On Zermelo-like problems: a Gauss-Bonnet inequality and an E. Hopf theorem

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Abstract

The goal of this paper is to describe Zermelo's navigation problem on Riemannian manifolds as a time-optimal control problem and give an efficient method in order to evaluate its control curvature. We will show that up to changing the Riemannian metric on the manifold the control curvature of Zermelo's problem has a simple to handle expression which naturally leads to a generalization of the classical Gauss-Bonnet formula in an inequality. This Gauss-Bonnet inequality enables to generalize to Zermelo's problems a theorem by E. Hopf establishing the flatness of Riemannian tori without conjugate points.

Keywords: Conjugate points, control curvature, feedback transformation, Gauss-Bonnet formula, Riemannian manifold, Zermelo's navigation problem.

1 Introduction

In the present paper we study a special class of time-optimal control problems on two-dimensional manifolds: the Zermelo-like problems. By Zermelo-like problems we mean the class of time-optimal control problems formed by the classical Zermelo's navigation problems on Riemannian manifolds and the corresponding co-problems.

Our first goal is to describe these two problems and give an explicit expression for their control curvature, the latter being the control analogue of the Gaussian curvature of surfaces. This is the purpose of Section 3.

Zermelo's navigation problem aims to find the minimum time trajectories in a Riemannian manifold (M, g) under the influence of a drift represented by a vector field $\boldsymbol{X}$. The study of Zermelo's navigation problem began in 1931 with the work by E. Zermelo [22] and a while later by C. Carathéodory in [12]. In a recent paper [10], Zermelo's navigation problem has been studied as a special case of Finslerian metric and has been an efficient tool in order to give a complete classification of strongly convex Randers metrics of constant flag curvature, the latter being the Finslerian analogue of the Riemannian sectional curvature.

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The co-Zermelo navigation problem on a Riemannian surface (M, g) with drift Υ, where Υ is a one-form on M, is a time-optimal problem for which the maximized Hamiltonian function h resulting from the Pontryagin Maximum Principle is such that $h^{-1}(1) = \cup_{q\in M}\{\Upsilon_q + \mathcal{S}_q^{g*}\}$, where $\mathcal{S}_q^{g*}$ is the unitary Riemannian cosphere of the metric g. What is surprising with this definition of the problem is that it naturally leads to a good system of coordinates in which the control curvature has a very nice and simple expression as a function of the drift one-form and the Gaussian curvature of the metric g. The curvature of the co-Zermelo problem is much more readable than the curvature of the Zermelo problem itself and, thus, much more exploitable.

Another surprising property of the co-Zermelo problem is that its flow is just a time rescaling of the magnetic flow of the pair (g, Υ), the latter being the solution of a fixed time variational problem. In particular, it implies that the curvature of the problem of a charged particle in a magnetic field is just a reparametrization of the curvature of the co-Zermelo problem.

We prove constructively that Zermelo's navigation problem on (M, g) with drift vector field $\boldsymbol{X}$ is feedback equivalent to a co-Zermelo problem but this time with respect to another Riemannian metric. This is the contents of Proposition 3.4 and its Corollary 3.5 in §3.4. This proposition is fundamental because it points out that there are two different Riemannian metrics canonically associated to a given Zermelo problem. In particular, it implies that the two problems have the same curvature and also allows to see a Zermelo's navigation problem as its dual co-Zermelo problem and vice versa.

This is of particular interest because the presentation of a given Zermelo's navigation problem as its feedback equivalent co-Zermelo problem has the serious advantage to allow the presentation of the curvature of the considered problem in a formula that is easier to handle . It also shows how the classical Zermelo's navigation problem is linked to magnetic flows.

The second goal of this paper is to show that there is a natural way to generalize the classical Gauss-Bonnet formula for Riemannian surfaces to an inequality for Zermelo-like problems. More precisely, we will see that, given a Zermelo-like problem on a surface M, there exists a canonically defined positive function ϕ such that $\int_{\mathcal{H}} \phi\kappa \, d\mathcal{L} \geqslant \chi(M)$, where $\chi(M)$ is the Euler characteristic of M, $\mathcal{H}$ is the hypersurface $h^{-1}(1)$ and $d\mathcal{L}$ is the Liouville volume on $\mathcal{H}$. Moreover, the function ϕ takes the constant value equal to one if and only if the Zermelo problem is indeed Riemannian, in which case the inequality turns out to be the classical Gauss-Bonnet formula (see Theorem 4.2).

Our last goal is to generalize to Zermelo's problems the E. Hopf's theorem which asserts that two-dimensional Riemannian tori without conjugate points are flat. This will be done in two steps following Hopf's method. First we show that, if a control system on a compact surface without boundary has no conjugate points then its total curvature $\int_{\mathcal{H}} \kappa \, d\mathcal{L}$ must be negative or zero and, in the latter case its curvature must be equal to zero identically. This is Theorem 5.1.The second step is to use the Gauss-Bonnet inequality together with Theorem 5.1 to deduce flatness. In the Riemannian situation Theorem 5.1 together with the Gauss-Bonnet inequality (which, in this case, reduces to the classical Gauss-Bonnet formula) imply straightforwardly flatness for tori without conjugate points. Of course Theorem 5.1 applies to Zermelo-like problems but, due to the presence of the function ϕ in the Gauss-Bonnet inequality, the situation is more delicate and essentially different. Indeed, a Zermelo-like problem without conjugate points on a Riemannian torus is not necessarily flat unless its total curvature is zero. This situation is described in Theorem 5.2 and its Corollaries 5.3 and 5.4.

To conclude our paper we discuss further generalizations of the presented results to more general situations than the Riemannian one. We will see that even in the special case of Landsberg surfaces not all results can be transposed.

2 Curvature of two-dimensional smooth control systems

In the present paper smooth objects are supposed to be of class $\mathcal{C}^\infty$. Let us fix some notations. For a two-dimensional smooth manifold M, $\pi : T^*M \to M$ is the projection of the cotangent bundle to M. We denote by s the canonical Liouville one-form on T^*M, $s_\lambda = \lambda \circ \pi_*$, $\lambda \in T^*M$.

If M is endowed with a Riemannian structure g, $\langle \cdot, \cdot \rangle_g$ and $| \cdot |_g$ denote the Riemannian scalar product and the Riemannian norm respectively. Since the Riemannian structure defines a canonical identification between the tangent and the cotangent bundle of M, we use the notations of the scalar product and norm indifferently for vectors and covectors, vector fields and one-forms. We denote by $\mathcal{S}^g$ and $\mathcal{S}^{g*}$ the unitary spherical bundle $\{v \in TM \,:\, |v|_g = 1\}$ and the unitary cospherical bundle $\{\xi \in T^*M \,:\, |\xi|_g = 1\}$ respectively.

We denote by $[\boldsymbol{X}, \boldsymbol{Y}]$ the Lie bracket (or commutator) $\boldsymbol{X} \circ \boldsymbol{Y} - \boldsymbol{Y} \circ \boldsymbol{X}$ of vector fields $\boldsymbol{X}$, $\boldsymbol{Y} \in \operatorname{Vec} M$. It is again a vector field and in local coordinates on M the Lie bracket reads

$$[\boldsymbol{X}, \boldsymbol{Y}](q) = \frac{\partial \boldsymbol{Y}}{\partial q} \boldsymbol{X}(q) - \frac{\partial \boldsymbol{X}}{\partial q} \boldsymbol{Y}(q).$$

2.1 Definition of control curvature

We briefly recall some facts concerning the curvature of smooth control systems in dimension two. For more details on the subject we refer the reader to one of the following items [5, 19, 20].

Consider the following time-optimal smooth control problem

$$\begin{aligned} &\dot{q} = \boldsymbol{f}(q, u), \quad q \in M, \quad u \in U, \\ &q(0) = q_0, \quad q(t_1) = q_1, \\ &t_1 \to \min, \end{aligned} \tag{2.1}$$

where M and U are connected smooth manifolds of respective dimension two and one. For the above time-optimal control problem we denote by $h = \max_{u \in U} \langle \lambda, \boldsymbol{f}(q, u) \rangle$, $\lambda \in T_q^*M$, $q \in M$, the (normal) Hamiltonian function resulting from the PMP (Pontryagin Maximum Principle), by $\mathcal{H}$ the level set $h^{-1}(1) \subset T^*M$, and by $\vec{\boldsymbol{h}}$ the Hamiltonian field associated with the restriction of h to $\mathcal{H}$. Recall that the maximized Hamiltonian

h is a function on the cotangent bundle T^*M that is one-homogeneous on fibers and non-negative. Under the regularity assumptions of strong convexity

$$\boldsymbol{f}(q,u)\wedge\frac{\partial \boldsymbol{f}(q,u)}{\partial u}\neq 0,\quad \frac{\partial \boldsymbol{f}(q,u)}{\partial u}\wedge\frac{\partial^2 \boldsymbol{f}(q,u)}{\partial u^2}\neq 0,\quad q\in M,\quad u\in U, \tag{2.2}$$

the curve $\mathcal{H}_q = \mathcal{H}\cap T_q^*M$ admits, up to sign and translation, a natural parameter providing us with a vector field $\boldsymbol{v}_q$ on $\mathcal{H}_q$ and by consequence with a vertical vector field $\boldsymbol{v}$ on $\mathcal{H}$. The vector field $\boldsymbol{v}$ is characterized by the fact that it is, up to sign, the unique vector field on $\mathcal{H}$ such that

$$L^2_{\boldsymbol{v}}s|_{\mathcal{H}} = -s|_{\mathcal{H}} + bL_{\boldsymbol{v}}s|_{\mathcal{H}}, \tag{2.3}$$

where b is a smooth function on the level $\mathcal{H}$. The function b, which is by definition a feedback-invariant, is called the centro-affine curvature.

The vector fields $\vec{\boldsymbol{h}}$ and $\boldsymbol{v}$ which are, by definition, feedback-invariant satisfy the nontrivial commutator relation

$$\left[\vec{\boldsymbol{h}},\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right]\right] = \kappa\boldsymbol{v}, \tag{2.4}$$

where the coefficient κ is defined to be *the control curvature* or simply *the curvature* of the optimal control problem (2.1)-(2.2). The control curvature is by definition a feedback-invariant of the control system and a function on $\mathcal{H}$ (and not on M as the Gaussian one). Moreover, κ is the Gaussian curvature (lifted on $\mathcal{H}$) if the control system defines a Riemannian geodesic problem.

Example 2.1. Consider the time-optimal control problem corresponding to the geodesic problem on a two-dimensional Riemannian surface (M,g):

$$\begin{aligned}
&\dot q = u,\quad q\in M,\quad u\in T_qM,\quad |u|_g=1,\\
&q(0)=q_0,\quad q(t_1)=q_1\\
&t_1\to\min.
\end{aligned}$$

In this case, the Hamiltonian function of PMP is given by

$$h_g(\lambda) = |\lambda|_g,\quad \lambda\in T^*M,$$

and the vectors fields $\vec{\boldsymbol{h}}_g$ and $\boldsymbol{v}_g$ on $h_g^{-1}(1)$ by

$$\vec{\boldsymbol{h}}_g = \cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{e}_2 + (c_1\cos\theta + c_2\sin\theta)\frac{\partial}{\partial\theta},\quad \boldsymbol{v}_g = \frac{\partial}{\partial\theta},$$

where $(\boldsymbol{e}_1,\boldsymbol{e}_2)$ is a local g-orthonormal frame whose structural constants c_1, $c_2\in C^\infty(M)$, are defined by

$$[\boldsymbol{e}_1,\boldsymbol{e}_2] = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2,$$

and θ is the parameter on the fiber $h_g^{-1}(1)\cap T_q^*M = \mathcal{S}_q^{g*}$ defined by

$$\langle\lambda,\boldsymbol{e}_1(q)\rangle = \cos\theta,\quad \langle\lambda,\boldsymbol{e}_2(q)\rangle = \sin\theta.$$

The Gaussian curvature κ_g of the surface (M, g) is evaluated as follows:

$$\kappa_g = -c_1^2 - c_2^2 + L_{\boldsymbol{e}_1} c_2 - L_{\boldsymbol{e}_2} c_1. \tag{2.5}$$

Of course, for the Riemannian problem the curvature depends only on the base point $q \in M$ as one can see from formula (2.5) but in general this is not the case: the control curvature depends also on the coordinate in the fiber $\mathcal{H}_q$ and thus is a function on the whole three-dimensional manifold $\mathcal{H}$.

2.2 Reparametrization

Let $\varphi \in C^\infty(\mathcal{H})$ be a positive function. Define $\hat{\boldsymbol{h}}$, $\hat{\boldsymbol{v}} \in \operatorname{Vec}\mathcal{H}$ by

$$\hat{\boldsymbol{h}} = \varphi \vec{\boldsymbol{h}}, \quad \hat{\boldsymbol{v}} = \varphi^{-\frac{1}{2}} \boldsymbol{v}.$$

For any $f, g \in C^\infty(\mathcal{H})$ and $\boldsymbol{X}, \boldsymbol{Y} \in \operatorname{Vec}\mathcal{H}$ the general relation $[f\boldsymbol{X}, g\boldsymbol{Y}] = fg[\boldsymbol{X}, \boldsymbol{Y}] + fL_{\boldsymbol{X}} g\boldsymbol{Y} - gL_{\boldsymbol{Y}} f\boldsymbol{X}$ and an easy calculation imply that vector fields $\hat{\boldsymbol{h}}$ and $\hat{\boldsymbol{v}}$ satisfy the nontrivial commutator relation

$$\left[\hat{\boldsymbol{h}}, \left[\hat{\boldsymbol{v}}, \hat{\boldsymbol{h}}\right]\right] = \hat{\kappa}\hat{\boldsymbol{v}} + \xi\hat{\boldsymbol{h}}, \quad \hat{\kappa},\ \xi \in C^\infty(\mathcal{H}), \tag{2.6}$$

where the function $\hat{\kappa}$ satisfies

$$\kappa = \frac{\hat{\kappa} - \mathcal{S}(\varphi)}{\varphi^2}, \quad \mathcal{S}(\varphi) = \varphi L_{\vec{\boldsymbol{h}}}\left(\frac{L_{\vec{\boldsymbol{h}}}\varphi}{2}\right) - \left(\frac{L_{\vec{\boldsymbol{h}}}\varphi}{2}\right)^2. \tag{2.7}$$

The function $\hat{\kappa}$ defined by the relation (2.6) is called the *φ-reparametrization* of the curvature κ.

