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Hamiltonian tools for the analysis of optical polarization control

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The study of the polarization dynamics of two counterpropagating beams in optical fibers has recently been the subject of a growing renewed interest, from both the theoretical and experimental points of view. This system exhibits a phenomenon of polarization attraction, which can be used to achieve a complete polarization of an initially unpolarized signal beam, almost without any loss of energy. Along the same way, an arbitrary polarization state of the signal beam can be controlled and converted into any other desired state of polarization, by adjusting the polarization state of the counterpropagating pump beam. These properties have been demonstrated in various different types of optical fibers, i.e., isotropic fibers, spun fibers, and telecommunication optical fibers. This article is aimed at providing a rather complete understanding of this phenomenon of polarization attraction on the basis of new mathematical techniques recently developed for the study of Hamiltonian singularities. In particular, we show the essential role that play the peculiar topological properties of singular tori in the process of polarization attraction. We provide here a pedagogical introduction to this geometric approach of Hamiltonian singularities and give a unified description of the polarization attraction phenomenon in various types of optical fiber systems. © 2012 Optical Society of America

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1. INTRODUCTION

Achieving the repolarization of an optical field without loss is a fundamental physical phenomenon that can find a large variety of applications in telecommunication optical systems. In 2000, Heebner et al. proposed a “universal polarizer” performing polarization of unpolarized light with 100% efficiency, in contrast with conventional polarizers that unavoidably waste 50% of unpolarized light [1]. By using a photorefractive two-beam coupling, it was shown in Ref. [1] that unpolarized light can be converted to a polarized state with essentially a unit efficiency. This phenomenon has been termed “lossless polarization attraction,” in the sense that all input polarization configurations are transformed into a particular polarization state, without any loss of energy. This kind of polarization attraction has been also identified in different nonlinear systems [2] and, in particular, in an optical fiber system pumped by two counterpropagating beams [3]. This latter phenomenon has been the subject of a growing interest these last years, from both the theoretical [4–12] and experimental [3,13–16] points of view. It finds its origin in pioneering studies of polarization dynamics of optical beams that counterpropagate in optical fibers and whose nonlinear interaction is mediated by the Kerr effect [17–27]. The recent works on polarization attraction in optical fibers reveal that the signal wave can be attracted toward a particular state of polarization (SOP) via a suitable choice of the injected SOP of the counterpropagating pump beam. An efficient polarization attraction has been shown to occur in different types of optical fibers, such as isotropic fibers (IFs) [3,5,6,13], highly birefringent spun fibers (HBSF) [8,11], as well as randomly birefringent fibers (RBFS) used in optical telecommunication systems [9,11,14]. It is important to note that these phenomena of polarization attraction exhibit different properties that depend on the characteristics of the considered optical fiber. More precisely, when one considers IFs and RBFS, polarization attraction can take place either on a specific SOP, or on an ensemble of distinct SOPs [6,11]. On the other hand, in HBSFs, the attraction process has been recently shown to also occur on a specific line of polarization states that lie on the surface of the Poincaré sphere [11]. This remarkable and unexpected result is illustrated in Fig. 1. We considered here 64 different input states of polarization of the signal uniformly distributed over the Poincaré sphere (green points), while the SOP of the counterpropagating pump is kept fixed at the fiber output (yellow point). We see that all SOPs of the signal are attracted toward a specific line of polarization states at the fiber output (red points).

So far, most of the theoretical works aimed at describing this process of polarization attraction have focused on the derivation of the equations governing the evolution of the counterpropagating beams in different types of optical fibers, as well as on intensive numerical simulations of their spatiotemporal dynamics. However, little has been done in order to provide a theoretical description of the phenomenon of polarization attraction. Our aim in this article is to make a step in this direction by introducing a new set of tools that find their origin in recently developed mathematical techniques, in particular for the study of Hamiltonian singularities [28,29]. We have successfully used this mathematical approach in Refs. [5,6] to study polarization attraction in IFs. We have recently extended this work to the polarization control in HBSFs and RBFS, the latter ones being used in optical
telecommunication systems \cite{11}. In this article we provide a
unified geometrical description of these phenomena of polariza-
tion attraction. In particular, we show the essential role that
the peculiar topological properties of singular tori play
in the process of polarization attraction. This role is similar
to the one played by the separatrix in purely one-dimensional
systems \cite{30}. We remark that the existence of these singular
structures had been essentially ignored in the physics litera-
ture until their recent introduction in the domain of atomic
and molecular systems \cite{29,31,32}. The aim of the present
article is to render these new mathematical tools accessible to
a broad audience in the context of nonlinear optics. In this
respect, this paper can also be viewed as a pedagogical intro-
duction to this geometric approach of Hamiltonian singulari-
 ties, which may subsequently help the interested reader to
enter into a more specialized literature of Hamiltonian
dynamics \cite{28,29,33}.

The paper is organized as follows. In Sec. 2, we study the
Hamiltonian structure of the stationary states of the spatio-
temporal equations of wave propagation in nonlinear optics.
In particular, we show that the corresponding Hamiltonian
dynamics is integrable for the three different examples of
optical fibers considered in this article. Section 3 is devoted
to the derivation of the properties of the different singular
tori, which allow a geometrical description of the station-
ary states. We show that the existence and the position of
these singularities crucially depend on the characteristics
of the optical fiber. In Sec. 4, we present the phenomenon of
polarization attraction and the key role played by the
singular tori. Some conclusions and discussions are pre-
sented in Sec. 5. Finally, considering the example of IFs,
we show in the Appendix how a stationary soliton solution
can be explicitly determined on the surface of a singular
torus.

2. INTEGRABLE HAMILTONIAN ON THE
SPHERE

The aim of this section is to introduce the specific Hamiltonian
structure required to describe the stationary solutions of
the phenomenon of polarization attraction in optical fibers.
Before entering into the details, we underline that this math-
ematical approach is based on the following observation.
Numerical simulations have revealed that, under rather gen-
eral conditions, the spatiotemporal dynamics of the counter-
propagating optical beams relaxes, after a complex transient,
toward a stationary state. The theoretical approach discussed
in this article provides a geometrical study of these stationary
states. It reveals that they lie in the neighborhood of a singular
torus of the corresponding phase-space representation. We
shall see that it is the peculiar topological property inherent
to the singular torus that enables a complete control of the
polarization process.

The Hamiltonian formalism in a flat phase space (such as
\( \mathbb{R}^2 \)) is well known and has been widely employed in nonlinear
optics in a large variety of systems. However, here the phase-
space representation of the dynamics of polarization is the
Poincaré sphere, which refers to the nonstandard phase space
\( S^2 \times S^2 \), i.e., a given point in this phase space corresponds
to two vectors, \( \vec{S} \) and \( \vec{J} \) on the signal and pump Poincaré spheres.
We have thus to adapt the conventional Hamiltonian formal-
ism used in a flat phase space so as to use it with a spherical
gometry.

