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EXTREME VALUES OF RANDOM OR CHAOTIC DISCRETIZATION STEPS*

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Abstract

By sorting independent random variables and considering the difference between two consecutive order statistics, we get random variables, called *steps* or *spacings*, that are neither independent nor identically distributed. We characterize the probability distribution of the maximum value of these steps, in three ways: i/ with an exact formula; ii/ with a simple and finite approximation whose error tends to be controlled; iii/ with asymptotic behavior when the number of random variables drawn (and therefore the number of steps) tends towards infinity.

The whole approach can be applied to chaotic dynamical systems by replacing the distribution of random variables by the invariant measure of the attractor when it is set.

The interest of such results is twofold. In practice, for example in the telecommunications domain, one can find a lower bound for the number of antennas needed in a phone network to cover an area. In theory, our results take place inside the extreme value theory extended to random variables that are neither independent nor identically distributed.

1. Introduction. Ask hundred people to give a number between zero and a thousand, and then sort their answers. What is the maximum step size between two successive values? This apparently simple question, depending of course on the probability distribution governing the choice of the people, has a concise solution, highlighting the probability of the maximum size of a step. The present article attempts to answer this question in a more general context.

We can introduce the problem differently in order to identify its practical interest. We consider a grid consisting of a certain number of independent, identically distributed random variables. This grid can be used for various purposes such as, for instance, the reconstruction of a signal. Indeed, imagine a mixing dynamical system and noting the state of the system at random observation times, we want to reconstruct the entire attractor if it exists. In order to solve this problem, it is useful to link the sample size to the size of the greatest step between two successive values of the ordered variables drawn from the

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invariant measure of the attractor. This will provide an idea of the approximation made in the reconstruction of the attractor. A practical question may be: How many observations do we need to have a 95% probability (we are particularly interested in high quantiles) of getting a maximum discretization step size below a certain threshold?

In this paper, we present three main results concerning the problem of extreme step. First, we provide an exact formula for the distribution function of the maximum step size, given in Proposition 1. This result being not easy to use in practice, we introduce some approximations: a first approximation as given in Theorem 1 and easy to calculate, and an asymptotic distribution as given in Theorem 2, when the number of random variables tends towards infinity.

This discretization grid approach seems to have some similarities with Extreme Value Theory and Monte Carlo or Quasi-Monte Carlo methods, nevertheless it presents some conceptual differences. We list some below.

- ▷ Monte Carlo or Quasi-Monte Carlo methods: the core of the method involves calculating expectations of a certain function using a discretization grid but it is quite different from the method presented here. Indeed, authors use several kinds of discretization grids for Monte Carlo. Some of them [11] transform the probability distribution of the signal in order to draw more variables in significant areas and use this importance sampling to the detriment of a small maximum step size in the grid. Other authors [25][23] use deterministic low discrepancy sequences of variables in order to reduce the step sizes. In contrast to these works, the discretization grid in our study is simply composed of random variables drawn by a given distribution.
- ▷ Extreme Value Theory: the aim of such a theory is to describe the limit probability distribution of the maximum (scaled and translated) of a set of independent and identically distributed random variables [10][12][22]. Examples of non-independent variables have been proposed to generalize the concept, for example by replacing independence by a strong mixing condition [20][19][18][3]. Similarly, in our study, we seek the probability distribution of the maximum of a set of random variables – the discretization steps – which are neither independent nor identically distributed. We provide two results, one with finite samples, the other one in an asymptotic setting. Nevertheless, the problem of maximum step size, also known as maximum spacing, is not new. Indeed, some theoretical articles, such as those of Lévy [21], Deheuvels [4][5][6] and Devroye [7][8][9] address that problem of maximum spacing for some distributions. These authors impose an asymptotic framework. We use a different approach based on the countermonotonicity of the spacings which provides approximations suitable for small samples. Stevens [26] obtained the exact distribution for a finite sample, but only for a uniform distribution. Our results may be applied to more general probability distributions.

Some applications using approximations of the distribution function of a maximum step size exist: for instance, [1][2][24] consider an *ad hoc* network modelled by a set of identically distributed random variables, representing mobile antennas. Their goal is to find the optimal transmission range of these antennas, in order for the network to work with a high probability, in the sense that all the nodes would be connected. The maximum difference between two random variables is crucial in this context. These articles make some approximations and assumptions which are different from those in our approach. However, for the uniform distribution, we obtain similar results in the asymptotic case which is their framework. Our main contribution is thus the study of very general distributions in a non-asymptotic case, taking into account the dependence of the steps.

The article is divided into three parts, describing successively the framework (Section 2), the theory (Section 3) and some applications (Section 4).

2. Framework: building the discretization grid. We introduce some notations to define the framework:

- ▹ $U_1, ..., U_T$ are $T > 1$ independent and identically distributed real random variables;
- ▹ $U_{1:T} \leq U_{2:T} \leq ... \leq U_{T:T}$ are the corresponding order statistics;
- ▹ δ_X and Δ_X are the probability density function and the cumulative distribution function of a random vector X; in particular, δ_U (supposed to be continuous) and Δ_U are associated to the one-dimensional random variables $U_1, ..., U_T$;
- ▹ $Supp(f)$ is the support of the function f;
- ▹ $V_1, ..., V_T$ are the discretization steps, defined by:

$$V_t = \begin{cases} U_{t:T} - U_{(t-1):T} & \text{if } t > 1 \\ U_{1:T} - \inf(Supp(\delta_U)) & \text{else.} \end{cases}$$

A question may arise about the expression of δ_U in practice. We mentioned in the introduction the possibility that these random variables would be drawn according to the invariant measure of the attractor of a dynamical system, if such invariant measure exists. Despite the fact that such invariant measures are often difficult to identify, some attractors possess advantageous properties that allow us to find their invariant measure. The book [13] addresses this problem and obtains the invariant measure of some chaos via the Frobenius-Perron operator. We discuss these examples later.

Our goal is to determine the cumulative distribution function of the maximum step size, $\Delta_{\max\limits_{t\in\{1,...,T\}} V_t}$, and more specifically the probability that the maximum step size is below a certain threshold $v > 0$:

$$\Delta_{\max\limits_{t\in\{1,...,T\}} V_t}(v) = \Delta_{V_1,...,V_T}(v, ..., v).$$

Though we are able to identify $\delta_{V_1,...,V_T}$, such a cumulative distribution function, $\Delta_{V_1,...,V_T}$, is difficult to calculate because of the successive integrations of the density δ_U. We propose different ways of approaching this distribution function. The first one reduces the problem in determining a unique integral, and the other one adds a deterministic search for maximum, which is algorithmically much less costly than the calculation of nested integrals.

