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Combinatorial study of colored Hurwitz polyzêtas

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abstract
A combinatorial study discloses two surjective morphisms between generalized shuffle algebras and algebras generated by the colored Hurwitz polyzêtas. The combinatorial aspects of the products and co-products involved in these algebras will be examined.

1 Introduction

Classically, the Riemann zêta function is $\zeta(s) = \sum_{n>0} n^{-s}$, the Hurwitz zêta function is $\zeta(s;t) = \sum_{n>0} (n-t)^{-s}$ and the colored zêta function is $\zeta\binom{s}{q} = \sum_{n>0} q^s n^{-s}$, where q is a root of unit. The three previous functions are defined over $\mathbb{Z}_{>0}$ but can be generalized over any *composition* (sequence of positive integers) $\mathbf{s} = (s_1, \ldots, s_r)$, like, respectively, the Riemann polyzêta function $\zeta(\mathbf{s}) = \sum_{n_1 > \ldots > n_r > 0} n_1^{-s_1} \ldots n_r^{-s_r}$, the Hurwitz polyzêta function $\zeta(\mathbf{s};\mathbf{t}) = \sum_{n_1 > \ldots > n_r > 0} (n_1 - t_1)^{-s_1} \ldots (n_r - t_r)^{-s_r}$ and the colored polyzêta function $\zeta\binom{\mathbf{s}}{q^{\mathbf{i}}} = \sum_{n_1 > \ldots > n_r > 0} q^{i_1 n_1} \ldots q^{i_r n_r} n_1^{-s_1} \ldots n_r^{-s_r}$, with q a root of unit and $\mathbf{i} = (i_1, \ldots, i_r)$ a composition. These sums converge when $s_1 > 1$.

To study simultaneously these families of polyzêtas, the colored Hurwitz polyzêtas, for a composition $\mathbf{s} = (s_1, \ldots, s_r)$ and a tuple of complex numbers $\xi = (\xi_1, \ldots, \xi_r)$ and a tuple of parameters in $]-\infty; 1[$, $\mathbf{t} = (t_1, \ldots, t_r)$, are defined by [6]

$$\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s}) = \sum_{n_1 > \ldots > n_r > 0} \frac{\xi_1^{n_1} \ldots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \ldots (n_r - t_r)^{s_r}}. \tag{1}$$

Note that, for $l = 1 \ldots, r$, the numbers ξ_l are not necessary roots of unity q^{i_l}. We are working, in this note, with the condition

$$(E) \quad \forall i,\ |\prod_{k=1}^{i}\xi_k| \le 1 \text{ and } t_i \in]-\infty;1[.$$

Hence, $\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}};\mathbf{s})$ converges if $s_1 > 1$. We note $\mathcal{E}$ the set of $\mathbb{C}$-tuples verifying (E).

These polyzêtas are obtained as special values of iterated integrals[1] over singular differential 1-forms introduced in [10]. As iterated integrals, they are encoded by words or by non commutative formal power series [10] and are used to construct bases for asymptotic expanding [14] or symbolic integrating fuchian differential equations [11] exactly or approximatively [8]. The meromorphic continuation of the colored Hurwitz polyzêtas[2] is already studied in [5, 6]. In our studies, we constructed an integral representation[3] of colored Hurwitz polyzêtas and a *distribution* treating *simultanously two singularities* and our methods permit to make the meromorphic continuation *commutatively* over the variables $s_1,\ldots,s_r$ [5, 6]. Moreover, [6] gives another way to obtain the meromorphic continuation thanks to *translation equations* [4]. Our methods give the structure of multi-poles [5] (Theorem 4.2) and two ways to calculate algorithmically the multi-residus[4].

In this note, in continuation with our previous works [10, 11, 12, 13, 5, 6], we are focusing on *Hofp algebra*, for a class of products as *minusstuffle* ($\sqcup\!\!\!\!-$), *mulstuffle* ($\sqcup\!\!\!\!\!\cdot$), $\ldots$, and in particular for the *new* product *duffle* ($⧢\!\!\cdot$), obtained as "tensorial product" of $\sqcup\!\!\!\!\!\cdot$ and the well known stuffle ($\uplus$), of symbolic representations of these polyzêtas (see Definition 2.1 and Proposition 2.1 bellow).

2 Combinatorial objects

2.1 Some products and their algebraic structures

Let X be an encoding alphabet and the free monoid over X is denoted by X^*. The *length* of any word $w \in X^*$ is denoted by $|w|$ and the unit of X^* is denoted by 1_{X^*}. For any unitary commutative algebra A, a formal power series S over X with coefficients in A can be written as the infinite sum $\sum_{w\in X^*}\langle S|w\rangle w$. The set of polynomials (resp. formal power series) over X with coefficients in A is denoted by $A\langle X\rangle$ (resp. $A\langle\langle X\rangle\rangle$). The set of degree 1 monomials is $AX = \{ax/a \in A, x \in X\}$.

Definition 2.1 *We note $\mathcal{P}$ the set of products $\star$ over $A\langle X\rangle$ verifying the conditions :*

---

[1] They are presented as generalized Nielsen polylogarithms in [10] (Definition 2.3) and as generalized Lerch functions in [12] (Definition 3).

[2] See also references and a discussion about meromorphic continuation of Riemann polyzêtas in [5].

[3] This integral representation is obtained by applying successively the polylogarithmic transform [10]. It is an application of *non* commutative convolution as shown in [9] (Section 2.4). Other integral representations can be also deduced easily by change of variables, for example $t = zr$ and then $r = e^{-u}$ [5].

[4] Other meromorphic continuations can also be obtained by Mellin transform as already done in [17] or by classical estimation on the imaginary part [7] but these later work reccursively, depth by depth, and the commutativity of this process over the variables $s_1,\ldots,s_r$ must be proved. Unfortunately, the structure of multi-poles as well as multi-residus are missing in both works [7, 17]. In [16], to make the meromorphic continuation (giving the expression of non positive integers multi-residus via a generalization of Bernoulli numbers – but not of *all* multi-residus) of the specialization at roots of unity of colored Hurwitz polyzêtas $\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}};\mathbf{s})$, the author bases on the integral representation, on the contours, of the multiple Hurwitz-Lerch which corresponds *mutatis mutandis* to the integral representation of generalized Lerch functions introduced earlier in [5] (Corollary 3.3).

