \) is its rank, and \( G \) is its adjoint group.

1.1. The dual of \( g \) identifies to \( g \) by a non degenerate symmetric bilinear form on \( g \) extending the Killing form of the derived algebra of \( g \). Denote by \( (v, w) \mapsto \langle v, w \rangle \) this bilinear form and denote by \( I_g \) the ideal of \( k[g \times g] \) generated by the functions \((x, y) \mapsto \langle v, [x, y] \rangle\)'s with \( v \) in \( g \). The commuting variety \( C(g) \) of \( g \) is the subvariety of elements \((x, y)\) of \( g \times g \) such that \([x, y] = 0\). It is the underlying variety to the subscheme \( S(g) \) of \( g \times g \) defined by \( I_g \). It is a well known and long standing open question whether or not this scheme is reduced, that is \( C(g) = S(g) \). According to Richardson [Ri79], \( C(g) \) is irreducible and according to Popov [Po08, Theorem 1], the singular locus of \( S(g) \) has codimension at least 2 in \( C(g) \). Then, according to Serre’s normality criterion, arises the question to know whether or not \( C(g) \) is normal. There are many results about the commuting variety. A result of Dixmier [Di79] proves that \( I_g \) contains all the elements of the radical of \( I_g \) which have degree 1 in the second variable. In [Ga-Gi06], Gan and Ginzburg prove that for \( g \) simple of type A, the invariant elements under \( G \) of \( I_g \) is a radical ideal of the algebra \( k[g \times g]^G \) of invariant elements of \( k[g \times g] \) under \( G \). In [Gi12], Ginzburg proves that the normalisation of \( C(g) \) is Cohen-Macaulay.

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1.2. Main results and sketch of proofs. According to the identification of $\mathfrak{g}$ and its dual, $\mathbb{K}[\mathfrak{g} \times \mathfrak{g}]$ equals the symmetric algebra $S(\mathfrak{g} \times \mathfrak{g})$ of $\mathfrak{g} \times \mathfrak{g}$. The main result of this note is the following theorem:

**Theorem 1.1.** The subscheme of $\mathfrak{g} \times \mathfrak{g}$ defined by $I_\mathfrak{g}$ is Cohen-Macaulay and normal. Furthermore, $I_\mathfrak{g}$ is a prime ideal of $S(\mathfrak{g} \times \mathfrak{g})$.

According to Richardson’s result and Popov’s result, it suffices to prove that the scheme $\mathfrak{g}$ is Cohen-Macaulay. The main idea of the proof in the theorem uses the main argument of the Dixmier’s proof: for a finitely generated module $M$ over $S(\mathfrak{g} \times \mathfrak{g})$, $M = 0$ if the codimension of its support is at least $l + 2$ with $l$ the projective dimension of $M$ (see Appendix A).

Introduce the characteristic submodule of $\mathfrak{g}$, denoted by $B_\mathfrak{g}$. By definition, $B_\mathfrak{g}$ is a submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ and an element $\varphi$ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ is in $B_\mathfrak{g}$ if and only if for all $(x, y)$ in $\mathfrak{g} \times \mathfrak{g}$, $\varphi(x, y)$ is in the sum of subspaces $\mathfrak{g}^{ax+by}$ with $(a, b)$ in $\mathbb{K}^2 \setminus \{0\}$ and $\mathfrak{g}^{ax+by}$ the centralizer of $ax + by$ in $\mathfrak{g}$. According to a Bolsinov’s result, $B_\mathfrak{g}$ is a free $S(\mathfrak{g} \times \mathfrak{g})$-module of rank $b_\mathfrak{g}$, the dimension of the Borel subalgebras of $\mathfrak{g}$. Moreover, the orthogonal complement of $B_\mathfrak{g}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ is a free $S(\mathfrak{g} \times \mathfrak{g})$-module of rank $b_\mathfrak{g} - \ell$. These two results are fundamental in the proof of the following proposition:

**Proposition 1.2.** For $i$ positive integer, the submodule $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g})$ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \wedge^i \wedge^{b_\mathfrak{g}}(\mathfrak{g})$ has projective dimension at most $i$.

Denoting by $E$ the quotient of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ by $B_\mathfrak{g}$, let $E_i$ be the quotient of $\wedge^i(E)$ by its torsion module. The $S(\mathfrak{g} \times \mathfrak{g})$-modules $\wedge^i(\mathfrak{g}) \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g})$ and $E_i$ are isomorphic. Furthermore, for $i \geq 2$, $E_i$ is isomorphic to a direct factor of the quotient of $E \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_{i-1}$ by its torsion module. Denoting by $E_{i-1}$ this quotient, the projective dimension of $E_{i-1}$ is at most $d_{i-1} + 1$ if $d_{i-1}$ is the projective dimension of $E_{i-1}$, whence a proof of the proposition by induction on $i$.

Let $d$ be the $S(\mathfrak{g} \times \mathfrak{g})$-derivation of the algebra $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ such that for all $v$ in $\mathfrak{g}$, $dv$ is the function on $\mathfrak{g} \times \mathfrak{g}$, $(x, y) \mapsto \langle v, [x, y] \rangle$. Then the ideal of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$ generated by $\wedge^{b_\mathfrak{g}}(B_\mathfrak{g})$ is a graded submodule of the graded complex $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{K}} \mathfrak{g}$. The support of the homology of this complex is contained in $C(\mathfrak{g})$. Then we deduce from Proposition 1.2 that this complex has no homology in degree different from $b_\mathfrak{g}$ and $\mathbb{K}[C(\mathfrak{g})]$ is Cohen-Macaulay by Auslander-Buchsbaum’s theorem.

1.3. Notations. • For $V$ a module over a $\mathbb{K}$-algebra, its symmetric and exterior algebras are denoted by $S(V)$ and $\wedge(V)$ respectively. If $E$ is a subset of $V$, the submodule of $V$ generated by $E$ is denoted by $\text{span}(E)$. When $V$ is a vector space over $\mathbb{K}$, the grassmannian of all $d$-dimensional subspaces of $V$ is denoted by $\text{Gr}_d(V)$.

• All topological terms refer to the Zariski topology. If $Y$ is a subset of a topological space $X$, denote by $\overline{Y}$ the closure of $Y$ in $X$. For $Y$ an open subset of the algebraic variety $X$, $Y$ is called a big open subset if the codimension of $X \setminus Y$ in $X$ is at least 2. For $Y$ a closed subset of an algebraic variety $X$, its dimension is the biggest dimension of its irreducible components and its codimension in $X$ is the smallest codimension in $X$ of its irreducible components. For $X$ an algebraic variety, $\mathbb{K}[X]$ is the algebra of regular functions on $X$.

• All the complexes considered in this note are graded complexes over $\mathbb{Z}$ of vector spaces and their differentials are homogeneous of degree $-1$ and they are denoted by $d$. As usual, the gradation of the complex is denoted by $C_d$. 
The dimension of the Borel subalgebras of \( g \) is denoted by \( b_g \). Set \( n := b_g - \ell \) so that \( \dim g = 2b_g - \ell = 2n + \ell \).

- The dual \( g^* \) of \( g \) identifies with \( g \) by a given non degenerate, invariant, symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( g \times g \) extending the Killing form of \([g, g]\).

- For \( x \in g \), denote by \( g^* \) the centralizer of \( x \) in \( g \). The set of regular elements of \( g \) is

\[
\mathfrak{g}_{\text{reg}} := \{ x \in g \mid \dim g^* = \ell \}.
\]

The subset \( \mathfrak{g}_{\text{reg}} \) of \( g \) is a \( G \)-invariant open subset of \( g \). According to [V72], \( g \setminus \mathfrak{g}_{\text{reg}} \) is equidimensional of codimension 3.

- Denote by \( S(g)^0 \) the algebra of \( g \)-invariant elements of \( S(g) \). Let \( p_1, \ldots, p_\ell \) be homogeneous generators of \( S(g)^0 \) of degree \( d_1, \ldots, d_\ell \) respectively. Choose the polynomials \( p_1, \ldots, p_\ell \) so that \( d_1 \leq \cdots \leq d_\ell \).

For \( i = 1, \ldots, \ell \) and \( (x, y) \in g \times g \), consider a shift of \( p_i \) in direction \( y \): \( p_i(x + ty) \) with \( t \in \mathbb{K} \). Expanding \( p_i(x + ty) \) as a polynomial in \( t \), one obtains

\[
(1) \quad p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y) t^m; \quad \forall (t, x, y) \in \mathbb{K} \times g \times g
\]

where \( y \mapsto (m!) p_i^{(m)}(x, y) \) is the derivative at \( x \) of \( p_i \) at the order \( m \) in the direction \( y \). The elements \( p_i^{(m)} \) defined by (1) are invariant elements of \( S(g) \otimes_\mathbb{K} S(g) \) under the diagonal action of \( G \) in \( g \times g \). Remark that \( p_i^{(0)}(x, y) = p_i(x) \) while \( p_i^{(d_i)}(x, y) = p_i(y) \) for all \( (x, y) \in g \times g \).

**Remark 1.3.** The family \( \mathcal{P}_x := \{ p_i^{(m)}(x, \cdot) \}; \ 1 \leq i \leq \ell, 1 \leq m \leq d_i \} \) for \( x \in g \), is a Poisson-commutative family of \( S(g) \) by Mishchenko-Fomenko [MF78]. One says that the family \( \mathcal{P}_x \) is constructed by the argument shift method.

- Let \( i \in \{1, \ldots, \ell \} \). For \( x \) in \( g \), denote by \( e_i(x) \) the element of \( g \) given by

\[
\langle e_i(x), y \rangle = \frac{d}{dt} p_i(x + ty) \big|_{t=0}
\]

for all \( y \) in \( g \). Thereby, \( e_i \) is an invariant element of \( S(g) \otimes_\mathbb{K} g \) under the canonical action of \( G \). According to [Ko63, Theorem 9], for \( x \in g \), \( x \) is in \( \mathfrak{g}_{\text{reg}} \) if and only if \( e_1(x), \ldots, e_\ell(x) \) are linearly independent. In this case, \( e_1(x), \ldots, e_\ell(x) \) is a basis of \( g^* \).

