Subdiffusive behavior generated by irrational rotations

François Huveneers

Subdiffusive behavior generated by irrational rotations

F. Huveneers*
UcL, FYMA, 2 Chemin du Cyclotron,
B-1348 Louvain-la-Neuve, Belgium.
E-mail : francois.huveneers@uclouvain.be

Abstract

We study asymptotic distributions of the sums $y_n(x) = \sum_{k=0}^{n-1} \psi(x + k\alpha)$ with respect to the Lebesgue measure, where $\alpha \in \mathbf{R} - \mathbf{Q}$ and where ψ is the 1-periodic function of bounded variation such that $\psi(x) = 1$ if $x \in [0, 1/2[$ and $\psi(x) = -1$ if $x \in [1/2, 1[$. For every $\alpha \in \mathbf{R} - \mathbf{Q}$, we find a sequence $(n_j)_j \subset \mathbf{N}$ such that $y_{n_j}/\sqrt{j}$ is asymptotically normally distributed. For $n \geq 1$, let $z_n \in (y_m)_{m \leq n}$ be such that $||z_n||_{\mathrm{L}^2} = \max_{m \leq n} ||y_m||_{\mathrm{L}^2}$. If α is of constant type, we show that $z_n/||z_n||_{\mathrm{L}^2}$ is also asymptotically normally distributed. We give an heuristic link with the theory of expanding maps of the interval.

1 Introduction

Some purely deterministic, smooth and finite dimensional dynamical systems may generate diffusion process. Such a diffusion is due to uncertainty on initial conditions. If a distribution is initially concentrated in one point, it will remain so under the flow of such a system. But if the initial conditions are distributed on some larger set of the phase space, it may well be that the distribution evolves diffusively.

Some cases of deterministic diffusion have been successfully investigated [2]. Let us mention the theory of expanding maps of the interval [12], and the important result by Bunimovich and Sinai about the Lorentz gas [4]. In the two previous examples, the underlying dynamical system is hyperbolic ; and it has been suggested that macroscopic diffusion is generally due to microscopic chaos [9]. But numerical experiments with systems of zero Lyapunov exponents show that diffusion may happen even in the absence of hyperbolicity [7].

The rotation of the circle by an irrational angle is a well known example of ergodic non hyperbolic dynamical system. Burton and Denker [5] (see also [6]) have shown that one may find a function $\psi \in$

*Partially supported by the Belgian IAP program P6/02 and by the University of Helsinki.

$\mathrm{L}^2(\mathbf{T},\mathbf{R})$ such that $y_n/||y_n||_{\mathrm{L}^2}$ is asymptotically normally distributed. By Denjoy-Koksma inequality (see (4) below), this ψ is not a bounded variation function. Among other results, Liardet and Volný have shown (Theorems 1 and 2 in [14]) that, if $r \geq 0$, then there exist numbers $\alpha \in \mathbf{R}-\mathbf{Q}$ and a sequence $(d_n)_n \subset \mathbf{R}_0^+$ such that for every ψ in a dense G_δ set of $\mathcal{C}^r(\mathbf{T},\mathbf{R})$, the distributions of $d_n y_n$ form a dense set in the space of all probability measures on $\mathbf{R}$. Their results do not cover the case where α is of constant type (see (9) below) and ψ of bounded variation.

Let $\mathbf{T}=\mathbf{R}/\mathbf{Z}$. If $u \in \mathrm{L}^1(\mathbf{T},\mathbf{R})$, one defines

$$\mathrm{Var}(u)=\sup\{\int_0^1 uv'\mathrm{d}x : v \in \mathcal{C}^1(\mathbf{T},\mathbf{R}), ||v||_{\mathrm{L}^\infty} \leq 1\}. \tag{1}$$

One defines also the set $\mathrm{BV}(\mathbf{T},\mathbf{R})=\{u \in \mathrm{L}^1(\mathbf{T},\mathbf{R}) : \mathrm{Var}(u)<\infty\}$.

Let $\psi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ be such that $\int_0^1 \psi \mathrm{d}x = 0$. Let $\alpha \in \mathbf{R}-\mathbf{Q}$. We consider the map

$$\mathrm{F} : \mathbf{T}\times\mathbf{R} \to \mathbf{T}\times\mathbf{R} : (x,y) \mapsto (x+\alpha, y+\psi(x)). \tag{2}$$

If $n \in \mathbf{N}$, one defines implicitly the function $y_n \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ by the relation

$$\mathrm{F}^n(x,y)=(x+n\alpha, y+y_n(x)). \tag{3}$$

Explicitly, one has $y_n(x)=\sum_{k=0}^{n-1}\psi(x+k\alpha)$ for $n \geq 1$. Although y_n depends on ψ and α, one will not generally write it. Let m_{L} be the Lebesgue measure on $\mathbf{T}$. The space $(\mathbf{T}, m_{\mathrm{L}})$ is then a probability space, and $(y_n)_{n\geq 0}$ is a sequence of random variables on this space.

The sequence $(y_n)_{n\geq 0}$ has been widely studied [1][8][11][13]. Here are two important informations. First, the sequence $(y_n)_{n\geq 0}$ is bounded in $\mathrm{L}^2(\mathbf{T},\mathbf{R})$ if and only if there exists $u \in \mathrm{L}^2(\mathbf{T},\mathbf{R})$ such that $\mathrm{R}_\alpha u - u = \psi$ (where by definition $\mathrm{R}_\alpha u(x) = u(x+\alpha)$) ([11] p.183). Next, let p/q be an irreducible fraction such that $|\alpha - p/q| \leq 1/q^2$ (by Dirichlet theorem, there are infinitely many such fractions). Denjoy-Koksma inequality ([11] p.73) asserts that

$$||y_q||_{\mathrm{L}^\infty} = ||\sum_{k=0}^{q-1}\mathrm{R}_{k\alpha}\psi||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\psi). \tag{4}$$

Let us now present our results. We will actually only consider the function ψ_* defined by

$$\psi_*(x)=1 \quad \text{if} \quad 0 \leq x < 1/2, \qquad \psi_*(x)=-1 \quad \text{if} \quad 1/2 \leq x < 1. \tag{5}$$

It is known that there is no $u \in \mathrm{L}^2(\mathbf{T},\mathbf{R})$ that solves the equation $\mathrm{R}_\alpha u - u = \psi_*$ (Lemma 2, Section 2).

First, can we find an increasing sequence $(n_j)_{\geq 1} \subset \mathbf{N}$ such that $y_{n_j}/\sqrt{j}$ should be asymptotically normally distributed (with strictly positive variance) ? Proposition 1 answers this question positively. This means that, if we looked at the system at the times n_j only, we should observe a diffusion process. Next, how fast has to grow the sequence $(n_j)_{j\geq 1}$? If α is of constant type (see (9)), we will see that it may be taken to grow exponentially, but not slower (see Remark after Proposition 2, and Corollary 1).

It seems also natural to consider the sequence $(z_n)_{n\geq 0} \subset \mathrm{BV}(\mathbf{T},\mathbf{R})$, defined as follows :

$$z_n \in (y_m)_{0\leq m\leq n} \quad : \quad ||z_n||_{\mathrm{L}^2} = \max_{0\leq m\leq n} ||y_m||_{\mathrm{L}^2} \tag{6}$$

(one take the first element of $(y_m)_{0\leq m\leq n}$ if there is more than one possibility). In Proposition 2, we will see that the sequence $z_n/||z_n||_{\mathrm{L}^2}$ is asymptotically normally distributed.

Let $\mathrm{G}(\sigma)$ be the probability measure on $\mathbf{R}$ that admits the density $f(x) = e^{-x^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$ $(\sigma > 0)$.

Proposition 1 *Let $(y_n)_{n\geq 0}$ be defined in (3). If $\psi = \psi_*$ (see (5)), and if $\alpha \in \mathbf{R} - \mathbf{Q}$, there exists an increasing sequence $(n_j)_{j\geq 1} \subset \mathbf{N}$ such that $y_{n_j}/\sqrt{j} \overset{\mathrm{D}}{\to} \mathrm{G}(1)$ as $j \to \infty$.*

This result is quite weak, because the sequence $(n_j)_{j\geq 1}$ is completely unknown. Nevertheless, we believe it has some interest. First, the result is valid for any irrational number α. Next, the proof is not technical but contains the principal ideas we need for proving our second Proposition. Finally, it allows us to make an heuristic link between our case and the theory of expanding maps of the interval (see Section 2, after the proof of Proposition 1).

One will then need the theory of continued fractions. Let $(a_n)_{n\geq 0} \subset \mathbf{N}$ be the sequence of partial quotients of α (see for example [10] for definition and details). The sequence $(p_n/q_n)_{n\geq 0} \subset \mathbf{Q}$ of convergents of α is then defined as follows : $p_0/q_0 = a_0/1$, $p_1/q_1 = (a_0a_1 + 1)/a_1$, and, for $n \geq 1$,

$$q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}. \tag{7}$$

One will usually not write explicitly the dependence of a_n and p_n/q_n on α. Here is a fundamental result of the theory of continued fractions : for $n \geq 0$, one has

$$\frac{1}{q_n + q_{n+1}} \leq |q_n\alpha - p_n| \leq \frac{1}{q_{n+1}} \leq \frac{1}{a_{n+1}q_n}. \tag{8}$$

Let us introduce a particular class of numbers. One says that $\alpha \in \mathbf{R} - \mathbf{Q}$ is of *constant type* if

$$\exists \mathrm{C} > 0 : \forall q \in \mathbf{Z}_0, \forall p \in \mathbf{Z}, |q\alpha - p| \geq \frac{\mathrm{C}}{|q|}. \tag{9}$$

Equivalently, α is of constant type if

$$\exists d \geq 1 : \forall n \geq 0, a_n \leq d. \tag{10}$$

This implies that the sequence $(q_n)_{n\geq 0}$ grows only exponentially with n. These numbers form a set of zero Lebesgue measure.