An additional difficulty concerns the dependence between all the step sizes: indeed, if the step sizes were independent, the joint probability would be equal to the product of the marginal probabilities, and we would easily find a bound to the step size quantiles by changing each of these marginal probabilities by the one of an ideal step corresponding to the lowest density of the invariant measure (where step sizes are bigger). This dependence forces us to use the theory of copulas. However, the expression of the copula for the vector $(V_1, ..., V_T)$ requires explicit knowledge of the marginal probability of each component V_t. Since this is not trivial, we simply build our approximation using a classical result of the theory of copulas based on the Fréchet-Hoeffding theorem.

3. Main results. Before we turn specifically to the useful approximation of the probability distribution of the maximum step size, we present an intermediary result.

3.1. *Exact probability distribution of the greatest discretization step size.* The first proposition deals with the exact probability distribution function of the maximum of all the step sizes.

PROPOSITION 1. *Let $U_1, ..., U_T$ be $T > 1$ independent and identically distributed real random variables, with a probability density δ_U, and $V_1, ..., V_T$ be random variables (the discretization steps) defined by:*

$$V_t = \begin{cases} U_{t:T} - U_{(t-1):T} & \text{if } t > 1 \\ U_{1:T} - \inf(Supp(\delta_U)) & \text{else.} \end{cases} \tag{1}$$

Then, the probability density function of the maximum of the step sizes, $(V_1, ..., V_T)$, is, for $v \geq 0$:

$$\Delta_{\max\limits_{t\in\{1,...,T\}} V_t}(v) = T! \int_{[0,v]^T} \prod_{k=1}^{T} \delta_U \left(\sum_{m=0}^{k} v_m \right) dv_1...dv_T, \tag{2}$$

where v_0 denotes $\inf(Supp(\delta_U))$.

PROOF. First, we recall that the random variables $V_1, ..., V_T$ are neither independent nor identically distributed.

Let $\phi : \mathbb{R}^T \longrightarrow \mathbb{R}$ be a Borel function. Let $u_1, ..., u_T \in Supp(\delta_U)$ and $v_1, ..., v_T \geq 0$. We characterize the density $\delta_{V_1,...,V_T}$ of the vector of the steps, defined in equation (1), by:

$$\begin{array}{rcl} \mathbb{E}\left[\phi(V_1, ..., V_T)\right] & = & \int_{\mathbb{R}^T} \phi(v_1, ..., v_T) \mathbf{1}_{\{v_1 \geq 0, ..., v_T \geq 0\}} \delta_{V_1,...,V_T}(v_1, ..., v_T) dv_1...dv_T \\ & = & \int_{\mathbb{R}^T} \phi(v_1, ..., v_T) \mathbf{1}_{\{v_1 \geq 0, ..., v_T \geq 0\}} \delta_{V_1+v_0,...,V_T}(v_1 + v_0, ..., v_T) dv_1...dv_T \\ & = & \int_{\mathbb{R}^T} \phi(u_1 - v_0, u_2 - u_1, ..., u_T - u_{T-1}) \mathbf{1}_{\{v_0 \leq u_1 \leq ... \leq u_T\}} \delta_{U_{1:T},...,U_{T:T}}(u_1, ..., u_T) du_1...du_T. \end{array}$$

The integration is done with the following substitution in order to link order statistics and steps:

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{T-1} \\ u_T \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 + v_0 \\ v_2 \\ \vdots \\ v_{T-1} \\ v_T \end{pmatrix}.$$

Since the Jacobian of such a substitution is 1, we get:

$$\mathbb{E}\left[\phi(V_1, ..., V_T)\right] = \int_{\mathbb{R}^T} \phi(v_1, ..., v_T)\mathbf{1}_{\{v_1 \geq 0, ..., v_T \geq 0\}}\delta_{U_{1:T}, ..., U_{T:T}}\left(\sum_{m=0}^{1} v_m, ..., \sum_{m=0}^{T} v_m\right) dv_1...dv_T,$$

which leads to:

$$\delta_{V_1,...,V_T}(v_1, ..., v_T) = \delta_{U_{1:T},...,U_{T:T}}\left(\sum_{m=0}^{1} v_m, ..., \sum_{m=0}^{T} v_m\right). \tag{3}$$

We also know [14] that the distribution function of the vector $(U_{1:T}, ..., U_{T:T})$ of the order statistics is:

$$\delta_{U_{1:T},...,U_{T:T}}(u_1, ..., u_T) = T!\prod_{t=1}^{T}\delta_U(u_t)\mathbf{1}_{u_1 \leq ... \leq u_T}.$$

This relationship, used with (3), provides the joint probability density function of the vector of the steps, $(V_1, ..., V_T)$:

$$\delta_{V_1,...,V_T}(v_1, ..., v_T) = T!\prod_{k=1}^{T}\delta_U\left(\sum_{m=0}^{k} v_m\right). \tag{4}$$

Then we get:

$$\begin{aligned} \Delta_{\max\limits_{t\in\{1,...,T\}} V_t}(v) &= \Delta_{V_1,...,V_T}(v, ..., v) \\ &= \textstyle\int_{[0,v]^T}\delta_{V_1,...,V_T}(v_1, ..., v_T)dv_1...dv_T. \end{aligned} \tag{5}$$

Finally, the equations (4) and (5) provide the cumulative distribution function of the maximum of the step sizes, that is to say equation (2). □

The Proposition 1 shows that the calculation of the cumulative distribution function of the maximum step size, $\Delta_{\max\limits_{t\in\{1,...,T\}} V_t}(v)$, is difficult to compute because of the successive integrations of the density δ_U. Thus, we now provide specific approximations.

3.2. *Estimation of the distribution of the greatest discretization step size.* We emphasized the algorithmic difficulty of calculating the exact value of the cumulative distribution function of the maximum step size. Indeed, it amounts to the calculation of T nested integrals. Thanks to the theory of copulas, we propose a lower bound to the probability of the maximum step size. Theorem 1 provides such a lower bound, whose value is obtained by calculating a single integral.