Notice that the function φ can be seen as the derivative of a time reparametrization function over the flow of $\vec{\boldsymbol{h}}$. Indeed, if $e^{t\vec{\boldsymbol{h}}}$ denotes the flow of $\vec{\boldsymbol{h}}$, one easily checks that the time reparametrization $t = t(\tau) = \int_0^\tau \varphi \circ e^{s\vec{\boldsymbol{h}}}(\lambda) ds$ has the required property.

3 Zermelo-like problems

3.1 Zermelo's navigation problem

In his 1931 article [22] Zermelo formulates the following problem:

"In an unbounded plane where the wind distribution is given by a vector field as a function of position and time, a ship moves with constant velocity relative to the surrounding air mass. How must the ship be steered in order to come from a starting point to a given goal in the shortest time?"

For our purpose we assume that we are working on a Riemannian surface in the presence of a stationary wind distribution that we call drift. Zermelo's navigation problem thus consists of finding the quickest path (in time) of a point on a Riemannian surface (M, g) in the presence of a stationary drift modeled by an autonomous vector field $\boldsymbol{X} \in \operatorname{Vec} M$. This time-optimal control problem reads

$$\dot{q} = \boldsymbol{X}(q) + u, \quad q \in M, \quad u \in T_qM, \quad |u|_g = 1, \tag{3.1}$$
$$q(0) = q_0, \quad q(t_1) = q_1 \tag{3.2}$$
$$t_1 \to \min, \tag{3.3}$$

and we call it *Zermelo problem of the pair* $(g, \boldsymbol{X})$. The Hamiltonian function of PMP is

$$h(\lambda) = \max_{|u|_g \leqslant 1} \left(\langle \lambda, \boldsymbol{X} \rangle + \langle \lambda, u \rangle \right) = \langle \lambda, \boldsymbol{X}(q) \rangle + |\lambda|_g, \tag{3.4}$$

and the Hamiltonian vector field on $\mathcal{H} = h^{-1}(1)$ has the form

$$\vec{\boldsymbol{h}} = \boldsymbol{X} + \vec{\boldsymbol{h}}_g + \Big(\langle u_{\max}, [\boldsymbol{e}_1, \boldsymbol{e}_2] \rangle_g \, \langle u_{\max}, \boldsymbol{X} \rangle_g + L_{[u_{\max}, \boldsymbol{v}_g]} \, \langle u_{\max}, \boldsymbol{X} \rangle_g \Big) \boldsymbol{v}_g, \tag{3.5}$$

where the function $u_{\max} = u_{\max}(\lambda)$ is the restriction to $\mathcal{H}$ of the maximized control obtained in the maximization (3.4). Relation (3.5) leads naturally to an expression of the curvature of Zermelo's navigation problem (3.1)-(3.3) as a function of the drift $\boldsymbol{X}$ and the Gaussian curvature of the surface (M, g). We do not give here a precise formula for this expression of the curvature since it leads to a formula which is rather complicated and hardly exploitable except for very simple cases. We refer the reader to [20] for a detailed description and coordinate expression of the curvature of this problem.

Although abnormal extremals can occur for the Zermelo problem (for instance, they do exist starting from a point $q_0 \in M$ if $|\boldsymbol{X}(q_0)|_g \geqslant 1$, see [19]), only normal extremals will be considered. In order to ensure the non-existence of abnormal extremals, we assume that $|\boldsymbol{X}|_g < 1$.

3.2 Co-Zermelo's navigation problem

Let (M, g) be a Riemannian surface and Υ be a smooth one-form on M satisfying $|\Upsilon|_g < 1$.

Definition 3.1. *We call co-Zermelo problem of the pair* (g, Υ) *the following time-optimal control problem on* M

$$\dot{q} = \frac{u}{1 + \langle \Upsilon_q, u \rangle}, \quad q \in M, \quad u \in T_qM, \quad |u|_g = 1, \tag{3.6}$$
$$q(0) = q_0, \quad q(t_1) = q_1,$$
$$t_1 \to \min .$$

Let $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ be a local g-orthonormal frame. The reader can check that the maximization condition of PMP requires the maximizing control to be

$$u_{\max} = \frac{1}{|\lambda|_g^2} \big((\alpha \, \langle \lambda, \boldsymbol{e}_1 \rangle + \beta \, \langle \lambda, \boldsymbol{e}_2 \rangle) \boldsymbol{e}_1 + (\alpha \, \langle \lambda, \boldsymbol{e}_2 \rangle - \beta \, \langle \lambda, \boldsymbol{e}_1 \rangle) \boldsymbol{e}_2 \big),$$

where

$$\alpha = \sqrt{\left\langle \lambda, \Upsilon_{\pi(\lambda)} \right\rangle_g^2 + \left(1 - |\Upsilon_{\pi(\lambda)}|_g^2\right) |\lambda|_g^2}, \quad \beta = \langle \lambda, \boldsymbol{e}_1 \rangle \langle \Upsilon, \boldsymbol{e}_2 \rangle - \langle \lambda, \boldsymbol{e}_2 \rangle \langle \Upsilon, \boldsymbol{e}_1 \rangle ,$$

which implies that the maximized Hamiltonian reads

$$h(\lambda) = \frac{-\left\langle \lambda, \Upsilon_{\pi(\lambda)} \right\rangle_g + \sqrt{\left\langle \lambda, \Upsilon_{\pi(\lambda)} \right\rangle_g^2 + \left(1 - |\Upsilon_{\pi(\lambda)}|_g^2\right) |\lambda|_g^2}}{1 - |\Upsilon_{\pi(\lambda)}|_g^2}. \tag{3.7}$$

It easily follows from (3.7) that

$$\left\langle \lambda - h(\lambda)\Upsilon_{\pi(\lambda)}, \lambda - h(\lambda)\Upsilon_{\pi(\lambda)} \right\rangle_g^2 = h(\lambda)^2, \quad \lambda \in T^*M, \tag{3.8}$$

which, restricted to $\mathcal{H} = h^{-1}(1)$, can be (locally) rewritten as

$$\left\langle \lambda - \Upsilon_{\pi(\lambda)}, \boldsymbol{e}_1(\pi(\lambda)) \right\rangle^2 + \left\langle \lambda - \Upsilon_{\pi(\lambda)}, \boldsymbol{e}_2(\pi(\lambda)) \right\rangle^2 = 1, \quad \lambda \in \mathcal{H}.$$

Hence, the fiber $\mathcal{H}_{\pi(\lambda)} = \mathcal{H} \cap T^*_{\pi(\lambda)}M$ can be naturally parametrized by an angle θ defined by

$$\left\langle \lambda - \Upsilon_{\pi(\lambda)}, \boldsymbol{e}_1(\pi(\lambda)) \right\rangle = \cos\theta, \quad \left\langle \lambda - \Upsilon_{\pi(\lambda)}, \boldsymbol{e}_2(\pi(\lambda)) \right\rangle = \sin\theta.$$

Notice that equation (3.7) also implies that, on $\mathcal{H}$, the maximizing control $u_{\max}$ satisfies

$$\langle u_{\max}(\lambda), \boldsymbol{e}_1(\pi(\lambda)) \rangle_g = \cos\theta, \quad \langle u_{\max}(\lambda), \boldsymbol{e}_2(\pi(\lambda)) \rangle_g = \sin\theta. \tag{3.9}$$

Indeed, on $\mathcal{H}$, the maximizing condition of PMP reads

$$1 = \langle \lambda, \boldsymbol{f}(\pi(\lambda), u_{\max}(\lambda)) \rangle = \frac{\langle \lambda, u_{\max}(\lambda) \rangle}{1 + \left\langle \Upsilon_{\pi(\lambda)}, u_{\max}(\lambda) \right\rangle},$$

or, equivalently,

$$1 = \left\langle \lambda - \Upsilon_{\pi(\lambda)}, u_{\max}(\lambda) \right\rangle ,$$

from which (3.9) follows.

We now derive the equation of the Hamiltonian field associated to h on the level surface $\mathcal{H}$. This vector field (see [20]) is given by

$$\vec{\boldsymbol{h}}(\theta, q) = \pi_* \vec{\boldsymbol{h}}(\theta, q) + c(\theta, q)\frac{\partial}{\partial \theta} = \boldsymbol{f}(q, u_{\max}(\theta, q)) + c(\theta, q)\frac{\partial}{\partial \theta}.$$

According to (3.6) and (3.9) the horizontal part of the field $\vec{\boldsymbol{h}}$ on $\mathcal{H}$ is

$$\pi_* \vec{\boldsymbol{h}} = \frac{1}{\varphi_g^{\Upsilon}}(\cos\theta\, \boldsymbol{e}_1 + \sin\theta\, \boldsymbol{e}_2),$$

where

$$\varphi_g^{\Upsilon}(\theta, q) = 1 + \cos\theta \left\langle \Upsilon_q, \boldsymbol{e}_1(q) \right\rangle + \sin\theta \left\langle \Upsilon_q, \boldsymbol{e}_2(q) \right\rangle .$$

Since $\vec{\boldsymbol{h}}$ is the Hamiltonian field restricted to $\mathcal{H}$, we have $ds|_{\mathcal{H}}(\vec{\boldsymbol{h}},\cdot)=0$, from which we can deduce the $\frac{\partial}{\partial\theta}$ part of $\vec{\boldsymbol{h}}$. Let $(\boldsymbol{e}_1^*,\boldsymbol{e}_2^*)$ be the coframe dual to $(\boldsymbol{e}_1,\boldsymbol{e}_2)$ and denote $s|_{\mathcal{H}}=\omega$. In coordinates $\lambda=(\theta,q)$ on $\mathcal{H}$ the Liouville one-form ω takes the form

$$\begin{aligned}\omega &= \langle\lambda,\boldsymbol{e}_1\rangle\,\boldsymbol{e}_1^*+\langle\lambda,\boldsymbol{e}_2\rangle\,\boldsymbol{e}_2^*\\ &= (\langle\lambda-\Upsilon,\boldsymbol{e}_1\rangle+\langle\Upsilon,\boldsymbol{e}_1\rangle)\,\boldsymbol{e}_1^*+(\langle\lambda-\Upsilon,\boldsymbol{e}_2\rangle+\langle\Upsilon,\boldsymbol{e}_2\rangle)\,\boldsymbol{e}_2^*\\ &= \cos\theta\boldsymbol{e}_1^*+\sin\theta\boldsymbol{e}_2^*+\Upsilon,\end{aligned}\tag{3.10}$$

so that its exterior derivative is

$$d\omega=-\sin\theta d\theta\wedge\boldsymbol{e}_1^*+\cos\theta d\theta\wedge\boldsymbol{e}_2^*+\cos\theta d\boldsymbol{e}_1^*+\sin\theta d\boldsymbol{e}_2^*+d\Upsilon.$$

Using Cartan's formula for one-forms $d\xi(\boldsymbol{X},\boldsymbol{Y})=L_{\boldsymbol{X}}\langle\xi,\boldsymbol{Y}\rangle-L_{\boldsymbol{Y}}\langle\xi,\boldsymbol{X}\rangle-\langle\xi,[\boldsymbol{X},\boldsymbol{Y}]\rangle$, one easily see that

$$d\boldsymbol{e}_1^*=-c_1dV_g,\quad d\boldsymbol{e}_2^*=-c_2dV_g,$$

where, as in Section 2.1, c_1, c_2, are the structural constants of the frame $(\boldsymbol{e}_1,\boldsymbol{e}_2)$ and, $dV_g=\boldsymbol{e}_1^*\wedge\boldsymbol{e}_2^*$ denotes the Riemannian volume element on M. Let $\Omega\in\mathcal{C}^\infty(M)$ be the function defined by $d\Upsilon=-\Omega\,dV_g$ and denote $c_g=c_1\cos\theta+c_2\sin\theta$. Summing up, we have

$$d\omega=-\sin\theta d\theta\wedge\boldsymbol{e}_1^*+\cos\theta d\theta\wedge\boldsymbol{e}_2^*-(c_g+\Omega)dV_g,\tag{3.11}$$

from which we get

$$0=d\omega(\vec{\boldsymbol{h}},\cdot)=-c\sin\theta\boldsymbol{e}_1^*+c\cos\theta\boldsymbol{e}_2^*+\frac{c_g+\Omega}{\varphi_g^\Upsilon}\sin\theta\boldsymbol{e}_1^*-\frac{c_g+\Omega}{\varphi_g^\Upsilon}\cos\theta\boldsymbol{e}_2^*.$$

Hence,

$$c=\frac{c_g+\Omega}{\varphi_g^\Upsilon}.$$

Summing up, the Hamiltonian of the co-Zermelo problem reads

$$\vec{\boldsymbol{h}}(\theta,q)=\frac{1}{\varphi_g^\Upsilon(\theta,q)}\left(\cos\theta\,\boldsymbol{e}_1(q)+\sin\theta\,\boldsymbol{e}_2(q)+(c_g(\theta,q)+\Omega(q))\frac{\partial}{\partial\theta}\right)$$

or, equivalently

$$\vec{\boldsymbol{h}}=\frac{1}{\varphi_g^\Upsilon}\left(F_*^\Upsilon\vec{\boldsymbol{h}}_g+\Omega F_*^\Upsilon\boldsymbol{v}_g\right),\quad \varphi_g^\Upsilon(\lambda)=1+\left\langle\lambda-\Upsilon_{\pi(\lambda)},\Upsilon_{\pi(\lambda)}\right\rangle_g,\quad \lambda\in\mathcal{H},\tag{3.12}$$

where $\vec{\boldsymbol{h}}_g$ and $\boldsymbol{v}_g$ are defined as in Section 2.1 and F^Υ is the diffeomorphism

$$\begin{aligned}F^\Upsilon:\mathcal{S}^{g*}&\to\mathcal{H}\\ \lambda&\mapsto\lambda+\Upsilon_{\pi(\lambda)}.\end{aligned}\tag{3.13}$$

Notice that $(F^\Upsilon)^{-1}=F^{-\Upsilon}$.