Proposition 2 *Let $(z_n)_{n\geq 0}$ be defined in (6). Let $\psi = \psi^*$ be defined in (5). Let α be a number of constant type. One has $z_n/||z_n||_{\mathrm{L}^2} \overset{\mathrm{D}}{\to} \mathrm{G}(1)$ as $n \to \infty$. Moreover, there exist $\mathrm{C}, \epsilon > 0$ such that, if $q_j \leq n < q_{j+1}$, one has $\epsilon\sqrt{j} \leq ||z_n||_{\mathrm{L}^2} \leq \mathrm{C}\sqrt{j}$.*

Remark. Let $\psi = \psi^*$, and let α be a number of constant type. Let n_j be such that $z_{q_j} = y_{n_j}$. One has $n_j \leq q_j$. Let $\sigma_j = ||y_{n_j}||_{\mathrm{L}^2}/\sqrt{j}$. By Proposition 2, one has $\epsilon \leq \sigma_j = ||z_{q_j}||_{\mathrm{L}^2}/\sqrt{j} \leq \mathrm{C}$, and $y_{n_j}/\sqrt{j}\sigma_j = z_{q_j}/||z_{q_j}||_{\mathrm{L}^2} \xrightarrow{\mathrm{D}} \mathrm{G}(1)$.

Corollary 1 *Let $(y_n)_{n\geq 0}$ be defined in (3). Let $\psi = \psi_*$ be defined in (5). Let α be a number of constant type. Let $(n_j)_{j\geq 1} \subset \mathbf{N}$ and let $(\sigma_j)_{j\geq 1} \subset \mathbf{R}_0^+$ be such that $y_{n_j}/\sigma_j\sqrt{j} \xrightarrow{\mathrm{D}} \mathrm{G}(1)$. Moreover, suppose that there exist $\mathrm{C} > \epsilon > 0$ such that $\epsilon \leq \sigma_j \leq \mathrm{C}$ for every $j \geq 1$. Then, the sequence $(n_j)_{j\geq 1}$ does not grow slower than exponentially with j.*

Question. What happens when $\psi \neq \psi_*$? The choice $\psi = \psi_*$ is only needed to prove $||z_n||_{\mathrm{L}^2} \geq \epsilon\sqrt{j}$ when $n \geq q_j$. (Lemma 11, Section 4). It follows from the proof of this Lemma that other choices should be possible.

The organization of the paper is as follows. Proposition 1 is shown in Section 2. In Section 3, one shows an abstract central limit theorem ; this Section is independent of the others. One proves Proposition 2 and Corollary 1 in Section 4.

The letter C is used to denote a strictly positive constant that may vary from place to place.

2 Proof of Proposition 1

Let $\alpha \in \mathbf{R}-\mathbf{Q}$. Let $(p_n/q_n)_{n\geq 0}$ be its convergents, and $(a_n)_{n\geq 0}$ its partial quotients. Let $\psi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ be such that $\int_0^1 \psi \mathrm{d}x = 0$.

Lemma 1 *Let $n \geq 0$.*

1) *Of the fractions p_n/q_n et p_{n+1}/q_{n+1}, one at least satisfies $|\alpha - p/q| < 1/2q^2$.*

2) *If q_n is even, then q_{n+1} is odd.*

3) *If q_n and q_{n+2} are even, then $|\alpha - p_{n+1}/q_{n+1}| < 1/2q_{n+1}^2$.*

4) *From four consecutive convergents, one at least has an odd denominator and satisfies the inequality $|\alpha - p/q| < 1/2q^2$.*

Proof. For 1), see [10] p.152. Let us show 2) by contradiction. Let us suppose we have found a smallest $j \in \mathbf{N}$ such that q_j and q_{j+1} are even. We have $j \geq 1$ and therefore $q_{j+1} = a_{j+1}q_j + q_{j-1}$. Because q_{j-1} is odd and q_j is even, q_{j+1} should also be odd. Let us show 3). By 2), q_{n+1} is odd, and on the other hand we have that $q_{n+2} = a_{n+2}q_{n+1} + q_n$. The number a_{n+2} has to be even, and therefore $a_{n+2} \geq 2$. The result follows from (8). Finally, 4) is obtained by considering all the possibilities. □

If $u \in \mathrm{L}^2(\mathbf{T},\mathbf{R})$, if $k \in \mathbf{Z}$, one writes $\hat{u}(k) = \int_0^1 u(x)e^{-2i\pi kx}\mathrm{d}x$. If $u \in \mathrm{BV}(\mathbf{T},\mathbf{R})$, it follows from (1) that $|\hat{u}(k)| \leq \mathrm{Var}(u)/2\pi|k|$ for $k \neq 0$. One has

$$\widehat{y_n}(k) = \frac{1 - e^{2i\pi nk\alpha}}{1 - e^{2i\pi k\alpha}}\hat{\psi}(k), \quad (n \geq 1, k \in \mathbf{Z}_0). \tag{11}$$

Let us also introduce the following notation : if $x \in \mathbf{R}$, one writes

$$|x|_{\mathbf{T}} = \inf_{p \in \mathbf{Z}} |x - p|. \tag{12}$$

One checks the two following inequalities : for all $x, y \in \mathbf{R}$, one has

$$4|x|_{\mathbf{T}} \leq |1 - e^{2i\pi x}| \leq 2\pi |x|_{\mathbf{T}}, \tag{13}$$

$$|x + y|_{\mathbf{T}} \leq |x|_{\mathbf{T}} + |y|_{\mathbf{T}} \quad \text{and} \quad |1 - e^{2i\pi(x+y)}| \leq |1 - e^{2i\pi x}| + |1 - e^{2i\pi y}|. \tag{14}$$

Therefore, for every $m \in \mathbf{Z}$,

$$|1 - e^{2i\pi m x}| \leq |m|.|1 - e^{2i\pi x}|. \tag{15}$$

Moreover, if $n \geq 1$, $|q_n\alpha - p_n| = |q_n\alpha|_{\mathbf{T}}$ ([11] p.63).

Lemma 2 *Let $\psi = \psi_*$ given by (5). There exists no $u \in \mathrm{L}^2(\mathbf{T}, \mathbf{R})$ such that $\mathrm{R}_\alpha u - u = \psi_*$.*

Proof. A solution u should be such that $\hat{u}(k) = \hat{\psi}_*(k)/(e^{2i\pi k\alpha} - 1) = -2i/\pi k(e^{2i\pi k\alpha} - 1)$ if k is odd. By point 2) of Lemma 1, for infinitely many odd k, one may write $k = q_j$ for some $j \geq 1$. But one has $|k\alpha|_{\mathbf{T}} \leq 1/|k|$ for those k. Therefore $\hat{u}(k)$ should not go to 0 as $k \to \infty$. $\square$

Lemma 3 *One has $y_{q_n} \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{T}, \mathbf{R})$ as $n \to \infty$.*

Proof. By Denjoy-Koksma inequality (4), $||y_{q_n}||_{\mathrm{L}^2} \leq ||y_{q_n}||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\psi)$. Therefore, we only need to check that, if $k \in \mathbf{Z}_0$, $\widehat{y_{q_n}}(k) \to 0$ as $n \to \infty$. By (11),

$$|\widehat{y_{q_n}}(k)| \leq \frac{\mathrm{Var}(\psi)}{2\pi|k|} \frac{1}{|1 - e^{2i\pi k\alpha}|} |1 - e^{2i\pi q_n k\alpha}|$$

if $k \neq 0$. But by (15)

$$|1 - e^{2i\pi q_n k\alpha}| \leq |k|.|1 - e^{2i\pi q_n \alpha}| \leq 2\pi|k|.|q_n\alpha|_{\mathbf{T}} \to 0 \quad \text{as} \quad n \to \infty. \quad \square$$

A direct consequence of this Lemma is that, for every $\beta \in \mathbf{R}$, $\mathrm{R}_\beta y_{q_n} \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{T}, \mathbf{R})$ as $n \to \infty$.

If $x \in \mathbf{R}$, one sets $\overline{x} = x - \lfloor x \rfloor$. Following [11] p.64, we give some informations about some finite sequences $(\overline{n\alpha})_n$. If $p/q \in \mathbf{Q}$ is irreducible, one has $\{\overline{j.p/q}\}_{0 \leq j \leq q-1} = \{j/q\}_{0 \leq j \leq q-1}$. We say that $p/q \in \mathbf{Q}$ (p/q irreducible) is a *rational approximation* of α for the constant $0 < \beta \leq 1$ if the inequality $|\alpha - p/q| < \beta/q^2$ is satisfied. Let us write $\{\overline{j\alpha}\}_{0 \leq j \leq q-1} = \{\alpha_j\}_{0 \leq j \leq q-1}$, where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{q-1} < 1$.

If $\alpha > p/q$, one has $k\alpha - k.p/q < k\beta/q^2 < 1/q$ if $1 \leq k \leq q-1$. Therefore, if $0 \leq j \leq q-1$, there exists some $l(j) \in \mathbf{N}$ such that $0 \leq \alpha_j - l(j)/q \leq 1/q$. But the sequence $\{\alpha_j\}_{0 \leq j \leq q-1}$ is ordered, and so $l(j) = j$. One may thus write

$$0 = \alpha_0 < \frac{1}{q} < \alpha_1 < \frac{2}{q} < \alpha_2 < \cdots < \frac{q-1}{q} < \alpha_{q-1} < 1. \tag{16}$$

Similarly, if $\alpha < p/q$, one has

$$\alpha_0 = 0 < \alpha_1 < \frac{1}{q} < \alpha_2 < \frac{2}{q} < \cdots < \alpha_{q-1} < \frac{q-1}{q} < 1. \tag{17}$$

In both cases one has

$$|\alpha_j - j/q| < \beta/q \quad (1 \leq j \leq q-1). \tag{18}$$

The following Lemma gives a slight improvement of Denjoy-Koksma (4) inequality when $\psi = \psi_*$ (5).