Denote by \( e_i^{(m)} \), for \( 0 \leq m \leq d_i - 1 \), the elements of \( S(g \times g) \otimes_\mathbb{K} g \) defined by the equality:

\[
(2) \quad e_i(x + ty) = \sum_{m=0}^{d_i-1} e_i^{(m)}(x, y) t^m; \quad \forall (t, x, y) \in \mathbb{K} \times g \times g
\]

and set:

\[
V_{x,y} := \text{span}(\langle e_i(x, y)^{(0)}, \ldots, e_i(x, y)^{(d_i-1)} \rangle, i = 1, \ldots, \ell)
\]

for \( (x, y) \) in \( g \times g \).
2. Characteristic module

For \((x, y)\) in \(\mathfrak{g} \times \mathfrak{g}\), set:

\[
V_{x,y}' = \sum_{(a,b)\in \mathbb{K}^{2}\setminus\{0\}} g^{ax+by},
\]

and denote by \(P_{x,y}\) the span of \(x\) and \(y\). By definition, the characteristic module \(B\) of \(\mathfrak{g}\) is the submodule of elements \(\varphi\) of \(S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathfrak{g}} \mathfrak{g}\) such that \(\varphi(x, y)\) is in \(V_{x,y}'\) for all \((x, y)\) in \(\mathfrak{g} \times \mathfrak{g}\). In this section, some properties of \(B\) are given.

2.1. Denote by \(\Omega_{\mathfrak{g}}\) the subset of elements \((x, y)\) of \(\mathfrak{g} \times \mathfrak{g}\) such that \(P_{x,y}\) has dimension 2 and such that \(P_{x,y} \setminus \{0\}\) is contained in \(g_{\text{reg}}\). According to [CM08, Corollary 10], \(\Omega_{\mathfrak{g}}\) is a big open subset of \(\mathfrak{g} \times \mathfrak{g}\).

**Proposition 2.1.** Let \((x, y)\) be in \(\mathfrak{g} \times \mathfrak{g}\) such that \(P_{x,y} \cap g_{\text{reg}}\) is not empty.

(i) Let \(O\) be an open subset of \(\mathbb{K}^{2}\) such that \(ax + by\) is in \(g_{\text{reg}}\) for all \((a, b)\) in \(O\). Then \(V_{x,y}\) is the sum of the \(g^{ax+by}\)'s, \((a, b) \in O\).

(ii) The spaces \([x, V_{x,y}]\) and \([y, V_{x,y}]\) are equal.

(iii) The space \(V_{x,y}\) has dimension at most \(b_{\mathfrak{g}}\) and the equality holds if and only if \((x, y)\) is in \(\Omega_{\mathfrak{g}}\).

(iv) The space \([x, V_{x,y}]\) is orthogonal to \(V_{x,y}\). Furthermore, \((x, y)\) is in \(\Omega_{\mathfrak{g}}\) if and only if \([x, V_{x,y}]\) is the orthogonal complement of \(V_{x,y}\) in \(g\).

(v) The space \(V_{x,y}\) is contained in \(V_{x,y}'\). Moreover, \(V_{x,y} = V_{x,y}'\) if \((x, y)\) is in \(\Omega_{\mathfrak{g}}\).

(vi) For \(i = 1, \ldots, \ell\) and for \(m = 0, \ldots, d_{i} - 1\), \(e_{i}^{(m)}\) is a \(G\)-equivariant map.

**Proof.** (i) For pairwise different elements \(t_{1}, \ldots, t_{\ell}, d_{i} - 1\), \(i = 1, \ldots, \ell\) of \(\mathbb{K} \setminus \{0\}\), the \(e_{i}^{(m)}(x, y)'s, m = 0, \ldots, d_{i} - 1\) are linear combinations of the \(e_{i}(x + t_{i}y)'s, j = 1, \ldots, d_{i} - 1\) for \(i = 1, \ldots, \ell\). Furthermore, for all \(z\) in \(g_{\text{reg}}\), \(e_{1}(z), \ldots, e_{\ell}(z)\) is a basis of \(g^{z}\) by [Ko63, Theorem 9], whence the assertion since the maps \(e_{1}, \ldots, e_{\ell}\) are homogeneous.

(ii) Let \(O\) be an open subset of \((\mathbb{K} \setminus \{0\})^{2}\) such that \(ax + by\) is in \(g_{\text{reg}}\) for all \((a, b)\) in \(O\). For all \((a, b)\) in \(O\), \([x, g^{ax+by}] = [y, g^{ax+by}]\) since \([ax + by, g^{ax+by}] = 0\) and since \(ab \neq 0\), whence the assertion by (i).

(iii) According to [Bou02, Ch. V, §5, Proposition 3],

\[
d_{1} + \cdots + d_{\ell} = b_{\mathfrak{g}},
\]

So \(V_{x,y}\) has dimension at most \(b_{\mathfrak{g}}\). By [Bol91, Theorem 2.1], \(V_{x,y}\) has dimension \(b_{\mathfrak{g}}\) if and only if \((x, y)\) is in \(\Omega_{\mathfrak{g}}\).

(iv) According to [Bol91, Theorem 2.1], \(V_{x,y}\) is a totally isotropic subspace with respect to the skew bilinear form on \(\mathfrak{g}\)

\[
(v, w) \longmapsto \langle ax + by, [v, w] \rangle
\]

for all \((a, b)\) in \(\mathbb{K}^{2}\). As a result, by invariance of \(\langle \cdot, \cdot \rangle\), \(V_{x,y}\) is orthogonal to \([x, V_{x,y}]\). If \((x, y)\) is in \(\Omega_{\mathfrak{g}}\), \(\mathfrak{g}^{x}\) has dimension \(\ell\) and it is contained in \(V_{x,y}\). Hence, by (iii),

\[
\dim [x, V_{x,y}] = b_{\mathfrak{g}} - \ell = \dim \mathfrak{g} - \dim V_{x,y}
\]

so that \([x, V_{x,y}]\) is the orthogonal complement of \(V_{x,y}\) in \(\mathfrak{g}\). Conversely, if \([x, V_{x,y}]\) is the orthogonal complement of \(V_{x,y}\) in \(\mathfrak{g}\), then

\[
\dim V_{x,y} + \dim [x, V_{x,y}] = \dim \mathfrak{g}.
\]
Since $P_{x,y} \cap \mathfrak{g}_{\text{reg}}$ is not empty, $g^{ax+by} \cap V_{x,y}$ has dimension $\ell$ for all $(a, b)$ in a dense open subset of $\mathbb{C}^2$. By continuity, $g \cap V_{x,y}$ has dimension at least $\ell$ so that

$$2\dim V_{x,y} - \ell \geq \dim g.$$ 

Hence, by (iii), $(x, y)$ is in $\Omega_g$.

(iii) According to [Ko63, Theorem 9], for all $z$ in $g$ and for $i = 1, \ldots, \ell$, $e_i(z)$ is in $g^i$. Hence for all $t$ in $\mathbb{C}$, $e_i(x + ty)$ is in $V'_{x,y}$. So $e_i^{(m)}(x, y)$ is in $V'_{x,y}$ for all $m$, whence $V_{x,y} \subseteq V'_{x,y}$.

Suppose that $(x, y)$ is in $\Omega_g$. According to [Ko63, Theorem 9], for all $(a, b)$ in $\mathbb{C}^2 \setminus \{0\}$, $e_i(ax + by), \ldots, e_i(ax + by)$ is a basis of $g^{ax+by}$. Hence $g^{ax+by}$ is contained in $V_{x,y}$, whence the assertion.

Let $i$ be in $\{1, \ldots, \ell\}$. Since $p_i$ is $G$-invariant, $e_i$ is a $G$-equivariant map. As a result, its 2-polarizations $e_i^{(0)}, \ldots, e_i^{(d-1)}$ are $G$-equivariant under the diagonal action of $G$ in $g \times g$. □

Theorem 2.2. (i) The module $B_{\mathfrak{g}}$ is a free module of rank $b_{\mathfrak{g}}$ whose a basis is the sequence $e_i^{(0)}, \ldots, e_i^{(d-1)}$, $i = 1, \ldots, \ell$.

(ii) For $\varphi$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{C} \mathfrak{g}$, $\varphi$ is in $B_{\mathfrak{g}}$ if and only if $p\varphi \in B_{\mathfrak{g}}$ for some $p$ in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$.

(iii) For all $\varphi$ in $B_{\mathfrak{g}}$ for all $(x, y)$ in $\mathfrak{g} \times \mathfrak{g}$, $\varphi(x, y)$ is orthogonal to $[x, y]$.