Remark 3.2. Since Ω and κ_g depend only on the basepoint, we can naturally identify Ω with $\Omega\circ\pi|_{\mathcal{S}^{g*}}$ or $\Omega\circ\pi|_{\mathcal{H}}$ and κ_g with $\kappa_g\circ\pi|_{\mathcal{S}^{g*}}$ or $\kappa_g\circ\pi|_{\mathcal{H}}$, depending on the context. Notice that we have already used this identification in relation (3.12).

3.3 Curvature of the co-Zermelo problem

In order to get the expression of the curvature of the co-Zermelo problem, we first of all need to find the expression of the vertical field that satisfies relation (2.3).

According to (3.10) and (3.11),

$$\omega \wedge \frac{\partial \omega}{\partial \theta} = \varphi_g^{\Upsilon}\, \boldsymbol{e}_1^* \wedge \boldsymbol{e}_2^* = \varphi_g^{\Upsilon}\, dV_g \neq 0, \tag{3.14}$$

which shows that $(\omega, \frac{\partial \omega}{\partial \theta})$ forms a frame of horizontal one-forms on $\mathcal{H}$. The decomposition of the second derivative $\frac{\partial^2 \omega}{\partial \theta^2}$ in this frame reads

$$\frac{\partial^2 \omega}{\partial \theta^2} = -\frac{1}{\varphi_g^{\Upsilon}} \omega + \frac{\frac{\partial \varphi_g^{\Upsilon}}{\partial \theta}}{\varphi_g^{\Upsilon}} \frac{\partial \omega}{\partial \theta}.$$

Considering a reparametrization $\vartheta = \vartheta(\theta)$ satisfying $\frac{d\vartheta}{d\theta} = (\varphi_g^{\Upsilon})^{-\frac{1}{2}}$ in the above equation, we deduce (see [5, page 365]), that the vertical vector field $\boldsymbol{v}$ that satisfies (2.3) has the coordinate expression

$$\boldsymbol{v} = \sqrt{\varphi_g^{\Upsilon}} \frac{\partial}{\partial \theta}. \tag{3.15}$$

We can now compute the curvature of the co-Zermelo problem applying relation (2.4).

Proposition 3.3. *The curvature of the co-Zermelo problem of the pair* (g, Υ) *is*

$$\kappa_{\mathrm{co}\mathcal{Z}}^{(g,\Upsilon)} = (\varphi_g^{\Upsilon})^{-2} \big(\kappa_g + \Omega^2 + L_{F_*^{\Upsilon}[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} \Omega - \mathcal{S}(\varphi_g^{\Upsilon}) \big). \tag{3.16}$$

Proof. According to (3.12) and (3.15),

$$\begin{aligned} \vec{\boldsymbol{h}} &= \frac{\hat{\boldsymbol{h}}}{\varphi_g^{\Upsilon}}, & \hat{\boldsymbol{h}} &= F_*^{\Upsilon}\big(\vec{\boldsymbol{h}}_g + \Omega \boldsymbol{v}_g\big), \\ \boldsymbol{v} &= \sqrt{\varphi_g^{\Upsilon}}\, \hat{\boldsymbol{v}}, & \hat{\boldsymbol{v}} &= F_*^{\Upsilon} \boldsymbol{v}_g, \end{aligned}$$

which implies that it is enough for this problem to compute the φ_g^{Υ}-reparametrized curvature (defined in Section 2.2). We have

$$\begin{aligned} F_*^{-\Upsilon}\Big[\hat{\boldsymbol{h}}, \Big[\hat{\boldsymbol{v}}, \hat{\boldsymbol{h}}\Big]\Big] &= \Big[\vec{\boldsymbol{h}}_g + \Omega \boldsymbol{v}_g, \Big[\boldsymbol{v}_g, \vec{\boldsymbol{h}}_g + \Omega \boldsymbol{v}_g\Big]\Big] \\ &= \Big[\vec{\boldsymbol{h}}_g, \Big[\boldsymbol{v}_g, \vec{\boldsymbol{h}}_g\Big]\Big] + \Omega \Big[\boldsymbol{v}_g, \Big[\boldsymbol{v}_g, \vec{\boldsymbol{h}}_g\Big]\Big] + L_{[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} \Omega \boldsymbol{v}_g \\ &= \kappa_g \boldsymbol{v}_g - \Omega \vec{\boldsymbol{h}}_g + L_{[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} \Omega \boldsymbol{v}_g \\ &= (\kappa_g + \Omega^2 + L_{[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} \Omega) F_*^{-\Upsilon} \hat{\boldsymbol{v}} - \Omega F_*^{-\Upsilon} \hat{\boldsymbol{h}}. \end{aligned}$$

The result follows according to (2.7). ■

We refer the reader to [18] for a detailed presentation of the co-Zermelo problem with linear drift on the Euclidean plane $\mathbb{R}^2$. In particular, using the reparametrized curvature, the author studies in detail the occurrence of conjugate points.

3.4 Duality between Zermelo and co-Zermelo problems

In this section we prove a proposition which asserts the feedback equivalence between the Zermelo and the co-Zermelo navigation problems. Although this proposition is simple indeed, it will have a fundamental role in the sequel due to the fact that the curvature is much simpler to handle for the co-Zermelo problem than for the Zermelo navigation problem itself.

Let (M, g) be a Riemannian manifold and fix a g-orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$.

Given $\boldsymbol{X} \in \operatorname{Vec} M$, we define the g-orthonormal discontinuous frame associated to the vector field $\boldsymbol{X}$ with respect to the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ by

$$\begin{aligned} \boldsymbol{e}_1^{\boldsymbol{X}} &= \cos\theta^{\boldsymbol{X}} \boldsymbol{e}_1 + \sin\theta^{\boldsymbol{X}} \boldsymbol{e}_2 \\ \boldsymbol{e}_2^{\boldsymbol{X}} &= -\sin\theta^{\boldsymbol{X}} \boldsymbol{e}_1 + \cos\theta^{\boldsymbol{X}} \boldsymbol{e}_2 , \end{aligned}$$

where $q \mapsto \theta^{\boldsymbol{X}}(q)$ is the angle defined by

$$\begin{cases} \theta^{\boldsymbol{X}}(q) = 0 & \text{if } \boldsymbol{X}(q) = 0, \\ \cos\theta^{\boldsymbol{X}}(q) = \dfrac{\langle \boldsymbol{X}(q), \boldsymbol{e}_1(q)\rangle_g}{|\boldsymbol{X}(q)|_g}, \quad \sin\theta^{\boldsymbol{X}}(q) = \dfrac{\langle \boldsymbol{X}(q), \boldsymbol{e}_2(q)\rangle_g}{|\boldsymbol{X}(q)|_g} & \text{if } \boldsymbol{X}(q) \neq 0. \end{cases} \tag{3.17}$$

Similarly, we define the g-orthonormal discontinuous frame associated to the one-form $\Upsilon \in \Lambda^1(M)$ with respect to the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ by

$$\begin{aligned} \boldsymbol{e}_1^{\Upsilon} &= \cos\theta^{\Upsilon} \boldsymbol{e}_1 + \sin\theta^{\Upsilon} \boldsymbol{e}_2 \\ \boldsymbol{e}_2^{\Upsilon} &= -\sin\theta^{\Upsilon} \boldsymbol{e}_1 + \cos\theta^{\Upsilon} \boldsymbol{e}_2 , \end{aligned}$$

where $q \mapsto \theta^{\Upsilon}(q)$ is the angle defined by

$$\begin{cases} \theta^{\Upsilon}(q) = 0 & \text{if } \Upsilon_q = 0, \\ \cos\theta^{\Upsilon}(q) = \dfrac{\langle \Upsilon_q, \boldsymbol{e}_1(q)\rangle}{|\Upsilon_q|_g}, \quad \sin\theta^{\Upsilon}(q) = \dfrac{\langle \Upsilon_q, \boldsymbol{e}_2(q)\rangle}{|\Upsilon_q|_g} & \text{if } \Upsilon_q \neq 0. \end{cases}$$

Notice that in these discontinuous frames

$$\boldsymbol{X} = \langle \boldsymbol{X}, \boldsymbol{e}_1^{\boldsymbol{X}}\rangle_g \, \boldsymbol{e}_1^{\boldsymbol{X}} = |\boldsymbol{X}|_g \boldsymbol{e}_1^{\boldsymbol{X}}, \quad \Upsilon = \langle \Upsilon, \boldsymbol{e}_1^{\Upsilon}\rangle \, \boldsymbol{e}_1^{\Upsilon *} = |\Upsilon|_g \boldsymbol{e}_1^{\Upsilon *}.$$

Proposition 3.4. *Let (M, g) be a Riemannian surface. For every $\boldsymbol{X} \in \operatorname{Vec} M$ with $|\boldsymbol{X}|_g < 1$, there exists on M a Riemannian metric $\tilde{g} = \tilde{g}(g, \boldsymbol{X})$ and a one-form $\tilde{\Upsilon} = \tilde{\Upsilon}(g, \boldsymbol{X})$ such that the Zermelo problem of the pair $(g, \boldsymbol{X})$ and the co-Zermelo problem of the pair $(\tilde{g}, \tilde{\Upsilon})$ have the same Hamiltonians.*

Conversly, for every $\Upsilon \in \Lambda^1(M)$ with $|\Upsilon|_g < 1$, there exists on M a Riemannian metric $\hat{g} = \hat{g}(g, \Upsilon)$ and a vector field $\hat{\boldsymbol{X}} = \hat{\boldsymbol{X}}(g, \Upsilon)$ such that the co-Zermelo problem of the pair (g, Υ) and the Zermelo problem of the pair $(\hat{g}, \hat{\boldsymbol{X}})$ have the same Hamiltonians.

Proof. Consider Zermelo's navigation problem (3.1)-(3.3) and let $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ be an orthonormal frame for the metric g. Define some polar coordinates (ρ, θ) on the fiber T_q^*M by

$$\rho = |\lambda|_g, \quad \langle \lambda, \boldsymbol{e}_1 \rangle = \rho \cos\theta, \quad \langle \lambda, \boldsymbol{e}_2 \rangle = \rho \sin\theta,$$

so that the Hamiltonian (3.4) takes the form

$$h(\rho, \theta, q) = \rho \left(|\boldsymbol{X}(q)|_g \cos(\theta - \theta^{\boldsymbol{X}}(q)) + 1 \right),$$

where $\theta^{\boldsymbol{X}}(q)$ is the angle defined by (3.17). Thus, the curve $\mathcal{H}_q = h^{-1}(1) \cap T_q^*M$ has the polar equation

$$\rho(\theta) = \frac{1}{|\boldsymbol{X}(q)|_g \cos(\theta - \theta^{\boldsymbol{X}}(q)) + 1}. \tag{3.18}$$

Since $|\boldsymbol{X}|_g < 1$, the curve $\mathcal{H}_q$ is an ellipse centered at a focus. Moreover, this ellipse has for g a focal distance $c = (\rho(\pi + \theta^{\boldsymbol{X}}) - \rho(\theta^{\boldsymbol{X}}))/2 = |\boldsymbol{X}|_g(1 - |\boldsymbol{X}|_g^2)^{-1}$, a semimajor distance $a = \rho(\theta^{\boldsymbol{X}}) + \rho(\pi + \theta^{\boldsymbol{X}}) = (1 - |\boldsymbol{X}|_g^2)^{-1}$, and a semiminor distance $b = \sqrt{a^2 - c^2} = (1 - |\boldsymbol{X}|_g^2)^{-1/2}$.

In order to transform Zermelo navigation problem in a co-Zermelo problem, we consider the curve $\mathcal{H}_q$ as the drifted Riemannian cosphere at point q for a new Riemannian structure $\tilde{g}$ on the manifold. In other words, we ask the one-forms

$$\tilde{\boldsymbol{e}}_1^* = \frac{1}{1 - |\boldsymbol{X}|_g^2} \boldsymbol{e}_1^{\boldsymbol{X}*}, \quad \tilde{\boldsymbol{e}}_2^* = \frac{1}{\sqrt{1 - |\boldsymbol{X}|_g^2}} \boldsymbol{e}_2^{\boldsymbol{X}*}$$

to form an orthonormal coframe for the new Riemannian structure $\tilde{g}$ on the manifold and the one-form

$$\tilde{\Upsilon} = -c\, \boldsymbol{e}_1^{\boldsymbol{X}*} = -\frac{|\boldsymbol{X}|_g}{1 - |\boldsymbol{X}|_g^2} \boldsymbol{e}_1^{\boldsymbol{X}*} \tag{3.19}$$

to be the drift one-form of the co-Zermelo problem on $(M, \tilde{g})$. The corresponding (new) orthonormal frame $(\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2)$ is characterized by

$$\langle (\tilde{\boldsymbol{e}}_1^*, \tilde{\boldsymbol{e}}_2^*), (\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2) \rangle = \mathrm{Id},$$

which leads to

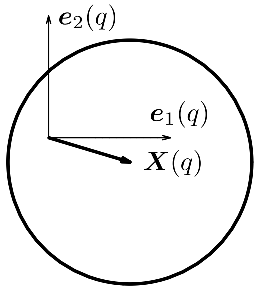
$$\tilde{\boldsymbol{e}}_1 = \left(1 - |\boldsymbol{X}|_g^2\right) \boldsymbol{e}_1^{\boldsymbol{X}}, \quad \tilde{\boldsymbol{e}}_2 = \sqrt{1 - |\boldsymbol{X}|_g^2}\, \boldsymbol{e}_2^{\boldsymbol{X}}. \tag{3.20}$$

Notice that we have $(\tilde{\boldsymbol{e}}_1, \tilde{\boldsymbol{e}}_2) = (\tilde{\boldsymbol{e}}_1^{-\tilde{\Upsilon}}, \tilde{\boldsymbol{e}}_2^{-\tilde{\Upsilon}})$ which shows in particular that $|\boldsymbol{X}|_g = |\tilde{\Upsilon}|_{\tilde{g}}$.