Theorem 1. *Let $v \geq 0$ and $U_1, ..., U_T$ be $T > 1$ independent and identically distributed real random variables, with a probability density δ_U. We consider the random variables $V_1, ..., V_T$ (the discretization steps that are neither independent nor identically distributed) defined in equation* (1)*:*

$$V_t = \begin{cases} U_{t:T} - U_{(t-1):T} & \text{if } t > 1 \\ U_{1:T} - \inf(Supp(\delta_U)) & \text{else.} \end{cases}$$

Then, there exists a lower bound Δ^-*, defined by:*

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}(v) \leq \Delta_{\max_{t\in\{1,\dots,T\}} V_t}(v), \tag{6}$$

or, equivalently, for $p \in [0,1]$*:*

$$\left[\Delta_{\max_{t\in\{1,\dots,T\}} V_t}\right]^{-1}(p) \leq \left[\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}\right]^{-1}(p),$$

such that:

(7)

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}(v) = \max\left\{0, 1 - T\int_{\inf(Supp(\delta_U))+v}^{\sup(Supp(\delta_U))} \delta_U(u)\left[\Delta_U(u-v) + 1 - \Delta_U(u)\right]^{T-1} du\right\}.$$

In this theorem, we use the inequality of Fréchet-Hoeffding. We assume that the lower bound is very close to the probability that we are interested in, because of the countermonotonicity of the variables studied, especially for high quantiles for which this effect is more obvious and which corresponds to the objective of this article. Indeed, it is well known that the lower Fréchet-Hoeffding bound does not correspond to any existing copula in any dimensions higher than two. Kettler [15] proposes a realistic lower bound, corresponding to the copula of step sizes when the support of δ_U is bounded. That seems to be the best example of countermonotonic variables. However, since the marginal distribution functions in our case are too complicated, we would rather use the Fréchet-Hoeffding inequality. The approach proposed by Kettler about step sizes encourages us to believe that the Fréchet-Hoeffding bound is close to the reality that we are studying.

Proof. Let $v \geq 0$. We recall that, according to the Fréchet-Hoeffding theorem, we can write:

$$\max\left\{0, -(T-1) + \sum_{t=1}^{T}\mathbb{P}(V_t \leq v)\right\} \leq \mathbb{P}\left(V_1 \leq v, ..., V_T \leq v\right) \leq \min_{t\in\{1,\dots,T\}} \mathbb{P}(V_t \leq v). \tag{8}$$

Moreover

$$\begin{array}{rcl}\mathbb{P}(V_t \leq v) & = & \int_{\mathbb{R}} \mathbb{P}(V_t \leq v, U_{t:T} = u)du \\ & = & \int_{\mathbb{R}} \mathbb{P}(V_t \leq v | U_{t:T} = u)\delta_{U_{t:T}}(u)du,\end{array}$$

which, by inverting the finite sum and the integral and using the Lemma 1 postponed at the end of the article, leads to:

(9)

$$\begin{array}{rcl}\sum_{t=1}^T \mathbb{P}(V_t \leq v) & = & \int_{\inf(Supp(\delta_U))+v}^{\sup(Supp(\delta_U))} \left[\sum_{t=1}^T \delta_{U_{t:T}}(u) - \sum_{t=1}^T \left(\frac{\Delta_U(u-v)}{\Delta_U(u)}\right)^{t-1}\delta_{U_{t:T}}(u)\right]du \\ & & + \int_{\inf(Supp(\delta_U))}^{\inf(Supp(\delta_U))+v} \left[\sum_{t=1}^T \delta_{U_{t:T}}(u) - \sum_{t=2}^T \left(\frac{\Delta_U(u-v)}{\Delta_U(u)}\right)^{t-1}\delta_{U_{t:T}}(u)\right]du.\end{array}$$

We know the expression of the probability density function of the order statistic, given in the proof of the Lemma 1 in equation (29):

$$\delta_{U_{t:T}}(u) = \frac{T!}{(T-t)!(t-1)!}[\Delta_U(u)]^{t-1}[1-\Delta_U(u)]^{T-t}\delta_U(u).$$

Thus:

$$\begin{array}{rcl} \sum_{t=1}^{T} \delta_{U_{t:T}}(u) & = & \delta_U(u) T \sum_{s=0}^{T-1} \frac{(T-1)!}{s!(T-1-s)!} [\Delta_U(u)]^s [1-\Delta_U(u)]^{T-1-s} \\ & = & \delta_U(u) T \left[\Delta_U(u) + 1 - \Delta_U(u)\right]^{T-1} \\ & = & \delta_U(u) T, \end{array} \tag{10}$$

and, for any $\rho > 0$:

$$\sum_{t=1}^{T} \rho^{t-1} \delta_{U_{t:T}}(u) \quad = \quad \delta_U(u) T \left[\rho \Delta_U(u) + 1 - \Delta_U(u)\right]^{T-1}. \tag{11}$$

On the other hand, for $u < v + \inf(Supp(\delta_U))$, $\Delta_U(u-v) = 0$ and therefore:

$$\sum_{t=2}^{T} \left(\frac{\Delta_U(u-v)}{\Delta_U(u)}\right)^{t-1} \delta_{U_{t:T}}(u) \quad = \quad 0. \tag{12}$$

Thus, using equations (9) to (12), the sum of the cumulative distributions of the step sizes becomes:

$$\sum_{t=1}^{T} \mathbb{P}(V_t \leq v) \quad = \quad T - T \int_{\inf(Supp(\delta_U))+v}^{\sup(Supp(\delta_U))} \delta_U(u) \left[\Delta_U(u-v) + 1 - \Delta_U(u)\right]^{T-1} du. \tag{13}$$

Finally, we use the lower Fréchet-Hoeffding bound (8) with the equation (13) and we get (6) and (7). $\square$

Even though the exact probability is much closer to the lower Fréchet-Hoeffding bound than to the upper one, we can provide an upper bound to the probability of the maximum step size, for comparison, thanks to the upper Fréchet-Hoeffding bound:

$$\Delta_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) \leq \Delta^{+}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v),$$

with:

$$\Delta^{+}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) = \min_{t\in\{1,\dots,T\}} \left[\frac{T!}{(T-t)!(t-1)!} \int_{\mathbb{R}} \left(\Delta_U(u)^{t-1} - \Delta_U(u-v)^{t-1}\right) \left(1-\Delta_U(u)\right)^{T-t} \delta_U(u) du\right]. \tag{14}$$