Proof. (i) and (ii) According to Proposition 2.1.(v), $e_i^{(m)}$ is in $B_{\mathfrak{g}}$ for all $(i, m)$. Moreover, according to Proposition 2.1.(iii), these elements are linearly independent over $S(\mathfrak{g} \times \mathfrak{g})$. Let $\varphi$ be an element of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{C} \mathfrak{g}$ such that $p\varphi$ is in $B_{\mathfrak{g}}$ for some $p$ in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$, for all $(x, y)$ in a dense open subset of $\Omega_{\mathfrak{g}}$, $\varphi(x, y)$ is in $V_{x,y}$ by Proposition 2.1.(v). According to Proposition 2.1.(iii), the map

$$\Omega_{\mathfrak{g}} \rightarrow \text{Gr}_{B_{\mathfrak{g}}}(\mathfrak{g}), \quad (x, y) \mapsto V_{x,y}$$

is regular. So, $\varphi(x, y)$ is in $V_{x,y}$ for all $(x, y)$ in $\Omega_{\mathfrak{g}}$ and for some regular functions $a_{i,m}$, $i = 1, \ldots, \ell$, $m = 0, \ldots, d_i - 1$ on $\Omega_{\mathfrak{g}}$,

$$\varphi(x, y) = \sum_{i=1}^{\ell} \sum_{m=0}^{d_i-1} a_{i,m}(x, y)e_i^{(m)}(x, y)$$

for all $(x, y)$ in $\Omega_{\mathfrak{g}}$. Since $\Omega_{\mathfrak{g}}$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and since $\mathfrak{g} \times \mathfrak{g}$ is normal, the $a_{i,m}$’s have a regular extension to $\mathfrak{g} \times \mathfrak{g}$. Hence $\varphi$ is a linear combination of the $e_i^{(m)}$’s with coefficients in $S(\mathfrak{g} \times \mathfrak{g})$. As a result, the sequence $e_i^{(m)}$, $i = 1, \ldots, \ell$, $m = 0, \ldots, d_i - 1$ is a basis of the module $B_{\mathfrak{g}}$ and $B_{\mathfrak{g}}$ is the subset of elements $\varphi$ of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{C} \mathfrak{g}$ such that $p\varphi$ is in $B_{\mathfrak{g}}$ for some $p$ in $S(\mathfrak{g} \times \mathfrak{g}) \setminus \{0\}$.

(iii) Let $\varphi$ be in $B_{\mathfrak{g}}$. According to (i) and Proposition 2.1.(iv), for all $(x, y)$ in $\Omega_{\mathfrak{g}}$, $[x, \varphi(x, y)]$ is orthogonal to $V_{x,y}$. Then, since $y$ is in $V_{x,y}$, $[x, \varphi(x, y)]$ is orthogonal to $y$ and $\langle \varphi(x, y), [x, y] \rangle = 0$, whence the assertion. □

2.2. Also denote by $\langle \langle .. \rangle \rangle$ the natural extension of $\langle .. \rangle$ to the module $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{C} \mathfrak{g}$.

Proposition 2.3. Let $C_{\mathfrak{g}}$ be the orthogonal complement of $B_{\mathfrak{g}}$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{C} \mathfrak{g}$.

(i) For $\varphi$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_\mathbb{C} \mathfrak{g}$, $\varphi$ is in $C_{\mathfrak{g}}$ if and only if $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all $(x, y)$ in a nonempty open subset of $\mathfrak{g} \times \mathfrak{g}$.

(ii) The module $C_{\mathfrak{g}}$ is free of rank $b_{\mathfrak{g}} - \ell$. Furthermore, the sequence of maps

$$(x, y) \mapsto [x, e_i^{(1)}(x, y)], \ldots, (x, y) \mapsto [x, e_i^{(d_i-1)}(x, y)], \quad i = 1, \ldots, \ell$$
is a basis of $C_\emptyset$.

(iii) The orthogonal complement of $C_\emptyset$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{k} \mathfrak{g}$ equals $B_{\emptyset}$.

Proof. (i) Let $\varphi$ be in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{k} \mathfrak{g}$. If $\varphi$ is in $C_\emptyset$, then $\varphi(x, y)$ is orthogonal to $V_{x,y}$ for all $(x, y)$ in $\Omega_\emptyset$. Then, according to Proposition 2.1,(iv), $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all $(x, y)$ in $\Omega_\emptyset$. Conversely, suppose that $\varphi(x, y)$ is in $[x, V_{x,y}]$ for all $(x, y)$ in a nonempty open subset $V$ of $\mathfrak{g} \times \mathfrak{g}$. By Proposition 2.1.(iv) again, for all $(x, y)$ in $V \cap \Omega_\emptyset$, $\varphi(x, y)$ is orthogonal to the $\varepsilon^{(m)}_i(x, y)$'s, $i = 1, \ldots, \ell$, $m = 0, \ldots, d_1 - 1$, whence the assertion by Theorem 2.1.

(ii) Let $C$ be the submodule of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{k} \mathfrak{g}$ generated by the maps

$$(x, y) \mapsto [x, \epsilon^{(1)}_i(x, y)], \ldots, (x, y) \mapsto [x, \epsilon^{(d_1-1)}_i(x, y)], \ i = 1, \ldots, \ell$$

According to (i), $C$ is a submodule of $C_\emptyset$. This module is free of rank $b_\emptyset - \ell$ since $[x, V_{x,y}]$ has dimension $b_\emptyset - \ell$ for all $(x, y)$ in $\Omega_\emptyset$ by Proposition 2.1. (iii) and (iv). According to (i), for $\varphi$ in $C_\emptyset$, for all $(x, y)$ in $\Omega_\emptyset$,

$$\varphi(x, y) = \sum_{i=1}^{\ell} \sum_{m=1}^{d_1-1} a_{i,m}(x, y)[x, \epsilon^{(m)}_i(x, y)]$$

with the $a_{i,m}$'s regular on $\Omega_\emptyset$ and uniquely defined by this equality. Since $\Omega_\emptyset$ is a big open subset of $\mathfrak{g} \times \mathfrak{g}$ and since $\mathfrak{g} \times \mathfrak{g}$ is normal, the $a_{i,m}$'s have a regular extension to $\mathfrak{g} \times \mathfrak{g}$. As a result, $\varphi$ is in $C$, whence the assertion.

(iii) Let $\varphi$ be in the orthogonal complement of $C_\emptyset$ in $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{k} \mathfrak{g}$. According to (ii), for all $(x, y)$ in $\Omega_\emptyset$, $\varphi(x, y)$ is orthogonal to $[x, V_{x,y}]$. Hence by Proposition 2.1,(iv), $\varphi(x, y)$ is in $V_{x,y}$ for all $(x, y)$ in $\Omega_\emptyset$. So, by Theorem 2.1, $\varphi$ is in $B_{\emptyset}$, whence the assertion. \hfill \Box

Denote by $B$ and $\ell$ the localizations of $B_{\emptyset}$ and $C_{\emptyset}$ on $\mathfrak{g} \times \mathfrak{g}$ respectively. For $(x, y)$ in $\mathfrak{g} \times \mathfrak{g}$, let $C_{x,y}$ be the image of $C_{\emptyset}$ by the evaluation map at $(x, y)$.

Lemma 2.4. There exists an affine open cover $O$ of $\Omega_\emptyset$ verifying the following condition: for all $O$ in $O$, there exist some subspaces $E$ and $F$ of $\mathfrak{g}$, depending on $O$, such that

$$\mathfrak{g} = E \oplus V_{x,y} = F \oplus C_{x,y}$$

for all $(x, y)$ in $O$. Moreover, for all $(x, y)$ in $O$, the orthogonal complement of $V_{x,y}$ in $\mathfrak{g}$ equals $C_{x,y}$.

Proof. According to Proposition 2.1,(iii) and (iv), for all $(x, y)$ in $\Omega_\emptyset$, $V_{x,y}$ and $C_{x,y}$ have dimension $b_\emptyset$ and $b_\emptyset - \ell$ respectively so that the maps

$$\Omega_\emptyset \longrightarrow \text{Gr}_{b_\emptyset}(\mathfrak{g}), \quad (x, y) \longmapsto V_{x,y}, \quad \Omega_\emptyset \longrightarrow \text{Gr}_{b_\emptyset - \ell}(\mathfrak{g}), \quad (x, y) \longmapsto C_{x,y}$$

are regular, whence the assertion. \hfill \Box

3. Torsion and projective dimension

Let $E$ and $E^\#$ be the quotients of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{k} \mathfrak{g}$ by $B_{\emptyset}$ and $C_{\emptyset}$ respectively. For $i$ positive integer, denote by $E_i$ the quotient of $\wedge^i(E)$ by its torsion module.
3.1. Let $B^*_g$ and $C^*_g$ be the duals of $B_g$ and $C_g$ respectively.

**Lemma 3.1.** (i) The $S(g \times g)$-modules $E$ and $E^#$ have projective dimension at most 1.
(ii) The $S(g \times g)$-modules $E$ and $E^#$ are torsion free.
(iii) The modules $C_g$ and $B_g$ are the duals of $E$ and $E^#$ respectively.
(iv) The canonical morphism from $E$ to $C^*_g$ is an embedding.

**Proof.** (i) By definition, the short sequences of $S(g \times g)$-modules,
\[
0 \to B_g \to S(g \times g) \otimes_k g \to E \to 0
\]
and
\[
0 \to C_g \to S(g \times g) \otimes_k g \to E^# \to 0
\]
are exact. Hence $E$ and $E^#$ have projective dimension at most 1 since $B_g$ and $C_g$ are free modules by Theorem 2.2 and Proposition 2.3.(ii).
(ii) The module $E$ is torsion free by Theorem 2.2.(ii). By definition, for $\varphi$ in $S(g \times g) \otimes_k g$, $\varphi$ is in $C_g$ if $p\varphi$ is in $C_g$ for some $p$ in $S(g \times g) \setminus \{0\}$, whence $E^#$ is torsion free.
(iii) According to the exact sequences of (i), the dual of $E$ in $S(g \times g) \otimes_k g$ and the dual of $E^#$ is the orthogonal complement of $C_g$ in $S(g \times g) \otimes_k g$, whence the assertion since $C_g$ is the orthogonal complement of $B_g$ in $S(g \times g) \otimes_k g$ by definition and since $B_g$ is the orthogonal complement of $C_g$ in $S(g \times g) \otimes_k g$ by Proposition 2.3.(iii).
(iv) Let $\bar{\omega}$ be in the kernel of the canonical morphism from $E$ to $C^*_g$. Let $\omega$ be a representative of $\bar{\omega}$ in $S(g \times g) \otimes_k g$. According to Proposition 2.3.(iii), $B_g$ is the orthogonal complement of $C_g$ in $S(g \times g) \otimes_k g$ so that $\omega$ is in $B_g$, whence the assertion. \qed

Set:
\[
E = \bigwedge_{i=1}^\ell \epsilon_i^{(d)} \wedge \cdots \wedge \epsilon_i^{(d-1)}
\]
and for $i$ positive integer, denote by $\theta_i$ the morphism
\[
S(g \times g) \otimes_k \bigwedge^i (g) \to \bigwedge^i (g) \cap \bigwedge^{b_i}(B_g), \quad \varphi \mapsto \varphi \wedge \epsilon.
\]

**Proposition 3.2.** Let $i$ be a positive integer.
(i) The morphism $\theta_i$ defines through the quotient an isomorphism from $E_i$ onto $\bigwedge^i (g) \cap \bigwedge^{b_i}(B_g)$.
(ii) The short sequence of $S(g \times g)$-modules
\[
0 \to B_g \otimes_{S(g \times g)} E_i \to g \otimes_k E_i \to E \otimes_{S(g \times g)} E_i \to 0
\]
is exact.