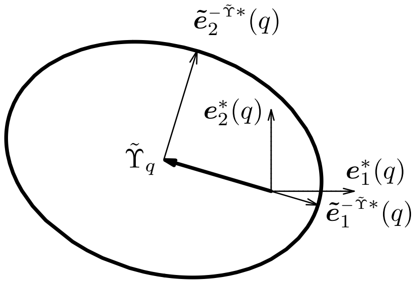
The situation described above is illustrated by the picture below.

At this point it is important to check that the metric $\tilde{g}$ and the one-form $\tilde{\Upsilon}$ are smooth. According to (3.17) and (3.19),

$$\begin{aligned} \tilde{\Upsilon} &= -\frac{|\boldsymbol{X}|_g}{1 - |\boldsymbol{X}|_g^2} \left(\frac{\langle \boldsymbol{X}, \boldsymbol{e}_1 \rangle_g}{|\boldsymbol{X}|_g} \boldsymbol{e}_1^* + \frac{\langle \boldsymbol{X}, \boldsymbol{e}_2 \rangle_g}{|\boldsymbol{X}|_g} \boldsymbol{e}_2^* \right) \\ &= -\left(1 - |\boldsymbol{X}|_g^2\right)^{-1} \left(\langle \boldsymbol{X}, \boldsymbol{e}_1 \rangle_g \boldsymbol{e}_1^* + \langle \boldsymbol{X}, \boldsymbol{e}_2 \rangle_g \boldsymbol{e}_2^* \right), \end{aligned}$$



$X(q) + \mathcal{S}_q^g \subset T_qM$



$\mathcal{H}_q = \tilde{\Upsilon}_q + \mathcal{S}_q^{\tilde{g}*} \subset T_q^*M$

or, equivalently,

$$\tilde{\Upsilon} = -\left(1 - |\boldsymbol{X}|_g^2\right)^{-1} \langle \boldsymbol{X}, \cdot \rangle_g \,,$$

which is clearly smooth if g and $\boldsymbol{X}$ are.

Smoothness of $\tilde{g}$ is also easily seen if one writes its matrix $\tilde{G}$ with respect to the g-orthonormal frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Let $\tilde{G}_{(\boldsymbol{e}_1,\boldsymbol{e}_2)}$ denote the matrix of the metric $\tilde{g}$ with respect to the frame $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. According to (3.17) and (3.20), $\tilde{G}_{(\boldsymbol{e}_1,\boldsymbol{e}_2)}$ is given by

$$\tilde{G}_{(\boldsymbol{e}_1,\boldsymbol{e}_2)} = \begin{pmatrix} \tilde{g}(\boldsymbol{e}_1,\boldsymbol{e}_1) & \tilde{g}(\boldsymbol{e}_1,\boldsymbol{e}_2) \\ \tilde{g}(\boldsymbol{e}_1,\boldsymbol{e}_2) & \tilde{g}(\boldsymbol{e}_2,\boldsymbol{e}_2) \end{pmatrix} = \frac{1}{\left(1 - |\boldsymbol{X}|_g^2\right)^2} \begin{pmatrix} 1 - \langle \boldsymbol{X}, \boldsymbol{e}_2 \rangle_g^2 & \langle \boldsymbol{X}, \boldsymbol{e}_1 \rangle_g \langle \boldsymbol{X}, \boldsymbol{e}_2 \rangle_g \\ \langle \boldsymbol{X}, \boldsymbol{e}_1 \rangle_g \langle \boldsymbol{X}, \boldsymbol{e}_2 \rangle_g & 1 - \langle \boldsymbol{X}, \boldsymbol{e}_1 \rangle_g^2 \end{pmatrix},$$

which is smooth since $\tilde{G}_{(\boldsymbol{e}_1,\boldsymbol{e}_2)} = \mathrm{Id}$ at points for which $|\boldsymbol{X}|_g = 0$.

In order to complete the proof it remains to check that the Hamiltonian function $h_{\mathcal{Z}}^{(g,\boldsymbol{X})}$ of the Zermelo problem of the pair $(g, \boldsymbol{X})$ and the Hamiltonian function $h_{\mathrm{co}\mathcal{Z}}^{(\tilde{g},\tilde{\Upsilon})}$ of the co-Zermelo problem of the pair $(\tilde{g}, \tilde{\Upsilon})$ coincide. For simplicity we denote $\tilde{c} =$

$|\tilde{\Upsilon}|_{\tilde{g}} = |\boldsymbol{X}|_g$. We have

$$
\begin{aligned}
h_{\mathcal{Z}}^{(g,\boldsymbol{X})}(\lambda) &= \langle \lambda, \boldsymbol{X}\rangle + |\lambda|_g = \langle \lambda, \tilde{c}\boldsymbol{e}_1^{\boldsymbol{X}}\rangle + \sqrt{\langle \lambda, \boldsymbol{e}_1^{\boldsymbol{X}}\rangle^2 + \langle \lambda, \boldsymbol{e}_2^{\boldsymbol{X}}\rangle^2} \\
&= \left\langle \lambda, \tilde{c}\frac{\tilde{\boldsymbol{e}}_1}{1-\tilde{c}^2}\right\rangle + \sqrt{\left\langle \lambda, \frac{\tilde{\boldsymbol{e}}_1}{1-\tilde{c}^2}\right\rangle^2 + \left\langle \lambda, \frac{\tilde{\boldsymbol{e}}_2}{\sqrt{1-\tilde{c}^2}}\right\rangle^2} \\
&= \frac{\langle \lambda, \tilde{c}\tilde{\boldsymbol{e}}_1\rangle + \sqrt{\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle^2 + (1-\tilde{c}^2)\,\langle \lambda, \tilde{\boldsymbol{e}}_2\rangle^2}}{1-\tilde{c}^2} \\
&= \frac{\langle \lambda, \tilde{c}\tilde{\boldsymbol{e}}_1\rangle + \sqrt{\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle^2 + \langle \lambda, \tilde{\boldsymbol{e}}_2\rangle^2 - \tilde{c}^2\,\langle \lambda, \tilde{\boldsymbol{e}}_2\rangle^2 - \tilde{c}^2\,\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle^2 + \tilde{c}^2\,\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle^2}}{1-\tilde{c}^2} \\
&= \frac{-\,\langle \lambda, -\tilde{c}\tilde{\boldsymbol{e}}_1\rangle + \sqrt{\big(\,\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle^2 + \langle \lambda, \tilde{\boldsymbol{e}}_2\rangle^2\,\big)(1-\tilde{c}^2) + \big(-\tilde{c}\,\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle\,\big)^2}}{1-\tilde{c}^2} \\
&= \frac{-\langle \lambda, \langle \tilde{\Upsilon}, \tilde{\boldsymbol{e}}_1\rangle \tilde{\boldsymbol{e}}_1^*\rangle_{\tilde{g}} + \sqrt{|\lambda|_{\tilde{g}}^2\,(1-\tilde{c}^2) + \big(\langle \tilde{\Upsilon}, \tilde{\boldsymbol{e}}_1\rangle\,\langle \lambda, \tilde{\boldsymbol{e}}_1\rangle\,\big)^2}}{1-\tilde{c}^2} \\
&= \frac{-\langle \lambda, \tilde{\Upsilon}\rangle_{\tilde{g}} + \sqrt{\big(1-|\tilde{\Upsilon}|_{\tilde{g}}^2\big)|\lambda|_{\tilde{g}}^2 + \langle \lambda, \tilde{\Upsilon}\rangle_{\tilde{g}}^2}}{1-|\tilde{\Upsilon}|_{\tilde{g}}^2} \\
&= h_{\mathrm{co}\mathcal{Z}}^{(g,\tilde{\Upsilon})}(\lambda).
\end{aligned}
$$

In order to prove the converse, one has just to permute the roles of vector fields and one-forms in the previous considerations. ■

Zermelo's navigation problem and co-Zermelo's navigation problem which have the same Hamiltonian are said to be *dual problems*. The above proposition implies in particular that two dual Zermelo problems have the same curvature. This proposition can be reformulated as follows.

Corollary 3.5. *Two dual Zermelo's problems are feedback equivalent.*

Proof. Let us use the same notations as in the proof of Proposition 3.4. A similar computation as the one made in the previous proof shows that two dual Zermelo's problems have the same sets of admissible velocities, i.e., that for every $q \in M$, $\{\boldsymbol{X}(q) + u \;:\; u \in \mathcal{S}_q^g\} = \{\tilde{u}(1-\langle \tilde{\Upsilon}_q, \tilde{u}\rangle)^{-1} \;:\; \tilde{u} \in \mathcal{S}_q^{\tilde{g}}\}$ (see equations (3.1) and (3.6) for the dynamics of Zermelo's problems). Thus, the feedback transformation $u \mapsto \tilde{u}(1-\langle \tilde{\Upsilon}_q, \tilde{u}\rangle)^{-1} - \boldsymbol{X}(q)$ gives the feedback equivalence. ■

3.5 Classical particle in a magnetic field on a Riemannian surface

The motion of a charged particle of unit mass under the presence of a magnetic field is modeled by what is called the magnetic flow. We will see here how the problem of a charged particle in a magnetic field is linked to the co-Zermelo problem. Magnetic flows were first considered by Arnold in [8] and by Anosov and Sinai in [7] but Sternberg was the first to give in [21] a formulation of this problem in terms of symplectic geometry.

Let (M, g) be a two-dimensional Riemannian manifold and $B \in \Lambda^2(M)$ a closed two-form thought as a magnetic field in which we have absorbed the electric charge of the particle as a parameter.

The magnetic flow of the pair (g, B) is the flow of the Hamiltonian

$$h_g^2(\lambda) = \langle \lambda, \lambda \rangle_g$$

with respect to the symplectic form $\sigma_B = ds + \pi^* B$ (see [21]). In the case where B derives from a magnetic potential, i.e., when $B = d\Upsilon$, $\Upsilon \in \Lambda^1(M)$, the magnetic flow is again Hamiltonian with respect to the canonical symplectic form ds but this time with the Hamiltonian function

$$h_{\text{mag}}(\lambda) = \frac{1}{2} \left\langle \lambda - \Upsilon_{\pi(\lambda)}, \lambda - \Upsilon_{\pi(\lambda)} \right\rangle_g = \frac{1}{2} h_g^2(\lambda - \Upsilon_{\pi(\lambda)}).$$

A straightforward computation shows that the Hamiltonian vector field $\vec{\boldsymbol{h}}_{\text{mag}}$ associated to h_{mag} restricted to $h_{\text{mag}}^{-1}(1)$ is given by

$$\vec{\boldsymbol{h}}_{\text{mag}} = F_*^{\Upsilon}(\vec{\boldsymbol{h}}_g + \Omega \boldsymbol{v}_g),$$

where $\Omega \in C^\infty(M)$ is defined in same way as the function Ω of the co-Zermelo problem. This shows that the equations of motion of a particle in a magnetic field are in fact the equations of motion of the reparametrized co-Zermelo problem. For this reason we define the curvature $\kappa_{\text{mag}}^{(g,\Upsilon)}$ of the magnetic flow to be the φ_g^{Υ}-reparametrized curvature of the co-Zermelo problem, i.e.,

$$\kappa_{\text{mag}}^{(g,\Upsilon)} = \kappa_g + \Omega^2 + L_{F_*^{\Upsilon}[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} \Omega, \tag{3.21}$$

so that,

$$\kappa_{\text{co}\mathcal{Z}}^{(g,\Upsilon)} = (\varphi_g^{\Upsilon})^{-2} \left(\kappa_{\text{mag}}^{(g,\Upsilon)} - \mathcal{S}(\varphi_g^{\Upsilon}) \right). \tag{3.22}$$

Remark 3.6. There is a theory on the reduction of the curvature of Hamiltonian flows by first integrals, see [3]. It turns out that our definition of curvature of the magnetic flow corresponds to the reduced curvature of the Hamiltonian h_{mag} on the level $h_{\text{mag}}^{-1}(1)$.

4 A Gauss-Bonnet inequality for Zermelo's problems

This section is dedicated to some global "Gauss-Bonnet properties" of Zermelo's problems which turn out to be key ingredients in the proof of Hopf's theorem for Zermelo problems (the purpose of the next section).

On the three-dimensional surface $\mathcal{H}$ there exists a canonical volume element, called Liouville volume element, defined by $d\mathcal{L} = -s|_{\mathcal{H}} \wedge ds|_{\mathcal{H}}$. Since the Liouville one-form $s|_{\mathcal{H}}$ is invariant by $\vec{\boldsymbol{h}}$ so is $d\mathcal{L}$, i.e.,

$$L_{\vec{\boldsymbol{h}}}\, d\mathcal{L} = 0. \tag{4.1}$$

In the case of a Riemannian surface (M, g) the Liouville volume element on $h_g^{-1}(1) = \mathcal{S}^{*g}$ is called Riemannian volume element and we denote it by $d\mathcal{R}_g$. In this particular case it is easy to check that $d\mathcal{R}_g$ is invariant by the vertical field $\boldsymbol{v}_g$ (actually the Riemannian case together with the Lorentzian case are the only ones satisfying the regularity assumptions (2.2) for which the canonical vector field $\boldsymbol{v}$, defined by relation (2.3), preserves the Liouville volume). Thus, being invariant by $\vec{\boldsymbol{h}}_g$ and $\boldsymbol{v}_g$, the Riemannian volume element is also invariant by their bracket, that is,

$$L_{[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} d\mathcal{R}_g = 0. \tag{4.2}$$

Using relation (3.14) one can easily check that for the co-Zermelo problem of the pair (g, Υ) the two volume elements $d\mathcal{L}$ and $d\mathcal{R}_g$ are linked by the relation

$$F^{\Upsilon *} d\mathcal{L} = \varphi_g^\Upsilon \circ F^{-\Upsilon} \, d\mathcal{R}_g, \tag{4.3}$$

where F^Υ is the diffeomorphism defined by relation (3.13).