Theorem 1 provides an interesting expression, but in certain cases it will be difficult to calculate it analytically. A good solution would be to discretize the integral which appears in the theorem, in a way that preserves the fact that our estimator is a lower bound. Thus instead of an integral, we can consider a sum together with a deterministic search for maximum. This is the subject of the following corollary:

Corollary 1. *Let $v \geq 0$ and $U_1, ..., U_T$ be $T > 1$ independent and identically distributed real random variables, with a probability density δ_U. We consider the random variables $V_1, ..., V_T$ defined in equation* (1)*. Assume that the support of δ_U is bounded. Let $\Theta \in \mathbb{N}$ and $(S_1, ..., S_\Theta)$ be a set of intervals forming a partition of* $[\inf(Supp(\delta_U)) + v, \sup(Supp(\delta_U))]$. *We define $\Delta^{-,(S_1,\dots,S_\Theta)}_{\max\limits_{t\in\{1,\dots,T\}} V_t}$ by:*

$$\Delta^{-,(S_1,\dots,S_\Theta)}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) = \max\left\{0, 1 - T\sum_{\theta=1}^{\Theta} \left[\sup(S_\theta) - \inf(S_\theta)\right] \rho_{S_\theta}(v)\right\}, \tag{15}$$

where:

$$\rho_{S_\theta}(v) = \max_{u\in S_\theta}\left(\delta_U(u)\left[\Delta_U(u-v)+1-\Delta_U(u)\right]^{T-1}\right).$$

Then:

$$\Delta^{-,(S_1,\dots,S_\Theta)}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) \leq \Delta^{-}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) \tag{16}$$

and:

$$\lim_{\max\limits_{\theta\in[1,\Theta]}[\sup(S_\theta)-\inf(S_\theta)]\to 0} \Delta^{-,(S_1,\dots,S_\Theta)}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) = \Delta^{-}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v).$$

Proof. All the results are a direct consequence of the discretization of the integral appearing in Theorem 1, more precisely in equation (7):

$$\begin{array}{rcl}\Delta^{-}_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) & = & \max\left\{0, 1-T\int_{\inf(Supp(\delta_U))+v}^{\sup(Supp(\delta_U))}\delta_U(u)\left[\Delta_U(u-v)+1-\Delta_U(u)\right]^{T-1}du\right\} \\ & = & \max\left\{0, 1-T\sum_{\theta=1}^{\Theta}\int_{S_\theta}\delta_U(u)\left[\Delta_U(u-v)+1-\Delta_U(u)\right]^{T-1}du\right\} \\ & \geq & \max\left\{0, 1-T\sum_{\theta=1}^{\Theta}\left[\sup(S_\theta)-\inf(S_\theta)\right]\max\limits_{u\in S_\theta}\left(\delta_U(u)\left[\Delta_U(u-v)+1-\Delta_U(u)\right]^{T-1}\right)\right\},\end{array}$$

because, for $u\in\mathbb{R}$, $\delta_U(u)\left[\Delta_U(u-v)+1-\Delta_U(u)\right]^{T-1}\geq 0$. Then, with the definition of $\Delta^{-,(S_1,\dots,S_\Theta)}_{\max\limits_{t\in\{1,\dots,T\}} V_t}$ in (15), we get (16). □

3.3. *Towards a limit distribution.* One objective of the extreme value theory is to find the limit distribution function of the maximum of a set of random variables. In the theory we developed above, we did not consider limit distributions, but we tried to describe precisely what happens for a given number of steps. Nevertheless, we can definitely present a limit theorem describing the shape of the lower bound of the probability when drawing an infinite number of variables.

Theorem 2. *Let $v\geq 0$ and $U_1,\dots,U_T$ be $T>1$ independent and identically distributed real random variables, with a probability density δ_U supposed to be differentiable. We consider the random variables $V_1,\dots,V_T$ (the discretization steps that are neither independent nor identically distributed) defined in equation* (1)*:*

$$V_t = \begin{cases} U_{t:T}-U_{(t-1):T} & \text{if } t>1 \\ U_{1:T}-\inf(Supp(\delta_U)) & \text{else.}\end{cases}$$

When v is greater than a certain threshold $\bar{v}(T)$ such that $\lim\limits_{T\to\infty} Tg\left(\bar{v}(T)\right)=1$, where

$$g : v \longmapsto \int_{\mathbb{R}}\delta_U(u)e^{-\delta_U(u)v}du,$$

then:

$$\Delta^{-}_{\max\limits_{t\in\{1,\dots,T\}} V_t}\left(\frac{v}{T}\right) \overset{T\to\infty}{\sim} \max(0; 1-Tg(v)). \tag{17}$$

In the equation (17), we used the notation $a(T) \overset{T\to\infty}{\sim} b(T)$, for functions a and b. This means that, when $\lim_{T\to\infty} b(T) \neq 0$, then $\lim_{T\to\infty} \frac{a(T)}{b(T)} = 1$.

PROOF. According to Theorem 1:

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}\left(\frac{v}{T}\right) = \max\left\{0, 1 - T\int_{\inf(Supp(\delta_U))+v/T}^{\sup(Supp(\delta_U))} \delta_U(u)\left[\Delta_U(u-v/T)+1-\Delta_U(u)\right]^{T-1} du\right\}. \tag{18}$$

Moreover, for $u \in \mathbb{R}$, we get the Taylor expansions:

$$\left[\Delta_U(u-v/T)+1-\Delta_U(u)\right] \overset{T\to\infty}{=} \left[1-\delta_U(u)\frac{v}{T}\right] + o\left(\frac{v}{T}\right), \tag{19}$$

because δ_U is differentiable. Thus:

$$\left[1-\delta_U(u)\frac{v}{T}\right]^{T-1} \overset{T\to\infty}{=} \exp\left(-v\delta_U(u)\right) + o\left(\frac{v}{T}\right). \tag{20}$$

Putting (19) and (20) together leads to:

$$\left[\Delta_U(u-v/T)+1-\Delta_U(u)\right]^{T-1} \overset{T\to\infty}{=} \exp\left(-v\delta_U(u)\right) + o\left(\frac{v}{T}\right). \tag{21}$$

We can now integrate the Taylor expansion (21) and use it in (18):

