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Combinatorial study of colored Hurwitz polyzetas

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\textbf{abstract}

A combinatorial study discloses two surjective morphisms between generalized shuffle algebras and algebras generated by the colored Hurwitz polyzetas. The combinatorial aspects of the products and co-products involved in these algebras will be examined.

\section{Introduction}

Classically, the Riemann z\‘eta function is $\zeta(s) = \sum_{n>0} n^{-s}$, the Hurwitz z\‘eta function is $\zeta(s; t) = \sum_{n>0} (n-t)^{-s}$ and the colored z\‘eta function is $\zeta(s; q) = \sum_{n>0} q^n n^{-s}$, where $q$ is a root of unit. The three previous functions are defined over $\mathbb{Z}_{>0}$ but can be generalized over any \textit{composition} (sequence of positive integers) $s = (s_1, \ldots, s_r)$, like, respectively, the Riemann polyz\‘eta function $\zeta(s) = \sum_{n_1>\ldots>n_r>0} n_1^{-s_1} \cdots n_r^{-s_r}$, the Hurwitz polyz\‘eta function $\zeta(s; t) = \sum_{n_1>\ldots>n_r>0} (n_1-t_1)^{-s_1} \cdots (n_r-t_r)^{-s_r}$ and the colored polyz\‘eta function $\zeta(s; q) = \sum_{n_1>\ldots>n_r>0} q^{i_1 n_1} \cdots q^{i_r n_r} n_1^{-s_1} \cdots n_r^{-s_r}$, with $q$ a root of unit and $i = (i_1, \ldots, i_r)$ a composition. These sums converge when $s_1 > 1$.

To study simultaneously these families of polyzetas, the colored Hurwitz polyzetas, for a composition $s = (s_1, \ldots, s_r)$ and a tuple of complex numbers $\xi = (\xi_1, \ldots, \xi_r)$ and a tuple of parameters $t = (t_1, \ldots, t_r)$, are defined by [6]

$$Di(F_{\xi,t}; s) = \sum_{n_1>\ldots>n_r>0} \frac{\xi_1^{n_1} \cdots \xi_r^{n_r}}{(n_1-t_1)^{s_1} \cdots (n_r-t_r)^{s_r}}. \tag{1}$$

Note that, for $l = 1, \ldots, r$, the numbers $\xi_l$ are not necessary roots of unity $q^{i_l}$. We are working, in this note, with the condition
\( \forall i, \prod_{k=1}^{i} \xi_k \leq 1 \) and \( t_i < -\infty \).

Hence, \( \text{Di}(\mathcal{F}_\xi, t; s) \) converges if \( s_1 > 1 \). We note \( \mathcal{E} \) the set of \( \mathbb{C} \)-tuples verifying \((E)\).

These polyzêtas are obtained as special values of iterated integrals\(^1\) over singular differential 1-forms introduced in [10]. As iterated integrals, they are encoded by words or by non commutative formal power series [10] and are used to construct bases for asymptotic expanding [14] or symbolic integrating fuchian differential equations [11] exactly or approximatively [8]. The meromorphic continuation of the colored Hurwitz polyzêtas\(^2\) is already studied in [5, 6]. In our studies, we constructed an integral representation\(^3\) of colored Hurwitz polyzêtas and a distribution treating simultaneously two singularities and our methods permit to make the meromorphic continuation commutatively over the variables \( s_1, \ldots, s_r \).\(^4\) Moreover, [6] gives another way to obtain the meromorphic continuation thanks to translation equations [4]. Our methods give the structure of multi-poles [5] (Theorem 4.2) and two ways to calculate algorithmically the multi-residus\(^4\).

In this note, in continuation with our previous works [10, 11, 12, 13, 5, 6], we are focusing on Hopf algebra, for a class of products as minusstuffle \((\mathbb{W})\), multistuffle \((\mathbb{W})\), \ldots, and in particular for the new product duffle \((\mathbb{W})\), obtained as “tensorial product” of \(\mathbb{W}\) and the well known stuffle \((\mathbb{W})\), of symbolic representations of these polyzêtas (see Definition 2.1 and Proposition 2.1 bellow).

\section{Combinatorial objects}

\subsection{Some products and their algebraic structures}

Let \( X \) be an encoding alphabet and the free monoid over \( X \) is denoted by \( X^* \). The length of any word \( w \in X^* \) is denoted by \( |w| \) and the unit of \( X^* \) is denoted by \( 1_X \). For any unitary commutative algebra \( A \), a formal power series \( S \) over \( X \) with coefficients in \( A \) can be written as the infinite sum \( \sum_{w \in X^*} \langle S | w \rangle w \). The set of polynomials (resp. formal power series) over \( X \) with coefficients in \( A \) is denoted by \( A[X] \) (resp. \( A[[X]] \)).

The set of degree 1 monomials is \( AX = \{ ax/a \in A, x \in X \} \).

\textbf{Definition 2.1} We note \( \mathcal{P} \) the set of products \( * \) over \( A[X] \) verifying the conditions :

\(^1\)They are presented as generalized Nielsen polylogarithms in [10] (Definition 2.3) and as generalized Lerch functions in [12] (Definition 3).

