Hahn-Banach theorems for convex functions

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HAHN-BANACH THEOREMS FOR CONVEX FUNCTIONS

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We start from a basic version of the Hahn-Banach theorem, of which we provide a proof based on Tychonoff's theorem on the product of compact intervals. Then, in the first section, we establish conditions ensuring the existence of affine functions lying between a convex function and a concave one in the setting of vector spaces — this directly leads to the theorems of Hahn-Banach, Mazur-Orlicz and Fenchel. In the second section, we caracterize those topological vector spaces for which certain convex functions are continuous — this is connected to the uniform boundedness theorem of Banach-Steinhaus and to the closed graph and open mapping theorems of Banach. Combining both types of results readily yields topological versions of the theorems of the first section.

In all the text, X stands for a real vector space. For $A \subset X$, we denote by $\mathrm{cor}(A)$ the core (algebraic interior) of A : $a \in \mathrm{cor}(A)$ if and only if $A - a$ is absorbing. Given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we call *domain of* f the set $\mathrm{dom}\, f = \{\, x \in X \mid f(x) < +\infty \,\}$ and we declare f *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all x, y in $\mathrm{dom}\, f$ and $0 \leq t \leq 1$. A real-valued function $p : X \to \mathbb{R}$ is said to be *sublinear* if it is convex and positively homogeneous.

We use standard abbreviations and notations: *TVS* for topological vector space, *LCTVS* for locally convex topological vector space, *lcs* for lower semicontinuous, X^* for the algebraic dual of X, X' for its topological dual, $\sigma(X^*, X)$ for the topology of pointwise convergence on X^*, etc.

Prologue

The following basic theorem is the starting point, and crucial part, of the theory. It retains the essence of both the Hahn-Banach theorem —

non-emptiness assertion — and the Banach-Alaoglu theorem — $\sigma(X^*, X)$-compactness assertion. Its proof combines the key arguments of the proofs of these theorems.

Basic Theorem *For any sublinear function $p : X \to \mathbb{R}$, the set $\{ x^* \in X^* \mid x^* \le p \}$ is non-empty and $\sigma(X^*, X)$-compact.*

Proof. In the space $E = \mathbb{R}^X$ supplied with the product topology, the set X^ν of all sublinear forms is closed and the set

$$K := \{ q \in X^\nu \mid q \le p \} = \prod_{x \in X} [-p(-x), p(x)] \cap X^\nu$$

is compact by Tychonoff's theorem. For $x \in X$, put

$$F(x) := \{ q \in K \mid q(x) + q(-x) = 0 \}.$$

Clearly $\bigcap_{x \in X} F(x) = \{ x^* \in X^* \mid x^* \le p \}$. Since each $F(x)$ is closed in the compact set K, to obtain the desired result it only remains to show that for any finite family $\{x_0, x_1, \ldots, x_n\}$ in X, the intersection of the $F(x_i)$'s is not empty. We first observe that $F(x_0)$ is not empty; that is, we observe that there exists $q_0 \in X^\nu$ verifying

$$q_0 \le p \quad \text{and} \quad q_0(x_0) + q_0(-x_0) = 0.$$

Indeed, it suffices to take for q_0 the sublinear hull of p and of the function equal to $-p(x_0)$ at $-x_0$ and to $+\infty$ elsewhere, namely:

$$q_0(x) := \inf_{\lambda \ge 0} \left(p(x + \lambda x_0) - \lambda p(x_0) \right).$$

We then apply the argument again, with q_0 and x_1 in lieu of p and x_0, to obtain q_1 in $F(x_0) \cap F(x_1)$, and so forth until obtaining q_n in $F(x_0) \cap F(x_1) \cap \ldots \cap F(x_n)$. □

The non-emptiness assertion is Theorem 1 in Banach [2]. Its original proof, as well as the proof given in most textbooks, relies on the axiom of choice. The fact that it can also be derived from Tychonoff's theorem on the product of compact intervals was observed for the first time by Łoś and Ryll-Nardzewski [9]. It is now well-known, after the works of Luxemburg [10] and Pincus [12], that Banach's theorem (and hence the Hahn-Banach theorem) is logically weaker than Tychonoff's theorem on the product of compact intervals (and hence the Banach-Alaoglu theorem and the above theorem), which itself is weaker than the axiom of choice. From this point of view, the above statement is therefore optimal with respect to its proof.