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}\left(\frac{v}{T}\right) \overset{T\to\infty}{=} \max\left\{0, 1 - T\int_{\inf(Supp(\delta_U))}^{\sup(Supp(\delta_U))} \delta_U(u)\exp\left(-v\delta_U(u)\right) du\right\} + o\left(1\right). \tag{22}$$

When v is greater than a certain threshold $\bar{v}(T)$ such that $\lim_{T\to\infty} Tg\left(\bar{v}(T)\right) = 1$, which implies that the right part of the equation (22) is greater than zero, we get (17). □

4. Examples for diverse invariant measures. We can apply the previous results to any type of distribution known analytically. We give examples for the attractors characterized by maps defined on $[0,1]$. See Guégan [13] and Lasota and Mackey [16][17] for interesting developments on the subject. We restrict to the logistic attractor and the 2-adic attractor and we provide the distribution of their maximum step size. We also study the case of a density with unbounded support: the exponential distribution. Finally, we discuss the building of the discretization grid.

In addition, all our theoretical work can be applied without any modification to more general probability density functions and thus to the search for the maximum step size for any random variable.

4.1. *Logistic attractor.* The logistic attractor of parameter α is defined by the recurrence relation:

$$U_{n+1} = \alpha U_n(1-U_n).$$

In a very general case, no theoretical result seems to reveal an invariant measure for the logistic attractor. However, if its parameter is 4, which is then literally a chaotic system, we know the existence and shape of the invariant measure, μ_4 :

$$\mu_4 : x \in [0,1] \longmapsto \frac{1}{\pi\sqrt{x(1-x)}},$$

in which we recognize a beta density function of parameters 1/2 and 1/2 – see Ulam and von Neumann [27]. We represent the density in Figure 1. We note M_4 as the associated cumulative distribution function:

$$M_4 : x \in [0,1] \longmapsto \int_0^x \frac{1}{\pi\sqrt{y(1-y)}} dy = \frac{1}{2} + \frac{1}{\pi}\arcsin(2x-1).$$

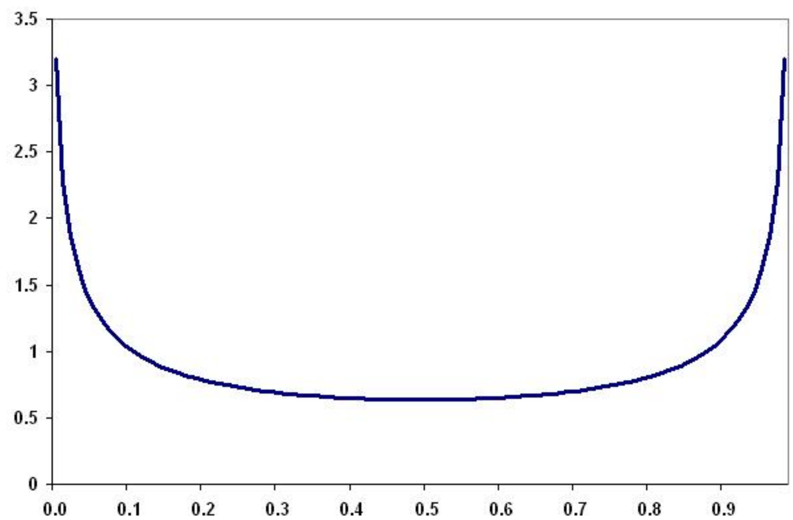

FIGURE 1. *Probability density function μ_4: invariant measure of the logistic attractor of parameter 4.*

We apply Corollary 1 to this attractor and represent the result in Figure 2. We note that the upper bound of quantiles (the corresponding curve is below the empirical distribution) approaches reality for high probabilities, especially when increasing the number Θ of intervals in the partition $(S_1, ..., S_\Theta)$. That proximity between the empirical quantiles and our approximation corroborates the intuition that the step sizes are countermonotonic and thus that the lower inequality of Fréchet-Hoeffding applied to the distribution of the step sizes V_t is close to reality. Indeed, when a step V_t is much larger than others, there is less room for other steps, therefore other steps will potentially be smaller.

For probabilities higher than 85%, the empirical quantile and its upper bound, calculated for a partition of the support in 100 intervals, are nearly equal, as we can see in Figure 3.

4.2. *2-adic attractor.* The r-adic attractor is defined by the recurrence relation:

$$U_{n+1} = rU_n \mod 1.$$

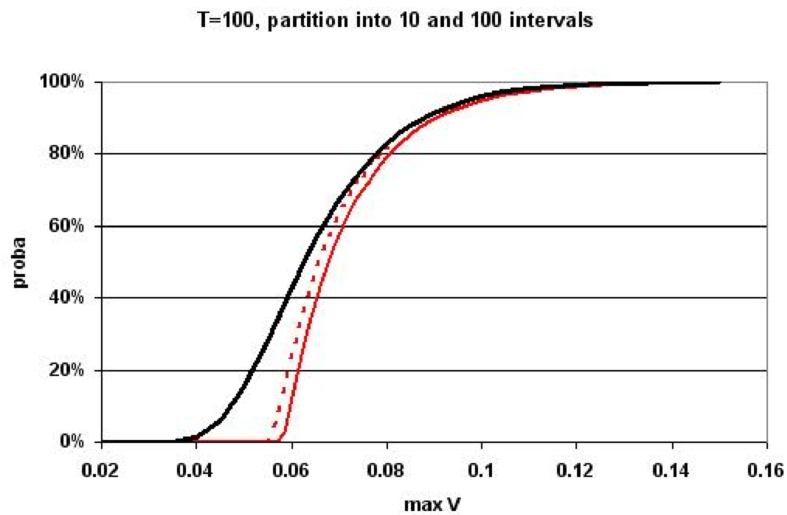

FIGURE 2. *Simulated cumulative distribution function (in black) of the maximum step size for the logistic attractor of parameter 4, with* $T = 100$*. The simulation has been carried out with* $10,000$ *sets of* T *variables drawn in the invariant measure of the attractor. In red, we represent the lower bound of the cumulative distribution function (the upper bound of quantiles), specified in Corollary 1, for partitions of 10 (solid line) or 100 (dotted line) intervals of equal size.*

probability	simulated quantile	approximated quantile	relative error
85%	0.0820	0.0825	0.6%
90%	0.0879	0.0881	0.2%
95%	0.0968	0.0977	0.9%
99%	0.1183	0.1199	1.4%

FIGURE 3. *Empirical and approximated quantile of the maximum step size for the logistic attractor of parameter 4, with* $T = 100$*. The simulation has been carried out with* $10,000$ *sets of* T *variables drawn in the invariant measure of the attractor. The approximated quantile is calculated with Corollary 1 and a partition of 100 intervals of equal size.*

Its invariant measure, when $r = 2$, is the constant function [13]:

$$\mu_2 : x \in [0,1] \longmapsto 1.$$

In fact, that is also the probability density function of a uniform random variable. We note M_2 as the corresponding cumulative distribution function, which is the identity function.