\(^2\)See also references and a discussion about meromorphic continuation of Riemann polyzêtas in [5].

\(^3\)This integral representation is obtained by applying successively the polylogarithmic transform [10]. It is an application of non commutative convolution as shown in [9] (Section 2.4). Other integral representations can be also deduced easily by change of variables, for example \( t = zr \) and then \( r = e^{-u} \) [5].

\(^4\)Other meromorphic continuations can also be obtained by Mellin transform as already done in [17] or by classical estimation on the imaginary part [7] but these later work recursively, depth by depth, and the commutativity of this process over the variables \( s_1, \ldots, s_r \) must be proved. Unfortunately, the structure of multi-poles as well as multi-residus are missing in both works [7, 17]. In [16], to make the meromorphic continuation (giving the expression of non positive integers multi-residus via a generalization of Bernoulli numbers – but not of all multi-residus) of the specialization at roots of unity of colored Hurwitz polyzêtas \( \text{Di}(\mathcal{F}_\xi, t; s) \), the author bases on the integral representation, on the contours, of the multiple Hurwitz-Lerch which corresponds mutatis mutandis to the integral representation of generalized Lerch functions introduced earlier in [5] (Corollary 3.3).
(i) the map $\ast : A(X) \times A(X) \to A(X)$ is bilinear,

(ii) for any $w \in X^*$, $1_{X^*} \ast w = w \ast 1_{X^*} = w$,

(iii) for any $a, b \in X$ and $u, v \in X^*$,

$$au \ast bv = a(u \ast bv) + b(au \ast v) + [a, b](u \ast v),$$

where $[., .] : AX \times AX \to AX$ is a function verifying:

- (S1) $\forall a \in AX$, $[a, 0] = 0$,
- (S2) $\forall (a, b) \in (AX)^2$, $[a, b] = [b, a]$,
- (S3) $\forall (a, b, c) \in (AX)^3$, $[[a, b], c] = [a, [b, c]]$.

**Example 1 (see [18])** Product of iterated integrals.

The shuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (a, b) \in X^2, \forall (u, v) \in X^{*2}, \quad au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v).$$

For example, for any letter $x_0, x$ and $x'$ in $X$,

$$x_0 x' \shuffle x_0^3 x = x_0 x'' x_0^2 x + 2x_0^2 x' x_0 x + 3x_0^2 x' x + 3x_0^2 x x' + x_0^2 x x_0 x'.$$

**Example 2 (see [15])** Product of quasi-symmetric functions.

Let $X$ be an alphabet indexed by $\mathbb{N}$.

The stuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (a, b) \in X^2, \forall (u, v) \in X^{*2}, \quad x_i u \shuffle x_j v = x_i (u \shuffle x_j v) + x_j (x_i u \shuffle v) + x_{i+j} (u \shuffle v).$$

In particular, with the alphabet $Y = \{y_1, y_2, y_3, \ldots \}$,

$$\{y_1 y_1\} \shuffle y_2 = y_3 y_1 y_2 + y_3 y_2 y_1 + y_3 y_3 y_1 + y_2 y_3 y_1 + y_3 y_1.$$

**Example 3 ([3])** Product of large multiple harmonic sums.

Let $X$ be an alphabet indexed by $\mathbb{N}$.

The minus-stuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \quad x_i u \shuffle x_j v = x_i (u \shuffle x_j v) + x_j (x_i u \shuffle v) - x_{i+j} (u \shuffle v).$$

**Example 4 ([16])** Product of colored sums.

Let $X$ be an alphabet indexed by a monoid $(\mathbb{I}, \times)$.

The multistuffle is a bilinear product such that:

$$\forall w \in X^*, \quad w \shuffle 1_{X^*} = 1_{X^*} \shuffle w = w \quad \text{and} \quad \forall (x_i, x_j) \in X^2, \forall (u, v) \in X^{*2}, \quad x_i u \shuffle x_j v = x_i (u \shuffle x_j v) + x_j (x_i u \shuffle v) + x_{i+j} (u \shuffle v).$$

For example, with $X$ indexed by $\mathbb{Q}^+$,

$$x_{\frac{3}{2} x_{-1}} \shuffle x_{\frac{3}{2}} = x_{\frac{3}{2} x_{-1} x_{\frac{3}{2}}} + x_{\frac{3}{2} x_{-1} x_{\frac{3}{2}}} + x_{\frac{3}{2} x_{-1} x_{\frac{3}{2}}} + x_{\frac{3}{2} x_{-1} x_{\frac{3}{2}}} + x_{\frac{3}{2} x_{-1} x_{\frac{3}{2}}}.$$
Remark 2.1 Thanks to the one-to-one correspondence \((i_1, \ldots, i_r) \mapsto x_{i_1} \ldots x_{i_r}\), between tuples of \(\mathcal{I}\) and word over \(X\), the calculus of \(x^3 x_{-1} \shuffle x^3\) can be written as \((\frac{2}{3}, -1) \shuffle (\frac{2}{3}) = (\frac{2}{3}, -1, \frac{1}{3}) + (\frac{2}{3}, \frac{2}{3}, -1) + (\frac{1}{3}, \frac{2}{3}, -1) + (\frac{1}{3}, -1)\).