The equations governing the polarization dynamics of the
counterpropagating beams can be written in the following
general form \cite{8,9,27}:

\[
\begin{align*}
\frac{\partial \vec{S}}{\partial t} &= \vec{\mathbf{S}} \times (\vec{I}_s \vec{S}) + \vec{\mathbf{S}} \times (\vec{I}_p \vec{J}) \\
\frac{\partial \vec{J}}{\partial t} &= \vec{\mathbf{J}} \times (\vec{I}_s \vec{J}) + \vec{\mathbf{J}} \times (\vec{I}_s \vec{S})
\end{align*}
\]

(1)

where \( \xi \) is the spatial coordinate along the fiber. The Stokes
vectors \( \vec{S} = (S_x, S_y, S_z) \) and \( \vec{J} = (J_x, J_y, J_z) \) describe,
respectively, the polarization states of the forward and backward
beams on the Poincaré sphere and “x” denotes the vector product.
The matrices \( \vec{I}_s \) and \( \vec{I}_p \) are diagonal and their coeffi-
cients depend on the type of fibers considered \cite{8,9}. The radii of the forward and backward spheres, \( S_b \) and \( J_b \), are
the signal and pump powers, which are conserved quantities
of the spatiotemporal dynamics. For convenience, we normal-
ized the problem with respect to the characteristic nonlinear
time \( \tau_0 = 1/(\gamma J_0) \) and length \( L_0 = \nu \tau_0 \), where \( \gamma \) is the non-
linear Kerr coefficient and \( \nu \) the group-velocity of light in
the optical fiber.

The introduction of the cylindrical coordinates

\[
\begin{align*}
S_x &= \sqrt{S_0^2 - I_0^2} \cos \varphi_f \\
S_y &= \sqrt{S_0^2 - I_0^2} \sin \varphi_f \\
S_z &= I_f \\
J_x &= \sqrt{J_b^2 - I_b^2} \cos \varphi_b \\
J_y &= \sqrt{J_b^2 - I_b^2} \sin \varphi_b \\
J_z &= -I_b
\end{align*}
\]

allows us to define the standard Hamiltonian structure that makes use of the conventional Poisson brackets defined by

\[
\{ g_1, g_2 \} = \sum_{\alpha, \beta} \left( \frac{\partial g_1}{\partial \varphi_\alpha} \frac{\partial g_2}{\partial \varphi_\beta} - \frac{\partial g_1}{\partial \varphi_\beta} \frac{\partial g_2}{\partial \varphi_\alpha} \right).
\]
With this definition we have the following relations, \( \{ I_{b}, I_{f} \} = \{ \phi_{b}, \phi_{f} \} = 0 \) and \( \{ \phi_{f}, I_{b} \} = \delta_{bf} \). In these coordinates the dynamics can be described by the usual Hamiltonian structure with a function \( H(I_{b}, \phi_{b}, I_{f}, \phi_{f}) \) and the well-known Hamilton equations where \( \xi \) plays here the role of time of the classical dynamical system:

\[
\begin{align*}
\frac{\partial I_{b}}{\partial t} &= \{ I_{b}, H \} = -\frac{\partial H}{\partial \phi_{b}}, \\
\frac{\partial \phi_{b}}{\partial t} &= \{ \phi_{b}, H \} = \frac{\partial H}{\partial I_{b}}.
\end{align*}
\] (2)

However, these cylindrical coordinates are not globally defined, in the sense that when the polarization is circular, i.e., when \( I_{b}^{2} \) (respectively \( I_{f}^{2} \)) is equal to \( S_{b}^{2} \) (respectively \( J_{b}^{2} \)), then the angle \( \phi_{b} \) (respectively \( \phi_{f} \)) is not defined. It turns out that the Stokes variables on the Poincaré sphere are globally defined variables. The Poisson brackets can be expressed in the Stokes coordinates in the following form:

\[
\{ S_{x}, S_{y} \} = \frac{\partial S_{x}}{\partial \xi} \frac{\partial S_{y}}{\partial \eta} - \frac{\partial S_{x}}{\partial \eta} \frac{\partial S_{y}}{\partial \xi} = S_{x} \quad \text{and} \quad \ldots \quad \{ J_{x}, J_{y} \} = \frac{\partial J_{x}}{\partial \xi} \frac{\partial J_{y}}{\partial \eta} - \frac{\partial J_{x}}{\partial \eta} \frac{\partial J_{y}}{\partial \xi} = -J_{y},
\]

and we obtain by a circular permutation of the indices, the relations \( \{ S_{x}, S_{y} \} = \epsilon_{ijk} S_{k} \) and \( \{ J_{x}, J_{y} \} = -\epsilon_{ijk} J_{k} \), \( \epsilon_{ijk} \) being the completely antisymmetric tensor, i.e., it changes sign under the exchange of any pair of indices. Note that the minus sign in the \( J \)-equation is due to the counterpropagation of the two waves. These expressions of the Poisson brackets on the sphere in the Stokes coordinates are valid for any point of the sphere, which is not the case for the cylindrical coordinates.

The corresponding expression of the Hamiltonian also needs to be determined in the Stokes coordinates. We recall that, for a given Hamiltonian \( H(\vec{S}, \vec{J}) \), the dynamics of any functional (polynomial) \( G(\vec{S}, \vec{J}) \) is given by \( \partial G = (H \cdot G) \). Accordingly, the stationary system associated to Eq. (1) has the form

\[
\frac{dS_{i}}{dt} = \{ S_{i}, H \} \quad \text{and} \quad \frac{dJ_{i}}{dt} = \{ J_{i}, H \}.
\] (3)

A constant of motion \( K \) is independent of the time coordinate \( \xi \) and is therefore characterized by the fact that it Poisson commutes with the Hamiltonian, i.e., \( \{ K, H \} = 0 \). From Eq. (3) and the above definition of the Poisson brackets on the sphere, it is straightforward to check that the Hamiltonian

\[
H = -\vec{S} \cdot \vec{J} \cdot \frac{1}{2} (\vec{S} \cdot \vec{S} + \vec{J} \cdot \vec{J})
\] (4)

describes the stationary system of Eq. (1). We will now apply this formalism to three different models: the IF, the RBF, and the HBSF models.