In this simple case, we can apply directly Theorem 1 without using a partition of the support. We get the following expression for the lower bound of the distribution function:

$$\begin{aligned} \Delta^-_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) &= \max\left\{0, 1 - T\int_{[v,1]}[1-v]^{T-1}\,du\right\} \\ &= \max\left\{0, 1 - T(1-v)^T\right\}. \end{aligned} \tag{23}$$

The limit distribution function is obtained thanks to Theorem 2:

$$\Delta^-_{\max\limits_{t\in\{1,\dots,T\}} V_t}\left(\frac{v}{T}\right) \overset{T\to\infty}{\sim} \max(0; 1 - Te^{-v}).$$

For this attractor, the upper bound is obtained from equation (14) by successive integrations by parts or by recognizing Beta functions:
(24)

$$\begin{aligned} \Delta^+_{\max\limits_{t\in\{1,\dots,T\}} V_t}(v) &= \min_{t\in\{1,\dots,T\}}\left[\tfrac{T!}{(T-t)!(t-1)!}\int_{\mathbb{R}}\left(M_2(u)^{t-1} - M_2(u-v)^{t-1}\right)(1-M_2(u))^{T-t}\mu_2(u)du\right] \\ &= \min_{t\in\{1,\dots,T\}}\left[\tfrac{T!}{(T-t)!(t-1)!}\left(\int_{[0,1]}u^{t-1}(1-u)^{T-t}du - \int_{[v,1]}(u-v)^{t-1}(1-u)^{T-t}du\right)\right] \\ &= \min_{t\in\{1,\dots,T\}}\left[1-(1-v)^T\right] \\ &= 1-(1-v)^T. \end{aligned}$$

We represent both the upper and lower bounds in Figure 4. We note that the upper bound of quantiles, corresponding to the lower Fréchet-Hoeffding inequality, converges on the empirical quantiles for high probabilities. On the other hand, the lower bound of quantiles, corresponding to the upper Fréchet-Hoeffding inequality, is very far from the empirical quantiles. These two facts are consistent with our intuition that the step sizes are countermonotonic.

For probabilities higher than 85%, the empirical quantile and its upper bound are nearly equal, as we can see in Figure 5.

4.3. *Exponential distribution.* In order to give an example of unbounded support of probability density, we study the case of exponential random variables of parameter λ, defined by the probability density function:

$$\mu_\lambda : x \geq 0 \longmapsto \lambda e^{-\lambda x},$$

and by the cumulative distribution function:

$$M_\lambda : x \geq 0 \longmapsto 1 - e^{-\lambda x}.$$

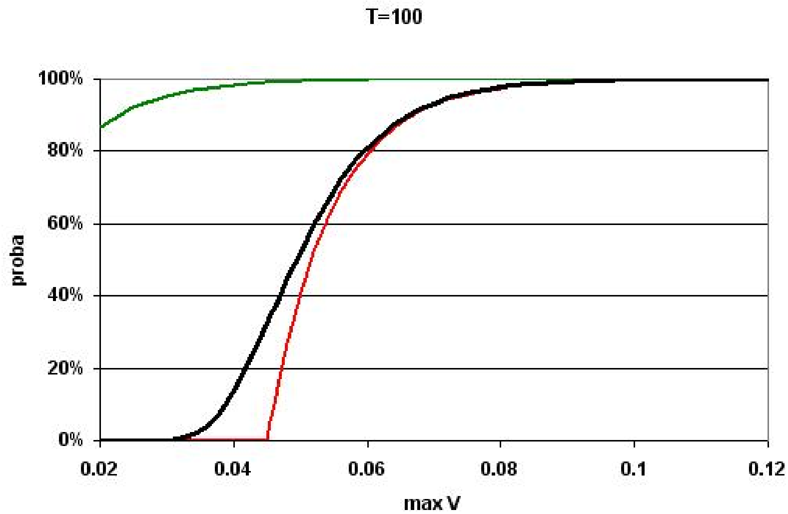

FIGURE 4. *Simulated cumulative distribution function (in black) of the maximum step size for the 2-adic attractor, with $T = 100$. The simulation has been carried out with $10,000$ sets of T variables drawn in the invariant measure of the attractor. In red, we represent the lower bound of the cumulative distribution function (the upper bound of quantiles, equation (23)). In green, we represent the upper bound of the cumulative distribution function (the lower bound of quantiles, equation (24)).*

probability	simulated quantile	approximated quantile	relative error
85%	0.0624	0.0630	1.0%
90%	0.0664	0.0667	0.5%
95%	0.0724	0.0732	1.1%
99%	0.0867	0.0880	1.5%

FIGURE 5. *Empirical and approximated quantile of the maximum step size for the 2-adic attractor, with $T = 100$. The simulation has been carried out with $10,000$ sets of T variables drawn in the invariant measure of the attractor.*

We can apply directly Theorem 1, without using a partition of the support. We obtain the following expression for the lower bound of the distribution function:

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}(v) = \max\left\{0, 1 - T\int_{[v,+\infty)} \lambda e^{-\lambda u}\left[1 + e^{-\lambda u}\left(1 - e^{\lambda v}\right)\right]^{T-1} du\right\}. \tag{25}$$

Thanks to the substitution $x = e^{-\lambda u}$ in (25), we get:

$$\begin{aligned} \Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}(v) &= \max\left\{0, 1 + T\int_{[e^{-\lambda v},0)}\left[1 + x\left(1 - e^{\lambda v}\right)\right]^{T-1} dx\right\} \\ &= \max\left\{0, 1 - \frac{1-e^{-\lambda v T}}{1-e^{-\lambda v}} e^{-\lambda v}\right\}. \end{aligned} \tag{26}$$

In [9] we have an exact asymptotic formula for the cumulative distribution function of the maximum step of an infinite set of exponential random variables which is:

$$\Delta_{\max_{t\geq 1} V_t}(v) = \prod_{k=1}^{\infty}\left(1 - e^{-\lambda v k}\right).$$

We note that $\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}$ may be written as:

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}(v) = \max\left\{0, 1 - \sum_{k=1}^{T}\left(e^{-\lambda v k}\right)\right\},$$

which is the first-order expansion of:

$$\prod_{k=1}^{T}\left(1 - e^{-\lambda v k}\right) \tag{27}$$

when v is big. We find empirically that this product, inspired by the asymptotic formula of [9], is an excellent approximation of the true distribution function. For this reason, we can assert, for the example of exponential random variables, that high quantiles of discretization steps are well approximated by our formula. We can check that in Figure 6.