**Proof.** (i) For $j$ positive integer, denote by $\pi_j$ the canonical map from $S(g \times g) \otimes_k \bigwedge^j (g)$ to $\bigwedge^j (E)$. Let $\omega$ be in the kernel of $\pi_i$. Let $O$ be an element of the affine open cover of $\Omega_g$ of Lemma 2.4 and let $W$ be a subspace of $g$ such that
\[
g = W \oplus V_{x,y}
\]
for all $(x, y)$ in $O$ so that $\pi_1$ induces an isomorphism
\[
\mathbb{k}[O] \otimes_k W \to \mathbb{k}[O] \otimes_{S(g \times g)} E
\]
Moreover, $B_\emptyset$ is the kernel of $\pi_1$. Then, from the equality

$$\mathcal{O}[\mathfrak{g}] \otimes_{\mathfrak{k}} \Lambda^i(\emptyset) = \bigoplus_{j=0}^i \Lambda^j(W) \otimes \mathcal{O}[\mathfrak{g}] \otimes_{\mathfrak{k}} \Lambda^{i-j}(B_\emptyset)$$

it results that the restriction of $\omega$ to $O$ is in $\mathcal{O}[\mathfrak{g}] \otimes_{\mathfrak{k}} \Lambda^{i-1}(\emptyset) \wedge B_\emptyset$. Hence the restriction of $\omega \wedge \varepsilon$ to $O$ equals 0 and $\omega$ is in the kernel of $\theta_i$ since $\bigwedge^i(\emptyset) \wedge b_i(B_\emptyset)$ has no torsion as a submodule of a free module.

As a result, $\theta_i$ defines through the quotient a morphism from $\bigwedge^i(E)$ to $\bigwedge^i(\emptyset) \wedge \bigwedge^{b_i}(B_\emptyset)$. Denote it by $\theta_i'$. Since $\bigwedge^i(\emptyset) \wedge b_i(B_\emptyset)$ is torsion free, the torsion submodule of $\bigwedge^i(E)$ is contained in the kernel of $\theta_i'$. Hence $\theta_i'$ defines through the quotient a morphism from $E_i$ to $\bigwedge^i(\emptyset) \wedge \bigwedge^{b_i}(B_\emptyset)$. Denoting it by $\theta_i$, $\theta_i'$ and $\theta_i$ are surjective since too is $\theta_i$.

Let $\overline{\omega}$ be in the kernel of $\theta_i'$ and let $\omega$ be a representative of $\overline{\omega}$ in $S(\emptyset \times \emptyset) \otimes_{\mathfrak{k}} \bigwedge^i(\emptyset)$. Then $\omega \wedge \varepsilon = 0$ so that the restriction of $\omega$ to the above open subset $O$ is in $\mathcal{O}[\mathfrak{g}] \otimes_{\mathfrak{k}} \bigwedge^{i-1}(\emptyset) \wedge B_\emptyset$. As a result, the restriction of $\overline{\omega}$ to $O$ equals 0. So, $\overline{\omega}$ is in the torsion submodule of $\bigwedge^i(E)$, whence the assertion.

(ii) By definition, the sequence

$$0 \rightarrow B_\emptyset \rightarrow S(\emptyset \times \emptyset) \otimes_{\mathfrak{k}} \emptyset \rightarrow E \rightarrow 0$$

is exact. Then the sequence

$$\text{Tor}_1^{S(\emptyset \times \emptyset)}(E, E_i) \rightarrow B_\emptyset \otimes_{S(\emptyset \times \emptyset)} E_i \rightarrow \emptyset \otimes_{\mathfrak{k}} E_i \rightarrow E \otimes_{S(\emptyset \times \emptyset)} E_i \rightarrow 0$$

is exact. By definition, $E_i$ is torsion free. As a result, $B_\emptyset \otimes_{S(\emptyset \times \emptyset)} E_i$ is torsion free since $B_\emptyset$ is a free module. Then, since $\text{Tor}_1^{S(\emptyset \times \emptyset)}(E, E_i)$ is a torsion module, its image in $B_\emptyset \otimes_{S(\emptyset \times \emptyset)} E_i$ equals 0, whence the assertion. \hfill\Box

3.2. For $i$ positive integer, $\langle \cdot, \cdot \rangle$ has a canonical extension to $S(\emptyset \times \emptyset) \otimes_{\mathfrak{k}} \bigwedge^i(\emptyset)$ denoted again by $\langle \cdot, \cdot \rangle$.

**Lemma 3.3.** Let $i$ be a positive integer. Let $T_i$ be the torsion module of $E \otimes_{S(\emptyset \times \emptyset)} E_i$ and let $T'_i$ be its inverse image by the canonical morphism $\emptyset \otimes_{\mathfrak{k}} E_i \rightarrow E \otimes_{S(\emptyset \times \emptyset)} E_i$.

(i) The canonical morphism from $\bigwedge^i(E)$ to $\bigwedge^i(C^*_\emptyset)$ defines through the quotient an embedding of $E_i$ into $\bigwedge^i(C^*_\emptyset)$.

(ii) The module of $T'_i$ is the intersection of $\emptyset \otimes_{\mathfrak{k}} E_i$ and $B_\emptyset \otimes_{S(\emptyset \times \emptyset)} \bigwedge^i(C^*_\emptyset)$.

(iii) The module $T'_i$ is isomorphic to $\text{Hom}_{S(\emptyset \times \emptyset)}(E^\emptyset, E_i)$.

**Proof.** (i) According to Lemma 3.1.(iii), there is a canonical morphism from $\bigwedge^i(E)$ to $\bigwedge^i(C^*_\emptyset)$. Let $\overline{\omega}$ be in its kernel and let $\omega$ be a representative of $\overline{\omega}$ in $S(\emptyset \times \emptyset) \otimes_{\mathfrak{k}} \bigwedge^i(\emptyset)$. Then $\omega$ is orthogonal to $\Lambda^i(C^*_\emptyset)$ with respect to $\langle \cdot, \cdot \rangle$. So for $O$ as in Lemma 2.4, the restriction of $\omega$ to $O$ is in $\mathcal{O}[\mathfrak{g}] \otimes_{S(\emptyset \times \emptyset)} \Lambda^{i-1}(\emptyset) \wedge B_\emptyset$. Hence the restriction of $\overline{\omega}$ to $O$ equals 0. In other words, $\overline{\omega}$ is in the torsion module of $\bigwedge^i(E)$, whence the assertion since $\bigwedge^i(C^*_\emptyset)$ is a free module.

(ii) Since $\bigwedge^i(C^*_\emptyset)$ is a free module, by Proposition 3.2.(ii), there is a morphism of short exact sequences

$$0 \rightarrow B_\emptyset \otimes_{S(\emptyset \times \emptyset)} E_i \rightarrow \emptyset \otimes_{\mathfrak{k}} E_i \rightarrow E \otimes_{S(\emptyset \times \emptyset)} E_i \rightarrow 0$$

$$0 \rightarrow B_\emptyset \otimes_{S(\emptyset \times \emptyset)} \bigwedge^i(C^*_\emptyset) \rightarrow \emptyset \otimes_{\mathfrak{k}} \bigwedge^i(C^*_\emptyset) \rightarrow E \otimes_{S(\emptyset \times \emptyset)} \bigwedge^i(C^*_\emptyset) \rightarrow 0$$
Moreover, the two first vertical arrows are embeddings. Hence $T'_j$ is the intersection of $g \otimes_k E$ and $B_\mathfrak{g} \otimes_{S(g \times g)} \wedge^1(C^*_g)$ in $g \otimes_k \wedge^1(C^*_g)$.

(iii) According to the identification of $g$ with its dual, $g \otimes_k E_i = \text{Hom}_k(g, E_i)$. Moreover, according to the short exact sequence of $S(g \times g)$-modules

$$0 \to C_g \to S(g \times g) \otimes_k g \to E^\# \to 0$$

the sequence of $S(g \times g)$-modules

$$0 \to \text{Hom}_{S(g \times g)}(E^\#, E_i) \to \text{Hom}_k(g, E_i) \to \text{Hom}_{S(g \times g)}(C_g, E_i) \to \text{Ext}_{S(g \times g)}^1(E^\#, E_i)$$

is exact. For $\varphi$ in $\text{Hom}_{S(g \times g)}(S(g \times g) \otimes_k g, E_i)$, $\varphi$ is in the kernel of the third arrow if and only if $C_g$ is contained in the kernel of $\varphi$. On the other hand, according to the identification of $g$ and its dual, $\text{Hom}_k(g, \wedge^1(C^*_g))$ identifies with $g \otimes_k \wedge^1(C^*_g)$. By Proposition 2.3, $B_\mathfrak{g}$ is the orthogonal complement of $C_g$ in $S(g \times g) \otimes_k g$ and $C_g^*$ is a free $S(g \times g)$-module. So, for $\psi$ in $\text{Hom}_{S(g \times g)}(S(g \times g) \otimes_k g, \wedge^1(C^*_g))$, $\psi$ equals $0$ on $C_g$ if and only if it is in $B_\mathfrak{g} \otimes_{S(g \times g)} \wedge^1(C^*_g)$, whence the assertion by (ii).