**Isotropic fiber.** The diagonal matrices read \( \mathcal{I}_{s} = \text{diag}(-1, -1, 0) \) and \( \mathcal{I}_{b} = \text{diag}(-2, -2, 0) \). The stationary system is given by

\[
\begin{align*}
\frac{\partial S_{x}}{\partial t} &= S_{2}S_{y} + 2S_{y}J_{y}, \\
\frac{\partial S_{y}}{\partial t} &= -S_{2}S_{x} - 2S_{x}J_{x}, \\
\frac{\partial S_{z}}{\partial t} &= 2J_{y}S_{y} - 2S_{y}J_{y}, \quad \text{and} \\
\frac{\partial J_{x}}{\partial t} &= -J_{y}J_{y} - 2J_{y}S_{y}, \\
\frac{\partial J_{y}}{\partial t} &= J_{y}J_{x} + 2J_{y}S_{x}, \\
\frac{\partial J_{z}}{\partial t} &= 2J_{y}J_{y} - 2S_{y}J_{y}.
\end{align*}
\]

Up to an additive constant, the Hamiltonian thus takes the form

\[
H = 2(S_{y}J_{y} + S_{x}J_{x}) \cdot \frac{1}{2} (S_{2}^{2} + J_{2}^{2}).
\] (5)

and we also remark that \( K = S_{z} - J_{z} \) is a constant of motion since \( \{ H, K \} = 0 \).

**Randomly birefringent fiber.** The diagonal matrices read \( \mathcal{I}_{s} = \text{diag}(-1, 0, 0) \) and \( \mathcal{I}_{b} = \text{diag}(-1, -1, -2) \). The stationary system can be written as

\[
\begin{align*}
\frac{\partial S_{x}}{\partial t} &= -S_{0}J_{x} - S_{0}J_{y}, \\
\frac{\partial S_{y}}{\partial t} &= -S_{0}J_{x} + S_{0}J_{y}, \\
\frac{\partial S_{z}}{\partial t} &= J_{0}S_{x} + J_{0}S_{y}, \\
\frac{\partial J_{x}}{\partial t} &= S_{0}J_{x} + J_{0}S_{y}, \\
\frac{\partial J_{y}}{\partial t} &= -S_{0}J_{x} - J_{0}S_{y}, \\
\frac{\partial J_{z}}{\partial t} &= -J_{0}S_{x} - J_{0}S_{y}.
\end{align*}
\]

The Hamiltonian becomes

\[
H = S_{x}J_{x} - S_{y}J_{y} - S_{z}J_{z},
\] (6)

and it Poisson commutes with the three constants of motion \( K_{1} = S_{x} + J_{x}, K_{2} = S_{y} - J_{y}, \) and \( K_{3} = S_{z} + J_{z} \).

**Highly birefringent spun fiber.** The diagonal matrices read \( \mathcal{I}_{s} = \text{diag}(0, 0, \beta) \) and \( \mathcal{I}_{b} = \text{diag}(1, -1, -2) \), where \( \alpha = \cos^{2} \varphi, \beta = 2 \sin^{2} \varphi - \cos^{2} \varphi \), and \( \varphi \) is the ellipticity of the fiber [8]. The stationary system reads

\[
\begin{align*}
\frac{\partial S_{x}}{\partial t} &= a(S_{y}J_{y} - 2S_{0}J_{z}) + \beta S_{x}S_{y}, \\
\frac{\partial S_{y}}{\partial t} &= a(S_{x}J_{x} + 2S_{0}J_{z}) - \beta S_{x}S_{y}, \\
\frac{\partial S_{z}}{\partial t} &= -a(J_{y}S_{x} + J_{0}S_{y}), \quad \text{and} \\
\frac{\partial J_{x}}{\partial t} &= a(2S_{y}J_{y} - S_{0}J_{z}) - \beta J_{y}S_{y}, \\
\frac{\partial J_{y}}{\partial t} &= -a(2S_{x}J_{x} + S_{0}J_{z}) + \beta J_{y}S_{y}, \\
\frac{\partial J_{z}}{\partial t} &= a(J_{y}S_{x} + J_{0}S_{y}).
\end{align*}
\]

leading to the Hamiltonian

\[
H = a(S_{y}J_{y} - S_{0}J_{z} + 2S_{x}J_{x}) \cdot \frac{\beta}{2} (S_{2}^{2} + J_{2}^{2})
\] (7)

that Poisson commutes with \( K = S_{z} + J_{z} \).

These three models are Liouville-integrable [34], i.e., they possess at least as many invariants as the dimension of the phase space divided by two. In our study the phase space
is $S^2 \times S^2$, which is described locally by $(I_x, \varphi_1, I_y, \varphi_2)$, and each model discussed here above has at least two invariants, $H$ and $K$. One of the important consequences of the integrability is the fact that the dynamics curls around a torus in the phase space. This means that there exists a set of action-angle coordinates $(J, \psi, K, \chi)$ such that the two actions are constant on the corresponding torus and the dynamics is described only by the two angles $(\psi, \chi)$. In these coordinates, the two actions are associated to the two radii of a torus and the two angles to the angles needed to describe a trajectory on the surface of the torus. This torus can be either regular or singular, as illustrated in Fig. 2. The regular torus generically leads to an oscillating behavior of the dynamics, while the singular torus describes more specific trajectories that exhibit a “monotonous” behavior. In this respect a singular torus can be regarded as a two-dimensional generalization of the well-known separatrix in purely one-dimensional systems. The singular tori play a key role in the phenomenon of polarization attraction and the next section is devoted to their characterization.

3. CONSTRUCTION OF THE SINGULAR TORI

In order to study the Hamiltonian systems established in the previous section, we introduce here the singular reduction technique. Its objective is to reduce the number of dimensions of the system by using the constant of motion $K$, which gives information on the nature of the singularities of the system. As we will see, this analysis provides all the material required to understand the polarization attraction phenomenon in the different types of optical fiber systems. A simple introduction to the reduction theory is presented below. We refer the reader to more specialized mathematical books for a more rigorous mathematical presentation of the theory [28].

In simple terms, the reduction theory can be viewed as a change of variables that exploits the constant of the motion $K$ to reduce the dimensionality of the problem. The lower dimensional phase space obtained by the reduction theory will be called reduced phase space. We reported in Fig. 3 a schematic representation of the original phase space $S^2 \times S^2$ and of the reduced phase space. The new variables of the reduced phase space need to be defined globally and in a way as simple as possible. This explains why we first reduce the dimension of the system with respect to the constant $K$, which has a simpler form than $H$. The new coordinates have, by definition, a dynamic that belongs to a given $K$-invariant subspace, which means that they should Poisson commute with $K$. These coordinates called invariant polynomials are denoted by $(x_0, x_1, x_2, x_3)$. Note that, in general, such coordinates are not constants for the $H$ dynamics, but only for the $K$ dynamics defined by

$$\frac{dS}{dz} = \{S, K\} \quad \text{and} \quad \frac{dJ}{dz} = \{J, K\}. \quad (8)$$

which is just another way to express the fact that they Poisson commute with $K$. The dynamical system defined by Eq. (8) is different from the physical system of interest. We may just consider it as a procedure to visualize the influence of the constant of motion $K$ on the dynamics.