1. Separation of convex functions

We first extend the basic theorem to the case of convex functions. For $f : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $p_f : x \mapsto \inf_{t>0} \frac{1}{t} f(tx)$ the *homogeneous hull* of f, and we set

$$\begin{aligned} S(f) &:= \{ x^* \in X^* \mid x^* \leq f \} \\ &= \{ x^* \in X^* \mid 0 \leq \inf(f - x^*) \}. \end{aligned}$$

Because of the following elementary facts, the extension is a straightforward consequence of the basic theorem.

Lemma 1 *If $f : X \to \mathbb{R} \cup \{+\infty\}$ is convex such that $f(0) \geq 0$ and $0 \in \operatorname{cor}(\operatorname{dom} f)$, then the function p_f is real-valued and sublinear.*

Lemma 2 *For any $f : X \to \mathbb{R} \cup \{+\infty\}$, $S(f) = S(p_f)$.*

Theorem 1 (Minoration of convex functions) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex such that $f(0) \geq 0$. If $0 \in \operatorname{cor}(\operatorname{dom} f)$, then $S(f)$ is non-empty and $\sigma(X^*, X)$-compact.*

Proof. Apply the basic theorem to p_f. □

Theorem 1' (ε-subdifferential) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex. If $x_0 \in \operatorname{cor}(\operatorname{dom} f)$, then for every $\varepsilon \geq 0$ the set*

$$\{ x^* \in X^* \mid x^*(x - x_0) \leq f(x) - f(x_0) + \varepsilon \quad \text{for all } x \in X \}$$

is non-empty and $\sigma(X^*, X)$-compact.

Proof. Apply Theorem 1 to the function $\tilde{f}(x) := f(x + x_0) - f(x_0) + \varepsilon$. □

Corollary *If in Theorem 1 we suppose further that X is a TVS and that f is continuous at some point of its domain, then $S(f)$ is non-empty, equicontinuous and $\sigma(X', X)$-compact.*

Proof. The set $S(f)$ is clearly equicontinuous, hence contained in X', and it is also clearly $\sigma(X', X)$-closed. □

More generally, we now search for separating a convex function from a concave one by an affine form. For $f, g : X \to \mathbb{R} \cup \{+\infty\}$, we denote by $f +_e g : x \mapsto \inf_{y \in X} (f(y) + g(x - y))$ the *epi-sum* (or *inf-convolution*) of f and g, and we set

$$\begin{aligned} S(f, g) &:= \{ x^* \in X^* \mid -g \leq x^* + r \leq f \quad \text{for some } r \in \mathbb{R} \} \\ &= \{ x^* \in X^* \mid 0 \leq \inf(f - x^*) + \inf(g + x^*) \}. \end{aligned}$$

As above, two elementary facts similarly reduce the argument to a simple invocation of the previous theorem.

Lemma 3 *If $f, g : X \to \mathbb{R} \cup \{+\infty\}$ are convex such that $(f +_e g)(0)$ is finite and $0 \in \operatorname{cor}(\operatorname{dom} f + \operatorname{dom} g)$, then $f +_e g$ takes its values in $\mathbb{R} \cup \{+\infty\}$ and is convex.*

Lemma 4 *For any $f, g : X \to \mathbb{R} \cup \{+\infty\}$, $S(f, g) = S(f +_e g^-)$, where $g^- : x \mapsto g(-x)$.*

Theorem 2 (Separation of convex functions) *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be convex such that $-g \leq f$. If $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$, then $S(f, g)$ is non-empty and $\sigma(X^*, X)$-compact.*

Proof. Apply Theorem 1 to $f +_e g^-$. □

Theorem 2' (Decomposition of the infimum of a sum) *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be convex such that $\inf(f + g)$ is finite. If $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$, then for every $\varepsilon \geq 0$ the set*

$$\{ x^* \in X^* \mid \inf(f + g) \leq \inf(f - x^*) + \inf(g + x^*) + \varepsilon \}$$

is non-empty and $\sigma(X^*, X)$-compact.