Lemma 4 *Let $\psi = \psi_*$ given by (5). Let p/q be a rational approximation of α for the constant $\beta \leq 1/2$, and suppose that q is odd. Then the function y_q takes only the values ± 1.*

Proof. Let $\phi = \sum_{k=0}^{q-1} \mathrm{R}_{kp/q}\psi = \sum_{k=0}^{q-1} \mathrm{R}_{k/q}\psi$. One has $\phi(x) = \psi(qx)$; indeed, one has $\mathrm{R}_{1/q}\phi = \phi$ and $\phi|_{[0,1/q[} = (q-1)/2 + (\mathrm{R}_{(q-1)/2}\psi)|_{[0,1/q[} - (q-1)/2$. Let us then write $\{\overline{j\alpha}\}_{0\leq j\leq q-1} = \{\alpha_j\}_{0\leq j\leq q-1}$, where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{q-1} < 1$. One has $y_q = \sum_{k=0}^{q-1} \mathrm{R}_{k\alpha}\psi = \sum_{k=1}^{q-1}(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi + \phi$.

By (16) and if $\alpha > p/q$, one has, for $0 \leq k \leq q-1$,

$$(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi(x) = \begin{cases} +2 & \text{if} \quad x \in [1-\alpha_k, 1-k/q[, \\ -2 & \text{if} \quad x \in [1/2-\alpha_k, 1/2-k/q[\quad (\mathrm{mod} 1), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, if $\alpha < p/q$, one has by (17) that, for $0 \leq k \leq q-1$,

$$(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi(x) = \begin{cases} -2 & \text{if} \quad x \in [1-k/q, 1-\alpha_k[, \\ +2 & \text{if} \quad x \in [1/2-k/q, 1/2-\alpha_k[\quad (\mathrm{mod} 1), \\ 0 & \text{otherwise.} \end{cases}$$

One now computes y_q. To fix the ideas, let us consider the case $\alpha > p/q$. If $0 \leq j \leq q-1$, one has

$$\begin{aligned} y_q\big|_{[j/q,(j+1)/q[} &= \sum_{k=1}^{q-1}(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi\big|_{[j/q,(j+1)/q[} + \phi\big|_{[j/q,(j+1)/q[} \\ &= (2\chi_{[\frac{j+1}{q}-\delta_1(j),\frac{j+1}{q}[} - 2\chi_{[\frac{j}{q}+\frac{1}{2q}-\delta_2(j),\frac{j}{q}+\frac{1}{2q}[}) + (\chi_{[\frac{j}{q},\frac{j}{q}+\frac{1}{2q}[} - \chi_{[\frac{j}{q}+\frac{1}{2q},\frac{j+1}{q}[}), \end{aligned}$$

where, by (18), $0 \leq \delta_1(j), \delta_2(j) < 1/2q$. $\square$

One checks that, if $u : \mathbf{T} \to \mathbf{R}$, if $n \geq 1$, if $c_0 = 0$, if $c_1, \ldots, c_n \geq 1$, then

$$\sum_{k=0}^{c_1+\cdots+c_n-1} \mathrm{R}_{k\alpha}u = \sum_{j=0}^{n} \mathrm{R}_{(c_0+\cdots+c_{j-1})\alpha} \sum_{k=0}^{c_j-1} \mathrm{R}_{k\alpha}u. \tag{19}$$

Proof of Proposition 1. By point 4) of Lemma 1, there exists a subsequence $(\tilde{p}_k/\tilde{q}_k)_{k\geq 1} \subset (p_n/q_n)_{n\geq 0}$ such that $|\alpha - \tilde{p}_k/\tilde{q}_k| < 1/2\tilde{q}_k^2$. Moreover, once $\tilde{q}_1, \ldots, \tilde{q}_k$ $(k \geq 1)$ are given, one may take $\tilde{q}_{k+1}$ as large as we please (by still taking a subsequence). So, by Lemma 4, $\mathrm{R}_\beta y_{\tilde{q}_k}$ takes only the values ± 1 $(\beta \in \mathbf{R})$. For $k \geq 1$, let $n_k = \tilde{q}_1 + \cdots + \tilde{q}_k$ and define $f_1 = y_{\tilde{q}_1}$ and $f_k = \mathrm{R}_{n_{k-1}\alpha}y_{\tilde{q}_k}$ (thus $f_k(x) = \pm 1$ and $f_k^2(x) = 1$ for every $x \in \mathbf{T}$ and every $k \geq 1$). By (19), one has $y_{n_k} = \sum_{j=1}^{k} f_j$.

Let $(\delta_k)_{k\geq 1} \subset \mathbf{R}_0^+$ be such that $\sum_{j=1}^k \delta_j/\sqrt{k} \to 0$ as $k \to \infty$. One may suppose that, for every $k \geq 1$, and for every $\gamma \in [-1,1]$

$$|\int_0^1 f_{n_{k+1}} e^{i\gamma(f_{n_1}+\cdots+f_{n_k})}\mathrm{d}x| \leq \delta_k. \tag{20}$$

Indeed, for some $m(k) \in \mathbf{N}$, one may write $[0,1] = \bigcup_{j=1}^{m(k)} \mathrm{I}_j$, in such a way that $e^{i\gamma(f_{n_1}+\cdots+f_{n_k})}$ is constant on each I_j $(1 \leq j \leq m(k))$. But, by Lemma 3, one may suppose that $|\int_{\mathrm{I}_j} f_{n_{k+1}}\mathrm{d}x| \leq \delta_k/m(k)$ $(1 \leq j \leq m(k))$; indeed one just needed to take $\tilde{q}_{k+1}$ large enough.

Let $\lambda \in \mathbf{R}$. For $k \geq 2$ and for $2 \leq j \leq k$, one has

$$e^{i\frac{\lambda}{\sqrt{k}}(f_{n_1}+\cdots+f_{n_j})} = e^{i\frac{\lambda}{\sqrt{k}}(f_{n_1}+\cdots+f_{n_{j-1}})}(1 + i\frac{\lambda}{\sqrt{k}}f_{n_j} - \frac{\lambda^2}{2k} + \mathcal{O}\frac{|\lambda|^3}{k^{3/2}}). \tag{21}$$

For $k \geq 1$ big enough, one has $|\lambda/\sqrt{k}| \leq 1$ and $|1 - \frac{\lambda^2}{2k}| \leq 1$. Therefore, using (21) recursively, and applying (20), one finds that

$$|\int_0^1 e^{i\frac{\lambda}{\sqrt{k}}(f_{n_1}+\cdots+f_{n_k})}\mathrm{d}x - (1 - \frac{\lambda^2}{2k})^k| \leq \frac{|\lambda|}{\sqrt{k}}\sum_{j=1}^{k-1}\delta_j + \mathcal{O}\frac{|\lambda|^3}{\sqrt{k}}.$$

So, for each $\lambda \in \mathbf{R}$, $\int_0^1 e^{i\frac{\lambda}{\sqrt{k}}y_{n_k}}\mathrm{d}x \to e^{-\lambda^2/2}$ as $k \to \infty$. $\square$

We now give an heuristic link between Proposition 1 and the theory of expanding maps of the interval [12]. If $k \geq 2$ is an integer, one defines the map $\mathrm{T}_k : \mathbf{T} \to \mathbf{T}, x \mapsto \overline{kx}$ (with the notation $\overline{x} = x - \lfloor x \rfloor$). For $n \geq 1$, $\mathrm{T}_k^n = \mathrm{T}_k \circ \cdots \circ \mathrm{T}_k = \mathrm{T}_{k^n}$.

Let ψ_* be the function given by (5). If $k \geq 2$, $(\psi_* \circ \mathrm{T}_k^n)_{n\geq 1}$ is a sequence of random variables on $(\mathbf{T}, m_{\mathrm{L}})$. One shows that there exists $\sigma_k > 0$ such that

$$\frac{1}{\sqrt{n}}\sum_{j=1}^n \psi_* \circ \mathrm{T}_k^j = \frac{1}{\sqrt{n}}\sum_{j=1}^n \psi_* \circ \mathrm{T}_{k^j} \overset{\mathrm{D}}{\to} \mathrm{G}(\sigma_k) \quad \text{as} \quad n \to \infty. \tag{22}$$

Indeed, if k is even, the random variables $(\psi_* \circ \mathrm{T}_k^n)_{n\geq 1}$ are actually independent and equidistributed (so $\sigma_k = 1$). In general, one may use Theorem 5 of [12] : one checks that T_k is mixing with respect to the invariant measure m_{L}, and that the equation $u \circ \mathrm{T}_k - u = \psi_*$ admits no solution $u \in \mathrm{L}^2(\mathbf{T}, \mathbf{R})$ (by Fourier expansion for example), so that $\sigma_k > 0$.

Let us now consider the sequence $(f_k)_{k\geq 1}$ constructed in the proof of Proposition 1 (we keep the notations of this part). One has $f_k = \mathrm{R}_{n_{k-1}\alpha} y_{\tilde{q}_k}$ $(k \geq 2)$. First, one may expect the rotation $\mathrm{R}_{n_{k-1}\alpha}$ to play no essential role in the decorrelation properties of the variables f_k $(k \geq 1)$. Next, the proof of Lemma 4 was entirely based on the fact that $y_{\tilde{q}_k}$ may be approximated by $\psi_* \circ \mathrm{T}_{\tilde{q}_k}$. For each irrational number α, the sequence $(q_k)_{k\geq 0}$ grows at least exponentially with k (a superexponential growth improves actually the decorrelations).