**Isotropic fiber.** Let us begin to illustrate the reduction technique with the example of the IF system. Equation (8) and the constant of motion $K = S_z - J_z$ lead to the differential system

$$\begin{align*}
S_y &= S_x, \\
S_x &= -S_y, \\
S_z &= 0.
\end{align*} \quad \begin{align*}
J_y &= J_x, \\
J_x &= -J_y, \\
J_z &= 0.
\end{align*} \quad (9)$$

Fig. 2. (Color online) Examples of tori of an integrable Hamiltonian phase space of dimension four. A system can freely oscillate around a regular (standard) torus (a), but its evolution can also be blocked by the presence of a pinched point in a singly (b) or doubly (c) pinched torus. The singular pinched tori can be viewed as a two-dimensional generalization of the concept of separatrix, well-known for systems with 1 degree of freedom. A bitorus and a curled torus are represented in (d) and (e). They can be constructed by gluing two regular tori along a circle, with an additional twist in the case of a curled torus.

Fig. 3. (Color online) Schematic illustration of the reduction process, which maps the main phase space, i.e., the two Poincaré spheres, $S^2 \times S^2$, toward the reduced phase space, which has the form of a deformed sphere with two conical singularities. The reduced phase space is defined by Eq. (10) for the IF and Eq. (12) for the RBF and HBSF.
These equations simply describe circles in the main phase space: they correspond to a simultaneous rotation of $\bar{S}$ and $\bar{J}$ around the $z$ axis with the same angular velocity. The idea of the reduction theory is to get rid of this trivial circular dynamic by projecting it to a point in the reduced phase space, as illustrated in Fig. 3 and Fig. 5. In order to determine the coordinates of the reduced phase space, we have to consider variables that are invariant under Eq. (9), i.e., such that they Poisson commute with $K$. It can be shown that the reduced phase space can be expressed in terms of four coordinates $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ with a relation between them, which leads, as expected, to a three-dimensional space. $K$ being the Hamiltonian of the system (9), it can be chosen as one of the invariant polynomials, $\bar{x}_0$. The other three can be found by analyzing Eq. (9). It is straightforward to check that a possible set of invariant polynomials is

$$
\begin{align*}
\bar{x}_0 &= K = S_z - J_z, \\
\bar{x}_1 &= S_z + J_z, \\
\bar{x}_2 &= S_y J_z, \\
\bar{x}_3 &= S_y J_y - S_j J_x.
\end{align*}
$$

The new coordinates obey the relation

$$
\bar{x}_3 + \left(\bar{x}_2 + \frac{1}{4}(\bar{x}_0^2 - 2\bar{x}_1^2)\right)^2
- \left(S_z^2 - \frac{1}{4}(\bar{x}_0 + \bar{x}_1)^2\right)\left(J_z^2 - \frac{1}{4}(\bar{x}_0 - \bar{x}_1)^2\right) = 0, \quad (10)
$$

with the constraint $-S_z - J_z \leq \bar{x}_1 \leq S_z + J_z$, which simply originates in the definitions of the Stokes coordinates. It can be shown that the variables $(\bar{x}_0, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ constitute a basis for the set of polynomial functions commuting with $K^2$. In particular, since $\{H, K\} = 0$, one can write the Hamiltonian as

$$
H = 2\bar{x}_2 + \frac{1}{4}\bar{x}_0^2 - \frac{3}{4}\bar{x}_1^2, \quad (11)
$$

For each fixed $K$, Eq. (10) defines a surface in the space $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$, which is called the reduced phase space. It is depicted in red in Figs. 4 and 5. For each value of $(H, K)$, Eq. (11) defines a second surface, called Hamiltonian surface, and is represented in blue in Fig. 4. The intersection of these two surfaces gives the trajectories of the system. To recover the dynamics in the main phase space from the dynamics in the reduced phase space, we have to make the Cartesian product of the reduced dynamics by the circle that comes from Eq. (9). For example, if the dynamics follow a circle in the reduced phase space, a Cartesian product with a second circle leads to a regular torus [see Fig. 2(a)]. The fact that each point of the reduced phase space corresponds to a circle in the main phase space holds true as long as the derivative on the reduced phase space is continuous, i.e., as long as the surface is smooth. This is illustrated in Fig. 4. The surface is plotted with $K = 0$ and $S_0 = J_0$, thus $S_z = J_z$. The two nonsmooth points are of coordinates $(\bar{x}_1 = \pm 2, \bar{x}_2 = 1, \bar{x}_3 = 0)$. This entails $S_z = J_z = S_0$ or $S_z = J_z = -S_0$. If we report this result in Eq. (9), we see that the motion is no longer circular but stationary, i.e., it refers to a point. This clearly illustrates the following important aspect: when a point of the reduced phase space is not smooth, then this point does not correspond to a circle in the main phase space $S^2 \times S^2$, but to a stationary point.

Let us illustrate this fact with the concrete example of the IF, in which, for simplicity, we consider the case $S_0 = J_0 = 1$. The surfaces of Eq. (10) and Eq. (11) are represented in Fig. 4 for $(H = -1, K = 0)$. The intersection contains two nonsmooth points $(\bar{x}_1 = \pm 2, \bar{x}_2 = 1, \bar{x}_3 = 0)$; thus, the...
corresponding set in the original phase space is a singular doubly pinched torus (see Fig. 5). One can plot these two surfaces for different values of \((H, K)\), and verify that there is no other isolated singularity. These results are summarized in the so-called energy-momentum diagram of Fig. 6. Such a diagram is the ensemble of all the possible values of \(H\) and \(K\) for some given fixed values of the radii \(S_0\) and \(J_0\) of the Stokes spheres. It provides a global view of the stationary states of the system. Following the previous geometric analysis, and according to the Liouville–Arnold theorem, any point of this set will be associated either to a regular or to a singular torus.

The equation of the boundary of this diagram can be obtained from the property that, at the boundary, the intersection of the reduced phase space and of the Hamiltonian surface reduces to a point. The computation goes as follows. First, one notices that the symmetry of the two surfaces implies that such an intersection occurs when \(x_3 = x_1 = 0\). We then express from Eq. (10) the polynomial \(x_3^2(x_1)\) and we compute its roots:

\[
x_1 = \pm \sqrt{\frac{2S_0^4 - S_0^2 K^2 - K^2 x_2 - 2 x_2^2}{S_0^2 - x_2}}.
\]

Finally, using Eq. (11) and the fact that \(x_1 = 0\) we get

\[
H = 2S_0^2 - \frac{3}{4} K^2 \quad \text{and} \quad H = -2S_0^2 + \frac{1}{4} K^2.
\]

Note that for the IF model this diagram presents only one isolated singularity. This interesting property has a direct influence of the polarization attraction phenomenon, as explained in the next section.