Proof. Apply Theorem 2 to the functions $\tilde{f} := f - \inf(f + g) + \varepsilon$ and g. □

Corollary *If in Theorem 2 we suppose further that X is a TVS and that $f +_e g$ is continuous at some point of its domain (this is the case if f is continuous at some point of its domain), then $S(f, g)$ is non-empty, equicontinuous and $\sigma(X', X)$-compact.*

Proof. If f is continuous at some point of its domain, it is actually continuous on the non-empty set $\operatorname{int}(\operatorname{dom} f)$, and since 0 belongs to $\operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g) = \operatorname{int}(\operatorname{dom} f) - \operatorname{dom} g$, we infer that f is continuous at some point of $\operatorname{dom} f \cap \operatorname{dom} g$, which implies at once that $f +_e g^-$ is continuous at 0. The result now follows from the corollary of Theorem 1 because $S(f, g) = S(f +_e g^-)$. □

The literature on the Hahn-Banach theorem is too broad to give any fair account in this short article. We refer to Buskes [5] for a comprehensive survey and an extensive bibliography, and to König [8] for a deep discussion on the theorem and its various applications. For variants of the above results, see, e.g., Holmes [6, p. 23 and p. 42], Vangeldère [21], Théra [19].

Before proceeding, we mention several simple consequences.

Banach-Alaoglu Theorem – algebraic version *If $C \subset X$ is convex and absorbing, then the set $\{ x^* \in X^* \mid x^*(x) \leq 1$ for all $x \in C \}$ is $\sigma(X^*, X)$-compact.*

Proof. Apply Theorem 1 to the function equal to 1 on C and to $+\infty$ elsewhere. □

When X is a TVS and C a convex neighborhood of 0, we recover the classical Banach-Alaoglu Theorem.

Hahn-Banach Theorem – sandwich version *Let $p : X \to \mathbb{R}$ be sublinear, $C \subset X$ be convex and $\tau : C \to \mathbb{R}$ be concave. If $\tau \leq p|C$, then there exists $x^* \in X^*$ such that $\tau \leq x^*|C$ and $x^* \leq p$.*

Proof. Apply Theorem 2, with $f = p$ and g equal to $-\tau$ on C and to $+\infty$ elsewhere, to obtain $x^* \in X^*$ et $r \in \mathbb{R}$ such that

$$\tau \leq x^*|C + r \quad \text{and} \quad x^* + r \leq p.$$

The second inequality implies $r \leq 0$, so that $\tau \leq x^*|C$. Since on the other hand p is positively homogeneous, we also have $x^* \leq p$. □

The result above is Theorem 1.7 in König [8]. Of course, if C is a vector subspace and if τ is linear, we get the classical Hahn-Banach Theorem.

Mazur-Orlicz Theorem – convex version *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be convex, $A \subset X$ and $\beta : A \to \mathbb{R}$. If $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{conv} A)$, then the following two statements are equivalent :*

(1) *There exist $x^* \in X^*$ and $r \in \mathbb{R}$ such that*

$$\beta(a) \leq x^*(a) + r, \text{ for all } a \in A, \quad \text{and} \quad x^* + r \leq f\,;$$

(2) *There exists $\gamma : \operatorname{conv} A \to \mathbb{R}$ such that*

$$\sum_{k=1}^{n} \lambda_k \beta(a_k) \leq \gamma(\sum_{k=1}^{n} \lambda_k a_k) \leq f(\sum_{k=1}^{n} \lambda_k a_k),$$

whenever $\{a_1, \ldots, a_n\} \subset A$, $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$, $\sum_{k=1}^{n} \lambda_k = 1$.