Lemma 4.1. *Let (M, g) be a compact, connected, orientable, two-dimensional Riemannian manifold without boundary. Let Υ be a smooth one-form on M. Then,*

$$\frac{1}{4\pi^2} \int_{\mathcal{S}^{g*}} \kappa_{\mathrm{mag}}^{(g,\Upsilon)} \circ F^\Upsilon \, d\mathcal{R}_g \geqslant \chi(M), \tag{4.4}$$

$$\frac{1}{4\pi^2} \int_{\mathcal{H}} \varphi_g^\Upsilon \kappa_{\mathrm{co}\mathcal{Z}}^{(g,\Upsilon)} \, d\mathcal{L} \geqslant \chi(M), \tag{4.5}$$

where $\chi(M)$ is the Euler characteristic of the surface M.

Proof. According to (3.21),

$$\begin{aligned} \frac{1}{4\pi^2} \int_{\mathcal{S}^{g*}} \kappa_{\mathrm{mag}}^{(g,\Upsilon)} \circ F^\Upsilon \, d\mathcal{R}_g &= \int_{\mathcal{S}^{g*}} \kappa_g \, d\mathcal{R}_g + \int_{\mathcal{S}^{g*}} \Omega^2 \, d\mathcal{R}_g + \int_{\mathcal{S}^{g*}} L_{[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} \Omega \, d\mathcal{R}_g \\ &= 2\pi \int_M \kappa_g \, dV_g + 2\pi \int_M \Omega^2 \, dV_g - \int_{\mathcal{S}^{g*}} \Omega \, L_{[\vec{\boldsymbol{h}}_g, \boldsymbol{v}_g]} d\mathcal{R}_g, \end{aligned}$$

which, according to (4.2) and to the classical Gauss-Bonnet formula, is equivalent to

$$\frac{1}{4\pi^2} \int_{\mathcal{S}^{g*}} \kappa_{\mathrm{mag}}^{(g,\Upsilon)} \circ F^\Upsilon \, d\mathcal{R}_g = 4\pi^2 \chi(M) + 2\pi \int_M \Omega^2 \, dV_g \geqslant 4\pi^2 \chi(M). \tag{4.6}$$

This proves relation (4.4). According to (3.22) and (4.1), we have

$$\begin{aligned} \int_{\mathcal{H}} \varphi_g^\Upsilon \mathcal{S}(\varphi_g^\Upsilon) \, d\mathcal{L} &= -\int_{\mathcal{H}} L_{\vec{\boldsymbol{h}}} \left(\frac{L_{\vec{\boldsymbol{h}}} \varphi_g^\Upsilon}{2} \right) d\mathcal{L} + \int_{\mathcal{H}} \left(\frac{L_{\vec{\boldsymbol{h}}} \varphi_g^\Upsilon}{2} \right)^2 \frac{d\mathcal{L}}{\varphi_g^\Upsilon} \\ &= \int_{\mathcal{H}} \left(\frac{L_{\vec{\boldsymbol{h}}} \varphi_g^\Upsilon}{2} \right)^2 \frac{d\mathcal{L}}{\varphi_g^\Upsilon} \geqslant 0. \end{aligned} \tag{4.7}$$

Relation (4.5) now follows from (4.6) and (4.7). ∎

Theorem 4.2. *Let M be a compact, connected, orientable, two-dimensional Riemannian manifold without boundary. If κ is the curvature of a Zermelo-like problem, then there exists a canonically defined positive function ϕ which is identically equal to one if and only if the problem is Riemannian and such that*

$$\frac{1}{4\pi^2}\int_{\mathcal{H}} \phi\kappa \, d\mathcal{L} \geqslant \chi(M). \tag{4.8}$$

Moreover, when ϕ is identically equal to one relation (4.8) is the classical Gauss-Bonnet formula.

Proof. It follows straightforwardly from the previous lemma and Proposition 3.4. ■

It immediately follows from the above theorem that

Theorem 4.3. *Zermelo's problems having nonpositive and not identically equal to zero curvature do not exist on two-dimensional tori.*

Proof. We prove the result by contradiction. Let κ be the curvature of a Zermelo-like problem on a two-dimensional Riemannian torus and let ϕ be the function of Theorem 4.2. Suppose that $\kappa \leqslant 0$. Since κ does not vanish identically, there exists a point $\lambda \in \mathcal{H}$ such that $\kappa(\lambda) < 0$. Since moreover ϕ is a strictly positive function it follows that $\int_{\mathcal{H}} \phi\kappa \, d\mathcal{L} < 0$. But this contradicts the Gauss-Bonnet inequality of Theorem 4.2 which, in this case, reads $\int_{\mathcal{H}} \phi\kappa \, d\mathcal{L} \geqslant 4\pi^2\chi(\mathbb{T}^2) = 0$. ■

Remark 4.4. Although the previous theorem is an immediate consequence of inequality (4.8), we want to point out that it can also be seen as a consequence of a more general fact if "nonpositive" is replaced by "negative" in its formulation. Indeed, the flow generated by the Hamiltonian of a smooth control system having negative curvature is Anosov (see [2]). Moreover, in the appendix to the paper by Anosov and Sinai [7], Margulis proved that, if an Anosov flow operates on a three-dimensional manifold, then its fundamental group has exponential growth. Therefore, an Anosov flow cannot be carried by a three-dimensional torus since the fundamental group of the latter is the free abelian group $\mathbb{Z}^3$ which is known to have polynomial and not exponential growth (see e.g. [15]). Finally, one easily checks that the hypersurface $\mathcal{H}$ of a Zermelo-like problem (of course, whose drift has Riemannian norm strictly smaller that one) over a two-dimensional torus is diffeomorphic to a three-dimensional torus.

It's worth mentioning that the Gauss-Bonnet inequality (4.8) becomes an equality *not only* if the problem is Riemannian. Indeed,

Proposition 4.5. *The Gauss-Bonnet inequality of Theorem 4.2 is an equality if and only if the drift is identically equal to zero or the Gaussian curvature of the manifold is zero and the drift is constant in any system of local coordinates in which $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ commute.*

Proof. It follows from Proposition 3.4 that it is enough to prove the result for the co-Zermelo problem of the pair (g, Υ). Let $M = \cup_\alpha O_\alpha$ where the O_α's are domains of local g-orthonormal frames and let $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ be such a frame. From relation (4.6) we know that

$$\int_{\mathcal{H}} \varphi_g^\Upsilon \kappa_{\mathrm{co}\mathcal{Z}}^{(g,\Upsilon)}\, d\mathcal{L} = 4\pi^2\chi(M) + 2\pi \int_M \Omega^2\, dV_g + \int_{\mathcal{H}} \left(\frac{L_{\vec{\boldsymbol{h}}_{\mathrm{mag}}} \varphi_g^\Upsilon}{2\varphi_g^\Upsilon} \right)^2 F^{-\Upsilon *} d\mathcal{R}_g \tag{4.9}$$

so that the Gauss-Bonnet inequality becomes an equality if and only if

$$\Omega = 0 \quad \text{and} \quad L_{\vec{\boldsymbol{h}}_{\mathrm{mag}}} \varphi_g^\Upsilon = 0 \tag{4.10}$$

identically. On the one hand, the condition $\Omega = 0$ means that the drift form Υ is closed (recall that Ω is defined by $d\Upsilon = \Omega\, dV_g$), which implies

$$0 = d\Upsilon(\boldsymbol{e}_1, \boldsymbol{e}_2) = L_{\boldsymbol{e}_1}\Upsilon_2 - L_{\boldsymbol{e}_2}\Upsilon_1 - \Upsilon_1 c_1 - \Upsilon_2 c_2, \tag{4.11}$$

where $\Upsilon_1 = \langle \Upsilon, \boldsymbol{e}_1 \rangle$ and $\Upsilon_2 = \langle \Upsilon, \boldsymbol{e}_2 \rangle$.

On the other hand, assuming that $\Omega = 0$ holds true, condition $L_{\vec{\boldsymbol{h}}_{\mathrm{mag}}} \varphi_g^\Upsilon = 0$ reads $L_{F_*^\Upsilon \vec{\boldsymbol{h}}_g} \varphi_g^\Upsilon = 0$. According to the notations of Example 2.1,

$$\begin{aligned} 0 &= L_{\cos\theta \boldsymbol{e}_1 + \sin\theta \boldsymbol{e}_2 + (c_1\cos\theta + c_2\sin\theta)\frac{\partial}{\partial\theta}} (1 + \Upsilon_1\cos\theta + \Upsilon_2\sin\theta) \\ &= (L_{\boldsymbol{e}_1}\Upsilon_1 + c_1\Upsilon_2)\cos^2\theta + (L_{\boldsymbol{e}_2}\Upsilon_2 - c_2\Upsilon_1)\sin^2\theta \\ &\quad + (L_{\boldsymbol{e}_1}\Upsilon_2 + L_{\boldsymbol{e}_2}\Upsilon_1 - c_1\Upsilon_1 + c_2\Upsilon_2)\cos\theta\sin\theta. \end{aligned} \tag{4.12}$$

Equations (4.11) and (4.12) are thus equivalent to the system of equations

$$\begin{aligned} L_{\boldsymbol{e}_1}\Upsilon_2 - L_{\boldsymbol{e}_2}\Upsilon_1 - c_1\Upsilon_1 - c_2\Upsilon_2 &= 0 \\ L_{\boldsymbol{e}_1}\Upsilon_1 + c_1\Upsilon_2 &= 0 \\ L_{\boldsymbol{e}_2}\Upsilon_2 - c_2\Upsilon_1 &= 0 \\ L_{\boldsymbol{e}_1}\Upsilon_2 + L_{\boldsymbol{e}_2}\Upsilon_1 - c_1\Upsilon_1 + c_2\Upsilon_2 &= 0. \end{aligned}$$

Replacing the first and last equations by their sum and difference respectively we equivalently get

$$L_{\boldsymbol{e}_1}\Upsilon_2 - c_1\Upsilon_1 = 0 \tag{4.13}$$
$$L_{\boldsymbol{e}_1}\Upsilon_1 + c_1\Upsilon_2 = 0 \tag{4.14}$$
$$L_{\boldsymbol{e}_2}\Upsilon_2 - c_2\Upsilon_1 = 0 \tag{4.15}$$
$$L_{\boldsymbol{e}_2}\Upsilon_1 + c_2\Upsilon_2 = 0. \tag{4.16}$$

Now we differentiate equation (4.16) along $\boldsymbol{e}_1$ and subtract to the result the differentiation along $\boldsymbol{e}_2$ of equation (4.14). According to (2.5), we get

$$
\begin{aligned}
0 &= L_{\boldsymbol{e}_1}(4.16) - L_{\boldsymbol{e}_2}(4.14) \\
&= L_{\boldsymbol{e}_1} \circ L_{\boldsymbol{e}_2}\Upsilon_1 + c_2 L_{\boldsymbol{e}_1}\Upsilon_2 + \Upsilon_2 L_{\boldsymbol{e}_1} c_2 - L_{\boldsymbol{e}_2} \circ L_{\boldsymbol{e}_1}\Upsilon_1 - c_1 L_{\boldsymbol{e}_2}\Upsilon_2 - \Upsilon_2 L_{\boldsymbol{e}_2} c_1 \\
&= L_{[\boldsymbol{e}_1,\boldsymbol{e}_2]}\Upsilon_1 + \Upsilon_2(L_{\boldsymbol{e}_1}c_2 - L_{\boldsymbol{e}_2}c_1) + (\Upsilon_1 c_1)c_2 - (\Upsilon_1 c_2)c_1 \\
&= c_1 L_{\boldsymbol{e}_1}\Upsilon_1 + c_2 L_{\boldsymbol{e}_2}\Upsilon_1 + \Upsilon_2(L_{\boldsymbol{e}_1}c_2 - L_{\boldsymbol{e}_2}c_1) \\
&= \Upsilon_2(-c_1^2 - c_2^2 + L_{\boldsymbol{e}_1}c_2 - L_{\boldsymbol{e}_2}c_1) \qquad (4.17) \\
&= \Upsilon_2 \kappa_g. \qquad (4.18)
\end{aligned}
$$

In the same way, using this time equations (4.13) and (4.15), we get

$$0 = L_{\boldsymbol{e}_2}(4.13) - L_{\boldsymbol{e}_1}(4.15) = \Upsilon_1 \kappa_g. \qquad (4.19)$$

If the Gaussian curvature is identically equal to zero then the Riemannian manifold is a flat torus. In this case we can choose a system of local coordinates (q_1, q_2) on M in which $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ commute. In these coordinates equations (4.13), (4.14), (4.15) and (4.16) read

$$L_{\boldsymbol{e}_i}\Upsilon_j = 0, \quad i, j = 1, 2,$$

which obviously implies that the coefficients Υ_1 and Υ_2 are constant, or, equivalently that Υ is constant. In particular, Υ has constant Riemannian norm.