One comes thus to the conclusion that the sequence $y_{n_k}/\sqrt{k} = \sum_{j=1}^k f_j/\sqrt{k}$ is likely to have a statistical behavior analogous to (22). The proof of Proposition 1 was greatly simplified by the fact that one allowed $\tilde{q}_k$ to grow arbitrarily fast with k. In the two next Sections, we prove basically that an exponential growth is enough in some cases.

3 Central Limit Theorem

Let μ be a probability measure on $\mathbf{T}$. In this Section, $\mathrm{L}^p(\mathbf{T},\mathbf{R}) = \mathrm{L}^p(\mathbf{T},\mathbf{R},\mathrm{d}\mu)$ $(p \geq 1)$.

Proposition 3 *Let $(q_k)_{k\geq 1} \subset \mathbf{N}_0$, and suppose there exists $\rho > 0$ such that for every $k \geq 1$,*

$$q_{k+1} \geq e^{2\rho} q_k. \tag{23}$$

Let $(f_{jk})_{j,k\geq 1} \subset \mathrm{BV}(\mathbf{T},\mathbf{R})$ be random variables on $(\mathbf{T},\mu)$ such that $\int_0^1 f_{jk}\mathrm{d}\mu = 0$ $(j,k \geq 1)$. Let $\mathrm{S}_n = f_{n1} + \cdots + f_{nn}$. Suppose that there exists $\mathrm{C} > 0$ such that

1) *for every $j,k \geq 1$, $||f_{jk}||_{\mathrm{L}^\infty} \leq \mathrm{C}$ and $\mathrm{Var}(f_{jk}) \leq \mathrm{C}q_k$,*

2) *for some $\beta \in \mathbf{R}$, for every $\phi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ such that $\int_0^1 \phi\mathrm{d}\mu = 0$, for $j \geq 1$ and for $t \geq s \geq 1$,*

$$|\int_0^1 \phi . f_{js}\mathrm{d}\mu| \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{s^\beta}{q_s}, \tag{24}$$

$$|\int_0^1 \phi . f_{js} f_{jt}\mathrm{d}\mu| \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{s^\beta}{q_s}, \tag{25}$$

3) *if $\sigma_n = ||\mathrm{S}_n/\sqrt{n}||_{\mathrm{L}^2}$, there exists $\epsilon > 0$ such that $\epsilon \leq \sigma_n \leq \mathrm{C}$ for each $n \geq 1$.*

Then, $\mathrm{S}_n/\sqrt{n}\sigma_n \overset{\mathrm{D}}{\to} \mathrm{G}(1)$ as $n \to \infty$.

Proof. To simplify notations, one will only consider the case where $f_{jk} = f_{lk}$ for all $j,l,k \geq 1$, and write $f_k = f_{1k}$ $(k \geq 1)$. The hypotheses of the Proposition only involve estimates which are independent of the index j of f_{jk} $(j,k \geq 1)$. Therefore, the proof of the general case is a straightforward adaptation of this one. We begin by a Lemma.

Lemma 5 *Under the hypothesis of Proposition 3, there exists $\mathrm{C} > 0$ such that for every $m,n \geq 1$, $||\sum_{k=m}^{m+n} f_k||_{\mathrm{L}^4} \leq \mathrm{C}(n\ln(n+m))^{1/2}$.*

Proof. One has

$$||\sum_{k=m}^{m+n} f_k||_{\mathrm{L}^4}^4 = \sum_{m\leq s,t,u,v\leq m+n} \int_0^1 f_s f_t f_u f_v \mathrm{d}\mu \leq 4! \sum_{m\leq s\leq t\leq u\leq v\leq m+n} |\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu|. \tag{26}$$

Until the end of this proof, one assumes $m \leq s \leq t \leq u \leq v \leq m+n$. One defines

$$\mathrm{S}_{stuv} = |\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu| \quad \text{and} \quad \mathrm{S}_{tu} = \sum_{s,v} \mathrm{S}_{stuv}. \tag{27}$$

By hypothesis 1), one has always $\mathrm{S}_{stuv} \leq \mathrm{C}$. Let us obtain two others estimates of this quantity. If $f,g \in \mathrm{BV}(\mathbf{T},\mathbf{R})$, one has $\mathrm{Var}(fg) \leq ||f||_{\mathrm{L}^\infty}\mathrm{Var}(g) + ||g||_{\mathrm{L}^\infty}\mathrm{Var}(f)$. Therefore, by 1), one has

$$\mathrm{Var}(f_s f_t f_u) \leq \mathrm{C}(q_s + q_t + q_u) \leq 3\mathrm{C}q_u.$$

By (24) and (23), one first obtains that

$$\mathrm{S}_{stuv} = |\int_0^1 (f_s f_t f_u).f_v \mathrm{d}\mu| \leq \mathrm{C}\frac{q_u}{q_v} v^\beta \leq \mathrm{C}e^{-2\rho(v-u)} v^\beta. \tag{28}$$

Similarly, by (25), (24) and (23), one then gets

$$\begin{aligned} \mathrm{S}_{stuv} &\leq |\int_0^1 \Big(f_s f_t - \int_0^1 f_s f_t \mathrm{d}\mu\Big).f_u f_v \mathrm{d}\mu| + |\int_0^1 f_s.f_t \mathrm{d}\mu|.|\int_0^1 f_u f_v \mathrm{d}\mu| \\ &\leq \mathrm{C}(\frac{q_t}{q_u}u^\beta + \frac{q_s}{q_t}t^\beta) \leq \mathrm{C}(e^{-2\rho(u-t)}u^\beta + e^{-2\rho(t-s)}t^\beta). \end{aligned} \tag{29}$$

Set $\kappa = (\beta/\rho)\ln(n+m)$. One assumes $\kappa \geq 1$. Inequalities (28) and (29) imply respectively

$$\mathrm{S}_{stuv} \leq \mathrm{C}e^{-\rho(v-u)} \quad \text{if} \quad v-u \geq \kappa, \tag{30}$$

$$\mathrm{S}_{stuv} \leq \mathrm{C}(e^{-\rho(u-t)} + e^{-\rho(t-s)}) \quad \text{if} \quad u-t \geq \kappa \quad \text{and} \quad t-s \geq \kappa. \tag{31}$$

We now estimate S_{tu} in (26) for fixed t, u. First we consider the case $u - t < \kappa$. By (30) one gets (setting $v - u = k$ after the second inequality)

$$\begin{aligned} \mathrm{S}_{tu} &= \sum_{s,v:v-u<\kappa} \mathrm{S}_{stuv} + \sum_{s,v:v-u\geq\kappa} \mathrm{S}_{stuv} \\ &\leq \sum_{s,v:v-u<\kappa} \mathrm{C} + \sum_{s,v:v-u\geq\kappa} \mathrm{C}e^{-\rho(v-u)} \leq \sum_{k<\kappa,s} \mathrm{C} + \sum_{k\geq\kappa,s} \mathrm{C}e^{-\rho k} \leq \mathrm{C}n\kappa + \mathrm{C}n \leq \mathrm{C}n\kappa. \end{aligned} \tag{32}$$

Next, we consider the case $u - t \geq \kappa$. We write the decomposition $\mathrm{S}_{tu} = \mathrm{S}_{tu}(1) + \mathrm{S}_{tu}(2) + \mathrm{S}_{tu}(3)$. Those three terms will be defined one by one. First, in the same way as (32), one gets

$$\mathrm{S}_{tu}(1) \triangleq \sum_{s,v:u-t\leq v-u} \mathrm{S}_{stuv} \leq \sum_{s,v:u-t\leq v-u} \mathrm{C}e^{-\rho(v-u)} \leq \mathrm{C}ne^{-\rho(u-t)}. \tag{33}$$

Then, if $v - u < u - t \leq t - s$, one uses (31) to get $\mathrm{S}_{stuv} \leq \mathrm{C}e^{-\rho(u-t)}$, and so

$$\mathrm{S}_{tu}(2) \triangleq \sum_{s,v:v-u<u-t\leq t-s} \mathrm{S}_{stuv} \leq \mathrm{C}e^{-\rho(u-t)} \sum_{k<u-t,s} 1 \leq \mathrm{C}n(u-t)e^{-\rho(u-t)}. \tag{34}$$

Finally, let $\mathrm{B} = \{(s,v) : v-u < u-t, t-s < u-t\}$. For at most κ^2 elements $(s,v) \in \mathrm{B}$, one has $v - u < \kappa$ and $t - s < \kappa$, and so only the estimate $\mathrm{S}_{stuv} \leq \mathrm{C}$. For all the others, one has $\mathrm{S}_{stuv} \leq \min\{\mathrm{C}e^{-\rho(v-u)}, \mathrm{C}e^{-\rho(t-s)}\}$ by (30) and (31) (to use (31) one uses the fact that $t - s < u - t$). Therefore,

$$\mathrm{S}_{tu}(3) = \sum_{s,v\in\mathrm{B}} \mathrm{S}_{stuv} \leq \mathrm{C}\kappa^2 + \mathrm{C}\sum_{j,k\geq 0} \min\{e^{-\rho j}, e^{-\rho k}\} \leq \mathrm{C}\kappa^2 + \mathrm{C} \leq \mathrm{C}\kappa^2. \tag{35}$$

By (26 - 27) and (32 - 35) one obtains

$$||\sum_{k=m}^{m+n} f_k||_{\mathrm{L}^4}^4 \leq \mathrm{C}\sum_{t,u:u-t<\kappa} n\kappa + \mathrm{C}\sum_{t,u}\Big(ne^{-\rho(u-t)} + n(u-t)e^{-\rho(u-t)} + \kappa^2\Big).$$