Example 5 (61) Product of colored Hurwitz polyzetas.
Let \(Y\) and \(E\) be two alphabets and consider the alphabet \(A = Y \times E\) with the concatenation defined recursively by \((y, e). (w_Y, w_E) = (y w_Y, e w_E)\) for any letters \(y \in Y\), \(e \in E\), and any word \(w_Y \in Y^*, w_E \in E^*\). The unit of the monoid \(A^*\) is given by \(1_{A^*} = (1_Y, 1_E^*)\). If \(Y\) is indexed by \(\mathbb{N}\) and \(E\) by a monoid \((\mathcal{I}, \times)\), the shuffle is a \(\times\) product such that \(\forall w \in A^*, \quad w \shuffle 1_{A^*} = 1_{A^*} \shuffle w = w\), \(\forall (y_i, e_i) \in Y^2, \forall (e_i, e_k) \in E^2\), \((u, v) \in A^{2*}\), \((y_i, e_i) \shuffle (y_j, e_k) = (y_i y_j, e_i e_k)\), \(u \shuffle (y_j, e_k) = (y_j, e_k) u\) thanks to \((ii)\) since \((i)\).

Proposition 2.1 The shuffle, the stuffle, the minus-stuffle and the mulstuffle are elements of \(P\), with respectively, \([x_i, x_j] = 0, [x_i, x_j] = x_{i+j}, [x_i, x_j] = -x_{i+j}, [x_i, x_j] = x_{i\times j}\) for any letters \(x_i\) and \(x_j\) of \(X\).
The shuffle is in \(P\), with \([y_i, e_i], (y_j, e_j)] = (y_i y_j, e_i e_k)\) for all \(y_i, y_j\) in \(Y\), \(e_i, e_k\) in \(E\).

Proposition 2.2 Let \(* \in P\), then \((A^\langle X \rangle, *)\) is a commutative algebra.

Proof. We just have to show the commutativity and the associativity of 
* . To obtain \(w_1 * w_2 = w_2 * w_1\) for all \(w_1, w_2\) in \(X^*\), we use an induction on \(|w_1| + |w_2|\).
It is true when \(|w_1| + |w_2| \leq 1\) thanks to \((i)\) since \(w_1\) or \(w_2\) is \(1_X\). The equality \((iii)\), the condition \((S2)\) and the commutative of + give the induction. In the same way, an induction on \(|w_1| + |w_2| + |w_3|\) gives \(w_1 * (w_2 * w_3) = (w_1 * w_2) * w_3\) thanks to \((iii)\) and \((S3)\).

If we associate to each letter of \(X\) an integer numbered called weight, the weight of a word is the sum of the weight of its letters. In this case \(X\) is graduated.
In [15], Hoffman works over \(\overline{X} = X \cup \{0\}\) with \([,,]\) \(\overline{X} \times \overline{X} \rightarrow \overline{X}\) and call quasi-product any product in \(P\) with the additional condition :

\[\text{(S4)} \quad \text{Either } [a, b] = 0 \text{ for all } a, b \text{ in } X; \text{ or the weight of } [a, b] \text{ is the sum of the weight of } a \text{ and the weight of } b \text{ for all } a, b \text{ in } X.\]

Example 6 1. The shuffle is a quasi-product.

2. Let \(X\) be an alphabet indexed by \(\mathbb{N}\) and define the weight of \(x_i, i \in \mathbb{N}\), by \(i\).
Then the stuffle is a quasi-product.

Theorem 2.1 (155) If \(X\) is graduated and has a quasi-product *, then \((A^\langle X \rangle, *)\) is a commutative graduated \(A\)-algebra.

We can define \((i)\) a comultiplication \(\Delta : A^\langle X \rangle \rightarrow A^\langle X \rangle \otimes A^\langle X \rangle\),
(ii) a counit \(\epsilon : A^\langle X \rangle \rightarrow A,\)
by : \(\forall w \in X^*, \quad \Delta w = \sum_{u|v=w} u \otimes v \quad \text{and} \quad \epsilon(w) = \begin{cases} 1 \text{ if } w = 1_X, \\ 0 \text{ otherwise.} \end{cases}\)

The coproduct \(\Delta\) is coassociative so \((A^\langle X \rangle, \Delta, \epsilon)\) is a coalgebra.
Lemma 2.1 For any \( w \in X^* \) and \( x \in X \), \((x \otimes 1_{X^*}) \Delta w + 1_{X^*} \otimes x w = \Delta x w.\)

Proof. \( \forall w \in X^*, \forall x \in X, \Delta x w = \sum_{u \otimes v = w} u \otimes v = \sum_{x u' \otimes v = w} x u' \otimes v + 1_{X^*} \otimes x w \) 

so \( \Delta x w = x \otimes 1_{X^*} \left( \sum_{u' \otimes v = w} u' \otimes v \right) + 1_{X^*} \otimes x w = (x \otimes 1_{X^*}) \Delta w + 1_{X^*} \otimes x w. \) \( \square \)

Proposition 2.3 If \( \ast \in \mathcal{P} \), then \((A(X), \ast, \Delta, \epsilon)\) is a bialgebra.