(i) *the map* $\star : A\langle X\rangle \times A\langle X\rangle \to A\langle X\rangle$ *is bilinear,*

(ii) *for any* $w \in X^*$, $1_{X^*} \star w = w \star 1_{X^*} = w$,

(iii) *for any* $a, b \in X$ *and* $u, v \in X^*$,

$$au \star bv = a(u \star bv) + b(au \star v) + [a,b](u \star v),$$

where $[.,.] : AX \times AX \to AX$ *is a function verifying :*

(S1) $\forall a \in AX,\ [a,0] = 0$,

(S2) $\forall (a,b) \in (AX)^2,\ [a,b] = [b,a]$,

(S3) $\forall (a,b,c) \in (AX)^3,\ [[a,b],c] = [a,[b,c]]$.

Example 1 (see [18]) Product of interated integrals.
The shuffle is a bilinear product such that :

$$\begin{array}{rl} \forall w \in X^* & w \sqcup\!\sqcup 1_{X^*} = 1_{X^*} \sqcup\!\sqcup w = w \qquad \textit{and} \\ \forall (a,b) \in X^2, \forall (u,v) \in {X^*}^2, & au \sqcup\!\sqcup vb = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v). \end{array}$$

For example, for any letter x_0, x *and* x' *in* X,

$$x_0x' \sqcup\!\sqcup x_0^2x = x_0x'x_0^2x + 2x_0^2x'x_0x + 3x_0^3x'x + 3x_0^3xx' + x_0^2xx_0x'.$$

Example 2 (see [15]) Product of quasi-symmetric functions.
Let X *be an alphabet indexed by* $\mathbb{N}$.
The stuffle is a bilinear product such that :

$$\begin{array}{l} \forall w \in X^*, \qquad w \uplus 1_{X^*} = 1_{X^*} \uplus w = w \qquad \textit{and} \\ \forall (x_i,x_j) \in X^2, \forall (u,v) \in {X^*}^2, \\ \qquad x_iu \uplus x_jv = x_i(u \uplus x_jv) + x_j(x_iu \uplus v) + x_{i+j}(u \uplus v). \end{array}$$

In particular, with the alphabet $Y = \{y_1, y_2, y_3, \ldots\}$,

$$(y_3y_1) \uplus y_2 \quad = \quad y_3y_1y_2 + y_3y_2y_1 + y_3y_3 + y_2y_3y_1 + y_5y_1.$$

Example 3 ([3]) Product of large multiple harmonic sums.
Let X *be an alphabet indexed by* $\mathbb{N}$.
The minus-stuffle is a bilinear product such that :

$$\begin{array}{l} \forall w \in X^*, \qquad w \boxminus 1_{X^*} = 1_{X^*} \boxminus w = w \qquad \textit{and} \\ \forall (x_i,x_j) \in X^2, \forall (u,v) \in {X^*}^2, \\ \qquad x_iu \boxminus x_jv = x_i(u \boxminus x_jv) + x_j(x_iu \boxminus v) - x_{i+j}(u \boxminus v). \end{array}$$

Example 4 ([6]) Product of colored sums.
Let X *be an alphabet indexed by a monoid* $(\mathcal{I}, \times)$.
The mulstuffle is a bilinear product such that :

$$\begin{array}{l} \forall w \in X^* \qquad w \boxdot 1_{X^*} = 1_{X^*} \boxdot w = w \qquad \textit{and} \\ \forall (x_i,x_j) \in X^2, \forall (u,v) \in {X^*}^2, \\ \qquad x_iu \boxdot x_jv = x_i(u \boxdot x_jv) + x_j(x_iu \boxdot v) + x_{i\times j}(u \boxdot v). \end{array}$$

For example, with X *indexed by* $\mathbb{Q}^*$,

$$x_{\frac{2}{3}}x_{-1} \boxdot x_{\frac{1}{2}} = x_{\frac{2}{3}}x_{-1}x_{\frac{1}{2}} + x_{\frac{2}{3}}x_{\frac{1}{2}}x_{-1} + x_{\frac{2}{3}}x_{\frac{-1}{2}} + x_{\frac{1}{2}}x_{\frac{2}{3}}x_{-1} + x_{\frac{1}{3}}x_{-1}.$$

Remark 2.1 *Thanks to the one-to-one correspondence* $(i_1, \ldots, i_r) \mapsto x_{i_1} \ldots x_{i_r}$ *between tuples of* $\mathcal{I}$ *and word over* X*, the calculus of* $x_{\frac{2}{3}} x_{-1} ⧢ x_{\frac{1}{2}}$ *can be written as* $\left(\frac{2}{3}, -1\right) ⧢ \left(\frac{1}{2}\right) = \left(\frac{2}{3}, -1, \frac{1}{2}\right) + \left(\frac{2}{3}, \frac{1}{2}, -1\right) + \left(\frac{2}{3}, \frac{-1}{2}\right) + \left(\frac{1}{2}, \frac{2}{3}, -1\right) + \left(\frac{1}{3}, -1\right)$.

Example 5 ([6]) Product of colored Hurwitz polyzêtas.
Let Y *and* E *be two alphabets and consider the alphabet* $A = Y \times E$ *with the concatenation defined recursively by* $(y, e).(w_Y, w_E) = (yw_Y, ew_E)$ *for any letters* $y \in Y$*,* $e \in E$*, and any word* $w_Y \in Y^*$*,* $w_E \in E^*$*. The unit of the monoide* A^* *is given by* $1_{A^*} = (1_{Y^*}, 1_{E^*})$*. If* Y *is indexed by* $\mathbb{N}$ *and* E *by a monoid* $(\mathcal{I}, \times)$*, the duffle is a bilinear product such that* $\forall w \in A^*, \quad w \uplus 1_{A^*} = 1_{A^*} \uplus w = w$*,*
$\forall (y_i, y_j) \in Y^2, \forall (e_l, e_k) \in E^2, \forall (u, v) \in {A^*}^2, \quad (y_i, e_l).u \uplus (y_j, e_k).v = (y_i, e_l).(u \uplus (y_j, e_k).v) + (y_j, e_k).\left((y_i, e_l).u \uplus v\right) + (y_{i+j}, e_{l \times k}).(u \uplus v)$.