The upper bound of quantiles has the advantage of being very fast to compute, and although it is an approximation, we know where it stands in relation to reality: we chose to overestimate the quantiles. Moreover, for probabilities higher than 90%, the empirical quantile matches its upper bound.

4.4. *Back to the building of the discretization grid.* One of the motivations of this article was: How many observations do we need to have a 95% probability of getting a maximum discretization step size below a certain threshold?

As we are able to provide an upper bound of the quantiles (even more accurate for large probabilities), we can get a minimum number of observations (T) that allows the grid to

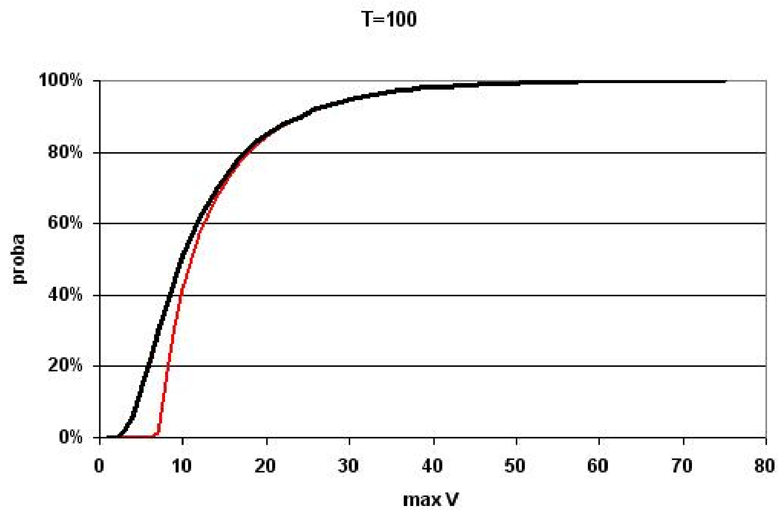

FIGURE 6. *Simulated cumulative distribution function (in black) of the maximum step size for exponential random variables, with intensity* $\lambda = 0.1$ *and* $T = 100$. *The simulation has been carried out with* $10,000$ *sets of* T *variables drawn in the exponential probability distribution. In red, we represent the lower bound of the cumulative distribution function (the upper bound of quantiles, equation* (26)*). Matching the black line, we represent the approximation by the truncated product of the analytic asymptotic cumulative distribution function, equation* (27)*), which coincides with the simulated distribution.*

reach a target of proximity for all the nodes. As far as our approximation is an overestimation of the quantile, then the minimum number of observations claimed by the same approximation is also slightly higher than in reality. Thus, our approximation allows us to build a discretization grid which is at least as accurate as we want.

For example, for a draw of random variables in the invariant measure of the 2-adic attractor, we recall the estimation obtained in equation (23):

$$\Delta^-_{\max_{t\in\{1,\dots,T\}} V_t}(v) = \max\left\{0, 1 - T(1-v)^T\right\}.$$

For a number of observations T and a given probability p, this leads to an upper bound of the maximum step size:

$$v \leq 1 - \left(\frac{1-p}{T}\right)^{\frac{1}{T}}. \tag{28}$$

We present this result in Figure 7, for various values of p.

As an illustration, with a 95% probability, a maximum step size of 0.02 is guaranteed for $T > 450$. If the probability is 99.9%, then the condition becomes $T > 663$.

5. Conclusion. We introduced the problem of extreme values of random or chaotic discretization steps with the facetious question:

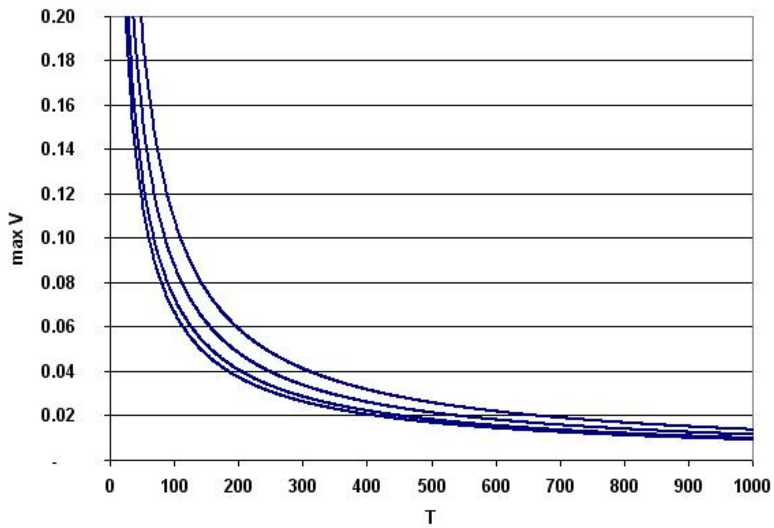

Figure 7. *Upper bounds of the maximum step size (for the 2-adic attractor) as a function of the number of observations (T), as in equation* (28)*. Each curve is for one probability, from the top right to the bottom left, 99.9%, 99%, 95% et 90%.*

> Ask hundred people to give a number between zero and a thousand, and then sort their answers. What is the maximum step size between two successive values?

The answer can be read, scaled, in Figure 7 if the distribution of such numbers is uniform: the maximum step size is less than 67 with a probability greater than 90%, less than 74 with a probability greater than 95%, less than 88 with a probability greater than 99% and less than 109 with a probability greater than 99.9%.