Remember that \( \ast \) acts over \( A(X) \otimes A(X) \) by \((u \otimes v) \ast (u' \otimes v') = (u \ast u') \otimes (v \ast v').\)

Proof. \( \epsilon \) is obviously a \( \ast \)-homomorphism. It still has to be show \( \Delta(w_1) \ast \Delta(w_2) = \Delta(w_1 \ast w_2) \) over \( X^*.\) This equality is true if \( w_1 \) or \( w_2 \) is equal to \( 1_{X^*}.\)

Assume now that \( \Delta(u) \ast \Delta(v) = \Delta(u \ast v) \) for any word \( u \) and \( v \) such that \(|u| + |v| \leq n, n \in \mathbb{N}, \) and \( w_1 \) and \( w_2 \) be in \( X^* \) with \(|w_1| + |w_2| = n + 1.\) We note \( w_1 = au \) and \( w_2 = bv \) with \( a \) and \( b \) two letters of \( X, \) \( u \) and \( v \) two words of \( X^*.\) Thus, by definition, 

\[
\Delta(w_1) \ast \Delta(w_2) = \sum_{u_1 u_2 = u, v_1 v_2 = v} (au_1 \ast bv_1) \otimes (u_2 \ast v_2) + \sum_{u_1 u_2 = u} (au_1) \otimes (u_2 \ast bv) + \sum_{v_1 v_2 = v} (bv_1) \otimes (au \ast v_2) + 1_{X^*} \otimes (au \ast bv)
\]

Using the induction hypothesis then the lemma 2.1 (since \([a, b] \in AX\)) gives

\[
\Delta(w_1) \ast \Delta(w_2) = \Delta(a(u \ast w_2)) + \Delta(b(w_1 \ast v)) + \Delta([a, b](u \ast v))
\]

\[
= \Delta(a(u \ast w_2)) + b(w_1 \ast v) + [a, b](u \ast v)
\]

\[
= \Delta(w_1 \ast w_2).
\] \( \square \)
Remark 2.2 In particular, $\Delta$ is a $\omega$-homomorphism, a $\boxplus$-homomorphism and a $\bullet$-homomorphism.

Let $C_n$ be the set of positive integer sequences $(i_1, \ldots, i_k)$ such that $i_1 + \ldots + i_k = n$.

**Theorem 2.2** Define $a_\ast$ by, for all $x_1, \ldots, x_n$ in $X$,

$$a_\ast(x_1 \ldots x_n) = \sum_{(i_1, \ldots, i_k) \in C_n} (-1)^k x_1 \ldots x_{i_1} \ast x_{i_1+1} \ldots x_{i_1+i_2} \ast \ldots \ast x_{i_1+i_2+\ldots+i_{k-1}+1} \ldots x_n$$

then, if $\ast \in \mathcal{P}$, $(A(X), \ast, \Delta, \epsilon, a_\ast)$ is a Hopf algebra.

**Proof.** With the applications:

$\mu : A \rightarrow A(X)$ and $m : A(X) \otimes A(X) \rightarrow A(X)$

$\lambda \mapsto \lambda 1_X$, $u \otimes v \mapsto u \ast v$,

the antipode must verify $m \circ (a_\ast \otimes Id) \circ \Delta = \mu \circ \epsilon$, or, in equivalent terms

$$\sum_{u,v \in A} a_\ast(u) \ast v = \langle w|1_X \cdot \rangle 1_X \ast$$

i.e. $\{a_\ast(1_X \ast) = 1_X \ast n \forall x \in X, a_\ast(x) = -x$ and, if $w = x_1 \ldots x_n$ with $n \geq 2, x_1, \ldots, x_n \in X$,

$$a_\ast(w) = -\sum_{k=1}^{n-1} a_\ast(x_1 \ldots x_k) \ast x_{k+1} \ldots x_n.$$ An induction over the length $n$ shows that $a_\ast$ defined in theorem verifies these equalities, and, in the same way, $a_\ast$ verifies $m \circ (Id \otimes a_\ast) \circ \Delta = \mu \circ \epsilon$. \hfill \Box

**Corollary 2.1** If $\ast$ is $\omega$ or $\boxplus$ or $\bullet$ or $\bullet\boxplus$, then this construction gives an Hopf algebra. Moreover, for $\omega$ or $\boxplus$, we obtain a graduated Hopf algebra.