If the Gaussian curvature is not identically equal to zero then it follows from equations (4.17) and (4.19) that the form Υ must be zero wherever κ_g is different from zero. Consider the closed set $A = \{q \in M \,:\, \kappa_g(q) = 0\}$. If the interior of A is empty it follows from its continuity that Υ vanishes identically on M. If the interior of A is nonempty, by continuity of Υ, we first get that $\Upsilon|_{M \setminus \operatorname{int} A} = 0$. Then, reasoning as above on each domain O_α, implies that Υ has constant Riemannian norm in restriction to every connected component of the closure of the interior of A. These constants are necessarily zero since they should agree with the value of $|\Upsilon|_g$ on the boundary $\partial(M \setminus \operatorname{int} A) = \partial(\operatorname{clo} \operatorname{int} A)$. ■

Remark 4.6. The two equations appearing in (4.10) are actually equivalent to the unique equation $L_{\vec{\boldsymbol{h}}_{\text{mag}}}\varphi_g^\Upsilon = 0$. Indeed, $L_{\vec{\boldsymbol{h}}_{\text{mag}}}\varphi_g^\Upsilon$ is a polynomial of degree two in $\cos\theta$, $\sin\theta$. In particular we have

$$
\begin{aligned}
0 &= L_{\vec{\boldsymbol{h}}_{\text{mag}}}\varphi_g^\Upsilon(\pi/2, q) + L_{\vec{\boldsymbol{h}}_{\text{mag}}}\varphi_g^\Upsilon(-\pi/2, q) = \Omega\Upsilon_1 \\
0 &= L_{\vec{\boldsymbol{h}}_{\text{mag}}}\varphi_g^\Upsilon(0, q) + L_{\vec{\boldsymbol{h}}_{\text{mag}}}\varphi_g^\Upsilon(\pi, q) = \Omega\Upsilon_2,
\end{aligned}
$$

which obviously implies that $\Omega = 0$.

5 A Hopf theorem for control systems

It is well known that Riemannian tori without conjugate points are flat. This result was first proved by E. Hopf in 1943 for the two-dimensional case (see [17]) and then for higher dimensional manifolds by D. Burago and S. Ivanov in 1994 (see [11]). We give in this section a generalization of Hopf's for control systems.

5.1 Jacobi curves

We introduce here the *Jacobi curves* which are a generalization of the space of Jacobi fields along Riemannian geodesics. Since the construction of Jacobi curves does not depend on the dimension of the manifold, it will be carried out in the general case. Most of the material presented in this section can be fund in great details in the papers [1, 4, 6].

Let h be the Hamiltonian function of a time-optimal smooth control problem and $\mathcal{H}$ its hypersurface $h^{-1}(1)$. Let $e^{t\vec{\boldsymbol{h}}} : \mathcal{H} \to \mathcal{H}$ denote the flow generated by the Hamiltonian field $\vec{\boldsymbol{h}}$. This flow defines a one-dimensional foliation $\mathcal{F}$ of $\mathcal{H}$ whose leaves, the trajectories of $\vec{\boldsymbol{h}}$, are transverse to the fibers T_q^*M, $q \in M$. This foliation enable us to make the following symplectic reduction.

Consider the canonical projection

$$\bar{\pi} : \mathcal{H} \to \Sigma = \mathcal{H}/\mathcal{F}.$$

The quotient space Σ, space of trajectories of $\vec{\boldsymbol{h}}$, is, at least locally, a well-defined smooth manifold and carries a structure of symplectic manifold with symplectic form $\bar{\sigma}$ characterized by the property that its pull-back to $\mathcal{H}$ is the restriction $\sigma|_{\mathcal{H}}$.

Let $\Pi \subset T\mathcal{H}$ denote the vertical distribution, i.e., $\Pi_\lambda = T_\lambda \mathcal{H}_{\pi(\lambda)}$, $\lambda \in \mathcal{H}$. The curve

$$\begin{aligned} J_\lambda : \mathbb{R} &\to T_{\bar{\pi}(\lambda)}\Sigma \\ t &\mapsto J_\lambda(t) = \bar{\pi}_* \circ e_*^{-t\vec{\boldsymbol{h}}} \Pi_{e^{t\vec{\boldsymbol{h}}}(\lambda)}, \end{aligned}$$

is called *Jacobi curve at* λ. Because the Hamiltonian flow preserves the symplectic structure, it is easy to check that the spaces $J_\lambda(t)$, $t \in \mathbb{R}$, are Lagrangian subspaces of the symplectic space $T_{\bar{\pi}(\lambda)}\Sigma$ so that the Jacobi curves are curves in the Lagrangian Grassmannian $L(T_{\bar{\pi}(\lambda)}\Sigma)$.

Recall that the Lagrangian Grassmannian $L(T_{\bar{\pi}(\lambda)}\Sigma)$ of the symplectic space $T_{\bar{\pi}(\lambda)}\Sigma$ is defined by:

$$L(T_{\bar{\pi}(\lambda)}\Sigma) = \{\Lambda \subset T_{\bar{\pi}(\lambda)}\Sigma \mid \Lambda^{\angle} = \Lambda\}, \quad \Lambda^{\angle} = \{\xi \in T_{\bar{\pi}(\lambda)}\Sigma \mid \bar{\sigma}(\xi, \Lambda) = 0\}.$$

The Lagrangian Grassmannian of a symplectic space is a well-defined smooth and compact manifold. In our particular case of a two-dimensional manifold M, the Lagrangian Grassmannian $L(T_{\pi(\lambda)}\Sigma)$ is diffeomorphic to the one-dimensional real projective space $\mathbb{RP}(1)$. Moreover, since the vertical distribution Π is spanned by the vertical vector field $\boldsymbol{v}$, the Jacobi curve can written as

$$J_\lambda(t) = \mathbb{R}\left(\bar{\pi}_* e^{t\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda)\right). \tag{5.1}$$

We say that a point $e^{t\vec{\boldsymbol{h}}}(\lambda)$ is *conjugate* to λ (or time t is conjugate to zero) if

$$J_\lambda(t) \cap J_\lambda(0) \neq \{0\}.$$

5.2 A Hopf theorem

In this section we prove the following result.

Theorem 5.1. *Consider a control system $\dot{q} = \boldsymbol{f}(q,u)$ on a compact surface M without boundary. Assume that the curves of admissible velocities are strongly convex curves surrounding the origin. If there is no conjugate points on M, then the total curvature $\int_{\mathcal{H}} \kappa \, d\mathcal{L}$ must be negative or zero. In the latter case κ must be zero.*

Proof. Notice that because the curves of admissible velocities are strongly convex curves surrounding the origin, the manifold $\mathcal{H}$ is compact. Although the proof proposed here essentially follows the one given by Hopf in [17], it will however be presented in a more intrinsic and geometrical manner. The first step in the proof consists in the construction of a well-defined function on any extremal of our system, i.e., a function that does not depend on time but only on the point of the extremal. To do so we use the notion of Jacobi curve described in the previous section.

Let λ be a point on the hypersurface $\mathcal{H} \subset T^*M$ and let $J_\lambda(t)$ be the Jacobi curve associated with the extremal $e^{t\vec{\boldsymbol{h}}}(\lambda)$. We have

$$J_\lambda(t) = \mathbb{R}\left(\bar{\pi}_* e^{t\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda)\right) \in \mathbb{RP}(1),$$

with

$$e^{t\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda) = \beta(t,\lambda)\boldsymbol{v}(\lambda) + \gamma(t,\lambda)\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda). \tag{5.2}$$

Considering $(\beta : \gamma)$ as homogeneous coordinate in $\mathbb{RP}(1)$, we can identify the Jacobi curve with the curve

$$t \mapsto (\beta(t,\lambda) : \gamma(t,\lambda)).$$

From the nonexistence of conjugate points it follows that $\gamma(t,\lambda) \neq 0$ for $t \neq 0$. We can thus use the chart $(\beta : \gamma) \mapsto \frac{\beta}{\gamma}$ and make the identification

$$J_\lambda(t) = y_t(\lambda) = \frac{\beta(t,\lambda)}{\gamma(t,\lambda)}, \quad t \neq 0.$$

It turns out (see e.g. [5, 19]) that the coefficients β and γ are solutions of the Cauchy problems

$$\ddot{\beta} + \kappa_t\beta = 0, \quad \beta(0,\lambda) = 1, \quad \dot{\beta}(0,\lambda) = 0, \quad \kappa_t = \kappa(e^{t\vec{\boldsymbol{h}}}(\lambda)), \tag{5.3}$$

$$\ddot{\gamma} + \kappa_t\gamma = 0, \quad \gamma(0,\lambda) = 0, \quad \dot{\gamma}(0,\lambda) = -1, \tag{5.4}$$

which shows in particular that β and γ are two linearly independent solutions of the Hill equation $\ddot{x} + \kappa_t x = 0$. Since β and γ are functions on $\mathbb{R} \times \mathcal{H}$, the dot in the previous equations has to be understood as the partial derivative with respect to the first variable. The derivative with respect to time of the function y_t is

$$\frac{dy_t}{dt} = \frac{\dot{\beta}\gamma - \beta\dot{\gamma}}{\gamma^2}.$$

Since the Wronskian $\dot\beta\gamma - \beta\dot\gamma$, which is constant with respect to time, equals to

$$\dot\beta(0,\lambda)\gamma(0,\lambda) - \beta(0,\lambda)\dot\gamma(0,\lambda) = 1,$$

the function y_t is strictly increasing or, equivalently the Jacobi curve is strictly increasing in $\mathbb{RP}(1)$. Therefore, the limit of y_t as t goes to infinity exists. Moreover, because of the nonexistence of conjugate points, this limit is finite. Indeed, notice that because of the initial conditions $\beta(0,\lambda) = 1$, $\gamma(0,\lambda) = 0$ and $\dot\gamma(0,\lambda) = -1$ we have for t small enough

$$y_t(\lambda) < 0, \quad y_{-t}(\lambda) > 0. \tag{5.5}$$

So if we suppose that

$$\lim_{t\to+\infty} y_t(\lambda) = +\infty, \tag{5.6}$$

it would follow from (5.5) and from the strict monotonicity of y_t the existence of $t^- < 0 < t^+$ such that $y_{t^-}(\lambda) = y_{t^+}(\lambda)$. Then, the time reparametrization $\tau = t - t^-$ would imply that time $\tau = t^+ - t^-$ is conjugate to $\tau = 0$, which is a contradiction. Hence, the function y^+ defined by

$$y^+(\lambda) = \lim_{t\to+\infty} y_t(\lambda), \quad \lambda \in \mathcal{H},$$

is a well-defined function on the manifold $\mathcal{H}$. Equivalently, the distribution $\Pi^+_\lambda \in T\mathcal{H}$ defined by

$$\Pi^+_\lambda = \lim_{t\to+\infty} J_\lambda(t) = \mathbb{R}\Big(y^+\boldsymbol{v} + \Big[\boldsymbol{v},\vec{\boldsymbol{h}}\Big]\Big)$$

is a well defined distribution on $\mathcal{H}$ transverse to the vertical distribution. This distribution Π^+_λ is, by definition, invariant by the flow of $\vec{\boldsymbol{h}}$. Because the function y^+ is differentiable along the trajectories of $\vec{\boldsymbol{h}}$ (see the proof in Appendix), this invariance reads

$$\Big[\vec{\boldsymbol{h}}, y^+\boldsymbol{v} + \Big[\boldsymbol{v},\vec{\boldsymbol{h}}\Big]\Big] = \alpha\left(y^+\boldsymbol{v} + \Big[\boldsymbol{v},\vec{\boldsymbol{h}}\Big]\right),$$

or, equivalently,

$$L_{\vec{\boldsymbol{h}}}y^+\boldsymbol{v} + y^+\Big[\vec{\boldsymbol{h}},\boldsymbol{v}\Big] + \Big[\vec{\boldsymbol{h}},\Big[\boldsymbol{v},\vec{\boldsymbol{h}}\Big]\Big] = \alpha y^+\boldsymbol{v} + \alpha\Big[\boldsymbol{v},\vec{\boldsymbol{h}}\Big], \tag{5.7}$$

where α is a function on $\mathcal{H}$. Solving (5.7) for α gives

$$\alpha = -y^+ \quad \text{and} \quad L_{\vec{\boldsymbol{h}}}y^+ + \kappa - \alpha y^+ = 0,$$

which shows that y^+ satisfies the Riccati equation

$$L_{\vec{\boldsymbol{h}}}y^+ + y^{+2} + \kappa = 0. \tag{5.8}$$

Being a limit of smooth functions, y^+ is clearly measurable. It is also uniformly bounded (see [14, Lemma 2.1]) and thus it is integrable. If we integrate equation (5.8) over $\mathcal{H}$ with respect to the Liouville volume $d\mathcal{L}$, the first term in the left-hand side of (5.8) disappears since the Liouville volume is invariant by the flow of $\vec{\boldsymbol{h}}$. As a result we obtain

$$\int_{\mathcal{H}} \kappa \, d\mathcal{L} = -\int_{\mathcal{H}} y^{+2} \, d\mathcal{L} \tag{5.9}$$

which immediately proves the validity of the first part of the theorem. Suppose now that the total curvature $\int_{\mathcal{H}} \kappa \, d\mathcal{L}$ is equal to zero. Then, (5.9) implies that the function y^+ vanishes everywhere on $\mathcal{H}$. According to (5.8) κ is therefore identically equal to zero. ■

We say that a control system $\dot{q} = \boldsymbol{f}(q, u)$ *is flat* if it is feedback equivalent to a control system of the form $\dot{q} = \boldsymbol{f}(u)$.

In the Riemannian case, a direct consequence of the Gauss-Bonnet and Theorem 5.1 is that two-dimensional Riemannian tori without conjugate points are flat. Contrary to the Riemannian situation, we shall see that Zermelo-like problems without conjugate points on tori are not necessarily flat.

The following three theorems give a good understanding of the Zermelo-like situation. To simplify notations, we omit the pair (g, Υ) in the writing of curvature and, the diffeomorphism (3.13) in formulas since, anyway, its action is clear.