Proposition 2.1 *The shuffle, the stuffle, the minus-stuffle and the mulstuffle are elements of* $\mathcal{P}$*, with respectively,* $[x_i, x_j] = 0, [x_i, x_j] = x_{i+j}, [x_i, x_j] = -x_{i+j}, [x_i, x_j] = x_{i \times j}$ *for any letters* x_i *and* x_j *of* X*.*
The duffle is in $\mathcal{P}$*, with* $[(y_i, e_l), (y_j, e_k)] = (y_{i+j}, e_{l \times k})$ *for all* y_i, y_j *in* Y*,* e_l, e_k *in* E*.*

Proposition 2.2 *Let* $\star \in \mathcal{P}$*, then* $(A\langle X \rangle, \star)$ *is a commutative algebra.*

Proof. We just have to show the commutativity and the associativity of $\star$.
To obtain $w_1 \star w_2 = w_2 \star w_1$ for all w_1, w_2 in X^*, we use an induction on $|w_1| + |w_2|$. It is true when $|w_1| + |w_2| \leq 1$ thanks to (i) since w_1 or w_2 is 1_{X^*}. The equality (iii), the condition (S2) and the commutative of $+$ give the induction. In the same way, an induction on $|w_1| + |w_2| + |w_3|$ gives $w_1 \star (w_2 \star w_3) = (w_1 \star w_2) \star w_3$ thanks to (iii) and (S3). □
If we associate to each letter of X an integer number called weight, the weight of a word is the sum of the weight of its letters. In this case X is graduated.
In [15], Hoffman works over $\overline{X} = X \cup \{0\}$ with $[.,.] : \overline{X} \times \overline{X} \to \overline{X}$ and call *quasi-product* any product in $\mathcal{P}$ with the additional condition :

(S4) Either $[a, b] = 0$ for all a, b in X; or the weight of $[a, b]$ is the sum of the weight of a and the weight of b for all a, b in X.

Example 6

1. *The shuffle is a quasi-product.*
2. *Let* X *be an alphabet indexed by* $\mathbb{N}$ *and define the weight of* x_i*,* $i \in \mathbb{N}$*, by* i *. Then the stuffle is a quasi-product.*

Theorem 2.1 ([15]) *If* X *is graduated and has a quasi-product* $\star$*, then* $(A\langle X \rangle, \star)$ *is a commutative graduated* A*-algebra..*

We can define (i) a comultiplication $\Delta : A\langle X \rangle \to A\langle X \rangle \otimes A\langle X \rangle$,
(ii) a counit $\epsilon : A\langle X \rangle \to A$,

by : $\quad \forall w \in X^*, \quad \Delta w = \sum_{uv=w} u \otimes v \quad$ and $\quad \epsilon(w) = \begin{cases} 1 \text{ if } w = 1_{X^*} \\ 0 \text{ otherwise.} \end{cases}$
The coproduct Δ is coassociative so $(A\langle X \rangle, \Delta, \epsilon)$ is a coalgebra.

Lemma 2.1 *For any $w \in X^*$ and $x \in X$, $(x \otimes 1_{X^*})\Delta w + 1_{X^*} \otimes xw = \Delta xw$.*

Proof. $\forall w \in X^*, \forall x \in X, \Delta xw = \sum_{uv=xw} u \otimes v = \sum_{u'v=w} xu' \otimes v + 1_{X^*} \otimes xw$

so $\quad \Delta xw = x \otimes 1_{X^*}\Big(\sum_{u'v=w} u' \otimes v\Big) + 1_{X^*} \otimes xw = (x \otimes 1_{X^*})\Delta w + 1_{X^*} \otimes xw.$ □

Proposition 2.3 *If $\star \in \mathcal{P}$, then $(A\langle X\rangle, \star, \Delta, \epsilon)$ is a bialgebra.*

Remember that $\star$ acts over $A\langle X\rangle \otimes A\langle X\rangle$ by $(u \otimes v) \star (u' \otimes v') = (u \star u') \otimes (v \star v')$.
Proof. ϵ is obviously a $\star$-homomorphism. It still has to be show $\Delta(w_1) \star \Delta(w_2) = \Delta(w_1 \star w_2)$ over X^*. This equality is true if w_1 or w_2 is equal to 1_{X^*}.
Assume now that $\Delta(u)\star\Delta(v) = \Delta(u\star v)$ for any word u and v such that $|u|+|v| \leq n$, $n \in \mathbb{N}$, and let w_1 and w_2 be in X^* with $|w_1| + |w_2| = n + 1$. We note $w_1 = au$ and $w_2 = bv$, with a and b two letters of X, u and v two words of X^*. Thus, by definition, $\Delta w_1 = \sum_{u_1u_2=u} au_1 \otimes u_2 + 1_{X^*} \otimes au$ and $\Delta w_2 = \sum_{v_1v_2=v} bv_1 \otimes v_2 + 1_{X^*} \otimes bv$.

$$
\begin{aligned}
&\Delta(w_1) \star \Delta(w_2)\\
&= \sum_{u_1u_2=u, v_1v_2=v} (au_1 \star bv_1) \otimes (u_2 \star v_2) + \sum_{u_1u_2=u} (au_1) \otimes (u_2 \star bv)\\
&+ \sum_{v_1v_2=v} (bv_1) \otimes (au \star v_2) + 1_{X^*} \otimes (au \star bv)\\
&= \sum_{u_1u_2=u, v_1v_2=v} (a(u_1 \star bv_1) \otimes (u_2 \star v_2) + b(au_1 \star v_1) \otimes (u_2 \star v_2)\\
&+([a,b](u_1 \star v_1)) \otimes (u_2 \star v_2)) + \sum_{u_1u_2=u} (au_1) \otimes (u_2 \star bv)\\
&+ \sum_{v_1v_2=v} (bv_1) \otimes (au \star v_2) + 1_{X^*} \otimes a(u \star bv)\\
&+1_{X^*} \otimes b(au \star v) + 1_{X^*} \otimes [a,b](u \star v)\\
&= \sum_{u_1u_2=u, v_1v_2=v} a(u_1 \star bv_1) \otimes (u_2 \star v_2) + \sum_{u_1u_2=u} (au_1) \otimes (u_2 \star bv)\\
&+ \sum_{u_1u_2=u, v_1v_2=v} b(au_1 \star v_1) \otimes (u_2 \star v_2) + \sum_{v_1v_2=v} (bv_1) \otimes (au \star v_2)\\
&+[a,b] \otimes 1_{X^*} \sum_{\substack{u_1u_2=u\\ v_1v_2=v}} (u_1 \otimes u_2) \star (v_1 \otimes v_2)\\
&+(1_{X^*} \otimes a(u \star bv) + 1_{X^*} \otimes b(au \star v) + 1_{X^*} \otimes [a,b](u \star v))\\
&= (a \otimes 1_{X^*})(\Delta(u) \star \Delta(w_2)) + 1_{X^*} \otimes a(u \star bv) + (b \otimes 1_{X^*})(\Delta(w_1) \star \Delta(v))\\
&+1_{X^*} \otimes b(au \star v) + ([a,b] \otimes 1_{X^*})(\Delta(u) \star \Delta(v)) + 1_{X^*} \otimes [a,b](u \star v).
\end{aligned}
$$