### 2.2 Iterated integral

Let us associate to each letter $x_i$ in $X$ a 1-differential form $\omega_i$, defined in some connected open subset $U$ of $\mathbb{C}$. For all paths $Z_0 \sim z$ in $U$, the Chen iterated integral associated to $w = x_{i_1} \cdots x_{i_k}$ along $Z_0 \sim z$, noted is defined recursively as follows

$$\alpha_{Z_0}^z(w) = \int\limits_{Z_0 \sim z} \omega_i(z_1) \alpha_{Z_0}^z(x_{i_2} \cdots x_{i_k})$$

and $\alpha_{Z_0}^z(1_X \ast) = 1$, verifying the rule of integration by parts [2]:

$$\alpha_{Z_0}^z(u \cup v) = \alpha_{Z_0}^z(u) \alpha_{Z_0}^z(v).$$

We extended this definition over $A(X)$ (resp. $A(\langle X \rangle)$ by

$$\alpha_{Z_0}^z(S) = \sum_{w \in X} \langle S|w \rangle \alpha_{Z_0}^z(w).$$
2.3 Shuffle relations

2.3.1 First encoding for colored Hurwitz polyzêtas

Let \( \xi = (\xi_n) \) be a sequence of complex numbers and \( T \) a family of parameters. Put \( X' \) an alphabet indexed over \( \mathbb{N}^* \times \mathbb{C}^0 \times T \) and \( X = \{ x_0 \} \cup X' \). To each \( x \) in \( X \) we associate the differential form:

\[
\omega_i(z) = \begin{cases} \frac{dz}{z} & \text{if } x = x_0 \\ \frac{1}{\prod_{k=1}^{i} \xi_k} \frac{dz}{z^i} & \text{if } x = x_i \xi^i \text{ with } i \geq 1. \end{cases}
\]

(5)

For any \( T \)-tuple \( t = (t_1, \ldots, t_r) \) we associate the \( T \)-tuple \( \xi = (\xi_1, \ldots, \xi_r) \) given by

\[
\begin{align*}
t_1 &= t_1 - t_2, \\
t_2 &= t_2 - t_3, \\
&\vdots \\
t_r &= t_{r-1} - t_r
\end{align*}
\]

in this way

\[
\begin{align*}
t_1 &= \xi_1 + \xi_2 + \cdots + \xi_n, \\
t_2 &= \xi_2 + \xi_3 + \cdots + \xi_n, \\
&\vdots \\
t_r &= \xi_r
\end{align*}
\]

We choose the sequence \( \xi \) and the family \( t \) such that the condition \((E)\) is satisfied.

**Proposition 2.4** For any \( s = (s_1, \ldots, s_r) \) with \( s_1 > 1 \) if \( \xi = (\xi_1, \ldots, \xi_r) \in \mathcal{E} \) and \( t = (t_1, \ldots, t_r) \in T' \), then \( \text{Di}(\mathbf{F}_\xi; s) = \alpha_0^1(x_0^{s_1-1} x_1 \xi_1 \overline{\tau} \cdots x_0^{s_r-1} x_r \xi_r \overline{\tau}) \).

**Proof.** Since \( \omega_i(z) = \sum \prod_{n>0} k=1 \xi_k^r z^n \frac{dz}{z+1+t} \) then \( \alpha_0^1(x_0^{s_1-1} x_1 \xi_1 \overline{\tau} \cdots x_0^{s_r-1} x_r \xi_r \overline{\tau}) \)

\[
\frac{d}{dz} \sum \prod_{r>0} k=1 \xi_k^r z^n \frac{dz}{z+1+t}
\]

and \( \alpha_0^1(x_0^{s_1-1} x_1 \xi_1 \overline{\tau} \cdots x_0^{s_r-1} x_r \xi_r \overline{\tau}) \) gives

\[
\prod_{m_1, \ldots, m_r > 0} j=1 (m_1 + \cdots + m_r - t_j - \cdots - t_r) t_j,\quad \text{and then, by change of variables,}
\]

\[
\sum_{n_1 > \cdots > n_r > 0} (n_1 - t_1)^{s_1} \cdots (n_r - t_r)^{s_r},
\]

\( \square \)

**Theorem 2.3** Let \( T \) be the group of parameters generated by \( \langle T; + \rangle \), \( C \) be a subgroup of \( \langle C' \rangle \) and \( A \) a sub-ring of \( C \). Put \( C' = C^0 \cap E \) and \( T' \) the set of finite tuple with elements in \( T \). Then the algebra generated by \( \{ \text{Di}(\mathbf{F}_\xi; s) \}_{\xi \in C'; s \in T'} \) is the A modulus generated by \( \{ \text{Di}(\mathbf{F}_\xi; s) \}_{\xi \in C'; s \in T'} \).