The following corollary results from Lemma 3.3.

**Corollary 3.4.** Let $i$ be a positive integer and let $E_i$ be the quotient of $E \otimes_{S(g \times g)} E_i$ by its torsion module. Then the short sequence of $S(g \times g)$-modules

$$0 \to \text{Hom}_{S(g \times g)}(E^\#, E_i) \to g \otimes_k E_i \to \overline{E_i} \to 0$$

is exact.

3.3. Denote by $\text{Mod}_{S(g \times g)}$ the category of finite $S(g \times g)$-modules. Let $\iota$ be the morphism

$$S(g \times g) \otimes_k g \to C_g^*, \quad v \rightarrow (\mu \mapsto \langle v, \mu \rangle)$$

**Lemma 3.5.** Let $A_\mathfrak{g}$ be the quotient of $C_g^*$ by $\iota(S(g \times g) \otimes_k g)$. Then the two functors $A_\mathfrak{g} \otimes_{S(g \times g)} \bullet$ and $\text{Ext}_{S(g \times g)}^1(E^\#, \bullet)$ of the category $\text{Mod}_{S(g \times g)}$ are isomorphic.

**Proof.** For $d$ nonnegative integer, denote by $\text{Mod}_{S(g \times g)}(d)$ the full subcategory of $\text{Mod}_{S(g \times g)}$ whose objects are the modules of projective dimension at most $d$. Prove by induction on $d$ that the restrictions to $\text{Mod}_{S(g \times g)}(d)$ of the functors $\text{Ext}_{S(g \times g)}^1(E^\#, \bullet)$ and $A_\mathfrak{g} \otimes_{S(g \times g)} \bullet$ are isomorphic. Let $M$ be a finite $S(g \times g)$-module. Denoting by $d$ its projective dimension, there is a short exact sequence

$$0 \to Z \to P \to M \to 0$$

with $Z$ a module of projective dimension $d - 1$ if $d > 0$ and $Z = 0$ otherwise.

Suppose $d = 0$. Then, from the short exact sequence

$$0 \to C_g \to S(g \times g) \otimes_k g \to E^\# \to 0$$

one deduces the exact sequence

$$0 \to \text{Hom}_{S(g \times g)}(E^\#, M) \to \text{Hom}_k(g, M) \to \text{Hom}_{S(g \times g)}(C_g, M) \to \text{Ext}_{S(g \times g)}^1(E^\#, M) \to 0.$$
Since $C_\eta$ is a free module, $\text{Hom}_{S(\times g)}(C_\eta, M)$ is functorially isomorphic to $C_\eta^* \otimes_{S(\times g)} M$. Then by the right exactness of the functor $A_\eta \otimes_{S(\times g)} \bullet$, there is an isomorphism of exact sequences

$$
\begin{array}{ccccccccc}
\text{Hom}_k(g, M) & \longrightarrow & \text{Hom}_{S(\times g)}(C_\eta, M) & \longrightarrow & \text{Ext}^1_{S(\times g)}(E^#, M) & \longrightarrow & 0 \\
\delta_0 & & \delta_1 & & \delta & & \\
\eta \otimes_k M & \longrightarrow & C_\eta^* \otimes_{S(\times g)} M & \longrightarrow & A_\eta \otimes_{S(\times g)} M & \longrightarrow & 0 
\end{array}
$$

Since the two sequences depends functorially on $M$, from the isomorphisms of functors

$$
\eta \otimes_k \bullet \longrightarrow \text{Hom}_k(g, \bullet), \quad C_\eta^* \otimes_{S(\times g)} \bullet \longrightarrow \text{Hom}_{S(\times g)}(C_\eta, \bullet),
$$

we deduce that the restrictions to $\text{Mod}_{S(\times g)}(0)$ of the functors $\text{Ext}^1_{S(\times g)}(E^#, \bullet)$ and $A_\eta \otimes_{S(\times g)} \bullet$ are isomorphic.

Suppose the statement true for $d - 1$. Setting $Q := \iota(S(\times g) \otimes_k g)$, one has two short exact sequences

$$
0 \longrightarrow B_\eta \longrightarrow S(\times g) \otimes_k g \longrightarrow Q \longrightarrow 0, \quad 0 \longrightarrow Q \longrightarrow C_\eta^* \longrightarrow A_\eta \longrightarrow 0.
$$

Since $E^#$ has projective dimension at most 1 by Lemma 3.1, $\text{Ext}^2_{S(\times g)}(E^#, Z) = 0$. Then, by induction hypothesis, one has a commutative diagram

$$
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & \\
\text{Ext}^1_{S(\times g)}(E^#, Z) & \longrightarrow & \text{Ext}^1_{S(\times g)}(E^#, P) & \longrightarrow & \text{Ext}^1_{S(\times g)}(E^#, M) & \longrightarrow & 0 \\
\delta & & \delta & & \delta & & \\
Q \otimes_{S(\times g)} Z & \longrightarrow & C_\eta^* \otimes_{S(\times g)} P & \longrightarrow & C_\eta^* \otimes_{S(\times g)} M & \longrightarrow & 0 \\
\delta & & \delta & & \delta & & \\
Q \otimes_{S(\times g)} Z & \longrightarrow & Q \otimes_{S(\times g)} P & \longrightarrow & Q \otimes_{S(\times g)} M & \longrightarrow & 0 
\end{array}
$$

with exact lines and columns since $C_\eta^*$ is a free module. Let $a$ and $a'$ be in $C_\eta^* \otimes_{S(\times g)} P$ such that $da = da'$. Then $a - a' = da_1$ with $a_1$ in $C_\eta^* \otimes_{S(\times g)} Z$ so that

$$
d \cdot da - d \cdot da' = d \cdot \delta - d \cdot da_1 = 0
$$

whence a morphism

$$
C_\eta^* \otimes_{S(\times g)} M \xrightarrow{\delta_M} \text{Ext}^1_{S(\times g)}(E^#, M)
$$

uniquely defined by the equality $\delta_M \cdot d = d \cdot \delta$.

Let $a$ be in $\text{Ext}^1_{S(\times g)}(E^#, M)$. Then

$$
a = d \cdot \delta a_1 = \delta_M \cdot da_1 \quad \text{with} \quad a_1 \in C_\eta^* \otimes_{S(\times g)} P.
$$

Hence $\delta_M$ is surjective. Let $b$ be in the kernel of $\delta_M$. Then

$$
b = db_1, \quad d \cdot \delta b_1 = 0, \quad \delta b_1 = d \cdot \delta b_2 \quad \text{with} \quad b_1 \in C_\eta^* \otimes_{S(\times g)} P, \quad b_2 \in C_\eta^* \otimes_{S(\times g)} Z.
$$
so that \( b_1 - db_2 = \delta b_3 \) with \( b_3 \) in \( Q \otimes_{S(\mathfrak{g} \times \mathfrak{g})} P \), whence \( b = \delta \cdot db_3 \). As a result, the above diagram is canonically completed by an exact third column and one has an isomorphism of short exact sequences

\[
\begin{array}{c}
\Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, Z) \longrightarrow \Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, P) \longrightarrow \Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, M) \longrightarrow 0.
\end{array}
\]

Since the two sequences depends functorially on the short exact sequence

\[
0 \longrightarrow Z \longrightarrow P \longrightarrow M \longrightarrow 0
\]

and since the restrictions to \( \Mod_{S(\mathfrak{g} \times \mathfrak{g})}(d - 1) \) of the two functors \( \Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, \bullet) \) and \( A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} \bullet \) are isomorphic, too is their restrictions to \( \Mod_{S(\mathfrak{g} \times \mathfrak{g})}(d) \), whence the lemma since all object of \( \Mod_{S(\mathfrak{g} \times \mathfrak{g})} \) has a finite projective dimension.

□

From the exact sequence,

\[
0 \longrightarrow B_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{Z}} \mathfrak{g} \longrightarrow C^*_g \longrightarrow A_{\mathfrak{g}} \longrightarrow 0,
\]

we deduce the graded homology complex,

\[
0 \longrightarrow B_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow \mathfrak{g} \otimes_{\mathbb{Z}} E_i \longrightarrow C^*_g \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \longrightarrow 0
\]

denoted by \( C_* \). For \( i \) positive integer, let \( d_i \) and \( d'_i \) be the projective dimensions of \( E_i \) and \( \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \).

**Lemma 3.6.** Let \( Q \) be the space of cycles of degree 2 of the complex \( C_* \).

(i) Denoting by \( d''_i \) the projective dimension of \( \Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \), \( d'_i \) is at most \( \sup[d''_i - 2, d_i] \).

(ii) The complex \( C_* \) has no homology in degree 0, 1 and 3. Moreover, \( Q \) identifies with \( \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \).

(iii) The module \( \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \) has projective dimension at most \( d_i \).

**Proof.** (i) From the short exact sequence

\[
0 \longrightarrow C_{\mathfrak{g}} \longrightarrow S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathbb{Z}} \mathfrak{g} \longrightarrow E^# \longrightarrow 0
\]

one deduces the exact sequence

\[
0 \longrightarrow \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \longrightarrow \Hom_{\mathfrak{g}}(\mathfrak{g}, E_i) \longrightarrow \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(C_g, E_i) \longrightarrow \Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \longrightarrow 0
\]

whence the two short exact sequences

\[
0 \longrightarrow \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \longrightarrow \Hom_{\mathfrak{g}}(\mathfrak{g}, E_i) \longrightarrow Z \longrightarrow 0
\]

\[
0 \longrightarrow Z \longrightarrow \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(C_g, E_i) \longrightarrow \Ext^1_{S(\mathfrak{g} \times \mathfrak{g})}(E^#, E_i) \longrightarrow 0
\]

with \( Z \) the image of the arrow

\[
\Hom_{\mathfrak{g}}(\mathfrak{g}, E_i) \longrightarrow \Hom_{S(\mathfrak{g} \times \mathfrak{g})}(C_g, E_i)
\]

Denoting by \( d \) the projective dimension of \( Z \), one deduces the inequalities

\[
d'_i \leq \sup[d - 1, d_i], \quad d \leq \sup[d''_i - 1, d_i]
\]

since \( C_{\mathfrak{g}} \) is a free module, whence the assertion.
(ii) By right exactness of the functor • ⊗_{S(\mathfrak{g} \times \mathfrak{g})} E_i, C_{\mathfrak{g}} has no homology in degree 0 and 1. Moreover, its space of cycles of degree 3 is a torsion submodule of C_3. Since E_i is torsion free and since B_{\mathfrak{g}} is free, C_3 has no torsion. Hence C_{\mathfrak{g}} has no homology in degree 3. According to Lemma 3.3, (ii) and (iii), Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) identifies with a submodule of \mathfrak{g} \otimes_{\mathfrak{k}} E_i. According to these identifications, Q is the space of morphisms from S(\mathfrak{g} \times \mathfrak{g}) \otimes_{\mathfrak{k}} \mathfrak{g} to E_i, equal to 0 on C_{\mathfrak{g}}, that is Q = Hom_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i).