Using the induction hypothesis then the lemma 2.1 (since $[a,b] \in AX$) gives

$$
\begin{aligned}
\Delta(w_1) \star \Delta(w_2) &= \Delta(a(u \star w_2)) + \Delta(b(w_1 \star v)) + \Delta([a,b](u \star v))\\
&= \Delta(a(u \star w_2) + b(w_1 \star v) + [a,b](u \star v))\\
&= \Delta(w_1 \star w_2).
\end{aligned}
$$

□

Remark 2.2 *In particular, Δ is a $\shuffle$-homomorphism, a $\stuffle$-homomorphism and a $\dot{\sqcup}$-homomorphism.*

Let $\mathcal{C}_n$ be the set of positive integer sequences $(i_1, \ldots, i_k)$ such that $i_1 + \ldots + i_k = n$.

Theorem 2.2 *Define $a_\star$ by, for all $x_1, \ldots, x_n$ in X,*

$$\begin{aligned} & a_\star(x_1 \ldots x_n) \\ & = \sum_{(i_1,\ldots,i_k)\in\mathcal{C}_n} (-1)^k x_1 \ldots x_{i_1} \star x_{i_1+1} \ldots x_{i_1+i_2} \star \ldots \star x_{i_1+\ldots+i_{k-1}+1} \ldots x_n \end{aligned}$$

then, if $\star \in \mathcal{P}$, $(A\langle X\rangle, \star, \Delta, \epsilon, a_\star)$ is a Hopf algebra.

Proof. With the applications :

$$\begin{array}{cccc} \mu: & A & \to & A\langle X\rangle \\ & \lambda & \mapsto & \lambda 1_{X^*} \end{array} \quad \text{and} \quad \begin{array}{cccc} m: & A\langle X\rangle \otimes A\langle X\rangle & \to & A\langle X\rangle \\ & u \otimes v & \mapsto & u \star v \end{array},$$

the antipode must verify $m \circ (a_\star \otimes Id) \circ \Delta = \mu \circ \epsilon$, or, in equivalent terms

$$\sum_{uv=w} a_\star(u) \star v = \langle w | 1_{X^*} \rangle 1_{X^*}.$$

i.e. $\begin{cases} a_\star(1_{X^*}) = 1_{X^*}n \\ \forall x \in X, a_\star(x) = -x \end{cases}$ and, if $w = x_1 \ldots x_n$ with $n \geq 2$, $x_1, \ldots, x_n \in X$,

$$a_\star(w) = -\sum_{k=1}^{n-1} a_\star(x_1 \ldots x_k) \star x_{k+1} \ldots x_n.$$

An induction over the length n shows that $a_\star$ defined in theorem verifies these equalities, and, in the same way, $a_\star$ verifies $m \circ (Id \otimes a_\star) \circ \Delta = \mu \circ \epsilon$. $\square$

Corollary 2.1 *If $\star$ is $\shuffle$ or $\stuffle$ or $\dot{\sqcup}$ or $\stuffle\!\dot{\sqcup}$, then this construction gives an Hopf algebra. Moreover, for $\shuffle$ or $\stuffle$, we obtain a graduated Hopf algebra.*

2.2 Iterated integral

Let us associate to each letter x_i in X a 1-differential form ω_i, defined in some connected open subset $\mathcal{U}$ of $\mathbb{C}$. For all paths $z_0 \rightsquigarrow z$ in $\mathcal{U}$, the *Chen iterated integral* associated to $w = x_{i_1} \cdots x_{i_k}$ along $z_0 \rightsquigarrow z$, noted is defined recursively as follows

$$\alpha_{z_0}^z(w) = \int_{z_0 \rightsquigarrow z} \omega_{i_1}(z_1) \alpha_{z_0}^{z_1}(x_{i_2} \cdots x_{i_k}) \quad \text{and} \quad \alpha_{z_0}^z(1_{X^*}) = 1, \tag{2}$$

verifying the *rule of integration by parts* [2] :

$$\alpha_{z_0}^z(u \shuffle v) = \alpha_{z_0}^z(u) \alpha_{z_0}^z(v). \tag{3}$$

We extended this definition over $A\langle X\rangle$ (resp. $A\langle\langle X\rangle\rangle$) by

$$\alpha_{z_0}^z(S) = \sum_{w\in X^*} \langle S|w\rangle \alpha_{z_0}^z(w). \tag{4}$$

2.3 Shuffle relations

2.3.1 First encoding for colored Hurwitz polyzêtas

Let $\xi = (\xi_n)$ be a sequence of complex numbers and T a family of parameters. Put X' an alphabet indexed over $\mathbb{N}^* \times \mathbb{C}^{\mathbb{N}} \times T$ and $X = \{x_0\} \cup X'$. To each x in X we associate the differential form :

$$\begin{cases} \omega_0(z) = \dfrac{dz}{z} & \text{si } x = x_0 \\ \omega_{i,\xi,t}(z) = \dfrac{\prod_{k=1}^{i} \xi_k}{1 - \prod_{k=1}^{i} \xi_k \, z} \times \dfrac{dz}{z^t} & \text{if } x = x_{i,\xi,t} \text{ with } i \geqslant 1. \end{cases} \tag{5}$$