Theorem 5.2. *Consider a co-Zermelo problem on a compact Riemannian surface without boundary. If there is no conjugate points then the total curvatures $\int_{\mathcal{H}} \kappa_{\mathrm{co}\mathcal{Z}} d\mathcal{L}$ and $\int_{\mathcal{S}^{g*}} \kappa_{\mathrm{mag}} d\mathcal{R}_g$ have to be negative or zero. In the latter case the considered co-Zermelo problem is flat.*

Proof. The part of the theorem concerning $\kappa_{\mathrm{co}\mathcal{Z}}$ is given by Theorem 5.1. In order to check that the same conclusion holds for the curvature κ_{mag}, let us see how the function y^+ constructed in the proof of Theorem 5.1 changes under a reparametrization. For simplicity, denote $\psi^2 = \varphi$. In a general manner, we have

$$\vec{\boldsymbol{h}} = \frac{\hat{\boldsymbol{h}}}{\psi^2} \quad \text{and} \quad \boldsymbol{v} = \psi \hat{\boldsymbol{v}},$$

and we compute the new function $\hat{y}^+$:

$$\begin{aligned} y^+ \boldsymbol{v} + \left[\boldsymbol{v}, \vec{\boldsymbol{h}}\right] &= y^+ \psi \hat{\boldsymbol{v}} + \left[\psi \hat{\boldsymbol{v}} + \frac{1}{\psi^2} \hat{\boldsymbol{h}}\right] = y^+ \psi \hat{\boldsymbol{v}} + \frac{1}{\psi}\left[\hat{\boldsymbol{v}}, \hat{\boldsymbol{h}}\right] - \frac{1}{\psi^2} L_{\hat{\boldsymbol{h}}} \psi \hat{\boldsymbol{v}} \quad \left(\mathrm{mod}\ \vec{\boldsymbol{h}}\right) \\ &= \left(y^+ \psi - L_{\vec{\boldsymbol{h}}} \psi\right) \hat{\boldsymbol{v}} + \frac{1}{\psi}\left[\hat{\boldsymbol{v}}, \hat{\boldsymbol{h}}\right] \quad \left(\mathrm{mod}\ \vec{\boldsymbol{h}}\right). \end{aligned}$$

We thus have

$$\Pi^+ = \mathbb{R}\left(\hat{y}^+ \hat{\boldsymbol{v}} + \left[\hat{\boldsymbol{v}}, \hat{\boldsymbol{h}}\right]\right), \quad \hat{y}^+ = y^+ \psi^2 - \psi L_{\vec{\boldsymbol{h}}} \psi. \tag{5.10}$$

In the same way as for the function y^+ it is easy to see that the function $\hat{y}^+$ satisfies the Riccati equation

$$L_{\hat{\boldsymbol{h}}} \hat{y}^+ + \hat{y}^{+2} + \hat{\kappa} = 0. \tag{5.11}$$

Notice that the Riemannian volume element $d\mathcal{R}_g$ is invariant by $\vec{\boldsymbol{h}}_{\mathrm{mag}}$ since

$$L_{\vec{\boldsymbol{h}}_{\mathrm{mag}}} d\mathcal{R}_g = L_{\Omega \boldsymbol{v}_g} d\mathcal{R}_g = d(\Omega dV_g) = d(d\Upsilon) = 0.$$

Therefore the integration of (5.11) leads to

$$\int_{\mathcal{S}^{g*}} \kappa_{\text{mag}} d\mathcal{R}_g = -\int_{\mathcal{S}^{g*}} \hat{y}^{+2} d\mathcal{R}_g \leqslant 0.$$

This proves the first part of the theorem and a similar argument as the one used in the proof of Theorem 5.1 shows that κ_{mag} is zero everywhere when $\int_{\mathcal{S}^{g*}} \kappa_{\text{mag}}\, d\mathcal{R}_g = 0$.

We now complete the proof showing that the co-Zermelo problem is flat when the total curvatures $\int_{\mathcal{H}} \kappa_{\text{co}\mathcal{Z}}\, d\mathcal{L}$ and $\int_{\mathcal{S}^{g*}} \kappa_{\text{mag}}\, d\mathcal{R}_g$ are both zero. In that case, we must have $\kappa_{\text{mag}} = 0$ and $\kappa_{\text{co}\mathcal{Z}} = 0$ everywhere. In particular it implies

$$0 = \int_{\mathcal{S}^{g*}} \kappa_{\text{mag}}\, d\mathcal{R}_g = \int_{\mathcal{H}} \varphi \kappa_{\text{co}\mathcal{Z}}\, d\mathcal{L} = 0,$$

i.e. (see the proof of Lemma 4.1),

$$0 = 4\pi^2\chi(M) + 2\pi \int_M \Omega^2 dV_g = 4\pi^2\chi(M) + 2\pi \int_M \Omega^2 dV_g + \int_{\mathcal{H}} \left(\frac{L_{\vec{\boldsymbol{h}}}\varphi}{2}\right)^2 \frac{d\mathcal{L}}{\varphi},$$

which is equivalent to $L_{\vec{\boldsymbol{h}}}\varphi = 0$. Therefore, according to Theorem 4.5 and Remark 4.6, either the form Υ is different from zero and in this case the conclusion is obtained, or the form Υ is identically zero and in this case the problem is Riemannian. In the latter case, we have $0 = \kappa_{\text{mag}} = \kappa_g$ which, on the one hand, implies that the Riemannian surface is flat and, on the other hand, according to the Gauss-Bonnet formula, implies that the surface is a torus. The proof is complete. ■

The following corollaries are direct consequences of Theorem 5.2.

Corollary 5.3. *If a co-Zermelo problem on a two-dimensional Riemannian torus has no conjugate points then the torus is flat and the drift one-form is closed. In particular, time-optimal trajectories are straight lines.*

Corollary 5.4. *Zermelo-like problems without conjugate points on two-dimensional Riemannian tori are flat if and only if their total curvature is zero.*

5.3 A natural question

In the proof of Theorem 5.1 we constructed a well-defined function y^+ on $\mathcal{H}$ that satisfies Riccati equation (5.8). This construction is valid along every regular extremal without conjugate points. Recall moreover that a control system with negative curvature does not admit conjugate points. A very natural question is thus the following:

Does a control system without conjugate points admit a non positive φ-reparametrized curvature?

Since the function $\hat{y}^+$ satisfies Riccati equation (5.11), the question can be reformulated in the following manner: *does there exists a positive function ψ such that $L_{\hat{\boldsymbol{h}}}\hat{y}^+ = 0$, or equivalently such that $L_{\vec{\boldsymbol{h}}}\hat{y}^+ = 0$?* According to relation (5.10),

$$L_{\vec{\boldsymbol{h}}}\hat{y}^+ = L_{\vec{\boldsymbol{h}}}(y^+\psi^2 - \psi L_{\vec{\boldsymbol{h}}}\psi) = \psi^2 L_{\vec{\boldsymbol{h}}} y^+ + 2y^+\psi L_{\vec{\boldsymbol{h}}}\psi - (L_{\vec{\boldsymbol{h}}}\psi)^2 - \psi L^2_{\vec{\boldsymbol{h}}}\psi,$$

so that (dividing by ψ^2) $L_{\vec{h}}\hat{y}^+ = 0$ is equivalent to

$$L_{\vec{h}}y^+ + 2y^+\left(\frac{L_{\vec{h}}\psi}{\psi}\right) - \left(\frac{L_{\vec{h}}\psi}{\psi}\right)^2 - \frac{L^2_{\vec{h}}\psi}{\psi} = 0,$$

i.e., to

$$L_{\vec{h}}y^+ + 2y^+L_{\vec{h}}\log\psi - (L_{\vec{h}}\log\psi)^2 - \frac{L^2_{\vec{h}}\psi}{\psi} = 0. \tag{5.12}$$

Denote $g = \log\psi$. We have

$$L^2_{\vec{h}}g = L_{\vec{h}}(L_{\vec{h}}\log\psi) = L_{\vec{h}}\left(\frac{L_{\vec{h}}\psi}{\psi}\right) = \frac{(L^2_{\vec{h}}\psi)\psi - (L_{\vec{h}}\psi)^2}{\psi^2} = \frac{L^2_{\vec{h}}\psi}{\psi} - (L_{\vec{h}}g)^2,$$

or equivalently

$$\frac{L^2_{\vec{h}}\psi}{\psi} = L^2_{\vec{h}}g + (L_{\vec{h}}g)^2.$$

This implies that equation (5.12) is equivalent to

$$L_{\vec{h}}y^+ + 2y^+L_{\vec{h}}g - 2(L_{\vec{h}}g)^2 - L^2_{\vec{h}}g = 0,$$

i.e., to the Riccati equation

$$L_{\vec{h}}z + 2z^2 - 2y^+z - L_{\vec{h}}y^+ = 0, \tag{5.13}$$

where we have set $z = L_{\vec{h}}g$.

The function $z = y^+$ is solution to Riccati equation (5.13). Thus we will have the required reparametrization of $\vec{h}$ if we can solve the equation

$$L^2_{\vec{h}}\log\psi = y^+ \tag{5.14}$$

globally on the three-dimensional manifold $\mathcal{H}$. The first thing we need for the resolution of equation (5.14) is the continuity of the function y^+ on $\mathcal{H}$. The function y^+ is well defined on $\mathcal{H}$ and smooth along the trajectories of $\vec{h}$. But, we have, a priori, no information concerning the regularity of y^+ (even its continuity) with respect to directions transversal to $\vec{h}$. In the case of hyperbolic systems the function y^+ is easily seen to be continuous due to some "exponential estimates" along the stable distribution (see [16]). Also, for such systems the function y^+ is in general never differentiable and even never Lipschitz continuous but only Hölder continuous (see [16, Theorem 19.1.6]). In the case of systems without conjugate points the situation is quite different because we do not have the exponential estimates and by consequence the continuity of the function y^+ is not so obvious. What we can ensure is the following.

Lemma 5.5. *The function y^+ defined above is lower semi-continuous.*

Proof. Let $(\lambda_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ be a sequence converging to $\lambda \in \mathcal{H}$. Since $y_t(\lambda_n)$ is increasing in t, it follows that

$$y_t(\lambda_n) \leqslant y^+(\lambda_n) = \lim_{t\to+\infty} y_t(\lambda_n).$$

Taking the lim inf as n tends to infinity in the previous relation, we get, since $y_t(\lambda)$ is continuous in (t, λ),

$$y_t(\lambda) \leqslant \liminf_{\lambda_n\to\lambda} y^+.$$

Letting t going to infinity in the previous inequality leads to

$$y^+(\lambda) \leqslant \liminf_{\lambda_n\to\lambda} y^+,$$

which proves the lower semi-continuity of y^+. ■

Suppose that the function y^+ is continuous. Hence, we can solve locally equation (5.14). The question of solving the equation globally is more delicate because the problem is closely related to the fact that the quotient manifold Σ (introduced in Section 5.1) is globally defined. It is not our scope to discuss this problem here. However we can say the following. Let $\tilde{M}$ be the universal covering of M. Because of the nonexistence of conjugate points, $\tilde{M}$ is diffeomorphic to $\mathbb{R}^2$. Let

$$\dot{\tilde{q}} = \tilde{\boldsymbol{f}}(\tilde{q}, u), \quad \tilde{q} \in \tilde{M}, \quad u \in U, \tag{5.15}$$

be the lift on $\tilde{M}$ of the control system $\dot{q} = \boldsymbol{f}(q, u)$ and $\tilde{\mathcal{H}}$ be the corresponding Hamiltonian hypersurface. The continuity of y^+ implies that if the control system $\dot{q} = \boldsymbol{f}(q, u)$ has no conjugate points then there exists a reparametrization of $\vec{\boldsymbol{h}}$ or, equivalently, a globally defined function ψ satisfying equation (5.14), such that the lifted system (5.15) has negative curvature. Unfortunately, y^+ is not, in general, a continuous function as shown by Ballmann, Brin and Burns in [9] where the authors give an example of a two-dimensional compact surface without conjugate points where y^+ fails to be continuous.

It would be interesting to characterize two-dimensional smooth control systems without conjugate points where this function fails to be continuous. Which are the geometrical properties that prevent y^+ from being continuous?

6 Conclusion

We conclude this paper with a brief discussion of the extension of our results to more general structures than Riemannian surfaces. Of course, Zermelo-like problems can be defined on any manifold equipped with a geometric structure defined by an optimal control problem of type (2.1)-(2.2). A natural class of geometric structures on which one can hope to generalize our result is the class of manifolds equipped with a Finsler metric (see the book of Chern and Shen [13] for a nice and brief presentation of Riemann-Finsler geometry). Unfortunately, since the Gauss-Bonnet formula is not true for all Finsler surfaces, the results from Section 4 cannot be extended to all these structures.

One has to limits oneself to Zermelo-like problems on Landsberg surfaces on which almost all results from Section 4 remain true. Roughly speaking, a Landsberg surface is a Finsler surface on which the Gauss-Bonnet formula remains true (up to changing the classical 2π factor in the formula by the centro-affine length ℓ of the curve $\mathcal{H}_q$, which is defined by $\ell = \int_{\mathcal{H}_q} \mu|_{\mathcal{H}_q}$ where μ is a one-form on the hypersurface $\mathcal{H}$ such that $\langle \mu, \boldsymbol{v} \rangle = 1$). Without entering into details one can see that the Gauss-Bonnet formula still holds on Landsberg surfaces due to the fact that the centro-affine length of the curve $\mathcal{H}_q$ does not depend on the base point q. This property is characterized by the fact that the invariant b that appears in relation (2.3) is a first integral of the vector field $\vec{\boldsymbol{h}}$ (see [19] for details). If we now consider Zermelo-like problems on Landsberg surfaces, on the one hand, the Gauss-Bonnet inequality (4.8) still holds true. The proof is the same but this time one has to be more careful because the Landsberg volume element $d\mathcal{L}_{\text{land}}$ is not invariant under the vertical Landsberg field $\boldsymbol{v}_{\text{land}}$. Indeed, one can easily check that $L_{\boldsymbol{v}_{\text{land}}} d\mathcal{L}_{\text{land}} = b\, d\mathcal{L}_{\text{land}}$. Anyway, $d\mathcal{L}_{\text{land}}$ is still invariant under the bracket $[\vec{\boldsymbol{h}}_{\text{land}}, \boldsymbol{v}_{\text{land}}]$ since

$$L_{\left[\vec{\boldsymbol{h}}_{\text{land}}, \boldsymbol{v}_{\text{land}}\right]} d\mathcal{L}_{\text{land}} = \left(L_{\vec{\boldsymbol{h}}_{\text{land}}} b\right) d\mathcal{L}_{\text{land}} = 0.$$

On the contrary, Theorem 5.2 and its Corollaries 5.3 and 5.4 do not generalize to Zermelo-like problems on Landsberg surfaces. The reason is that Landsberg surfaces of zero curvature are not necessary flat (see [19], Theorem 4.3.3).