Proof. Clearly, (1) implies (2). Conversely, let g be the convex hull of $-\beta$, namely : if $x \notin \operatorname{conv} A$, $g(x) := +\infty$, while if $x \in \operatorname{conv} A$,

$$g(x) := \inf\{ -\sum_{k=1}^{n} \lambda_k \beta(a_k) \mid x = \sum_{k=1}^{n} \lambda_k a_k,\ a_k \in A, \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k = 1 \}.$$

We derive from (2) that $g(x) \geq -\gamma(x)$ for every $x \in \operatorname{conv} A$, hence g takes its values in $\mathbb{R} \cup \{+\infty\}$ and $\operatorname{dom} g = \operatorname{conv} A$; moreover, $-g \leq f$. We then deduce from Theorem 2 that $S(f, g)$ is non-empty, which is equivalent to statement (1). □

In the classical Mazur-Orlicz Theorem, f is sublinear (with finite values), so we can take $r = 0$ in statement (1) and $\gamma(x) = f(x)$ in statement (2). See also Sikorski [18] and Pták [13] for other simple proofs of Mazur-Orlicz's theorem starting from Banach's theorem. Note that the above theorem immediately yields Theorem 2: if g is as in Theorem 2, put $A = \operatorname{dom} g$ and $\beta = -g|A$.

Fenchel Theorem – algebraic version *Let* $f, g : X \to \mathbb{R} \cup \{+\infty\}$ *be convex. If* $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$*, then there exists* $x^* \in X^*$ *such that*

$$\inf(f + g) = \inf(f - x^*) + \inf(g + x^*).$$

Proof. We always have $\inf(f+g) \geq \inf(f-x^*)+\inf(g+x^*)$. If $\inf(f+g) = -\infty$, the result is obvious; otherwise, it suffices to invoke Theorem 2' with $\varepsilon = 0$. □

The Fenchel Duality Theorem corresponds to the case $X = \mathbb{R}^n$. On the other hand, when X is a TVS and f is continuous at some point of its domain, we obtain the theorem of Moreau [11] and Rockafellar [16].

2. Continuity of convex functions

From now on, X denotes a TVS. It follows from the corollary of Theorem 1 (Theorem 2, resp.) that the continuity of f ($f +_e g$, resp.) on the core of its domain is a sufficient condition for the set $S(f)$ ($S(f, g)$, resp.) to be non-empty and equicontinuous. In general this is also a necessary condition: if $f : X \to \mathbb{R} \cup \{+\infty\}$ is lsc convex and if $S(f)$ is non-empty and equicontinuous, then f is continuous at 0 (see Moreau [11, Proposition 8.d]). It is therefore interesting to determine those TVS for which the continuity of certain classes of convex functions is automatic.

For example, for the finest locally convex topology, any convex function is continuous on the core of its domain: this topology is such that the core of any convex set is equal to its interior. Let us consider the following less drastic properties:

(T) For any closed convex set $C \subset X$, $\operatorname{cor}(C) = \operatorname{int}(C)$.

(T+) For any closed convex sets $C, D \subset X$, $\operatorname{cor}(C + D) = \operatorname{int}(C + D)$.

Property (T) is well known: a Hausdorff locally convex TVS verifying (T) is said to be *tonnelé* (barrelled).

Proposition 1 *Let X be a TVS. Then:*

(1) *X verifies* (T) *if and only if every lsc convex function on X is continuous on the core of its domain.*

(2) *X verifies* (T+) *if and only if every epi-sum of lsc convex functions on X is continuous on the core of its domain.*

Proof. Let us show (2) for example. Suppose first that X verifies (T+). If the core of the domain of $f +_e g$ is empty, there is nothing to prove. If $f +_e g$ takes the value $-\infty$ at some point of this core, it is equal to $-\infty$ everywhere on it and so continuous. Otherwise we may assume that 0 lies in $\mathrm{cor}(\mathrm{dom}\, f + \mathrm{dom}\, g)$ and that $f +_e g$ is finite at 0. We have to show that $f +_e g$ is continuous at 0. We first easily see that 0 lies in $\mathrm{cor}(f^{\leq r} + g^{\leq r})$ for some $r \in \mathbb{R}$, where $f^{\leq r} := \{x \in X \mid f(x) \leq r\}$ and $g^{\leq r} := \{x \in X \mid g(x) \leq r\}$ are closed convex sets. We therefore derive from (T+) that 0 belongs to $\mathrm{int}(f^{\leq r} + g^{\leq r})$. Now, the convex function $f +_e g$ being bounded above on this neighborhood of 0, we conclude that it is continuous at that point.