For any T-tuple $\mathbf{t} = (t_1, \ldots, t_r)$ we associate the T-tuple $\overline{\mathbf{t}} = (\overline{t_1}, \ldots, \overline{t_r})$ given by

$$\begin{cases} \overline{t_1} &= t_1 - t_2, \\ \overline{t_2} &= t_2 - t_3, \\ &\vdots \\ \overline{t_r} &= t_{r-1} - t_r \end{cases} \quad \text{in this way} \quad \begin{cases} t_1 &= \overline{t_1} + \overline{t_2} + \ldots + \overline{t_n}, \\ t_2 &= \overline{t_2} + \ldots + \overline{t_n}, \\ &\vdots \\ t_r &= \overline{t_r} \end{cases} \tag{6}$$

We choose the sequence ξ and the family $\mathbf{t}$ such that the condition (E) is satisfied.

Proposition 2.4 *For any* $\mathbf{s} = (s_1, \ldots, s_r)$ *with* $s_1 > 1$ *if* $\xi = (\xi_1, \ldots, \xi_r) \in \mathcal{E}^r$ *and* $\mathbf{t} = (t_1, \ldots, t_r) \in T^r$*, then* $\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s}) = \alpha_0^1(x_0^{s_1-1} x_{1,\xi,\overline{t_1}} \ldots x_0^{s_r-1} x_{r,\xi,\overline{t_r}})$.

Proof. Since $\omega_{i,\xi,t}(z) = \displaystyle\sum_{n>0} \prod_{k=1}^{i} \xi_k^n \frac{z^n dz}{z^{1+t}}$ then $\alpha_0^z(x_{r,\xi,\overline{t_r}}) = \displaystyle\sum_{n>0} \prod_{k=1}^{r} \xi_k^n \frac{z^{n-\overline{t_r}}}{n - \overline{t_r}}$ and $\alpha_0^z(x_0^{s_r-1} x_{r,\xi,\overline{t_r}}) = \displaystyle\sum_{n>0} \prod_{k=1}^{r} \xi_k^n \frac{z^{n-\overline{t_r}}}{(n - \overline{t_r})^{s_r}}$. Hence, $\alpha_0^1(x_0^{s_1-1} x_{1,\xi,\overline{t_1}} \ldots x_0^{s_r-1} x_{r,\xi,\overline{t_r}})$ gives $\displaystyle\sum_{m_1,\ldots,m_r>0} \prod_{j=1}^{r} \frac{\prod_{k_j=1}^{j} \xi_{k_j}^{m_j}}{(m_j + \ldots + m_r - \overline{t}_j - \ldots - \overline{t}_r)^{s_j}}$, and then, by change of variables, $\displaystyle\sum_{n_1 > \ldots > n_r > 0} \frac{\xi_1^{n_1} \ldots \xi_r^{n_r}}{(n_1 - t_1)^{s_1} \ldots (n_r - t_r)^{s_r}}$. □

Theorem 2.3 *Let* $\mathcal{T}$ *be the group of parameters generated by* $\langle T; + \rangle$*,* $\mathcal{C}$ *be a subgroup of* $(\mathbb{C}^*, .)$ *and* A *a sub-ring of* $\mathbb{C}$*. Put* $\mathcal{C}' = \mathcal{C}^{\mathbb{N}} \cap \mathcal{E}$ *and* $\mathcal{T}'$ *the set of finite tuple with elements in* $\mathcal{T}$*. Then the* A *algebra generated by* $\{\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s})\}_{\xi \in \mathcal{C}', \mathbf{t} \in \mathcal{T}'}$ *is the* A *modulus generated by* $\{\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s})\}_{\xi \in \mathcal{C}', \mathbf{t} \in \mathcal{T}'}$.

Proof. We have express the product $\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}}; \mathbf{s}) \, \mathrm{Di}(\mathbf{F}_{\xi',\mathbf{t}'}; \mathbf{s}')$, with $\mathbf{s} = (s_1, \ldots, s_r)$, $\mathbf{s}' = (s'_1, \ldots, s'_{r'})$, $\xi, \xi' \in \mathcal{C}'$ and $\mathbf{t} = (t_1, \ldots, t_r), \mathbf{t}' = (t'_1, \ldots, t'_r) \in \mathcal{T}'$, as linear combination of colored Hurwitz polyzêtas. This is an iterated integral associated to $x_0^{s_1-1} x_{1,\xi,\overline{t_1}} \ldots x_0^{s_r-1} x_{r,\xi,\overline{t_r}} \sqcup\!\sqcup \, x_0^{s'_1-1} x_{1,\xi',\overline{t'_1}} \ldots x_0^{s'_{r'}-1} x_{r',\xi',\overline{t'_{r'}}}$ which is a sum of terms of the form $x_0^{s_1''-1}x_{1,\xi^{(1)},\overline{t_{(1)}}}\dots x_0^{s_i''-1}x_{j_i,\xi^{(i)},\overline{t_{(i)}}}\dots x_0^{s_r''-1}x_{j_{r''},\xi^{(r'')},\overline{t_{(r'')}}}$, with $s_i''\in\mathbb{N}$, $\xi^{(i)}$ is ξ or ξ' and $t_{(i)}$ is t_{j_i} or t'_{j_i} for all i; and $r''=r+r''$. Note that