Because $\kappa = (\beta/\rho)\ln(n+m)$, this gives the result. $\square$

Let us now come to the proof of the Proposition. Like in [4], one defines a kind of coarse grained variables that get more and more decorrelated as $n \to \infty$. Let $\gamma_1, \gamma_2 \in]0,1[$ be such that $\gamma_1 > \gamma_2 + 1/2$ (and thus $\gamma_2 < 1/2$). If $n \geq 1$, set $n_1 = \lfloor n^{\gamma_1} \rfloor$ and $n_2 = \lfloor n^{\gamma_2} \rfloor$. In the sequel, we suppose that n is large enough to have $n_2 \geq 1$. One writes $\mathrm{S}_n = \sum_{k=1}^{p(n)}(\mathrm{X}_{nk} + \mathrm{Y}_{nk})$ where $p(n)$ is the smallest integer such that $p(n).(n_1 + n_2) \geq n$ and where

$$\begin{aligned} \mathrm{X}_{nk} &= f_{(k-1)(n_1+n_2)+1} + \cdots + f_{(k-1)(n_1+n_2)+n_1}, && (36) \\ \mathrm{Y}_{nk} &= f_{(k-1)(n_1+n_2)+n_1+1} + \cdots + f_{k(n_1+n_2)} && (37) \end{aligned}$$

($1 \leq k \leq p(n) - 1$; for $k = p(n)$ the definition is the same but one puts 0 instead of f_j whenever $j > n$). One has $p(n)/n^{1-\gamma_1} \to 1$ as $n \to \infty$.

Let $\lambda \in \mathbf{R}$. This number will be treated as a constant in all our estimates. For $n \geq 1$, one defines

$$\mathrm{J}_n(\lambda) = \int_0^1 e^{i\frac{\lambda}{\sqrt{n}\sigma_n}\mathrm{S}_n}\mathrm{d}\mu, \tag{38}$$

and, for $1 \leq k \leq p(n)$,

$$\mathrm{I}_{nk}(\lambda) = \int_0^1 e^{i\frac{\lambda}{\sqrt{n}\sigma_n}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{nk})}\mathrm{d}\mu. \tag{39}$$

One puts also $\mathrm{I}_{n0}(\lambda) = 1$. By hypothesis 3) and (37), one has

$$|\mathrm{J}_n(\lambda) - \mathrm{I}_{np(n)}(\lambda)| \leq ||\frac{\lambda}{\sqrt{n}\sigma_n}\sum_{k=1}^{p(n)}\mathrm{Y}_{nk}||_{\mathrm{L}^\infty} \leq \mathrm{C}\frac{p(n)n_2}{\sqrt{n}} \leq \mathrm{C}n^{-(\gamma_1-(\gamma_2+1/2))}. \tag{40}$$

Lemma 6 *Under the hypothesis of Proposition 3 and if $1 \leq k \leq p(n)$, one has*

$$\mathrm{I}_{nk}(\lambda) = (1 - \frac{\lambda^2}{2n\sigma_n^2}\int_0^1 \mathrm{X}_{nk}^2\mathrm{d}\mu)\mathrm{I}_{n(k-1)}(\lambda) + r_{nk}(\lambda)$$

with $|r_{nk}(\lambda)| \leq \mathrm{C}(n^{\beta+1}e^{-2\rho n_2} + n^{-\frac{3}{2}(1-\gamma_1)}\ln^{3/2} n)$ (where C depends on λ).

Proof. Let us only consider the most difficult case $k \geq 2$. To simplify formulas, one will assume that $\sigma_n = 1$ for all $n \geq 1$. By hypothesis 3), this does not change our estimates. One has

$$\mathrm{I}_{nk}(\lambda) = \int_0^1 (1 + i\frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk} - \frac{\lambda^2}{2n}\mathrm{X}_{nk}^2 + \mathcal{O}\frac{|\mathrm{X}_{nk}|^3}{n^{3/2}})e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu. \tag{41}$$

If $g \in \mathcal{C}^1(\mathbf{R},\mathbf{R}) \cap \mathrm{L}^\infty(\mathbf{R},\mathbf{R})$, and if $u \in \mathrm{BV}(\mathbf{T},\mathbf{R})$, then $\mathrm{Var}(g \circ u) \leq ||g'||_{\mathrm{L}^\infty}\mathrm{Var}(u)$. By (23), one has $q_1 + \cdots + q_n \leq \mathrm{C}q_n$. So, by (36) and hypothesis 1), one has

$$\mathrm{Var}(e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}) \leq \frac{\mathrm{C}}{\sqrt{n}}q_{(k-2)(n_1+n_2)+n_1}. \tag{42}$$

Therefore, first, using (36), (24), (42) and (23), one gets

$$\begin{aligned} |\int_0^1 \frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk}e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu| &\leq \frac{\lambda}{\sqrt{n}}\sum_{j=1}^{n_1}|\int_0^1 e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}.f_{(k-1)(n_1+n_2)+j}\mathrm{d}\mu| \\ &\leq \frac{\mathrm{C}}{n}\sum_{j=1}^{n_1}\frac{q_{(k-2)(n_1+n_2)+n_1}}{q_{(k-1)(n_1+n_2)+j}}((k-1)(n_1+n_2)+j)^\beta \\ &\leq \frac{\mathrm{C}}{n}n_1e^{-2\rho n_2}n^\beta \leq \mathrm{C}n^\beta e^{-2\rho n_2}. \end{aligned} \tag{43}$$

Then, similarly, using (25) instead of (24), and noticing that nor X_{nk}^2 nor $e^{i\frac{\lambda}{\sqrt{n}}(X_{n1}+\cdots+X_{n(k-1)})}$ have in general a zero integral, one gets

$$|\int_0^1 \frac{\lambda^2}{2n} X_{nk}^2 e^{i\frac{\lambda}{\sqrt{n}}(X_{n1}+\cdots+X_{n(k-1)})} d\mu - \frac{\lambda^2}{2n}\int_0^1 X_{nk}^2 d\mu . \int_0^1 e^{i\frac{\lambda}{\sqrt{n}}(X_{n1}+\cdots+X_{n(k-1)})} d\mu| \leq Cn^{\beta+1}e^{-2\rho n_2}. \quad (44)$$

Finally, by Lemma 5 and (36), one has

$$\begin{aligned} |\int_0^1 \frac{|X_{nk}|^3}{n^{3/2}} e^{i\frac{\lambda}{\sqrt{n}}(X_{n1}+\cdots+X_{n(k-1)})} d\mu| &\leq \frac{C}{n^{3/2}}||X_{nk}||_{L^3}^3 \leq \frac{C}{n^{3/2}}||X_{nk}||_{L^4}^3 \leq \frac{C}{n^{3/2}}(n_1 \ln n)^{3/2} \\ &\leq Cn^{-\frac{3}{2}(1-\gamma_1)} \ln^{3/2} n. \end{aligned} \quad (45)$$

Inserting (43-45) in (41), one gets the result. □

To prove Proposition 3, it is enough to show that $J_n(\lambda) \to e^{-\lambda^2/2}$ as $n \to \infty$. For n large enough, one has $|1-(\lambda^2/2n)\int_0^1 X_{nk}^2 dx| \leq 1$. Thus, by (40) and by recursive application of Lemma 6, one has

$$|J_n(\lambda) - (1 - \frac{\lambda^2}{2n\sigma_n^2}\int_0^1 X_{np(n)}^2 d\mu)\dots(1-\frac{\lambda^2}{2n\sigma_n^2}\int_0^1 X_{n1}^2 d\mu)| \leq$$

$$Cp(n)(n^{\beta+1}e^{-2\rho n_2} + n^{-\frac{3}{2}(1-\gamma_1)}\ln^{3/2} n) + Cn^{-(\gamma_1-(\gamma_2+1/2))}.$$

Because $p(n)/n^{1-\gamma_1} \to 1$ as $n \to \infty$, the right hand side of this inequality goes to 0 as $n \to \infty$.

Therefore, it is enough to show that $\ln\prod_{k=1}^{p(n)}(1-(\lambda^2/2n)\int_0^1 X_{nk} d\mu) \to -\lambda^2/2$ as $n \to \infty$. By hypothesis 3), one has

$$\ln\prod_{k=1}^{p(n)}(1-\frac{\lambda^2}{2n\sigma_n^2}\int_0^1 X_{nk}^2 d\mu) = -\frac{\lambda^2}{2n\sigma_n^2}\sum_{k=1}^{p(n)}\int_0^1 X_{nk}^2 d\mu + \mathcal{O}\frac{1}{n^2}\sum_{k=1}^{p(n)}(\int_0^1 X_{nk}^2 d\mu)^2.$$

First, Lemma 5 is still valid if $||.||_{L^2}$ is used in place of $||.||_{L^4}$ (because $||.||_{L^2} \leq ||.||_{L^4}$), and so, by (36),

$$\frac{1}{n^2}\sum_{k=1}^{p(n)}(\int_0^1 X_{nk}^2 d\mu)^2 = \frac{1}{n^2}\sum_{k=1}^{p(n)}||X_{nk}||_{L^2}^4 \leq \frac{C}{n^2}p(n)n_1^2\ln^2 n \leq \frac{C}{n^2}n^{1-\gamma_1}n^{2\gamma_1}\ln^2 n \to 0 \quad \text{as} \quad n \to \infty.$$

Next, one has

$$\int_0^1 (\sum_{k=1}^{p(n)} X_{nk})^2 d\mu = \int_0^1 \sum_{k=1}^{p(n)} X_{nk}^2 d\mu + 2\sum_{j=2}^{p(n)}\sum_{k=1}^{j-1}\int_0^1 X_{nj}X_{nk} d\mu.$$