Randomly birefringent fiber: We now analyze the example of RBFs by following the same procedure as that outlined here above for IFs. In the case of RBFs, we need to choose a constant of motion among the three available ones in order to carry out the reduction process. Note that this arbitrary choice has no consequence on the properties of the energy-momentum diagram. The choice \(K_1 = S_z + J_z\) and Eq. (8), leads to

\[
\left\{ \begin{array}{l}
\dot{S}_x = -S_y \\
\dot{S}_y = S_x \\
\dot{S}_z = 0
\end{array} \right.; \quad \left\{ \begin{array}{l}
\dot{J}_x = J_y \\
\dot{J}_y = -J_x \\
\dot{J}_z = 0
\end{array} \right.
\] (12)

and the following polynomials are constant under this dynamic:

\[
\left\{ \begin{array}{l}
x_0 = K_3 - S_z + J_z \\
x_1 = S_z - J_z \\
x_2 = S_0^2 J_y - S_x J_y \\
x_3 = S_x J_y + S_0^2 J_x
\end{array} \right.
\]

These invariant polynomials fulfill the relation:

\[
x_3^2 + x_2^2 + \left( S_0^2 - \frac{1}{4} (x_0 + x_1)^2 \right) \left( J_0^2 - \frac{1}{4} (x_0 - x_1)^2 \right) = 0.
\] (13)

The Hamiltonian can be rewritten in the form:

\[
H = \frac{1}{4} (x_0^2 - x_1^2) - x_2.
\] (14)

Here again, we consider \(S_0 = J_0\), which leads to a unique value of \((H, K_3) = (-S_0^2, 0)\) for which the intersection is not smooth. Furthermore, if we report this value in Eqs. (13) and (14), both give the relation \(x_2 = S_0^2 - x_0^2/4\), i.e., the intersection is a segment, as can be seen in Fig. 8. The two edges of the segment are nonsmooth points of the reduced phase space and correspond therefore to points in the main phase space, while the others are regular and are thus associated to circles in the main phase space. By collecting the different results, we finally obtain a singular torus, which is topologically equivalent to a sphere. Indeed, the two edges of the intersection produce the two poles of the sphere and the other points of the segment correspond to the different parallel circles of the sphere (see Fig. 8). This sphere plays the same role as the singular torus for the IF case.

Let us now construct the energy-momentum diagram by using the same symmetry as for the case of IFs, \(x_3 = x_1 = 0\). Straightforward computations lead to the diagram of Fig. 7, whose boundary is determined by the equation

\[
H = \frac{K_3^2}{2} - S_0^2 \quad \text{and} \quad H = S_0^2.
\]

A distinguished feature of this diagram is the fact that the singular torus is on the border of the domain. This property will be shown to influence the phenomenon of polarization attraction.

Highly birefringent spun fiber: In this case, the Hamiltonian depends on a real parameter \(\varphi\), which gives rise to a family of reduction procedures. The constant of motion \(K = S_z + J_z\) is the same as that used for RBFs. Accordingly, one can use the same invariant polynomials for this system, which thus leads to the same reduced phase space. The Hamiltonian now reads

\[
H = \alpha \left( x_2^2 + \frac{x_0^2}{2} \right) - \frac{\beta}{4} (x_0^2 + x_1^2).
\] (15)
Let us consider again the case $S_0 = J_0$. The intersection of the reduced phase space and the Hamiltonian surface is reported in Fig. 9. It exhibits an eight-shaped intersection, which corresponds in the main phase space to a bitorus [see Fig. 2(4)], which is the union of two tori glued along a circle. Indeed, if we rotate this eight-shaped intersection around the circle of Eq. (12) in a three-dimensional space, we obtain a bitorus. The upper torus of the bitorus is drawn by the upper loop of the eight, and the same goes for the lower part of the eight.

We now discuss the energy-momentum diagram. In contrast with the examples of the IF and the RBF, whose energy-momentum diagrams were characterized by an isolated singularity, here the diagram is characterized by a continuous line of singular bitori. In other terms, we have an infinite number of values of $(H, K)$ that produce a bitorus. These values draw a line in the $(H, K)$ plane and it is possible to derive the equation of this line explicitly. Here again the symmetries of the problem lead to $x_3 = x_1 = 0$. We first compute the roots of the polynomial $x_2^2(x_1)$:

$$x_1 = \pm \sqrt{K^2 + 4S_0^2} \pm 4\sqrt{S_0^2K^2 + x_2^2}.$$

Then we use $x_1 = 0$ and Eq. (15) to eliminate $x_2$ and after a few steps of simple calculations we obtain

$$\begin{align*}
H &= \epsilon \sqrt{\alpha^2 \left(\frac{K^2}{\alpha^2} + S_0^2\right)^2 - S_0^2K^2} + \frac{\alpha^2}{\epsilon} - \frac{\alpha^2}{\epsilon} \\
H &= -2aS_0^2 - S_0^2\beta - \frac{\beta^2}{\epsilon} + |\epsilon| \sqrt{4\alpha^2S_0^2 + S_0^2\beta^2 + 3S_0^2\alpha^2},
\end{align*}$$

(16)

where $\epsilon = \pm 1$. The first equation of (16) provides the red line ($\epsilon = -1$) and the upper ($\epsilon = +1$) blue line of Fig. 10, while the second relation provides the lower blue lines. The main line of singularities in the energy-momentum (red line in Fig. 10) diagram can be simplified into

$$H = \cos^2 \varphi (K^2 - S_0^2) - \frac{\sin^2 \varphi}{2} K^2.$$

(17)
4. POLARIZATION ATTRACTION

In this section we study the polarization attraction phenomenon in the light of the singular reduction theory exposed above for the three examples of fiber systems.

Isotropic fiber. This case has been the subject of a detailed study in [6] and [10], respectively, for the phenomenon of polarization attraction and the issue of the soliton stability. We thus refer the reader to these two articles for more details. Here, we illustrate how the previously introduced mathematical tools can be applied to analyze the process of polarization attraction. First, we consider the following change of variables:

\[
\begin{align*}
S_x &= \sqrt{S_0^2 - I_f^2} \cos \varphi_f, \\
S_y &= \sqrt{S_0^2 - I_f^2} \sin \varphi_f, \\
S_z &= I_f,
\end{align*}
\]

\[
\begin{align*}
J_x &= \sqrt{J_0^2 - I_b^2} \cos \varphi_b, \\
J_y &= \sqrt{J_0^2 - I_b^2} \sin \varphi_b, \\
J_z &= -I_b,
\end{align*}
\]

to rewrite the Hamiltonian in the following form

\[
H = 2\sqrt{(S_0^2 - I_f^2)(S_0^2 - I_b^2)} \cos(\varphi_f - \varphi_b) - \frac{I_f^2 + I_b^2}{2}.
\] (18)

Next, we denote by \(J_z(L) = \epsilon\) the ellipticity of the pump wave injected into the fiber at \(z = L\). We assume that the spatiotemporal system relaxes toward the only singularity of the energy-momentum diagram (see Fig. 6), thus \(K = S_z - J_z = 0\) and \(H = -S_0^2/2\). This implies that the signal is attracted toward the same ellipticity as the pump wave \(S_z(L) = \epsilon\). Finally, we use \(H = -S_0^2/2\) to calculate the orientation of the polarization ellipse to which the signal is attracted, i.e., the angle \(\varphi_f(L)\) in the plane \(S_z = \epsilon\):

\[
(S_0^2 - \epsilon^2)(2 \cos(\varphi_f - \varphi_b) + 1) = 0.
\]

We have to consider two different cases. If the polarization of the pump is circular (\(\epsilon = J_b\)), then the signal is attracted toward the same circular polarization state. On the other hand, if the polarization of the pump is elliptic, then the orientation of the polarization ellipse of the signal is related to that of the pump by

\[
\varphi_f = \varphi_b \pm \frac{2\pi}{3}.
\] (19)

This result was confirmed by the numerical simulations of the spatiotemporal Eqs. (1) in Ref. [6].