$$\begin{aligned}
&\alpha_0^z(x_{i,\xi,\overline{t_i}}x_0^{s-1}x_{j,\xi',\overline{t_j}})\\
&=\int_0^z\sum_{m>0}\prod_{k=1}^{i}\xi_k^m z_1^{m-\overline{t_i}-1}dz_1\int_0^{z_1}\frac{dz_2}{z_2}\dots\int_0^{z_{s+1}}\sum_{n>0}\prod_{k=1}^{i}\xi_k'^n z_{s+1}^{n-\overline{t'_j}-1}dz_{s+1}\\
&=\sum_{m,n>0}\frac{(\xi_1\dots\xi_i)^m\left(\xi'_1\dots\xi'_j\right)^n}{(m+n-\overline{t_i}-\overline{t'_j})(n-\overline{t'_j})^s}z^{n+m},\\
&\alpha_0^1\big(x_0^{s_1''-1}x_{1,\xi^{(1)},\overline{t_{(1)}}}\dots x_0^{s_i''-1}x_{j_i,\xi^{(i)},\overline{t_{(i)}}}\dots x_0^{s_r''-1}x_{j_{r''},\xi^{(r'')},\overline{t_{(r'')}}}\big)\\
&=\sum_{m_1,\dots,m_{r''}>0}\prod_{i=1}^{r''}\frac{(\xi_1^{(i)}\dots\xi_{j_i}^{(i)})^{m_i}}{(m_i+\dots+m_{r''}-\overline{t_{(i)}}-\dots-\overline{t_{(r'')}})^{s_i''}}\\
&=\sum_{n_1>\dots>n_{r''}>0}\prod_{i=1}^{r''}\frac{\xi_i''^{n_i}}{(n_i-t_i'')^{s_i''}}
\end{aligned}$$

with $n_i=m_i+\dots+m_{r''}$, $t_i''=\overline{t_{(i)}}+\dots+\overline{t_{(r'')}}$ for all i, so $\mathbf{t}''\in\mathcal{T}$; $\xi_1''=\xi_1^{(1)}$ and $\xi_i''=\frac{\xi_1^{(i)}\dots\xi_{j_i}^{(i)}}{\xi_1^{(i-1)}\dots\xi_{j_{i-1}}^{(i-1)}}$ for $i>1$ so $\xi''\in\mathcal{C}$: we can express each term of the shuffle product as $\mathrm{Di}(\mathbf{F}_{\xi'',\mathbf{t}''};\mathbf{s}'')$. □

Note that the shuffle product over two words of X^*X' acts separately over $(\mathcal{C}',.)$, $(\mathcal{T}',+)$ and the convergent compositions. We can describe the situation with the shuffle algebra[5] :

Theorem 2.4 *Let $\mathcal{H}$ be the $\mathbb{Q}$-algebra generated by the colored Hurwitz polyzêtas. The map $\zeta:(\mathbb{Q}\langle(x_0^*x_{i,\xi,t})^*\rangle,ш)\twoheadrightarrow(\mathcal{H},.)$, $x_0^{s_1}x_{1,\xi,\overline{t_1}}\dots x_0^{s_r}x_{r,\xi,\overline{t_r}}\mapsto\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}};\mathbf{s}+1)$ is a surjective algebra morphism.*

Example 7 *Since* $\mathrm{Di}(\mathbf{F}_{\xi,t};3)=\alpha_0^1(x_0^2x_{1,\xi,t})$ *and* $\mathrm{Di}(\mathbf{F}_{\xi',t'};2)=\alpha_0^1(x_0x_{1,\xi',t'})$ *then* $\mathrm{Di}(\mathbf{F}_{xi,t};3)\,\mathrm{Di}(\mathbf{F}_{xi',t'};2)=\alpha_0^1(x_0x_{1,\xi',t'}ш\, x_0^2x_{1,\xi,t})$. *Example 1 with* $x=x_{1,\xi,t}$ *and* $x'=x_{1,\xi',t'}$ *gives the expression of* $x_0x_{1,\xi',t'}ш\, x_0^2x_{1,\xi,t}$. *But the first term obtained is*

$$\begin{aligned}
&\alpha_0^1(x_0x_{1,\xi',t'}x_0^2x_{1,\xi,t})\\
&=\int_0^1\frac{dz_1}{z_1}\int_0^{z_1}\sum_{m>0}\xi'^{\,m}z_2^{m-t'-1}dz_2\int_0^{z_2}\frac{dz_3}{z_3}\int_0^{z_3}\frac{dz_4}{z_4}\int_0^{z_4}\sum_{n>0}\xi^n z_5^{n-t-1}dz_5\\
&=\sum_{n,m>0}\frac{\xi'^{\,m}\xi^n}{(m+n-t'-t)^2(n-t)^3}\\
&=\sum_{n_1>n_2>0}\frac{(\xi')^{n_1}(\xi/\xi')^{n_2}}{(n_1-t'-t)^2(n_2-t)^3}\\
&=\mathrm{Di}(\mathbf{F}_{(\xi,\xi/\xi');(t+t',t)};(2,3)).
\end{aligned}$$

[5]Working in $\mathbb{Q}\langle(x_0^*x_{i,\xi,t})^*\rangle$ implies working in the graduated Hopf algebra $(\mathbb{Q}\langle X^*\rangle,ш,\Delta,\epsilon,a_ш)$.

We can make similar calculus for the other terms and find :

$$\begin{aligned}&\mathrm{Di}(\mathbf{F}_{\xi,t};3)\,\mathrm{Di}(\mathbf{F}_{\xi',t'};2)\\&=\mathrm{Di}(\mathbf{F}_{(\xi',\xi/\xi');(t+t',t)};(2,3))+2\,\mathrm{Di}(\mathbf{F}_{(\xi',\xi/\xi');(t+t',t)};(3,2))\\&\quad+3\,\mathrm{Di}(\mathbf{F}_{(\xi',\xi/\xi');(t+t',t)};(4,1))+3\,\mathrm{Di}(\mathbf{F}_{(\xi,\xi'/\xi);(t+t',t')};(4,1))\\&\quad+\mathrm{Di}(\mathbf{F}_{(\xi,\xi'/\xi);(t+t',t')};(3,2)).\end{aligned}$$

2.3.2 Second encoding for colored Hurwitz polyzêtas

For the Hurwitz polyzêtas, we can obtain an encoding indexed by a finite alphabet. Let the alphabet $X=\{x_0;x_1\}$ and associate to x_0 the form $\omega_0(z)=z^{-1}dz$ and at x_1 the form $\omega_1(z)=(1-z)^{-1}dz$.

For each $x\in X$ and $\lambda\in\mathbb{C}$, we note $(\lambda x)^*=\sum_{k\geq 0}(\lambda x)^k$. Then, (see [10], [11]), $\alpha_0^1\left(x_0^{s_1-1}(t_1x_0)^{*s_1}x_1\ldots x_0^{s_r-1}(t_rx_0)^{*s_r}x_1\right)=\zeta(\mathbf{s};\mathbf{t})$.