By (36) and hypothesis 1), $\mathrm{Var}(X_{nk}) \leq Cq_{(k-1)(n_1+n_2)+n_1}$. So, by (36), (24) and (23), one obtains as for (43) that, if $1 \leq k < j \leq p(n)$,

$$|\int_0^1 X_{nj}X_{nk} d\mu| \leq \sum_{l=1}^{n_1}|\int_0^1 X_{nk} f_{(j-1)(n_1+n_2)+n_1} d\mu| \leq Cn^\beta e^{-2\rho n_2}.$$

Therefore

$$\int_0^1 \sum_{k=1}^{p(n)} X_{nk}^2 d\mu - \int_0^1 (\sum_{k=1}^{p(n)} X_{nk})^2 d\mu \to 0 \quad \text{as} \quad n \to \infty.$$

By (37),

$$||\frac{1}{\sqrt{n}}\sum_{k=1}^{p(n)} Y_{nk}||_{L^2} \leq ||\frac{1}{\sqrt{n}}\sum_{k=1}^{p(n)} Y_{nk}||_{L^\infty} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $\sum_{k=1}^{p(n)}\int_0^1 X_{nk}^2 d\mu/n\sigma_n^2 - ||S_n||_{L^2}^2/n\sigma_n^2 \to 0$ as $n \to \infty$. One concludes by hypothesis 3). □

4 Proof of Proposition 2 and Corollary 1

Proof of Proposition 2. Let α be a number of constant type (see (9)), and let d be the constant given by (10). Let $(p_n/q_n)_{n\geq 0}$ be its convergents, and $(a_n)_{n\geq 0}$ be its partial quotients. One may decompose an integer $r \geq 1$ according to the *Ostrowski system of numeration.* One has $q_n \leq r < q_{n+1}$ for some $n \geq 0$. One writes

$$r = b_n q_n + \cdots + b_0 q_0, \tag{46}$$

where $b_k \geq 0$ $(0 \leq k \leq n)$ are integers defined recursively : first $b_n \geq 1$ is the only integer such that $0 \leq r - b_n q_n < q_n$; then, if $b_{k+1} \geq 0$ $(0 \leq k \leq n-1)$ is given, $b_k \geq 0$ is the only integer such that $0 \leq r - (b_n q_n + \cdots + b_k q_k) < q_k$. One has $b_k \leq a_{k+1}$, and by (10), $b_k \leq d$ $(1 \leq k \leq n)$.

Now let us consider a sequence $(s_n)_{n\geq 1}$ such that $q_n \leq s_n < q_{n+1}$. One wants to show that $z_{s_n}/||z_{s_n}||_{\mathrm{L}^2} \xrightarrow{\mathrm{D}} \mathrm{G}(1)$ as $n \to \infty$. All our estimates will be independent of the choice of the sequence $(s_n)_{n\geq 1}$. Therefore, this will actually imply that $z_n/||z_n||_{\mathrm{L}^2} \xrightarrow{\mathrm{D}} \mathrm{G}(1)$ as $n \to \infty$. Indeed, for $n \geq q_1$, one finds a sequence $(s_k)_{k\geq 1}$ with $q_k \leq s_k < q_{k+1}$ for every k, and such that $n = s_j$ for some $j \geq 1$; one has $j \to \infty$ as $n \to \infty$.

For each $n \geq 1$, there exists one and only one $1 \leq r_n \leq s_n$ such that $y_{r_n} = z_{s_n}$. Let $(b_k(n))_{1\leq k\leq n}$ be the sequence associated to the canonical decomposition (46) of r_n ($b_k(n)$ may not be defined for large k, one set then $b_k(n) = 0$). Let us define $(f_{jk})_{j,k\geq 1} \subset \mathrm{BV}(\mathbf{T}, \mathbf{R})$. Let $j \geq 1$. If $1 \leq k \leq j$, one sets

$$f_{jk} = \mathrm{R}_{(b_0(j)q_0+\cdots+b_{k-1}(j)q_{k-1})\alpha} \sum_{l=0}^{b_k(j)q_k-1} \mathrm{R}_{l\alpha}\psi. \tag{47}$$

If $k > j$, one sets $f_{jk} = 0$. By (19), one has $z_{s_n} = y_{r_n} = \sum_{1\leq k\leq n} f_{nk}$.

Therefore, it is enough to show that the sequences $(q_k)_{k\geq 1}$ and $(f_{jk})_{jk\geq 1}$ satisfy the hypotheses of Proposition 3. The estimates on $||z_n||_{\mathrm{L}^2}$ will then be also proven. Indeed, we will need to show that hypothesis 3) is satisfied, and thus that $\epsilon\sqrt{n} \leq ||z_{s_n}||_{\mathrm{L}^2} \leq \mathrm{C}\sqrt{n}$ (where $q_n \leq s_n < q_{n+1}$).

First, (23) is satisfied. One has $q_2 > q_1$, and, for $n \geq 2$, $q_{n-1} \geq q_n/2d$ by (10). Therefore, for $n \geq 2$,

$$q_{n+1} \geq q_n + q_{n-1} \geq (1 + 1/2d)q_n.$$

Next, hypothesis 1) is satisfied. By (47), (19), (10) and Denjoy-Koksma inequality (4), one has for all $j, k \geq 1$

$$||f_{jk}||_{\mathrm{L}^\infty} \leq b_k(j)||\sum_{l=0}^{q_k-1} \mathrm{R}_{l\alpha}\psi||_{\mathrm{L}^\infty} \leq d.\mathrm{Var}(\psi).$$

And, by (47) and (10), one has, for all $j, k \geq 1$

$$\mathrm{Var}(f_{jk}) \leq \mathrm{Var}(\psi)b_k(j)q_k \leq d.\mathrm{Var}(\psi)q_k.$$

Let us show that hypothesis 2) is satisfied.

Lemma 7 *If α is of constant type, then $\forall p \in \mathbf{N}$, $\exists c_0 \in \mathbf{N} : (\forall c \in \mathbf{N} : c \geq c_0)$, $\forall n \in \mathbf{N}$, $q_{cn} \geq q_n^p$.*

Proof. Let d be given by (10). For c large enough and $n \geq 1$, one has by (23), $q_{cn} \geq e^{2\rho cn} \geq (2d)^{pn} \geq q_n^p$. $\square$

Like in [3], one defines the sets

$$\Gamma_1 = \{j \in \mathbf{N}_0 : 1/q_1 \leq |1-e^{2i\pi j\alpha}|\}, \qquad \Gamma_n = \{j \in \mathbf{N}_0 : 1/q_n \leq |1-e^{2i\pi j\alpha}| < 1/q_{n-1}\} \quad (n \geq 2). \tag{48}$$

One has $\mathbf{N}_0 = \bigcup_{n\geq 1} \Gamma_n$ and $\Gamma_m \cap \Gamma_n = \emptyset$ if $m \neq n$.

Lemma 8 *There exists* $\mathrm{C} > 0$ *such that, for every* $m \geq 1$,

$$j \in \Gamma_m \Rightarrow j \geq \mathrm{C}q_m \quad \text{and} \quad j,k \in \bigcup_{n\geq m} \Gamma_n \Rightarrow |k-j| \geq \mathrm{C}q_m. \tag{49}$$

Proof. If $j \in \Gamma_m$, $|1-e^{2i\pi j\alpha}| < 1/q_{m-1}$. By (9) and (13), there exists $\mathrm{C} > 0$ such that $4\mathrm{C}/|j| \leq 4|j\alpha|_{\mathbf{T}} \leq 1/q_{n-1}$. Let d be given by (10). One has $q_{m-1} \geq q_m/2d$, and so $|j| \geq \mathrm{C}q_m$. Next, if $j,k \in \bigcup_{n\geq m} \Gamma_n$, one has by (14) that

$$|1-e^{2i\pi(k-j)\alpha}| \leq |1-e^{2i\pi k\alpha}| + |1-e^{2i\pi j\alpha}| \leq \frac{2}{q_{m-1}},$$

and so one gets the second inequality (49) as the first. $\square$

Lemma 9 *There exist* $\mathrm{C}, \beta \in \mathbf{R}$ *such that, for every* $\phi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ *with* $\int_0^1 \phi \mathrm{d}x = 0$, *for* $l \geq 1$ *and for* $t \geq s \geq 1$,

$$|\int_0^1 \phi f_{ls} \mathrm{d}x| \leq \mathrm{C.Var}(\phi)\frac{s^\beta}{q_s}, \qquad |\int_0^1 \phi f_{ls} f_{lt} \mathrm{d}x| \leq \mathrm{C.Var}(\phi)\frac{s^\beta}{q_s}. \tag{50}$$

Proof. Both inequalities may be shown in the same way, but the first one is simpler, and we will only prove the second one. In this proof, we consider as constants, numbers that depend only on α or ψ. Let $l \geq 1$ and $t \geq s \geq 1$. One writes

$$g = f_{ls} \quad \text{and} \quad h = f_{lt}.$$

Let $\phi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$. One has $|\hat{\phi}(k)| \leq \mathrm{Var}(\phi)/2\pi|k|$ if $k \in \mathbf{Z}_0$, and $\hat{\phi}(0) = 0$ by hypothesis. Let us first simplify the problem in two ways.