**Proof.** We have express the product \( \text{Di}(\mathbf{F}_\xi; s) \text{Di}(\mathbf{F}_{\xi'}; s') \), with \( s = (s_1, \ldots, s_r) \), \( s' = (s'_1, \ldots, s'_r) \), \( \xi, \xi' \in C' \) and \( t = (t_1, \ldots, t_r), t' = (t'_1, \ldots, t'_r) \) \( T' \), as linear combination of colored Hurwitz polyzêtas. This is an iterated integral associated to \( x_0^{s_1-1} x_1 \xi_1 \overline{\tau} \cdots x_0^{s_r-1} x_r \xi_r \overline{\tau} \) which is a sum of
terms of the form \( x_0^{s''-1} x_1, \xi(i) \overline{\cdots} x_0^{s''-1} x_j, \xi(i) \overline{\cdots} x_0^{s''-1} x_{j', \xi(i') \overline{\cdots}} \), with \( s'' \in \mathbb{N} \), \( \xi(i) \) is \( \xi' \) or \( \xi'' \) and \( t_{i'} \) is \( t_j \) or \( t_{j'} \) for all \( i \) and \( r'' = r + r'' \). Note that

\[
\alpha_{0}^{\xi}(x_{i, \xi, t_{i'}} x_{j, \xi', t_{j'}}) = \int_{0}^{z} \prod_{m > 0} \xi_m z_{m - r - 1} \int_{0}^{z_1} \frac{dz}{z_3} \int_{0}^{z_2} \frac{dz}{z_4} \int_{0}^{z_3} \frac{dz}{z_5} \sum_{n > 0} \xi_n z_{n - t - 1} \int_{0}^{z_5} \frac{dz}{z_6}
\]

with \( n_i = m_i + \cdots + m_{r''}, t_{i'} = t_i + \cdots + t_{r''} \) for all \( i \), so \( t'' \in T; \xi'' = \xi_i \) and \( \xi'' = \frac{\xi_{i'}^{(1)} \cdots \xi_{i'}^{(1)}}{\xi_{j'}^{(1)} \cdots \xi_{j'}^{(1)}} \) for \( i > 1 \) so \( \xi'' \in C \) : we can express each term of the shuffle product as \( \text{Di}(F_{\xi''}; t''; s') \).

Note that the shuffle product over two words of \( X^*X' \) acts separately over \( (C', \cdot), (T', +) \) and the convergent compositions. We can describe the situation with the shuffle algebra\(^5\):

**Theorem 2.4** Let \( H = \mathbb{Q} \)-algebra generated by the colored Hurwitz polyzetas. The map \( \zeta : (\mathbb{Q}(x_0 x_1, \xi, t_{i'})', \cdot, \cdot) \rightarrow (H, \cdot) \), \( x_0^{m_0} x_1, \xi, t_{i'} \overline{\cdots} x_0^{m_r} x_r, \xi, t_{i'} \rightarrow \text{Di}(F_{\xi, t_{i'}}; s + 1) \) is a surjective algebra morphism.

**Example 7** Since \( \text{Di}(F_{\xi, t'}; 3) = \alpha_{0}^{\xi}(x_0^3 x_1, \xi, t') \) and \( \text{Di}(F_{\xi, t''}; 2) = \alpha_{0}^{\xi}(x_0, x_1, \xi, t'' \overline{\cdots} x_0, x_1, \xi, t'') \) then \( \text{Di}(F_{\xi, t'}; 3) \text{Di}(F_{\xi, t''}; 2) = \alpha_{0}^{\xi}(x_0, x_1, \xi, t' \overline{\cdots} x_0, x_1, \xi, t'') \). Example 1 with \( x = x_1, \xi, t' \) gives the expression of \( x_0, x_1, \xi, t' \overline{\cdots} x_0, x_1, \xi, t'' \). But the first term obtained is

\[
\alpha_{0}^{\xi}(x_0, x_1, \xi, t' \overline{\cdots} x_0, x_1, \xi, t'') = \int_{0}^{z_1} \frac{dz_1}{z_1} \int_{0}^{z_1} \frac{dz_2}{z_2} \int_{0}^{z_2} \frac{dz_3}{z_3} \int_{0}^{z_3} \frac{dz_4}{z_4} \int_{0}^{z_4} \frac{dz_5}{z_5} \sum_{n > 0} \xi_n z_{n - t - 1} \int_{0}^{z_5} \frac{dz_6}{z_6}
\]

\[
= \sum_{n > 0} (m + n - t' - t'')^2 (n - t)^3
\]

\[
= \sum_{n_1 > n_2 > 0} (n_1 - t' - t'')^2 (n_2 - t)^3
\]

\[
= \text{Di}(F_{\xi, t''}; (t' + t''); s (2, 3)).
\]

\(^5\text{Working in } \mathbb{Q}(x_0^{m_0} x_1, \xi, t_{i'})' \text{ implies working in the graduated Hopf algebra } (\mathbb{Q}(X^*), \overline{\cdots}, \Delta, \epsilon, a_{i'}) \).
Proposition 2.5
With this notation, 

\[ \alpha_\omega Q \text{form } \omega \]

For the Hurwitz polyzetas, we can obtain an encoding indexed by a finite alphabet. Let 

\[ X \]

the alphabet

2.3.2 Second encoding for colored Hurwitz polyzetas

r functions (with depth one : surjective morphism of algebras.

indexed by \( N \omega \) the differential form

Let \( z \in B \) \( (0; 1) \neq \sum_{k \geq 0}^\infty (t x_1)^{s_1 x_1} \ldots x_0^r = \zeta \( s; t \). \)

Theorem 2.5 Let \( \mathcal{H}' \) be the \( \mathbb{Q} \)-algebra generated by the Hurwitz polyzetas and \( X \) be the \( \mathbb{Q} \)-algebra generated by \((t_1 x_0)^{s_1 x_1} \ldots (t_r x_0)^{s_r x_r} \). Then, \( \zeta : (X, \omega) \twoheadrightarrow (\mathcal{H}', \) is a surjective morphism of algebras.