We have then obtained three main innovative theoretical results:

- ▷ in the Proposition 1, we give an exact formula for the distribution function of the maximum step size;
- ▷ in Theorem 1, we give an approximation based on the countermonotonicity of the step sizes and which is easy to calculate;
- ▷ in Theorem 2, we give a limit distribution, when the number of random drafts tends towards infinity.

The contribution is interesting for the extreme value theory, since we consider the maximum of non-independent and non-identically distributed random variables. We also proposed numerical applications mainly for invariant measures of chaos. More general probability distributions can be used too. We presented a practical application to quantify the accuracy of a random or chaotic discretization grid. In further work, we will invoke the results of this paper to estimate the maximum error made when reconstructing attractors of chaos by the technique of wavelets, applied to some observations forming a discretization grid.

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APPENDIX A: LEMMA

In Lemma 1, used in the demonstration of Theorem 1, we introduce the conditional probability density of a step, given the position on its right.

LEMMA 1. *Let $U_1, ..., U_T$ be $T > 1$ independent and identically distributed real random variables, with a probability density δ_U. We consider the random variables $V_1, ..., V_T$ (the discretization steps that are neither independent nor identically distributed) defined in equation (1). Let $t \in \{1, ..., T\}$, $v \geq 0$ and $u \in \mathbb{R}$. The probability density of V_t conditional to $U_{t:T}$ is:*

$$\delta_{V_t|U_{t:T}}(v,u) = (t-1)\delta_U(u-v)\frac{[\Delta_U(u-v)]^{t-2}}{[\Delta_U(u)]^{t-1}}$$

and the corresponding cumulative distribution function, defined by the conditional probability:

$$\Delta_{V_t|U_{t:T}}(v,u) = \mathbb{P}\left[V_t \leq v | U_{t:T} = u\right],$$

can be expressed by:

$$\Delta_{V_t|U_{t:T}}(v,u) = 1 - \left[\frac{\Delta_U(u-v)}{\Delta_U(u)}\right]^{t-1}.$$

Moreover, when $t = 1$, then $V_t = U_{t:T} - \inf(Supp(\delta_U))$ and the previous result can also be written:

$$\Delta_{V_1|U_{1:T}}(v,u) = \mathbf{1}_{\{u \leq v + \inf(Supp(\delta_U))\}}.$$

PROOF. We prove the lemma in four steps:

1. We first need to have the probability density function of any order statistic. This is a well-known result:

$$\delta_{U_{t:T}}(u) = \frac{T!}{(T-t)!(t-1)!}[\Delta_U(u)]^{t-1}[1-\Delta_U(u)]^{T-t}\delta_U(u). \tag{29}$$

   For the proof, see [14].
2. We also need to obtain the joint density function of two following order statistics, for $t \geq 2$. We first write the joint density function of all the order statistics:

$$\delta_{U_{1:T},...,U_{T:T}}(u_1, ..., u_T) = T! \prod_{t=1}^{T} \delta_U(u_t)\mathbf{1}_{u_1 \leq ... \leq u_T}.$$

   For the proof, see [14]. Then we integrate this equation:
   (30)

$$\delta_{U_{(t-1):T},U_{t:T}}(u_{t-1}, u_t) = \frac{T!}{(T-t)!(t-2)!}\left[1-\Delta_U(u_t)\right]^{T-t}\left[\Delta_U(u_{t-1})\right]^{t-2}\delta_U(u_t)\delta_U(u_{t-1}).$$

3. We now want to obtain the joint density function of a step and the corresponding order statistic on the right of the step. Let $t \geq 2$. Let $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a Borel function. We characterize the density $\delta_{U_{t:T},V_t}$ by:

$$\begin{array}{rcl} \mathbb{E}\left[\phi(U_{t:T}, V_t)\right] & = & \int_{\mathbb{R}^2} \phi(u,v)\mathbf{1}_{\{v \geq 0\}}\delta_{U_{t:T},V_t}(u,v)dudv \\ & = & \int_{\mathbb{R}^2} \phi(u_t, u_t - u_{t-1})\mathbf{1}_{\{u_{t-1} \leq u_t\}}\delta_{U_{(t-1):T},U_{t:T}}(u_{t-1}, u_t)du_{t-1}du_t. \end{array}$$

We implement the following substitution:

$$\begin{pmatrix} u_{t-1} \\ u_t \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

Since the Jacobian of such a substitution is 1, we get:

$$\mathbb{E}\left[\phi(U_{t:T}, V_t)\right] = \int_{\mathbb{R}^2} \phi(u,v)\mathbf{1}_{\{v\geq 0\}}\delta_{U_{(t-1):T},U_{t:T}}(u-v,u)dudv,$$

which leads to:

$$\delta_{U_{t:T},V_t}(u,v) = \delta_{U_{(t-1):T},U_{t:T}}(u-v,u).$$

This relationship, used with (30), provides the joint probability of $(U_{t:T}, V_t)$:
(31)

$$\begin{array}{rcl} \delta_{U_{t:T},V_t}(u,v) & = & \delta_{U_{(t-1):T},U_{t:T}}(u-v,u) \\ & = & \frac{T!}{(T-t)!(t-2)!}\left[1-\Delta_U(u)\right]^{T-t}\left[\Delta_U(u-v)\right]^{t-2}\delta_U(u)\delta_U(u-v). \end{array}$$

4. According to Bayes' theorem, the conditional density function $\delta_{V_t|U_{t:T}}$ is:

$$\delta_{V_t|U_{t:T}}(v,u) = \frac{\delta_{U_{t:T},V_t}(u,v)}{\delta_{U_{t:T}}(u)}.$$

With equations (29) and (31), we conclude for $t \geq 2$ that:

$$\delta_{V_t|U_{t:T}}(v,u) = (t-1)\delta_U(u-v)\frac{\left[\Delta_U(u-v)\right]^{t-2}}{\left[\Delta_U(u)\right]^{t-1}}.$$

The cumulative distribution function is then obtained by integration:

$$\Delta_{V_t|U_{t:T}}(v,u) = 1 - \left[\frac{\Delta_U(u-v)}{\Delta_U(u)}\right]^{t-1}.$$

The case $t=1$ is trivial anyway. Indeed, $U_{1:T} = V_1 + \inf(Supp(\delta_U))$, and we get:

$$\Delta_{V_1|U_{1:T}}(v,u) = \mathbf{1}_{\{u\leq v+\inf(Supp(\delta_U))\}}.$$

□

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