Theorem 2.5 *Let $\mathcal{H}'$ be the $\mathbb{Q}$-algebra generated by the Hurwitz polyzêtas and $\mathcal{X}$ the $\mathbb{Q}$-algebra generated by $(t_1x_0)^{*s_1}x_1\ldots(t_rx_0)^{*s_r}x_r$. Then, $\zeta:(\mathcal{X},\sqcup\!\sqcup)\twoheadrightarrow(\mathcal{H}',.)$ is a surjective morphism of algebras.*

Note that we can apply the idea of encoding of "simple" colored Hurwitz zetas functions (with depth one : $r=1$). Let $\xi=(\xi_n)$ be a sequence of complex numbers in the unit ball $\mathcal{B}(0;1)$ and T a family of parameters. Let $X=\{x_0,x_1,\ldots\}$ be a alphabet indexed by $\mathbb{N}$. Associate to x_0 the differential form $\omega_0(z)=z^{-1}dz$ and to x_i, $i\geq 1$, the differential form $\omega_i(z)=\xi_i(1-\xi_iz)^{-1}dz$.

Proposition 2.5 *With this notation,* $\alpha_0^1\left(((tx_0)^*x_0)^{s-1}\,(tx_0)^*x_i\right)=\sum_{n>0}\dfrac{\xi_i^n}{(n-t)^s}$.

Proof. Since $\dfrac{\xi_i dz_0}{1-\xi_iz_0}=\xi_i\sum_{n\geq 0}(\xi_iz_0)^n dz_0$, we can write

$$\alpha_0^z\left((tx_0)^kx_i\right)=t^k\int_0^z\frac{dz_k}{z_k}\int_0^{z_k}\ldots\int_0^{z_1}\xi_i\sum_{n\geq 0}(\xi_iz_0)^n dz_0=\sum_{n>0}t^k\frac{\xi_i^nz^n}{n^{k+1}},$$

for $z\in\mathcal{B}(0;1)$ and for $k\in\mathbb{N}$. Thanks to the absolute convergence,

$$\alpha_0^z\left((tx_0)^*x_i\right)=\sum_{n>0}\frac{\xi_i^nz^n}{n}\sum_{k\geq 0}\left(\frac{t}{n}\right)^k=\sum_{n>0}\frac{\xi_i^nz^n}{n-t}.$$

In the same way, if $z\in\mathcal{B}(0;1)$:

$$\begin{aligned}\forall k\in\mathbb{N},\qquad&\alpha_0^z\left((tx_0)^kx_0(tx_0)^*x_i\right)=\sum_{n>0}t^k\frac{\xi_i^n}{n-t}\frac{z^n}{n^{k+1}},\\ \text{so}\qquad&\alpha_0^z\left((tx_0)^*x_0(tx_0)^*x_i\right)=\sum_{n>0}\frac{\xi_i^nz^n}{(n-t)^2}\end{aligned}$$

$$\text{and} \qquad \alpha_0^z\left(((tx_0)^*x_0)^{s-1}\,(tx_0)^*x_i\right) = \sum_{n>0}\frac{\xi_i^n z^n}{(n-t)^s}.$$

$\square$

Remark 2.3 *Note that, with the same notation,*

$$\begin{aligned}\alpha_0^z\left(x_1\,((t_2x_0)^*x_0)^{s-1}\,(t_2x_0)^*x_2\right) &= \sum_{n,m>0}\frac{\xi_2^n\xi_1^m z^{n+m}}{(n-t_2)^s(m+n)}\\ &= \sum_{n_1>n_2>0}\frac{\xi_2^{n_2}\xi_1^{n_1-n_2}z^{n_1}}{n_1(n_2-t_2)^s}.\end{aligned}$$

In other words, this encoding appears to be widespread only as couples of the type $\xi = (1,1,\ldots,1,\xi_r)$: with $\xi_1 = 1$ and $\omega_1 = (1-z)^{-1}dz$,

$$\begin{aligned}&\alpha_0^1\left(x_0^{s_1-1}(t_1x_0)^{*s_1}x_1\ldots x_0^{s_{r-1}-1}(t_{r-1}x_0)^{*s_{r-1}}x_{r-1}x_0^{s_r-1}(t_rx_0)^{*s_r}x_r\right)\\ &= \sum_{n_1>\ldots>n_r}\frac{\xi_r^{n_r}}{(n_1-t_1)^{s_1}\ldots(n_r-t_r)^{s_r}}.\end{aligned}$$

2.4 Duffle relations

Let $\lambda = (\lambda_n)$ be a set of parameters, $\mathbf{s} = (s_1,\ldots,s_r)$ a composition, $\xi \in \mathbb{C}^r$. Then

$$\forall n \in \mathbb{Z}_{>0}, \quad M^n_{\mathbf{s},\xi}(\lambda) = \sum_{n>n_1>\ldots>n_r>0}\prod_{i=1}^r \xi_i^{n_i}\lambda_{n_i}^{s_i} \quad \text{and} \quad M^n_{(),()}(\lambda) = 1. \tag{7}$$

We can export the duffle over the tuples $\mathbf{s} = (s_1,\ldots,s_r) \in \mathbb{Z}^r_{>0}$ and $\xi \in \mathbb{C}^r$ with :

$$\begin{aligned}&(\mathbf{s},\xi) ⧢ ((),1) = ((),1) ⧢ (\mathbf{s},\xi) = (\mathbf{s},\xi) \qquad \text{and}\\ &(s_1,\mathbf{s};\xi_1,\xi) ⧢ (r_1,\mathbf{r};\rho_1,\rho)\\ &= (s_1;\xi_1).\,((\mathbf{s};,\xi) ⧢ (r_1,\mathbf{r};\rho_1,\rho)) + (r_1;\rho_1).\,((s_1,\mathbf{s};\rho_1,\xi) ⧢ (\mathbf{r};\rho))\\ &\quad +(s_1+r_1;\xi_1\rho_1).\,((\mathbf{s};\xi) ⧢ (\mathbf{r};\rho))\end{aligned} \tag{8}$$