Note that we can apply the idea of encoding of “simple” colored Hurwitz zeta functions (with depth one : \( r = 1 \)). Let \( \xi = (\xi_n) \) be a sequence of complex numbers in the unit ball \( B(0; 1) \) and \( T \) a family of parameters. Let \( X = \{ x_0, x_1, \ldots \} \) be a alphabet indexed by \( N \). Associate to \( x_0 \) the differential form \( \omega_0(z) = z^{-1} d z \) and to \( x_i, i \geq 1 \), the differential form \( \omega_i(z) = \xi_i(1 - \xi z)^{-1} d z. \)

Proposition 2.5 With this notation, \( \alpha_0^1 \left( ((t x_0)^{s_1} x_1) \ldots (t x_0)^{s_r} x_r \right) = \sum_{n \geq 0} \xi^n \left( n - t \right)^n. \)

Proof. Since \( \frac{\xi_d x_0}{1 - \xi_z} = \xi \sum_{n \geq 0} (\xi z)^n d z_0, \) we can write

\[ \alpha_0^z \left( ((t x_0)^{k} x_i) \right) = \xi^k \sum_{n \geq 0} (\xi z_0)^n d z_0 = \sum_{n \geq 0} \xi^n \left( t n \right)^k \]

for \( z \in B(0; 1) \) and for \( k \in \mathbb{N} \). Thanks to the absolute convergence,

\[ \alpha_0^z \left( ((t x_0)^{s_1} x_1) \right) = \sum_{n \geq 0} \xi^n \left( t \frac{z}{n} \right)^k = \sum_{n \geq 0} \xi^n \left( t \frac{z}{n} \right)^k. \]

In the same way, if \( z \in B(0; 1) \)

\( \forall k \in \mathbb{N}, \quad \alpha_0^z \left( ((t x_0)^k x_0 (t x_0)^{s_1} x_1 \right) = \sum_{n \geq 0} \xi^n \left( t \frac{z}{n} \right)^k. \)

so

\[ \alpha_0^z \left( ((t x_0)^{s_1} x_1 (t x_0)^{s_r} x_r \right) = \sum_{n \geq 0} \xi^n \left( t \frac{z}{n} \right)^r. \]
and \[ \alpha_0^z \left( ((tx_0)^s x_0)^{s-1} (tx_0)^s x_0 \right) = \sum_{n>0} \frac{\xi^n z^n}{(n-t)^s}. \]

\[ \square \]

**Remark 2.3** Note that, with the same notation,
\[ \alpha_0^z \left( x_1 ((t_2 x_0)^s x_0)^{s-1} (t_2 x_0)^s x_2 \right) = \sum_{n,m>0} \frac{\xi^n \xi^m z^{n+m}}{(n-t_2)^s (m+n)} = \sum_{n_1,n_2>0} \frac{\xi^n \xi^m z^{n-n_1} n_1 (n_2-t_2)^{s}}{n_2 (n_2-t_2)^{s}}. \]

In other words, this encoding appears to be widespread only as couples of the type \( \xi = (1, \ldots, 1, \xi_r) \): with \( \xi_1 = 1 \) and \( \omega_1 = (1-z)^{-1}dz \),
\[ \alpha_0^1 \left( x_0^{s_0-1} (t_1 x_0)^{s_1} \ldots x_0^{s_r-1} (t_r x_0)^{s_r} x_r \right) = \sum_{n_1 \ldots n_r} \frac{\xi^{s_r} z^{n_1 \ldots n_r}}{(n_1-t_1)^{s_1} \ldots (n_r-t_r)^{s_r}}. \]

### 2.4 Duffle relations

Let \( \lambda = (\lambda_n) \) be a set of parameters, \( s = (s_1, \ldots, s_r) \) a composition, \( \xi \in \mathbb{C}^r \). Then
\[ \forall n \in \mathbb{Z}_{>0}, \quad M_{s,\xi}^n (\lambda) = \sum_{n>n_1 \ldots n_r>0} \prod_{i=1}^r \xi^n_i \lambda^s_i \quad \text{and} \quad M_{r,\rho}^n (\lambda) = 1. \] (7)

We can export the duffle over the tuples \( s = (s_1, \ldots, s_r) \in \mathbb{Z}_{>0}^r \) and \( \xi \in \mathbb{C}^r \) with :
\[ (s, \xi) \mathcal{W} ((,), 1) = ((,), 1) \mathcal{W} (s, \xi) = (s, \xi) \quad \text{and} \]
\[ (s_1, s; \xi_1, \xi) \mathcal{W} (r_1, r; \rho_1, \rho) = (s_1; \xi_1) \cdot ((s_1; \xi_1) \mathcal{W} (r_1; \rho_1, \rho)) + (s_1; \xi_1) \cdot ((s_1; \xi_1) \mathcal{W} (r_1; \rho_1, \rho)) . \] (8)