(iii) By (ii), one has a short exact sequence

\[
0 \to \text{Hom}_{S(\mathfrak{g} \times \mathfrak{g})}(E^\#, E_i) \to C^*_\mathfrak{g} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \to \text{A}_\mathfrak{g} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i \to 0.
\]

So, A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i has projective dimension at most sup{d_i' + 1, d_i}. According to Lemma 3.5, A_{\mathfrak{g}} \otimes_{S(\mathfrak{g} \times \mathfrak{g})} E_i and Ext_{S(\mathfrak{g} \times \mathfrak{g})}^1(E^\#, E_i) are isomorphic. So by (i),

\[
d_i' \leq \text{sup}(d_i' - 1, d_i),
\]

whence \(d_i' \leq d_i\). □

The following corollary results from Corollary 3.4 and Lemma 3.6, (iii), since B_{\mathfrak{g}} is free.

**Corollary 3.7.** Let \(i\) be a positive integer. Then \(E_i\) has projective dimension at most \(d_i + 1\).

### 3.4. For \(i\) a positive integer and for \(M\) a \(S(\mathfrak{g} \times \mathfrak{g})\)-module, let consider on \(M^{\#i}\) the canonical action of the symmetric group \(\mathfrak{S}_i\). For \(\sigma\) in \(\mathfrak{S}_i\), denote by \(\epsilon(\sigma)\) its signature. Let \(M^{\#i}_{\text{sign}}\) be the submodule of elements \(a\) of \(M^{\#i}\) such that \(\sigma.a = \epsilon(\sigma)a\) for all \(\sigma\) in \(\mathfrak{S}_i\) and let \(\delta_i\) be the endomorphism of \(M^{\#i}_{\text{sign}}\),

\[
a \mapsto \delta_i(a) = \frac{1}{i!} \sum_{\sigma \in \mathfrak{S}_i} \epsilon(\sigma)\sigma.a.
\]

Then \(\delta_i\) is a projection of \(M^{\#i}\) onto \(M^{\#i}_{\text{sign}}\).

For \(L\) submodule of \(C^{\#}_{\mathfrak{g}}\), denote by \(L_i\) the image of \(L^{\#i}\) by the canonical map from \(L^{\#i}\) to \((C^{\#}_{\mathfrak{g}})^{\#i}\) and set \(L_{i,\text{sign}} := L_i \cap (C^{\#}_{\mathfrak{g}})^{\#i}_{\text{sign}}\). Let \(\wedge^i(L)\) be the quotient of \(\wedge^i(L)\) by its torsion module. For \(i \geq 2\), identify \(\mathfrak{S}_{i-1}\) with the stabilizer of \(i\) in \(\mathfrak{S}_i\) and denote by \(L_{i-1,\text{sign},1}\) the submodule of elements \(a\) of \(L_i\) such that \(\sigma.a = \epsilon(\sigma)a\) for all \(\sigma\) in \(\mathfrak{S}_{i-1}\).

**Lemma 3.8.** Let \(i\) be a positive integer and let \(L\) be a submodule of \(C^{\#}_{\mathfrak{g}}\).

(i) The module \(L_i\) is isomorphic to the quotient of \(L^{\#i}\) by its torsion module.

(ii) The module \(L_{i,\text{sign}}\) is isomorphic to \(\wedge^i(L)\).

(iii) For \(i \geq 2\), the module \(L_{i,\text{sign}}\) is a direct factor of \(L_{i-1,\text{sign},1}\).

(iv) For \(i \geq 2\), the module \(L_{i-1,\text{sign},1}\) is isomorphic to the quotient of \(\wedge^{i-1}(L) \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L\) by its torsion module.

**Proof.** (i) Let \(L_1\) and \(L_2\) be submodules of a free module \(F\) over \(S(\mathfrak{g} \times \mathfrak{g})\). From the short exact sequence

\[
0 \to L_2 \to F \to F/L_2 \to 0
\]

one deduces the exact sequence

\[
\text{Tor}^1_{S(\mathfrak{g} \times \mathfrak{g})}(L_1, L_2) \to L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} L_2 \to L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} F \to L_1 \otimes_{S(\mathfrak{g} \times \mathfrak{g})} (F/L_2) \to 0.
\]
Since $F$ is free, $L_1 \otimes_{S_1(q \times q)} F$ is torsion free. Hence the kernel of the second arrow is the torsion module of $L_1 \otimes_{S_1(q \times q)} L_2$ since $\text{Tors}_1^{S_1(q \times q)}(L_1, L_2)$ is a torsion module, whence the assertion by induction on $i$.

(ii) There is a commutative diagram

$$
\begin{array}{ccc}
L_{\text{sign}}^{\otimes i} & \longrightarrow & (C_0^*)^{\otimes i} \\
\delta_i & & \delta_i \\
(L_{\text{sign}}^{\delta_i}) & & (C_0^*)^{\delta_i}
\end{array}
$$

so that $L_{i, \text{sign}}$ is the image of $L_{\text{sign}}^{\delta_i}$ by the canonical morphism $L_{\text{sign}}^{\delta_i} \longrightarrow (C_0^*)^{\delta_i}$, whence a commutative diagram

$$
\begin{array}{ccc}
L_{\text{sign}}^{\delta_i} & \longrightarrow & \wedge i(L) \\
& & \\
(C_0^*)^{\delta_i}_{\text{sign}} & \longrightarrow & \wedge i(C_0^*)
\end{array}
$$

According to (i), the kernel of the left down arrow is the torsion module of $L_{\text{sign}}^{\delta_i}$ so that the kernel of the right down arrow is the torsion module of $\wedge i(L)$ since the horizontal arrows are isomorphisms. Moreover, the image of $L_{i, \text{sign}}$ in $\wedge i(C_0^*)$ is the image of $\wedge i(L)$. Hence $\wedge i(L)$ is isomorphic to $L_{i, \text{sign}}$.

(iii) Denote by $Q_i$ the kernel of the endomorphism $\delta_i$ of $(C_0^*)^{\otimes i}$. Since $\delta_i$ is a projection onto $(C_0^*)^{\otimes i}_{\text{sign}}$ such that $\delta_i(L_i)$ is contained in $L_{i, \text{sign}}$,

$$
(C_0^*)^{\otimes i}_{\text{sign}} = (C_0^*)^{\otimes i} \oplus Q_i,
$$

whence

$$
L_{i-1, \text{sign}, 1} = L_{i, \text{sign}} \oplus Q_i \cap L_{i-1, \text{sign}, 1}
$$

since $L_{i, \text{sign}}$ is a submodule of $L_{i-1, \text{sign}, 1}$.

(iv) Let $L_i'$ be the image of $L_{i-1, \text{sign}, 1}$ by the canonical morphism $(C_0^*)^{\otimes i} \rightarrow \wedge i-1(C_0^*) \otimes_{S_1(q \times q)} C_0^*$. Then $L_i'$ is contained in $\wedge i-1(C_0^*) \otimes_{S_1(q \times q)} L$ since $\wedge i-1(C_0^*) \otimes_{S_1(q \times q)} L$ is a submodule of $\wedge i-1(C_0^*) \otimes_{S_1(q \times q)} C_0^*$. Moreover, the morphism $L_{i-1, \text{sign}, 1} \rightarrow L_i'$ is an isomorphism since too is the morphism

$$
(C_0^*)^{\otimes(i-1)}_{\text{sign}} \otimes_{S_1(q \times q)} C_0^* \rightarrow \wedge i-1(C_0^*) \otimes_{S_1(q \times q)} C_0^*.
$$

From (ii), it results the commutative diagram

$$
\begin{array}{ccc}
L_{\text{sign}}^{\otimes(i-1)} \otimes_{S_1(q \times q)} L & \longrightarrow & \wedge i-1(L) \otimes_{S_1(q \times q)} L \\
& & \\
L_{i-1, \text{sign}, 1} & \longrightarrow & L_i'
\end{array}
$$

with the right down arrow surjective. According to (i), the kernel of the left down arrow is the torsion module of $L_{\text{sign}}^{\otimes(i-1)} \otimes_{S_1(q \times q)} L$. Hence the kernel of the right down arrow is the torsion module of $\wedge i-1(L) \otimes_{S_1(q \times q)} L$, whence the assertion. □
Lemma 4.1. \( \text{so that} \)

Then \( C \) is a cycle of the complex \( S(\mathfrak{g} \times \mathfrak{g}) \times \mathfrak{g} \). Let \( i \) be a positive integer. Then \( E \) has projective dimension at most \( i \).