Randomly birefringent fiber. The application of the preceding mathematical tools to the polarization attraction in the RBFs was briefly discussed in Ref. [11]. Here, we summarize the main results obtained in [11] and also discuss the influence of the powers of the beams on the robustness of the attraction process (note that, with the adopted normalization it is equivalent to increase the beam power or the fiber length).

We first consider the case \(S_0 = J_b\). Using the reduction theory of Sec. 3, we know by symmetry of the directions that, on the singular torus, we have \(K_1 = K_2 = K_3 = 0\). It is then straightforward to deduce the polarization attraction relations: \(S_1(L) = -J_1(L), S_2(L) = J_2(L),\) and \(S_3(L) = -J_3(L)\). This theoretical prediction was also confirmed by the numerical simulations of the spatiotemporal Eqs. (1) in Ref. [11]. Note that the main difference between the RBF and IF is the fact that the singular torus lies on the boundary of the energy-momentum diagram (see Fig. 7). This strongly influences the efficiency of the attraction process, as illustrated by the comparison of the distance of the stationary solution to the singular torus for IFs and RBFs. For example, the average distance \(\rho = \sqrt{(H + 1)^2 + K^2}\) is \(10^{-2}\) in the RBF (with \(L = 15\)), whereas it is \(10^{-7}\) in the IF case (with \(L = 5\)). Furthermore, when one considers IFs, the distance \(\rho\) decreases exponentially with the fiber length \(L\) [5]. Conversely, for RBFs, the distance \(\rho\) also decreases as \(L\) increases, but in a slower way. This is illustrated in Fig. 11. It can be interpreted by considering the fact that the singular torus lies on the boundary of the energy-momentum diagram for RBFs, while it is located inside the diagram for IFs. Then, contrary to IFs, convergence toward the singular torus cannot occur in an isotropic way in RBFs, which limits the accuracy of the polarization attraction phenomenon.

Finally note that a power difference \((S_0 - J_b)\) can strongly affect the attraction process. We performed numerical simulations with different values of \(\Delta = (S_0 - J_b)/S_0\). The numerical study reveals that for \(\Delta\) in the range of few percents, the system still relaxes toward a stationary state, as illustrated in Fig. 12. For larger values of \(\Delta\), the spatiotemporal dynamics no longer converge toward a stationary state. Note that there is not a clear threshold on \(\Delta\). When \(\Delta\) increases we...
numerically observe that the solution with \( \tilde{S}(t = 0, z = 0) \) close to \( \tilde{J}(t = 0, z = 0) \) cannot relax toward a stationary state while all the others do. When \( \Delta \) is large enough there is no stationary state, for any initial condition.

**Highly birefringent span fiber.** The application of the theoretical tools exposed in the preceding section to the process of polarization attraction in HBSFs was briefly discussed in Ref. [11]. Here, we complete our results and also suggest a possible application of this system.

From the theoretical point of view, the case of HBSFs is fundamentally different from IFs or RBFs. Indeed, as explained in Sec. 3, instead of an isolated singularity, the energy-momentum diagram of HSBF exhibits a continuous line of singularities. The direct consequence of this property is the fact that the signal beam is no longer attracted toward a single SOP, but instead toward a line of polarization states that lie on the surface of the Poincaré sphere.

Let us illustrate this result by considering the particular values \( \vartheta = \pi/4 \) and \( |J_z(z = L)| = 0, J_y(z = L) = 1, J_z(z = L) = 0| \). Indeed, in this particular case, one can obtain a simple analytical expression of the line of singularities on the Poincaré sphere. For this purpose, we substitute in Eq. (7) the expression \( S_x = \sqrt{S_0^2 - S_y^2 - S_z^2} \) and \( S_z = K - J_z \) to get

\[
S_y = K^2 - S_z^2.
\]

Then using \( J_z = 0 \) and \( S_x = \sqrt{S_0^2 - S_y^2 - S_z^2} \), we get

\[
S_z = \pm \sqrt{S_0^2 - K^2 - (S_0^2 - K^2)}.
\]

Since \( J_z = 0 \) gives \( S_z = K \), we obtain a parametric curve on the Poincaré sphere:

\[
\begin{align*}
S_x &= \pm \sqrt{S_0^2 - K^2 - (S_0^2 - K^2)} \\
S_y &= K^2 - S_0^2 \\
S_z &= K
\end{align*}
\]  

These equations draw an eight-shaped line on the surface of the Poincaré sphere, as reported in Fig. 15. Similar calculations can be done in the general case with an arbitrary choice of the parameters \((J_x, J_y, J_z)\), and one still obtains an eight-shaped line.

It is important to note that there are two different restrictions to this phenomenon of attraction. The first one is due to the limit \( J_z(L) = \pm J_0 \), where Eq. (7) does not depend on \( S_y(L) \) and \( S_y(L) \). In this limit, Eq. (7) and Eq. (17) are not compatible, which means that the system no longer can relax toward the singular line of bitori.

This aspect is illustrated in Fig. 13, where we have computed the domain of existence of the attraction process in terms of \( K \) in the special case \( J_z(L) = 0 \). In this particular example, the equation of the border of the domain can be derived analytically. We first obtain \( S_y \) as a function of \( J_z \) and \( K \) with the same method used for Eq. (20). Then the limit values \( S_y = \pm 1 \) lead to the following equation:

\[
K_{\text{lim}} = \frac{3}{2} J_z \pm \sqrt{4 - 3J_z^2} \pm \sqrt{4 - 1 - J_z^2}.
\]

It is clear in Fig. 13 that the domain of \( K \) gets smaller as \( J_z \) approaches its limit values.