**Proposition 2.6** Let \( s = (s_1, \ldots, s_l) \) and \( r = (r_1, \ldots, r_k) \) be two compositions, \( \xi \in \mathbb{C}^l \), \( \rho \in \mathbb{C}^k \). Then
\[ \forall n \in \mathbb{N}, \quad M_{s,\xi}^n (\lambda) M_{r,\rho}^n (\lambda) = M_{(s,\xi), (r,\rho)}^n (\lambda). \]

**Proof.** Put the compositions \( s' = (s_2, \ldots, s_l) \), \( r' = (r_2, \ldots, r_k) \), the tuples of complex numbers \( \xi' = (\xi_2, \ldots, \xi_l) \) and \( \rho' = (\rho_2, \ldots, \rho_k) \), then
\[ M_{s',\xi'}^n (\lambda) M_{r',\rho'}^n (\lambda) = \sum_{n>n_1, n>n'_1} \xi^{n_1} \lambda^{n_1} M_{s',\xi'}^n (\lambda) \rho^{n_1} \lambda^{n_1} M_{r',\rho'}^n (\lambda) \]
\[ = \sum_{n>n_1} \xi^{n_1} \lambda^{n_1} M_{s',\xi'}^n (\lambda) \rho^{n_1} \lambda^{n_1} M_{r',\rho'}^n (\lambda) + \sum_{n>n'_1} \rho^{n'_1} \lambda^{n'_1} M_{s',\xi'}^n (\lambda) \rho^{n'_1} \lambda^{n'_1} M_{r',\rho'}^n (\lambda) \]

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\[ + \sum_{n>m} (\xi_1 \rho_1)^m \lambda_n^{s_1+r_1} M_{\nu', \xi'}^n(\lambda) M_{\nu', \rho'}^m(\lambda). \]

A recurrence ended the demonstration. \(\square\)

\textbf{Theorem 2.6} Let \(s = (s_1, \ldots, s_I)\) and \(r = (r_1, \ldots, r_k)\) be two compositions, \(\xi\) a \(l\)-tuple and \(\rho\) a \(k\)-tuple of \(E\). \(t = (t_1, \ldots, t)\) a \(l\)-tuple and \(t' = (t_1, \ldots, t)\) a \(k\)-tuple, both formed by the same parameter \(t\) diagonally. Then

\[ \text{Di}(F_{\xi, t}; s) \text{Di}(F_{\xi', t'; s'}) = \text{Di}(F_{\xi'',(t_1,\ldots,t); s'',\xi'}, \text{Di}(F_{\xi',(t_1,\ldots,t); s'}), \text{Di}(F_{\xi,t}; s)). \]

with \((s'', \xi'')) = (s, \xi) \times (s', \xi').\)

\textbf{Proof.} With \(\lambda_n = \frac{1}{n-t}\) for all \(n \in \mathbb{N}\), \(M^n_{\nu, \xi}(\lambda) = \sum_{n>n_1>\ldots>n_i} \prod_{i=1}^r \frac{\xi_{n_i}}{(n_i-t)^t}.\) So

\[ \lim_{n \to \infty} M^n_{\nu, \xi}(\lambda) = \text{Di}(F_{\xi,t}; s) \text{ and taking the limit of Proposition 2.6 gives the result.} \] \(\square\)

\textbf{Example 8} The use of examples 2 and 4 gives

\[ \text{Di}(F_{(3,1)}; (3,1)) \text{Di}(F_{(2)}; (2)) \]

\[ = \text{Di}(F_{(3,1), (t, t, t); (3,1,2)}) + \text{Di}(F_{(1, 1, 1), (t, t, t); (2,3,1)}) + \text{Di}(F_{(2, 1), (t, t, t); (3,3)}) + \text{Di}(F_{(2, 1), (t, t, t); (2,3,1)}) + \text{Di}(F_{(2, 1), (t, t, t); (3,3)}) \]

\textbf{Remark 2.4} Extend the duffle product to triplets \((s, t, \xi) \in \cup_{r \in \mathbb{N}^+} \mathbb{N}^r \times \{ t \} \times \mathbb{C}^r\) by

\[ (s_1, s; t; \xi_1, \xi) \times (r_1, r; t; t'; \rho_1, \rho) = (s_1, s; t; \xi_1, \xi) \times (r_1, r; t; t'; \rho_1, \rho) \]

and define the function \(F\) over \(I = \cup_{r \in \mathbb{N}^+} \mathbb{N}^r \times \{ t \} \times \mathbb{C}^r\) by \(F(s, t, \xi) = \text{Di}(F_{s, t}; s, \xi)\).

Then, by Theorem 2.6, the function \(F : (I, \times) \to (\mathbb{C}, .)\) is morphism of algebras.

\textbf{References}


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