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Appendix

We prove, in three steps, the differentiability of the function y^+, constructed in the proof of Theorem 5.1, along the trajectories of $\vec{\boldsymbol{h}}$.

Let us give some notation first. If $x : \mathbb{R} \times \mathcal{H} \to \mathbb{R}$ is a smooth function, we set $x_{t,\lambda} = x(t, \lambda)$. By $\partial_1 x$, we denote the partial derivative of x with respect to the first variable t, and by $\partial_2 x_{t,\lambda} : T_\lambda \mathcal{H} \to \mathbb{R}$ the tangent application at point λ of the mapping $x_t \in \mathcal{C}^\infty(\mathcal{H})$ defined by $\lambda \mapsto x_{t,\lambda}$.

Variable splitting in y_{t,λ_s}

We use here the chronological calculus notation (see [5]). Let $\lambda_0 \in \mathcal{H}$ be fixed and denote by λ_s the trajectory $e^{s\vec{\boldsymbol{h}}}(\lambda_0) = \lambda_0 \circ e^{s\vec{\boldsymbol{h}}}$. According to equation (5.2), we have, on the one hand

$$e^{t\,\mathrm{ad}\,\vec{\boldsymbol{h}}} \boldsymbol{v}(\lambda_s) = \beta_{t,\lambda_s} \boldsymbol{v}(\lambda_s) + \gamma_{t,\lambda_s} \left[\boldsymbol{v}, \vec{\boldsymbol{h}}\right](\lambda_s),$$

and, on the other hand

$$\begin{aligned} e^{t\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda_s) &= \lambda_s \circ e^{t\vec{\boldsymbol{h}}} \circ \boldsymbol{v} \circ e^{-t\vec{\boldsymbol{h}}} = \lambda_0 \circ e^{(s+t)\vec{\boldsymbol{h}}} \circ \boldsymbol{v} \circ e^{-(s+t)\vec{\boldsymbol{h}}} \circ e^{s\vec{\boldsymbol{h}}} \\ &= e_*^{s\vec{\boldsymbol{h}}} e^{(s+t)\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda_0) = e_*^{s\vec{\boldsymbol{h}}}\left(\beta_{t+s,\lambda_0}\boldsymbol{v}(\lambda_0) + \gamma_{t+s,\lambda_0}\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda_0)\right) \\ &= \beta_{t+s,\lambda_0} e_*^{s\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda_0) + \gamma_{t+s,\lambda_0} e_*^{s\vec{\boldsymbol{h}}}\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda_0), \end{aligned}$$

with

$$\begin{aligned} e_*^{s\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda_0) &= \lambda_s \circ e^{-s\vec{\boldsymbol{h}}} \circ \boldsymbol{v} \circ e^{s\vec{\boldsymbol{h}}} = e^{-s\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda_s) \\ &= \beta_{-s,\lambda_s}\boldsymbol{v}(\lambda_s) + \gamma_{-s,\lambda_s}\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda_s), \end{aligned}$$

and

$$\begin{aligned} e_*^{s\vec{\boldsymbol{h}}}\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda_0) &= \lambda_s \circ e^{-s\vec{\boldsymbol{h}}} \circ \left[\boldsymbol{v},\vec{\boldsymbol{h}}\right] \circ e^{s\vec{\boldsymbol{h}}} = e^{-s\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda_s) \\ &= \tilde{\gamma}_{-s,\lambda_s}\boldsymbol{v}(\lambda_s) + \tilde{\beta}_{-s,\lambda_s}\left[\boldsymbol{v},\vec{\boldsymbol{h}}\right](\lambda_s), \end{aligned}$$

where, the functions $\tilde{\gamma}$ and $\tilde{\beta}$ are solutions of the Cauchy problems

$$\partial_1\tilde{\gamma}_{t,\lambda} = \kappa(e^{t\vec{\boldsymbol{h}}}(\lambda))\beta_{t,\lambda}, \quad \tilde{\gamma}_{0,\lambda} = 0, \tag{A.1}$$

$$\partial_1\tilde{\beta}_{t,\lambda} = \kappa(e^{t\vec{\boldsymbol{h}}}(\lambda))\gamma_{t,\lambda}, \quad \tilde{\beta}_{0,\lambda} = 1, \tag{A.2}$$

which, taking into account (2.4), can be directly checked by differentiating equation (5.2) with respect to t. Hence, by identification of the coefficients in the two expressions of $e^{t\,\mathrm{ad}\,\vec{\boldsymbol{h}}}\boldsymbol{v}(\lambda_s)$, we get

$$\begin{aligned} \beta_{t,\lambda_s} &= \beta_{s+t,\lambda_0}\beta_{-s,\lambda_s} + \gamma_{s+t,\lambda_0}\tilde{\gamma}_{-s,\lambda_s} \\ \gamma_{t,\lambda_s} &= \beta_{s+t,\lambda_0}\gamma_{-s,\lambda_s} + \gamma_{s+t,\lambda_0}\tilde{\beta}_{-s,\lambda_s}, \end{aligned}$$

from which we deduce the following formula for y_{t,λ_s}:

$$y_{t,\lambda_s} = \frac{\beta_{s+t,\lambda_0}\beta_{-s,\lambda_s} + \gamma_{s+t,\lambda_0}\tilde{\gamma}_{-s,\lambda_s}}{\beta_{s+t,\lambda_0}\gamma_{-s,\lambda_s} + \gamma_{s+t,\lambda_0}\tilde{\beta}_{-s,\lambda_s}} = (1+\varepsilon_3(s))\frac{y_{t+s,\lambda_0} + \varepsilon_1(s)}{1 + y_{t+s,\lambda_0}\varepsilon_2(s)}$$

where

$$\varepsilon_1(s) = \frac{\tilde{\gamma}_{-s,\lambda_s}}{\beta_{-s,\lambda_s}}, \quad \varepsilon_2(s) = \frac{\gamma_{-s,\lambda_s}}{\tilde{\beta}_{-s,\lambda_s}}, \quad \varepsilon_3(s) = \frac{\beta_{-s,\lambda_s}}{\tilde{\beta}_{-s,\lambda_s}} - 1.$$

Taylor expansions of β_{-s,λ_s} and γ_{-s,λ_s}

Differentiating the second boundary condition in the Cauchy problem (5.3) implies that $\partial_2^k\beta_{0,\lambda} = 0$ for all $k \geqslant 1$. Consequently a Taylor expansion of β_{-s,λ_s} at $s = 0$ takes the form

$$\beta_{-s,\lambda_s} = \sum_{k=0}^{n} \partial_1^k\beta_{0,\lambda_0}\frac{(-s)^k}{k!} + O(s^{n+1})$$

For $(t,\lambda)=(-s,\lambda_s)$, equation (5.3) reads

$$\partial_1^2\beta_{-s,\lambda_s}+\kappa(\lambda_0)\beta_{-s,\lambda_s}=0, \tag{A.3}$$

which implies

$$\partial_1^2\beta_{0,\lambda_0}=-\kappa(\lambda_0).$$

The differentiation of equation (A.3) enable to calculate all derivatives $\partial_1^k\beta_{0,\lambda_0}$:

$$-\partial_1^3\beta_{-s,\lambda_s}+\partial_2\partial_1^2\beta_{-s,\lambda_s}\dot\lambda_s-\kappa(\lambda_0)\partial_1\beta_{-s,\lambda_s}+\kappa(\lambda_0)\partial_2\beta_{-s,\lambda_s}\dot\lambda_s=0,$$

which, according to the boundary conditions of the Cauchy problem (5.3), reads at $s=0$

$$-\partial_1^3\beta_{0,\lambda_0}=0.$$

Iteratively, one gets

$$\partial_1^{k+2}\beta_{0,\lambda_0}+\kappa(\lambda_0)\partial_1^k\beta_{0,\lambda_0}=0,\quad \forall k\geqslant 0.$$

This shows that β_{-s,λ_s} has the same Taylor expansion as the solution to $\ddot x+\kappa(\lambda_0)x=0$, $x(0)=1$, $\dot x(0)=0$. In other words,

$$\beta_{-s,\lambda_s}=\sum_{k=0}^{n}(-\kappa(\lambda_0))^k\,\frac{s^{2k}}{(2k)!}+O(s^{2n+2}).$$

In the same way one shows that γ_{-s,λ_s} admits the same Taylor expansion as the solution to $\ddot x+\kappa(\lambda_0)x=0$, $x(0)=0$, $\dot x(0)=1$, i.e.,

$$\gamma_{-s,\lambda_s}=\sum_{k=0}^{n-1}(-\kappa(\lambda_0))^k\,\frac{s^{2k+1}}{(2k+1)!}+O(s^{2n+3}).$$

Consequently, taking into account that the Cauchy problems (A.1) and (A.2) reads, for $(t,\lambda)=(-s,\lambda_s)$, $\partial_1\tilde\gamma_{-s,\lambda_s}=\kappa(\lambda_0)\beta_{-s,\lambda_s}$ and $\partial_1\tilde\beta_{-s,\lambda_s}=\kappa(\lambda_0)\gamma_{-s,\lambda_s}$ respectively, one easily get the Taylor expansions of $\tilde\gamma_{-s,\lambda_s}$ and $\tilde\beta_{-s,\lambda_s}$. Then, straightforward computations imply

$$\lim_{s\to 0}\varepsilon_i(s)=\lim_{s\to 0}\frac{d^2}{ds^2}\varepsilon_i(s)=0,\quad i=1,2,3,$$
$$\lim_{s\to 0}\frac{d}{ds}\varepsilon_1(s)=-\kappa(\lambda_0),\quad \lim_{s\to 0}\frac{d}{ds}\varepsilon_2(s)=1,\quad \lim_{s\to 0}\frac{d}{ds}\varepsilon_3(s)=0.$$

Notice that in the analytic case, we have, $\tilde\gamma_{-s,\lambda_s}=\kappa(\lambda_0)\gamma_{-s,\lambda_s}$ and $\tilde\beta_{-s,\lambda_s}=\beta_{-s,\lambda_s}$.

End of the proof: pointwise convergence and equicontinuity of $\frac{d}{ds}y_{t,\lambda_s}$

We now show that for s small enough $\frac{d}{ds}y_{t,\lambda_s}$ is uniformly bounded. Taking into account that the Wronskian $\gamma\partial_1\beta - \beta\partial_1\gamma$ is constant and equals to $(\gamma\partial_1\beta - \beta\partial_1\gamma)_{0,\lambda_0} = 1$, a straightfoward computation leads to

$$\frac{dy_{t,\lambda_s}}{ds} = \frac{d\varepsilon_3}{ds}\frac{y_{t+s,\lambda_0}+\varepsilon_1}{1+y_{t+s,\lambda_0}\varepsilon_2} + (1+\varepsilon_3)\left(\frac{\gamma_{t+s,\lambda_0}^{-2}+\frac{d}{ds}\varepsilon_1}{1+y_{t+s,\lambda_0}\varepsilon_2}\right.$$
$$\left. - \frac{(y_{t+s,\lambda_0}+\varepsilon_1)\left(\gamma_{t+s,\lambda_0}^{-2}\varepsilon_2 + y_{t+s,\lambda_0}\frac{d}{ds}\varepsilon_2\right)}{(1+y_{t+s,\lambda_0}\varepsilon_2)^2}\right).$$

Because γ is the solution of the Cauchy problem (5.4), $\lim_{t\to+\infty}\gamma_{t,\lambda} = -\infty$ (see [14, Theorem 2.1]). Hence,

$$\lim_{t\to+\infty}\frac{dy_{t,\lambda_s}}{ds} = \frac{d\varepsilon_3}{ds}\frac{y^+(\lambda_0)+\varepsilon_1}{1+y^+(\lambda_0)\varepsilon_2} + (1+\varepsilon_3)\left(\frac{\frac{d}{ds}\varepsilon_1}{1+y^+(\lambda_0)\varepsilon_2} - \frac{(y^+(\lambda_0)+\varepsilon_1)y^+(\lambda_0)\frac{d}{ds}\varepsilon_2}{(1+y^+(\lambda_0)\varepsilon_2)^2}\right),$$

which shows that $\frac{d}{ds}y_{t,\lambda_s}$ is pointwise convergent. Letting s go to zero, we get

$$\lim_{s\to0}\lim_{t\to+\infty}\frac{dy_{t,\lambda_s}}{ds} = -\kappa(\lambda_0) - y^+(\lambda_0)^2, \tag{A.4}$$

which shows that, for s small enough and t large enough, $\frac{d}{ds}y_{t,\lambda_s}$ is uniformly bounded. Similarly one shows that $\frac{d^2}{ds^2}y_{t,\lambda_s}$ is also uniformly bounded in t, so that $\frac{d}{ds}y_{t,\lambda_s}$ is equicontinuous. Summing up, $\frac{d}{ds}y_{t,\lambda_s}$ converges uniformly. Consequently, y_{t,λ_s} converges uniformly to $y^+(\lambda_s)$ in a neighborhood of zero and $\lim_{t\to+\infty}\frac{d}{ds}y_{t,\lambda_s} = \frac{d}{ds}y^+(\lambda_s)$. Equivalently, y^+ is differentiable along the trajectories of $\vec{h}$ in a neighborhood of λ_0. Since λ_0 has been chosen arbitrarily on $\mathcal{H}$, the result follows.

Notice that (A.4) proves in a different way that y^+ satisfies Riccati equation (5.8).

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