The second restriction is the fact that the attraction process of the signal beam only takes place over half of the eight-shaped line on the Poincaré sphere, a feature that has been established by the numerical simulations of the spatiotemporal Eqs. (4). This fact can also be easily explained by the nature of the singular bitori associated to the line of polarization attraction. Owing to the peculiar topology inherent to bitori, the stationary solutions can freely rotate around the bitorus, in contrast with trajectories that evolve on an ordinary pinched singular torus. This property allows for the existence of non-monotonic stationary solutions, i.e., stationary solutions that exhibit an oscillatory behavior. This is illustrated in Fig. 14, which reports several stationary solutions chosen among the stationary solutions that lie on the closed singular curve on the Poincaré sphere. Figure 14 (top) reports the stable stationary solutions, i.e., those that play the role of attractors for the spatiotemporal dynamics, whereas Fig. 14 (bottom) reports the unstable stationary solutions. Only the monotonic stationary solutions are stable, while the oscillatory ones are unstable. This feature corroborates the general observation pointed out originally in Refs. [4,5], that only monotonic stationary solutions are attractors for the spatiotemporal dynamics.

Finally, another interesting consequence of the existence of this singular line of polarization attraction is the possibility to

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**Fig. 12.** (Color online) RBF model: Numerical simulations of the spatiotemporal system on the Poincaré sphere with \( \Delta = 0.05 \). The green and red dots denote respectively the initial \((S(0))\) and final \((\tilde{S}(L))\) SOPs of the signal. The yellow dot displays the fixed pump SOP: \( \tilde{J}(L) = (1, 0, 0) \) for \( L = 15 \).

**Fig. 13.** (Color online) HBSF model: The gray domain depicts the possible values of \( K \) as a function of \( J_z(L) \) when \( J_z(L) = 0 \).
the signal beam will be attracted toward a particular point on
the eight-shaped closed curve at \( z = L \), and its ellipticity will
depend on the two angles that determine the initial linear
SOPs of the pump and signal beams. A numerical study re-
veals that the angle that characterizes the linear SOP of the
signal \( S(0) \) controls the position \( S(L) \) on the eight-shaped
curve, while the angle of \( J(L) \) rotates the eight-curve around
the \( z \) axis. This is illustrated in Fig. 15, where the color code
links the linear polarizations of \( S(0) \) to the corresponding el-
liptic polarizations of \( S(L) \) for a given linear polarization
of \( J(L) \).

5. CONCLUSION

In summary, we have presented the mathematical techniques
recently developed in order to provide a geometric approach
for the study of Hamiltonian singularities of integrable sys-
tems. This theory sheds new light to the interpretation of
the phenomenon of polarization attraction that occurs in
counterpropagating optical beams in different types of optical
fibers. We exposed in simple terms the theory of singular re-
duction, which permits to construct the energy-momentum
diagram and characterizes the geometric properties of the
underlying tori of the stationary system. We showed that the
energy-momentum diagram allows us to determine the essential
properties that characterize the process of polarization attrac-
tion. A brief overview of the different steps of this approach
can be summarized as follows:

- Study of the stationary system (Hamiltonian, constant
  of the motion)
- Introduction of the invariant polynomials associated to
  the constant of the motion
- Analysis of the intersection of the energy surface with
  the reduced phase space
- Construction of the energy-momentum diagram and
determination of the nature and of the position of the
  Hamiltonian singularities

We apply these tools in three different types of optical fibers.
In the example of IFs, the energy-momentum diagram is char-
acterized by one or two singularities, which leads to an effi-
cient process of polarization attraction toward a defined set
of SOPs. In RBFs, the singularities lie on the boundary of the
energy-momentum diagram, which significantly reduces the ef-
ciciency of the attraction process. Finally, in HBSFs the
energy-momentum diagram is characterized by the presence
of a continuous line of singularities, which leads to a polariza-
tion attraction toward a continuous line of polarization states
on the Poincaré sphere. We have completed previous results
concerning the polarization attraction process in RBFs and
HBSFs. The theory exposed here thus provides a rather com-
plete understanding of the properties of the phenomenon of
polarization attraction in these fiber systems. Note that, un-
der rather general conditions, for both HBSFs and RBFs the effi-
ciency of the attraction process increases as the fiber length
and the powers of the beams increase. We also underline that
all the analysis can be extended to unequal signal-pump
powers: For HBSFs polarization attraction still occurs along
a line of polarization states on the Poincaré sphere, while for
RBFs the singular torus is shown to split into two distinct sin-
gular tori for \( S_0 \neq J_0 \), whose SOP coordinates read \( S_x = \pm \rho J_x \),
\( S_y = \pm \rho J_y \), \( S_z = -\rho J_z \), with \( \rho = S_0 / J_0 \). For \( \rho \approx 1 \) (within
10%), the simulations reveal an attraction toward the two

Fig. 14. (Color online) HBSF model: Monotonic (up) and nonmono-
tonic (bottom) behaviors of the stationary solutions corre-
sponding respectively to the stable and unstable parts of the eight-shaped line.
Only the monotonic stationary solutions are stable, whereas the sta-
tionary solutions that exhibit an oscillatory behavior are unstable.
This explains why only half of the eight-shaped closed curve plays
the role of an attractor in the spatiotemporal dynamics.

Fig. 15. (Color online) HBSF model: Two series of numerical simu-
lations with two different values of \( J(L) \) that show the possibility of
producing an elliptic polarization with two linear polarizations.
The plain and dashed black lines depict respectively the stable and
unstable parts of the figure eight. The two large yellow dots are
associated to the two values of the pump \( J(L) \). The small dots on
the equator depict the injected polarization signal \( S(0) \) and the small
dots outside the equator (on eight-shaped curves) the outgoing
polarization of the signal after the interaction.

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SOPs states, while for higher values of $\rho$ the system exhibits a complex dynamics, including periodic behaviors that will be the subject of future investigations. We also point out that the rigorous mathematical proof of all these phenomena still constitutes an unsolved problem. A first step in this direction has been done in [35], where the relaxation process to a stationary state has been proven rigorously in the purely linear regime of the counterpropagating wave dynamics in a Bragg grating. Work is in progress in order to generalize these results to the nonlinear regime of the counterpropagating interaction. Besides the characterization of the phenomenon of polarization attraction, the theory exposed here can also be extended to study the stability properties of soliton solutions in a medium of finite extension. In a recent work it was shown that soliton solutions can become unstable due to the finite extension of the medium (e.g., a finite optical fiber length) [8]. In this way, the spatiotemporal dynamics relaxes toward a Hamiltonian singular state whose nature is completely different than that of the soliton state. More precisely, we showed that when the singular torus of the Hamiltonian system is isolated (as in IFs), then the space-time dynamics is asymptotically attracted toward the soliton solution, which is thus stable. Conversely, in HBSFs the soliton becomes unstable because of the presence of a continuous family of Hamiltonian singular tori and the space-time dynamics relaxes toward another stationary state of this family [10]. Work is in progress in order to extend this preliminary work to more general soliton systems, such as gap-solitons and three-wave interaction solitons.