Proof. According to Proposition 3.2,(i), the modules \( E_i \) and \( \Lambda^i(\mathfrak{g}) \wedge \Lambda^b_i(\mathfrak{B}_g) \) are isomorphic. Prove by induction on \( i \) that \( E_i \) has projective dimension at most \( i \). By Lemma 3.1,(i), it is true for \( i = 1 \). Suppose that it is true for \( i - 1 \). According to Corollary 3.7, \( E_{i-1} \) has projective dimension at most \( i \). By Lemma 3.8, for \( L = E, E_i \) is a direct factor of \( E_{i-1} \) since \( E \) is a submodule of \( C^*_g \) by Lemma 3.1,(iv) and since \( E_i = \wedge^i(E) \). Hence \( E_i \) has projective dimension at most \( i \). \( \square \)

4. Main results

Let \( I_g \) be the ideal of \( S(\mathfrak{g} \times \mathfrak{g}) \) generated by the functions \( (x, y) \mapsto \langle v, [x, y] \rangle \) with \( v \) in \( \mathfrak{g} \). The nullvariety of \( I_g \) in \( \mathfrak{g} \times \mathfrak{g} \) is \( \mathcal{C}(\mathfrak{g}) \). Let \( d \) be the \( S(\mathfrak{g} \times \mathfrak{g}) \)-derivation of the algebra \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \) such that \( dv \) is the function \( (x, y) \mapsto \langle v, [x, y] \rangle \) on \( \mathfrak{g} \times \mathfrak{g} \). The gradation on \( \Lambda(\mathfrak{g}) \) induces a gradation on \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \) so that \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \) is a graded homology complex.

Lemma 4.1. Denote by \( C_*(\mathfrak{g}) \) the graded submodule \( \Lambda(\mathfrak{g}) \wedge \Lambda^b(\mathfrak{B}_g) \) of \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \).

(i) The graded module \( C_*(\mathfrak{g}) \) is a graded subcomplex of \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \).

(ii) The support of the homology of \( C_*(\mathfrak{g}) \) is contained in \( \mathcal{C}(\mathfrak{g}) \).

Proof. (i) Set:

\[ \mathcal{E} := \Lambda^\ell_i e_i^{(0)} \wedge \cdots \wedge e_i^{(d_i-1)}. \]

Then \( C_*(\mathfrak{g}) \) is the ideal of \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \) generated by \( \mathcal{E} \) since \( e_i^{(0)}, \ldots, e_i^{(d_i-1)}, i = 1, \ldots, \ell \) is a basis of \( \mathfrak{B}_g \) by Theorem 2.2,(i). According to Theorem 2.2,(iii), for \( i = 1, \ldots, \ell \) and for \( m = 0, \ldots, d_i - 1 \), \( e_i^{(m)} \) is a cycle of the complex \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \). Hence too is \( \mathcal{E} \) and \( C_*(\mathfrak{g}) \) is a subcomplex of \( S(\mathfrak{g} \times \mathfrak{g}) \otimes_k \wedge(\mathfrak{g}) \) as an ideal generated by a cycle.

(ii) Let \( (x_0, y_0) \) be in \( \mathfrak{g} \times \mathfrak{g} \setminus \mathcal{C}(\mathfrak{g}) \) and let \( v \) be in \( \mathfrak{g} \) such that \( \langle v, [x_0, y_0] \rangle \neq 0 \). For some affine open subset \( O \) of \( \mathfrak{g} \times \mathfrak{g} \), containing \( (x_0, y_0) \), \( \langle v, [x, y] \rangle \neq 0 \) for all \( (x, y) \) in \( O \). Then \( dv \) is an invertible element of \( k[O] \). For \( c \) a cycle of \( k[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_*(\mathfrak{g}) \),

\[ d(v \wedge c) = (dv)c \]

so that \( c \) is a boundary of \( k[O] \otimes_{S(\mathfrak{g} \times \mathfrak{g})} C_*(\mathfrak{g}) \). \( \square \)

Theorem 4.2. (i) The complex \( C_*(\mathfrak{g}) \) has no homology in degree bigger than \( b_\mathfrak{g} \).

(ii) The ideal \( I_g \) has projective dimension \( 2n - 1 \).

(iii) The algebra \( S(\mathfrak{g} \times \mathfrak{g}) / I_g \) is Cohen-Macaulay.

(iv) The projective dimension of the module \( \Lambda^n(\mathfrak{g}) \wedge \Lambda^{b_\mathfrak{g}}(\mathfrak{B}_g) \) equals \( n \).

Proof. (i) Let \( Z \) be the space of cycles of degree \( b_\mathfrak{g} + 1 \) of \( C_*(\mathfrak{g}) \). Then we deduce from \( C_*(\mathfrak{g}) \) the complex

\[ 0 \rightarrow C_{2n+\ell}(\mathfrak{g}) \rightarrow \cdots \rightarrow C_{n+\ell+2}(\mathfrak{g}) \rightarrow Z \rightarrow 0. \]

According to Lemma 4.1,(ii), the support of its homology is contained in \( \mathcal{C}_g \). In particular, its codimension in \( \mathfrak{g} \times \mathfrak{g} \) is

\[ 4n + 2\ell - (2n + 2\ell) = 2n = n + n - 1 + 1 \]
According to Proposition 3.9, for \( i = n + \ell + 2, \ldots, 2n + \ell \), \( C_\ell(\mathfrak{g}) \) has projective dimension at most \( n \). Hence, by Corollary A.3, this complex is acyclic and \( Z \) has projective dimension at most \( 2n - 2 \), whence the assertion.

(ii) and (iii) Since \( B_\mathfrak{g} \) is a free module of rank \( b_\mathfrak{g} \), \( \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \) is a free module of rank 1. By definition, the short sequence

\[
0 \to Z \to \mathfrak{g} \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \to I_\mathfrak{g} \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \to 0
\]

is exact, whence the short exact sequence

\[
0 \to Z \to \mathfrak{g} \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \to I_\mathfrak{g} \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \to 0.
\]

Moreover, by Proposition 3.9, \( \mathfrak{g} \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \) has projective dimension at most 1. Then, by (i), \( I_\mathfrak{g} \) has projective dimension at most \( 2n - 1 \). As a result, \( \mathfrak{S}(g \times \mathfrak{g})/I_\mathfrak{g} \) has projective dimension at most \( 2n \). Then by Auslander-Buchsbaum’s theorem [Bou98, §3, n°3, Théorème 1], the depth of the graded \( \mathfrak{S}(g \times \mathfrak{g}) \)-module \( \mathfrak{S}(g \times \mathfrak{g})/I_\mathfrak{g} \) is at least

\[
4b_\mathfrak{g} - 2\ell - 2n = 2b_\mathfrak{g}
\]

so that, according to [Bou98, §1, n°3, Proposition 4], the depth of the graded algebra \( \mathfrak{S}(g \times \mathfrak{g})/I_\mathfrak{g} \) is at least \( 2b_\mathfrak{g} \). In other words, \( \mathfrak{S}(g \times \mathfrak{g})/I_\mathfrak{g} \) is Cohen-Macaulay since it has dimension \( 2b_\mathfrak{g} \). Moreover, since the graded algebra \( \mathfrak{S}(g \times \mathfrak{g})/I_\mathfrak{g} \) has depth \( 2b_\mathfrak{g} \), the graded \( \mathfrak{S}(g \times \mathfrak{g}) \)-module \( \mathfrak{S}(g \times \mathfrak{g})/I_\mathfrak{g} \) has projective dimension \( 2n \). Hence \( I_\mathfrak{g} \) has projective dimension \( 2n - 1 \).

(iv) By (i), \( I_\mathfrak{g} \) has projective dimension \( 2n - 1 \). Hence, according to Proposition 3.9 and according to (ii) and Corollary A.3, \( \wedge^n(\mathfrak{g}) \wedge \wedge^{b_\mathfrak{g}}(B_\mathfrak{g}) \) has projective dimension \( n \). □

**Theorem 4.3.** The subscheme of \( g \times g \) defined by \( I_\mathfrak{g} \) is Cohen-Macaulay and normal. Furthermore, \( I_\mathfrak{g} \) is a prime ideal.

**Proof.** According to Theorem 4.2,(iii), the subscheme of \( g \times g \) defined by \( I_\mathfrak{g} \) is Cohen-Macaulay. According to [Po08, Theorem 1], it is smooth in codimension 1. So by Serre’s normality criterion [Bou98, §1, n°10, Théorème 4], it is normal. In particular, it is reduced and \( I_\mathfrak{g} \) is radical. According to [Ri79], \( \mathfrak{C}(g) \) is irreducible. Hence \( I_\mathfrak{g} \) is a prime ideal. □

**Appendix A. Projective dimension and cohomology**

Recall in this section classical results. Let \( X \) be a Cohen-Macaulay irreducible affine algebraic variety and let \( S \) be a closed subset of codimension \( p \) of \( X \). Let \( P_* \) be a complex of finite projective \( \mathbb{k}[X] \)-modules whose length \( l \) is finite and let \( \varepsilon \) be an augmentation morphism of \( P_* \), whose image is \( R \), whence an augmented complex of \( \mathbb{k}[X] \)-modules,

\[
0 \to P_l \to P_{l-1} \to \cdots \to P_0 \xrightarrow{\varepsilon} R \to 0.
\]

Denote by \( \mathcal{P}_* \), \( \mathcal{R} \), \( \mathcal{K}_0 \) the localizations on \( X \) of \( P_* \), \( R \), the kernel of \( \varepsilon \) respectively and denote by \( \mathcal{K}_i \) the kernel of the morphism \( \mathcal{P}_i \to \mathcal{P}_{i-1} \) for \( i \) positive integer.

**Lemma A.1.** Suppose that \( S \) contains the support of the homology of the augmented complex \( P_* \).

(i) For all positive integer \( i < p - 1 \) and for all projective \( \mathcal{O}_X \)-module \( \mathcal{P} \), \( H(X \setminus S, \mathcal{P}) \) equals zero.
(ii) For all nonnegative integer \( j \leq l \) and for all positive integer \( i < p - j \), the cohomology group \( H^i(X \setminus S, \mathcal{K}_{l-j}) \) equals zero.