First, by Lemma 7, there exists $c \in \mathbf{N}$ such that $q_{cn} \geq q_n^4$ for every $n \in \mathbf{N}$. Let $\mathrm{P} : \mathrm{L}^2(\mathbf{T},\mathbf{R}) \to \mathrm{L}^2(\mathbf{T},\mathbf{R})$ be the projector defined by $\mathrm{P}u(x) = \sum_{k\leq q_{ct}} \hat{u}(k)e^{2i\pi kx}$. Let $\mathrm{Q} = \mathrm{Id} - \mathrm{P}$. One has $||\mathrm{Q}g||_{\mathrm{L}^2} \leq \mathrm{C}/q_t$ and $||\mathrm{Q}h||_{\mathrm{L}^2} \leq \mathrm{C}/q_t$; indeed, by hypothesis 1), one has for example

$$||\mathrm{Q}h||^2_{\mathrm{L}^2} \leq \frac{\mathrm{Var}^2(h)}{4\pi^2} \sum_{k:|k|\geq q_{ct}} \frac{1}{k^2} \leq \mathrm{C}\frac{q_t^2}{q_{ct}} \leq \frac{\mathrm{C}}{q_t^2}. \tag{51}$$

But $|\int_0^1 \phi gh \mathrm{d}x| \leq |\int_0^1 \phi \mathrm{P}g\mathrm{P}h \mathrm{d}x| + \int_0^1 |\phi \mathrm{Q}g\mathrm{P}h| \mathrm{d}x + \int_0^1 |\phi g \mathrm{Q}h| \mathrm{d}x$. Because $\int_0^1 \phi \mathrm{d}x = 0$, $||\phi||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\phi)$. By hypothesis 1) and (51), one has

$$\int_0^1 |\phi g \mathrm{Q}h| \mathrm{d}x \leq ||\phi||_{\mathrm{L}^\infty} ||g||_{\mathrm{L}^\infty} ||\mathrm{Q}h||_{\mathrm{L}^2} \leq \mathrm{C.Var}(\phi)\frac{1}{q_t},$$

and, because $||\mathrm{P}h||_{\mathrm{L}^2} \leq ||h||_{\mathrm{L}^2} \leq ||h||_{\mathrm{L}^\infty}$, one has similarly

$$\int_0^1 |\phi \mathrm{Q}g\mathrm{P}h|\mathrm{d}x \leq ||\phi||_{\mathrm{L}^\infty}||\mathrm{Q}g||_{\mathrm{L}^2}||\mathrm{P}h||_{\mathrm{L}^2} \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{1}{q_t}.$$

Therefore, it will suffice to estimate $|\int_0^1 \phi \mathrm{P}g\mathrm{P}h\mathrm{d}x|$ instead of $|\int_0^1 \phi g h \mathrm{d}x|$.

Next, let us prove that, if $t \geq c's$ for some $c' \in \mathbf{N}$, then the first inequality (50) implies the second one (the proof of the first inequality makes obviously no use of this fact). If $u, v \in \mathrm{BV}(\mathbf{T}, \mathbf{R})$, one has $\mathrm{Var}(uv) \leq ||u||_{\mathrm{L}^\infty}\mathrm{Var}(v) + ||v||_{\mathrm{L}^\infty}\mathrm{Var}(u)$. Therefore, using again the fact that $||\phi||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\phi)$, hypothesis 1), and the first inequality (50), one gets

$$|\int_0^1 (\phi g).h\mathrm{d}x| \leq \mathrm{C}(||g||_{\mathrm{L}^\infty}\mathrm{Var}(\phi) + \mathrm{Var}(\phi)\mathrm{Var}(h))\frac{t^\beta}{q_t} \leq \mathrm{C}\frac{\mathrm{Var}(\phi)}{q_s}(\frac{q_s t^\beta}{q_t} + \frac{q_s^2 t^\beta}{q_t}).$$

By Lemma 7, there exists $c' \in \mathbf{N}$ such that $q_{c'n} \geq q_n^2$ for every $n \in \mathbf{N}$. Therefore, if $t \geq c's$, one may write $t = c's + u$ with $u \geq 0$, and so, using (23),

$$\frac{q_s t^\beta}{q_t} + \frac{q_s^2 t^\beta}{q_t} \leq \frac{2q_{c's}(c's+u)^\beta}{q_{c's+u}} \leq \mathrm{C}\frac{(c's+u)^\beta}{e^{2\rho u}} \leq \mathrm{C}c'^\beta s^\beta \frac{(1+u/cs)^\beta}{e^{2\rho u}} \leq \mathrm{C}s^\beta.$$

One now comes to the proof itself (and one supposes $t < c's$). After some algebra one gets

$$\begin{aligned}
|\int_0^1 \phi \mathrm{P}g\mathrm{P}h\mathrm{d}x| &= |\sum_{j,k:|j|,|k|\leq q_{ct}} \hat{g}(j)\hat{h}(-k)\hat{\phi}(k-j)| \\
&\leq \frac{4\mathrm{Var}(\phi)}{2\pi}\sum_{1\leq j,k\leq q_{ct}, k\neq j} |\hat{g}(j)|.|\hat{h}(k)|.\frac{1}{|k-j|} + \frac{\mathrm{Var}(\phi)}{2\pi}\sum_{1\leq k\leq q_{ct}} \frac{|\hat{g}(k)|.|\hat{h}(k)|}{k} \\
&\triangleq \frac{\mathrm{Var}(\phi)}{2\pi}(4\mathrm{S}_1 + \mathrm{S}_2). \qquad (52)
\end{aligned}$$

Let us estimate S_1 given by (52). The set Γ_n ($n \geq 1$) are defined in (48). By (49) and (23), there exists a constant $w \geq 0$ such that, if $j \in \Gamma_{m+w}$, then $j > q_m$ ($m \geq 1$), and therefore

$$\mathrm{S}_1 = \sum_{1\leq m,n\leq ct+w}\ \sum_{\substack{j\in\Gamma_m, j\leq q_{ct}\\ k\in\Gamma_n, k\leq q_{ct}, k\neq j}} |\hat{g}(j)|.|\hat{h}(k)|.\frac{1}{|k-j|} \triangleq \sum_{1\leq m,n\leq ct+w} \mathrm{S}(m,n). \qquad (53)$$

Let us fix $m, n \in \{1, \ldots, ct+w\}$ and estimate $\mathrm{S}(m,n)$. Let us first consider the case $m \leq n$. By (47), (11), (15), (10) and (48), one has

$$|\hat{g}(j)| = |\widehat{f_{ls}}(j)| \leq \frac{|1 - e^{2i\pi b_s(l) q_s j\alpha}|}{|1 - e^{2i\pi j\alpha}|}\frac{\mathrm{Var}(\psi)}{2\pi j} \leq \frac{dj.|1 - e^{2i\pi q_s\alpha}|}{1/q_m}\frac{\mathrm{Var}(\psi)}{2\pi j} \leq \mathrm{C}\frac{q_m}{q_s}. \qquad (54)$$

By (47), (11), the fact that $|1 - e^{2i\pi x}| \leq 2$ for all $x \in \mathbf{R}$, and (48), one has

$$|\hat{h}(k)| = |\widehat{f_{lt}}(k)| \leq \frac{|1 - e^{2i\pi b_t(l) q_t k\alpha}|}{|1 - e^{2i\pi k\alpha}|}\frac{\mathrm{Var}(\psi)}{2\pi k} \leq \mathrm{C}\frac{q_n}{k}. \qquad (55)$$

Therefore, by (53), (54) and (55), and then (49), one has

$$\begin{aligned}
\mathrm{S}(m,n) &\leq \mathrm{C}\frac{q_m q_n}{q_s}\sum_{\substack{j\in\Gamma_m, j\leq q_{ct}\\ k\in\Gamma_n, k\leq q_{ct}, k\neq j}} \frac{1}{k|k-j|} \leq \mathrm{C}\frac{q_m q_n}{q_s}\sum_{k\in\Gamma_n, k\leq q_{ct}} \frac{1}{k}\sum_{j\in\Gamma_m, j\leq q_{ct}, j\neq k}\frac{1}{|k-j|} \\
&\leq \mathrm{C}\frac{q_m q_n}{q_s}\Big(\frac{\mathrm{C}}{q_n}\sum_{1\leq u\leq q_{ct}}\frac{1}{u}\Big).\Big(\frac{\mathrm{C}}{q_m}\sum_{1\leq u\leq q_{ct}}\frac{1}{u}\Big) \leq \frac{\mathrm{C}}{q_s}\ln^2 q_{ct}.
\end{aligned}$$

The case $m \geq n$ is analogous : one uses the estimates $|\hat{g}(j)| \leq \mathrm{C}q_m/j$ and $|\hat{h}(k)| \leq \mathrm{C}q_n/q_t$, to obtain $\mathrm{S}(m,n) \leq (\mathrm{C}/q_t)\ln^2 q_{ct}$. Therefore, one has $\mathrm{S}_1 \leq \mathrm{C}(ct+w)^2(\ln^2 q_{ct})/q_s$.

The sum S_2 is estimated in the same way. One gets $\mathrm{S}_2 \leq \mathrm{C}(ct+w)(\ln q_{ct})/q_s$. To get the result, one uses then the inequality $q_{ct} \leq (2d)^{ct}$, where d is given by (10), and one takes $\beta = 4$. $\square$

Let us show that hypothesis 3) is satisfied.