Proposition 2.6 *Let $\mathbf{s} = (s_1,\ldots,s_l)$ and $\mathbf{r} = (r_1,\ldots,r_k)$ be two compositions, $\xi \in \mathbb{C}^l$, $\rho \in \mathbb{C}^k$. Then*

$$\forall n \in \mathbb{N}, \quad M^n_{\mathbf{s},\xi}(\lambda)\,M^n_{\mathbf{r},\rho}(\lambda) = M^n_{(\mathbf{s},\xi) ⧢ (\mathbf{r},\rho)}(\lambda).$$

Proof. Put the compositions $\mathbf{s}' = (s_2,\ldots,s_l)$, $\mathbf{r}' = (r_2,\ldots,r_k)$, the tuples of complex numbers $\xi' = (\xi_2,\ldots,\xi_l)$ and $\rho' = (\rho_2,\ldots,\rho_k)$, then

$$\begin{aligned}&M^n_{\mathbf{s},\xi}(\lambda)\,M^n_{\mathbf{r},\rho}(\lambda)\\ &= \sum_{n>n_1,n>n'_1}\xi_1^{n_1}\lambda_{n_1}^{s_1}\,M^{n_1}_{\mathbf{s}',\xi'}(\lambda)\,\rho_1^{n'_1}\lambda_{n'_1}^{r_1}\,M^{n'_1}_{\mathbf{r}',\rho'}(\lambda)\\ &= \sum_{n>n_1}\xi_1^{n_1}\lambda_{n_1}^{s_1}\,M^{n_1}_{\mathbf{s}',\xi'}(\lambda)\,M^{n_1}_{\mathbf{r},\rho}(\lambda) + \sum_{n>n'_1}\rho_1^{n'_1}\lambda_{n'_1}^{r_1}\,M^{n'_1}_{\mathbf{s},\xi}(\lambda)\,M^{n'_1}_{\mathbf{r}',\rho'}(\lambda)\end{aligned}$$

$$+\sum_{n>m}(\xi_1\rho_1)^m\lambda_m^{s_1+r_1}\,M^m_{\mathbf{s}',\xi'}(\lambda)\,M^m_{\mathbf{r}',\rho'}(\lambda).$$

A recurrence ended the demonstration. $\square$

Theorem 2.6 *Let $\mathbf{s}=(s_1,\dots,s_l)$ and $\mathbf{r}=(r_1,\dots,r_k)$ be two compositions, ξ a l-tuple and ρ a k-tuple of $\mathcal{E}$, $\mathbf{t}=(t,\dots,t)$ a l-tuple and $\mathbf{t}'=(t,\dots,t)$ a k-tuple, both formed by the same parameter t diagonally. Then*

$$\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}};\mathbf{s})\,\mathrm{Di}(\mathbf{F}_{\xi',\mathbf{t}'};\mathbf{s}')=\mathrm{Di}(\mathbf{F}_{\xi'',(t,\dots,t)};\mathbf{s}''),$$

with $(\mathbf{s}'';\xi'')=(\mathbf{s};\xi)\sqcup\!\sqcup(\mathbf{s}';\xi')$.

Proof. With $\lambda_n=\dfrac{1}{n-t}$ for all $n\in\mathbb{N}$, $\quad M^n_{\mathbf{s},\xi}(\lambda)=\displaystyle\sum_{n>n_1>\dots>n_r}\prod_{i=1}^{r}\frac{\xi_i^{n_i}}{(n_i-t)^{s_i}}$. So $\displaystyle\lim_{n\to\infty}M^n_{\mathbf{s},\xi}(\lambda)=\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}};\mathbf{s})$ and taking the limit of Proposition 2.6 gives the result. $\square$

Example 8 *The use of examples 2 and 4 gives*

$$\begin{aligned}&\mathrm{Di}(\mathbf{F}_{(\frac{2}{3},-1),\mathbf{t}};(3,1))\,\mathrm{Di}(\mathbf{F}_{(\frac{1}{2}),(t)};(2))\\&=\mathrm{Di}(\mathbf{F}_{(\frac{2}{3},-1,\frac{1}{2}),(t,t,t)};(3,1,2))+\mathrm{Di}(\mathbf{F}_{(\frac{2}{3},\frac{1}{2},-1),(t,t,t)};(3,2,1))\\&+\mathrm{Di}(\mathbf{F}_{(\frac{2}{3},-\frac{1}{2}),\mathbf{t}};(3,3))+\mathrm{Di}(\mathbf{F}_{(\frac{1}{2},\frac{2}{3},-1),(t,t,t)};(2,3,1))+\mathrm{Di}(\mathbf{F}_{(\frac{1}{3},-1),\mathbf{t}};(5,1))\end{aligned}$$

Remark 2.4 *Extend the duffle product to triplets $(\mathbf{s},\mathbf{t},\xi)\in\cup_{r\in\mathbb{N}^*}\mathbb{N}^r\times\{t\}^r\times\mathbb{C}^r$ by*

$$\begin{aligned}(s_1,\mathbf{s};t,\mathbf{t};\xi_1,\xi)\sqcup\!\sqcup(r_1,\mathbf{r};t,\mathbf{t}';\rho_1,\rho)&=(s_1;t;\xi_1).\left((\mathbf{s};\mathbf{t};\xi)\sqcup\!\sqcup(r_1,\mathbf{r};t,\mathbf{t}';\rho_1,\rho)\right)\\&+(r_1;t;\rho_1).\left((s_1,\mathbf{s};t,\mathbf{t};\rho_1,\xi)\sqcup\!\sqcup(\mathbf{r};\mathbf{t}';\rho)\right)\\&+(s_1+r_1;t;\xi_1\rho_1).\left((\mathbf{s};\mathbf{t};\xi)\sqcup\!\sqcup(\mathbf{r};\mathbf{t}';\rho)\right),\end{aligned}$$

and define the function F over $\mathcal{I}=\cup_{r\in\mathbb{N}^*}\mathbb{N}^r\times\{t\}^r\times\mathbb{C}^r$ by $F(\mathbf{s},\mathbf{t},\xi)=\mathrm{Di}(\mathbf{F}_{\xi,\mathbf{t}};\mathbf{s})$. Then, by Theorem 2.6, the function $F:(\mathcal{I},\sqcup\!\sqcup)\to(\mathbb{C},.)$ is morphism of algebras.

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