**Proof.** (i) Let \( i < p - 1 \) be a positive integer. Since the functor \( H^i(X \setminus S, \bullet) \) commutes with the direct sum, it suffices to prove \( H^i(X \setminus S, O_X) = 0 \). Since \( S \) is a closed subset of \( X \), one has the relative cohomology long exact sequence

\[
\cdots \to H^i_S(X, O_X) \to H^i(X, O_X) \to H^i(X \setminus S, O_X) \to H^{i+1}_S(X, O_X) \to \cdots .
\]

Since \( X \) is affine, \( H^i(X, O_X) \) equals zero and \( H^i(X \setminus S, O_X) \) is isomorphic to \( H^{i+1}_S(X, O_X) \). Since \( X \) is Cohen-Macaulay, the codimension \( p \) of \( S \) in \( X \) equals the depth of its ideal of definition in \( k[X] \) [MA86, Ch. 6, Theorem 17.4]. Hence, according to [Gro67, Theorem 3.8], \( H^{i+1}_S(X, O_X) \) equals zero since \( i + 1 < p \).

(ii) Let \( j \) be a nonnegative integer. Since \( S \) contains the support of the homology of the complex \( P_\bullet \), for all nonnegative integer \( j \), one has the short exact sequence of \( O_{X \setminus S} \)-modules

\[
0 \to \mathcal{K}_{j+1 \mid X \setminus S} \to P_{j+1 \mid X \setminus S} \to \mathcal{K}_j \mid X \setminus S \to 0
\]

whence the long exact sequence of cohomology

\[
\cdots \to H^i(X \setminus S, P_{j+1}) \to H^i(X \setminus S, \mathcal{K}_j) \to H^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \to H^{i+1}(X \setminus S, P_{j+1}) \to \cdots .
\]

Then, by (i), for \( 0 < i < p - 2 \), the cohomology groups \( H^i(X \setminus S, \mathcal{K}_j) \) and \( H^{i+1}(X \setminus S, \mathcal{K}_{j+1}) \) are isomorphic since \( P_{j+1} \) is a projective module. Since \( \mathcal{P}_i = 0 \) for \( i > l \), \( \mathcal{K}_l \) and \( \mathcal{P}_l \) have isomorphic restrictions to \( X \setminus S \). In particular, by (i), for \( 0 < i < p - 1 \), \( H^i(X \setminus S, \mathcal{K}_{l-j}) \) equals zero. Then, by induction on \( j \), for \( 0 < i < p-j \), \( H^i(X \setminus S, \mathcal{K}_{l-j}) \) equals zero. \( \square \)

**Proposition A.2.** Let \( R' \) be a \( k[X] \)-module containing \( R \). Suppose that the following conditions are verified:

1. \( p \) is at least \( l + 2 \),
2. \( X \) is normal,
3. \( S \) contains the support of the homology of the augmented complex \( P_\bullet \).

(i) The complex \( P_\bullet \) is a projective resolution of \( R \) of length \( l \).

(ii) Suppose that \( R' \) is torsion free and that \( S \) contains the support in \( X \) of \( R' \). Then \( R' = R \).

**Proof.** (i) Let \( j \) be a positive integer. One has to prove that \( H^0(X, \mathcal{K}_j) \) is the image of \( P_{j+1} \). By Condition (3), the short exact sequence of \( O_{X \setminus S} \)-modules

\[
0 \to \mathcal{K}_{j+1 \mid X \setminus S} \to P_{j+1 \mid X \setminus S} \to \mathcal{K}_j \mid X \setminus S \to 0
\]

is exact, whence the cohomology long exact sequence

\[
0 \to H^0(X \setminus S, \mathcal{K}_{j+1}) \to H^0(X \setminus S, P_{j+1}) \to H^0(X \setminus S, \mathcal{K}_j) \to H^1(X \setminus S, \mathcal{K}_{j+1}) \to \cdots .
\]

By Lemma A.1(ii), \( H^1(X \setminus S, \mathcal{K}_{j+1}) \) equals 0 since \( 1 < p - l + j + 1 \), whence the short exact sequence

\[
0 \to H^0(X \setminus S, \mathcal{K}_{j+1}) \to H^0(X \setminus S, P_{j+1}) \to H^0(X \setminus S, \mathcal{K}_j) \to 0.
\]

Since the codimension of \( S \) in \( X \) is at least 2 and since \( X \) is irreducible and normal, the restriction morphism from \( P_{j+1} \) to \( H^0(X \setminus S, \mathcal{P}_{j+1}) \) is an isomorphism. Let \( \varphi \) be in \( H^0(X, \mathcal{K}_j) \). Then there exists an element
\[ \psi \text{ of } P_{j+1} \text{ whose image } \psi' \text{ in } H^0(X, \mathcal{X}_j) \text{ has the same restriction to } X \setminus S \text{ as } \varphi. \] Since \( P_j \) is a projective module and since \( X \) is irreducible, \( P_j \) is torsion free. Then \( \varphi = \psi' \) since \( \varphi - \psi' \) is a torsion element of \( P_j \), whence the assertion.

(ii) Let \( \mathcal{R}' \) be the localization of \( R' \) on \( X \). Arguing as in (i), since \( S \) contains the support of \( R' / R \) and since \( 1 < p - l \), the short sequence

\[ 0 \rightarrow H^0(X \setminus S, \mathcal{X}_0) \rightarrow H^0(X \setminus S, \mathcal{P}_0) \rightarrow H^0(X \setminus S, \mathcal{R}') \rightarrow 0 \]

is exact. Moreover, the restriction morphism from \( P_0 \) to \( H^0(X / S, \mathcal{P}_0) \) is an isomorphism since the codimension of \( S \) in \( X \) is at least 2 and since \( X \) is irreducible and normal. Let \( \varphi \) be in \( \mathcal{R}' \). Then for some \( \psi \) in \( P_0 \), \( \varphi - \varepsilon(\psi) \) is a torsion element of \( R' \). So \( \varphi = \varepsilon(\psi) \) since \( R' \) is torsion free, whence the assertion. \[ \square \]

**Corollary A.3.** Let \( C_* \) be a homology complex of finite \( \mathbb{k}[X] \)-modules whose length \( l \) is finite and positive. For \( j = 0, \ldots, l \), denote by \( Z_j \) the space of cycles of degree \( j \) of \( C_* \). Suppose that the following conditions are verified:

1. \( S \) contains the support of the homology of the complex \( C_* \).
2. For all \( i \), \( C_i \) is a submodule of a free module,
3. For \( i = 1, \ldots, l \), \( C_i \) has projective dimension at most \( d \),
4. \( X \) is normal and \( l + d \leq p - 1 \).

Then \( C_* \) is acyclic and for \( j = 0, \ldots, l \), \( Z_j \) has projective dimension at most \( l + d - j - 1 \).

**Proof.** Prove by induction on \( l - j \) that the complex

\[ 0 \rightarrow C_l \rightarrow \cdots \rightarrow C_{j+1} \rightarrow Z_j \rightarrow 0 \]

is acyclic and that \( Z_j \) has projective dimension at most \( l + d - j - 1 \). For \( j = l \), \( Z_j \) equals zero since \( C_l \) is torsion free by Condition (2) and since \( Z_l \) a submodule of \( C_l \), supported by \( S \) by Condition (1). Suppose \( j \leq l - 1 \) and suppose the statement true for \( j + 1 \). By Condition (3), \( C_{j+1} \) has a projective resolution \( P_* \) whose length is at most \( d \) and whose terms are finitely generated. By induction hypothesis, \( Z_{j+1} \) has a projective resolution \( Q_* \) whose length is at most \( l + d - j - 2 \) and whose terms are finitely generated, whence an augmented complex \( R_* \) of projective modules whose length is \( l + d - j - 1 \),

\[ 0 \rightarrow Q_{l+d-j-2} \oplus P_{l+d-j-1} \rightarrow \cdots \rightarrow Q_0 \oplus P_1 \rightarrow P_0 \rightarrow Z_j \rightarrow 0. \]

Denoting by \( d \) the differentials of \( Q_* \) and \( P_* \), the restriction to \( Q_i \oplus P_{i+1} \) of the differential of \( R_* \) is the map

\[(x, y) \mapsto (dx, dy + (-1)^i \delta(x)),\]

with \( \delta \) the map which results from the injection of \( Z_{j+1} \) into \( C_{j+1} \). Since \( P_* \) and \( Q_* \) are projective resolutions, the complex \( R_* \) is a complex of projective modules having no homology in positive degree. Hence the support of the homology of the augmented complex \( R_* \) is contained in \( S \) by Condition (1). Then, by Proposition A.2 and Condition (4), \( R_* \) is a projective resolution of \( Z_j \) of length \( l + d - j - 1 \) since \( Z_j \) is a submodule of a free module by Condition (2), whence the corollary since \( Z_0 = C_0 \) by definition. \[ \square \]

**Remark A.4.** Let \( \text{D}(X) \) be the bounded derived category of finite \( \mathbb{k}[X] \)-modules. For \( E \) an object of \( \text{D}(X) \), denote by \( \text{Supp}(E) \) the union of the supports in \( X \) of the homology modules \( H_i(E) \) of \( E \). By definition, the homological dimension of \( E \), written \( \text{hd}(E) \), is the smallest integer \( s \) such that \( E \) is quasi-isomorphic
to a complex of projective $k[X]$-modules of length $s$. If no such integer exists, $\dim (E) = \infty$. Since $X$ is Cohen-Macaulay, according to [MA86, Ch. 6, Theorem 17.4], we have the following proposition:

**Proposition A.5.** [BM02, Corollary 5.5] *Let $E$ be a non trivial object of $D(X)$. Then for all irreducible component $\Gamma$ of $\text{Supp}(E)$, $\dim X - \dim \Gamma \leq \dim (E)$.***

Corollary A.3 is a little bit similar to Proposition A.5. But it is not a consequence of Proposition A.5 since its proof does not use the normality of $X$.

**References**


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