APPENDIX A: DERIVATION OF THE SOLITON SOLUTION

In this appendix, we derive explicitly the form of the soliton solution for IF by assuming that this kind of solution lies on a singular torus [4,10]. This completes our presentation of these new Hamiltonian tools by showing their efficiency in the computation of soliton solutions. Similar computations could be done for HSBF and RBF. We start from Eq. (18) and we introduce a new pair of canonically conjugate coordinates $(J, \psi)$ and $(K, \chi)$, which are related to the old coordinates by the generating function $F = (I_p + I_s)\chi + I_p\psi$, such that

$$
\phi_s = \frac{\partial F}{\partial I_s} = \chi
$$

$$
\phi_p = \frac{\partial F}{\partial I_p} = \chi + \psi
$$

$$
K = \frac{\partial F}{\partial \chi} = I_s + I_p
$$

$$
J = \frac{\partial F}{\partial \psi} = I_p.
$$

The Hamiltonian in these new coordinates can be written as

$$
\frac{dJ}{dz} = -\frac{\partial H}{\partial \psi} = 2 \sqrt{(S_0^2 - J^2)(J_0^2 - (K - J)^2)} \sin \psi,
$$

which can be expressed as

$$
\left(\frac{dJ}{dz}\right)^2 = e(J - \alpha)(J - \beta)(J - \gamma)(J - \delta),
$$

where $e > 0$ and $\beta \leq \gamma \leq \delta \leq \alpha$, and the variable $\xi$ plays here the role of time. The solution of this equation can be written in terms of a Jacobi elliptic function [36]:

$$
J(\xi) = \beta + \frac{\gamma - \beta}{1 - \eta \sin^2(\lambda \xi / \mu)}, \quad (A1)
$$

where $\eta = \frac{\gamma - \beta}{\beta - \delta} > 0$, $\mu = \frac{\beta - \alpha}{\gamma - \delta} > 0$, and $\lambda = \frac{1}{2} \sqrt{e(\gamma - \alpha)(\beta - \delta)}$ [37].

We recall that on a singular pinched torus, one of the fundamental periods is infinite, which entails $\mu = 1$ since $\omega = \pi \lambda / QP(\mu)$, where $QP$ is the quarter-period of the Jacobi elliptic function. We thus set $e = 1 - \mu$. We first have to compute $J(\xi + L/4)$, where $L = 2\pi / \omega$. Here, we consider that the length of the fiber corresponds to one period of the solution. Then, a first-order Taylor series expansion in $\epsilon$ gives

$$
J(\xi) = \frac{\delta(\beta - \gamma) - \delta(\delta - \gamma)e^{2\epsilon \xi \sqrt{2}}}{\beta - \gamma - (\delta - \gamma)e^{2\epsilon \xi \sqrt{2}}} + o(\epsilon). \quad (A2)
$$

As $\epsilon$ is a function of the four roots $\alpha, \beta, \gamma$, and $\delta$, this expression of $J$ only depends on these roots and on the fact that the system is close to the singular torus. One should also notice that this formula corresponds to a choice of specific boundary conditions and the whole family of soliton solutions can be obtained in the same way from Eq. (A1). The four roots have the form

$$
K = \frac{1}{2} \pm \frac{1}{6} (15K^2 + 12H - 16S_0^2K^2 + 4H^2 + 48S_0^2)^{1/2}.
$$

The next step of the computation consists in introducing the circular symmetry around the singular torus to write:

$$
K = \rho \cos \theta
$$

$$
H = \rho \sin \theta - S_0^2
$$

where $\rho$ is a small parameter and $\theta \in [0, 2\pi]$. Performing a Taylor expansion of $J(\xi)$ in $\rho$ finally gives

$$
J(\xi) = S_0 \tan h(\lambda \xi) + O(\rho). \quad (A4)
$$

Moreover, the first-order expansion of $L = 2\pi / \omega$ has the form

$$
L = \frac{2}{\sqrt{4 - k^2S_0^2}} \left(2 \ln 2 + \ln S_0^2 - \ln \left[2 \cos^2 \theta - 2S_0^2k^2 \cos^2 \theta - 8S_0^2 \cos^2 \theta \left(\frac{k - 2}{k + 2}\right)^2 - \ln \rho\right] + O(\rho)\right). \quad (A5)
$$
which has a leading term in $\ln \rho$ for $\rho \ll 1$. This result allows us to characterize the polarization attraction in IF [5]. Because of the attracting properties of the singular torus, the spatio-temporal dynamics relax toward a stationary state that converges exponentially to the singular torus when the length $L$ of the fiber goes to infinity.

Returning back to the Cartesian coordinates, we need to determine $\psi$ and $\chi$. Using the coordinates $(H = -S_0^2, K = 0)$ of the singular torus and the Hamiltonian

$$H = 2(S_0^2 - J^2) \cos \psi - J^2,$$

we get that $\psi = \pm \frac{s}{2} \pi$. The equation of motion on the singular torus $K = 0$ implies $\chi = \chi(0)$, which gives

$$S_x = S_0 \cos(\chi(0)) \text{sech}(\sqrt{3}S_0 \xi)$$
$$S_y = S_0 \tanh(2S_0 \xi)$$
$$S_z = S_0 \sin(\chi(0)) \text{sech}(\sqrt{3}S_0 \xi)$$
$$J_x = S_0 \cos \left( \chi(0) + s \frac{\pi}{2} \right) \text{sech}(\sqrt{3}S_0 \xi)$$
$$J_y = S_0 \tanh(\sqrt{3}S_0 \xi)$$
$$J_z = S_0 \sin \left( \chi(0) + s \frac{\pi}{2} \right) \text{sech}(\sqrt{3}S_0 \xi).$$

(A6)

with $s = \pm 1$. All these solutions are not symmetric with respect to $\xi$. The numerical simulations reveal that the spatio-temporal dynamics will select the symmetric solution. This leads us to choose the corresponding value of $\chi(0)$, i.e., $\chi(0) = \frac{\pi}{2}$ and $s = 1$. We thus recover the stable soliton solution considered in the numerical simulations in Ref. [10]:

$$S_x = S_0 \sqrt{3} \text{sech}(\sqrt{3}S_0 \xi)$$
$$S_y = S_0 \tanh(\sqrt{3}S_0 \xi)$$
$$S_z = S_0 \frac{\sqrt{3}}{2} \text{sech}(\sqrt{3}S_0 \xi)$$
$$J_x = -S_0 \frac{\sqrt{3}}{2} \text{sech}(\sqrt{3}S_0 \xi)$$
$$J_y = S_0 \tanh(\sqrt{3}S_0 \xi)$$
$$J_z = S_0 \frac{\sqrt{3}}{2} \text{sech}(\sqrt{3}S_0 \xi)$$

(A7)

REFERENCES