Lemma 10 *Let α be a number of constant type. Let $n \geq 0$ and $0 \leq m \leq q_n$. One has $||y_m||_{\mathrm{L}^2} \leq \mathrm{C}\sqrt{n}$.*

Proof. By Lemma 7, there exists $c \in \mathbf{N}$ such that $q_{cn} \geq q_n^2$ for every $n \in \mathbf{N}$. By (11), one has

$$\int_0^1 y_m^2 \mathrm{d}x \leq \frac{\mathrm{Var}^2(\psi)}{2\pi^2} \sum_{k=1}^{q_{cn}-1} \frac{1}{k^2} \frac{|1-e^{2i\pi mk\alpha}|^2}{|1-e^{2i\pi k\alpha}|^2} + \frac{1}{2\pi^2} \sum_{k \geq q_{cn}} \frac{\mathrm{Var}^2(y_m)}{k^2}. \tag{56}$$

Because $y_m = \sum_{j=0}^{m-1} \mathrm{R}_{j\alpha}\psi$ $(m \geq 1)$, one has $\mathrm{Var}(y_m) \leq \mathrm{Var}(\psi)m \leq \mathrm{Var}(\psi)q_n$, and thus the second term in (56) is bounded by a constant. The sets Γ_m $(m \geq 1)$ are defined in (48). By (49) and (23), there exists a constant $w \geq 0$ such that, if $j \in \Gamma_{m+w}$, then $j > q_m$ $(m \geq 1)$, and so

$$\sum_{k=1}^{q_{cn}-1} \frac{1}{k^2} \frac{|1-e^{2i\pi mk\alpha}|^2}{|1-e^{2i\pi k\alpha}|^2} \leq \sum_{m=1}^{cn+w} \sum_{k \in \Gamma_m} \frac{1}{k^2} \frac{2}{|1-e^{2i\pi k\alpha}|^2} \leq \sum_{m=1}^{cn+w} \sum_{k \in \Gamma_m} \frac{2q_m^2}{k^2} \leq \mathrm{C} \sum_{m=1}^{cn+w} \sum_{u \geq 1} \frac{2q_m^2}{u^2 q_m^2} \leq \mathrm{C}n. \quad \square$$

One has $\mathrm{S}_n = z_{s_n} = y_m$ for some $m \leq q_{n+1}$, and therefore, by Lemma 10, $||\mathrm{S}_n||_{\mathrm{L}^2} \leq \mathrm{C}\sqrt{n+1} \leq \mathrm{C}\sqrt{n}$. The estimate $||\mathrm{S}_n||_{\mathrm{L}^2} \geq \epsilon\sqrt{n}$ is obtain by the next Lemma.

Lemma 11 *Under the hypotheses of Proposition 2, there exits $\epsilon > 0$ such that $||z_{s_n}||_{\mathrm{L}^2} \geq \epsilon\sqrt{n}$ for every $n \geq 1$.*

Proof. Let $\delta > 0$. By (23), there exists $l \geq 1$ such that, for each $n \geq 0$, $q_n/q_{n+l} \leq \delta$. Therefore, for each $k \geq 0$, $q_n/q_{n+kl} \leq \delta^k$. Now let us construct a sequence $(t_n)_{n\geq 1} \subset \mathbf{N}$. For $n \geq 1$, $t_n = 0$ except in the following cases. If $n = 4kl$ for some $k \geq 1$, then, by point 4) of Lemma 1, there exists $m \in \{n, \dots, n+3\}$ such that q_m is odd and that $q_m|q_m\alpha|_{\mathbf{T}} < 1/2$; one sets then $t_m = q_m$ (and one takes the smallest m if there is more than one possibility).

Let us fix $n \geq 1$. Because, $q_0 + \cdots + q_n \leq q_{n+3}$, one has $t_1 + \cdots + t_n \leq q_{n+3}$. Therefore, if $r = t_1 + \cdots + t_n$, one has $||z_{s_{n+3}}||_{\mathrm{L}^2} \geq ||z_{q_{n+3}}||_{\mathrm{L}^2} \geq ||y_r||_{\mathrm{L}^2}$. But, by (11), (5) and (8), one has

$$\begin{aligned} ||y_r||_{\mathrm{L}^2}^2 &= \frac{8}{\pi^2} \sum_{k \geq 1, k\,\mathrm{odd}} \frac{1}{k^2} \frac{|1-e^{2i\pi rk\alpha}|^2}{|1-e^{2i\pi k\alpha}|^2} \geq \frac{8}{\pi^2} \sum_{1 \leq u \leq n : q_u \mathrm{odd}} \frac{1}{q_u^2} \frac{|1-e^{2i\pi rq_u\alpha}|^2}{|1-e^{2i\pi q_u\alpha}|^2} \\ &\geq \mathrm{C} \sum_{1 \leq u \leq n : q_u \mathrm{odd}} |1-e^{2i\pi rq_u\alpha}|^2 \geq \mathrm{C} \sum_{u=1}^{n} |rt_u\alpha|_{\mathbf{T}}^2. \end{aligned}$$

Let $u \in \{1, \dots, n\}$ be such that $t_u \neq 0$ (and thus $t_u = q_u$). One writes $rt_u\alpha = t_u^2\alpha + \tau$, where $\tau = (t_1 + \cdots + t_{u-1})t_u\alpha + t_u(t_{u+1}\alpha + \cdots + t_n\alpha)$. By (14), one has $|\tau|_{\mathbf{T}} \leq (t_1 + \cdots + t_{u-1})|t_u\alpha|_{\mathbf{T}} + t_u(|t_{u+1}\alpha|_{\mathbf{T}} + \cdots + |t_n\alpha|_{\mathbf{T}})$. Let us now adopt the convention that $1/t_n = 0$ when $t_n = 0$. One has

$$|\tau|_{\mathbf{T}} \leq \frac{t_1}{q_u} + \cdots + \frac{t_{u-1}}{q_u} + \frac{q_u}{t_{u+1}} + \cdots + \frac{q_u}{t_n} \leq 2\sum_{k=1}^{\infty} \delta^k. \tag{57}$$

So, taking δ small enough (and thus l big enough), $|\tau|_{\mathbf{T}}$ can be made arbitrarily small.

If $p \in \mathbf{N}$, if $x \in \mathbf{R}$ and if $p|x|_{\mathbf{T}} \leq 1/2$, then $|px|_{\mathbf{T}} = p|x|_{\mathbf{T}}$. Therefore, if d is given by (10), one has by (8), and because $t_u|t_u\alpha|_{\mathbf{T}} < 1/2$ by construction, that

$$|t_u^2\alpha|_{\mathbf{T}} = q_u|q_u\alpha|_{\mathbf{T}} \geq \frac{q_u}{q_u + q_{u+1}} \geq \frac{1}{3d}.$$

Therefore, there exists $\mathrm{C} > 0$ such that $|rt_u\alpha|^2_{\mathbf{T}} \geq \mathrm{C}$, and so $||y_r||^2_{\mathrm{L}^2} \geq \mathrm{C}n/l$. □

This ends the proof of Proposition 2. □

Proof of Corollary 1. For j large enough, one has $n_j \geq 1$, and one may thus find $\tau(j) \in \mathbf{N}$ such that $q_{\tau(j)} \leq n_j < q_{\tau(j)+1}$. Now, by Lemma 10 and the hypothesis on $(\sigma_j)_{j\geq 1}$, one has $||y_{n_j}/\sigma_j\sqrt{j}||_{\mathbf{L}^2} \leq \mathrm{C}\sqrt{(\tau(j)+1)/j}$. Because $y_{n_j}/\sigma_j\sqrt{j} \overset{\mathrm{D}}{\to} \mathrm{G}(1)$, there has to be a number $\mathrm{C} > 0$ such that $\tau(j) \geq \mathrm{C}j$ for j large enough. The result follows from the fact that $n_j \geq q_{\tau(j)}$ and that $(q_n)_{n\geq 0}$ grows exponentially with n. □

Acknowledgments. I am very grateful to Professors J. Bricmont and A. Kupiainen for useful discussions and comments. I thank Belgian Interuniversity Attraction Poles Program for financial support.

References

[1] B. Adamczewski, *Répartition des suites* $(n\alpha)_{n\in\mathbf{N}}$ *et substitutions*, Acta Arithmetica, **112**, 1-22 (2004).

[2] V. Baladi, *Positive Transfer Operators and Decay of Correlations*, Advanced Series in Nonlinear Dynamics, **16**, World Scientific, River Edge, 2000.

[3] J. Bricmont, K. Gawedzki, A. Kupiainen, *KAM Theorem and Quantum Field Theory*, Communications in Mathematical Physics, **201** (3), 699-727 (1999).

[4] L.A. Bunimovich, Ya.G. Sinai, *Statistical properties of Lorentz gas with a periodic configuration of scatterers*, Communication in Mathematical Physics, **72**, 479-497 (1981).

[5] R. Burton, M. Denker, *On the central limit theorem for dynamical systems*, Transactions of the American Mathematical Society, **302** (2), 715-726 (1987).

[6] T. De La Rue, S. Ladouceur, G. Peskir, M. Weber, *On the Central Limit Theorem for Aperiodic Dynamical Systems and Applications*, Theory of Probability and Mathematical Statistics, **57**, 140-159 (1997).

[7] C.P. Dettmann, E.G.D. Cohen, *Microscopic chaos and diffusion*, Journal of Statistical Physics, **101** (3-4), 775-817 (2000).

[8] M. Drmota, R.F. Tichy, *Sequences, Discrepancies and Applications*, Lecture Notes in Mathematics **1651**, Springer-Verlag, Berlin, 1997.

[9] P. Gaspard, M.E. Briggs, M.K. Francis, J.V. Sengers, R.W. Gammon, J.R Dorfman, R.V. Calabrese, *Experimental evidence for microscopic chaos*, Nature, **394**, 865-868 (1998).

[10] G.H. Hardy, E.M. Wright, *An introduction to the theory of numbers*, 4th edition, Oxford University Press, Oxford, 1960.

[11] M.R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations*, Publications mathématiques de l'I.H.E.S., **49**, 5-233 (1979).

[12] F. Hofbauer, G. Keller, *Ergodic Properties of Invariant Measures for Piecewise Monotonic Transformations*, Mathematische Zeitschrift, **180**, 119-140 (1982).

[13] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*, J. Wiley, New York, 1974.

[14] P. Liardet, D. Volný, *Sums of continuous and differentiable functions in dynamical systems*, Israel Journal of Mathematics, **98**, 29-60 (1997).