The converse is evident for property (T+) precisely expresses that the epi-sum of indicator functions of closed convex sets is continuous on the core of its domain. □

The next proposition provides examples of spaces enjoying these properties:

Proposition 2 (1) *Every Baire TVS verifies* (T)*.*

(2) *Every metrizable complete TVS verifies* (T+)*.*

Proof. (1) is a classical result. We briefly show (2) in the particular case of Fréchet spaces, the adaptations for the non locally convex case being left to the reader. As usual, we may assume $0 \in \mathrm{cor}(C + D)$ and $0 \in C \cap D$. We must show that 0 belongs to $\mathrm{int}(C + D)$. Denote by (U_n) a countable basis of closed convex neighborhoods of 0 such that $U_{n+1} \subset U_n$. From Baire's theorem, we rapidly obtain that 0 belongs to $\mathrm{int}(\overline{C \cap U_n + D \cap U_n})$ for every n. Let then (U_{k_n}) be a subsequence of (U_n) such that

$$2U_{k_n} \subset C \cap U_n + D \cap U_n + U_{k_{n+1}},$$

from which follows

$$U_{k_1} \subset \sum_{i=1}^{n} \frac{1}{2^i}(C \cap U_i + D \cap U_i) + \frac{1}{2^n} U_{k_{n+1}}.$$

Any point x_0 in U_{k_1} can therefore be written as

$$x_0 = \sum_{i=1}^{\infty} \frac{1}{2^i}(c_i + d_i)$$

where $c_i \in C \cap U_i$ and $d_i \in D \cap U_i$. Since the U_i's are convex, we have

$$\sum_{i=p}^{q} \frac{c_i}{2^i} \in U_p,$$

so, from the completeness of X we derive that the point

$$c := \sum_{i=1}^{\infty} \frac{c_i}{2^i}$$

exists and belongs to U_1. In the same manner we have

$$d := \sum_{i=1}^{\infty} \frac{d_i}{2^i} \in U_1.$$

Now, C and D being closed convex, we also have $c \in C$ and $d \in D$. Finally, $x_0 = c + d$ belongs to $C \cap U_1 + D \cap U_1$. We thus have shown that U_{k_1} is contained in $C \cap U_1 + D \cap U_1$, which proves that 0 belongs to $\operatorname{int}(C + D)$. □

The above proof is an adaptation of the original proof of Banach's open mapping theorem.

Before concluding, let us show how the theorem of Banach-Steinhaus and the closed graph and open mapping theorems of Banach can be derived from Propositions 1 and 2.

Banach-Steinhaus Theorem *Let X be a TVS verifying* (T)*, Y be a Hausdorff LCTVS, and $\mathcal{H}$ be a set of continuous linear mappings from X into Y. If for every $x \in X$ and every continuous seminorm p on Y we have*

$$\sup_{h \in \mathcal{H}} p\,(h(x)) < \infty,$$

then $\mathcal{H}$ is equicontinuous.

Proof. For every continuous seminorm p on Y, the function $f : X \to \mathbb{R} \cup \{+\infty\}$ given by

$$f(x) := \sup_{h \in \mathcal{H}} p\,(h(x))$$

is everywhere finite, lsc, and convex, hence, by Proposition 1 (1), it is everywhere continuous, which exactly means that $\mathcal{H}$ is equicontinuous. □

We recall that a relation $A \subset X \times Y$ between two topological spaces X and Y is said to be *lower semicontinuous* (LSC) at $x_0 \in \operatorname{dom} A := \{ x \in X \mid Ax \neq \emptyset \}$ if for every open set $U \subset Y$ which meets Ax_0 the set $A^{-1}(U) := \{ x \in X \mid Ax \cap U \neq \emptyset \}$ is a neighborhood of x_0. It is clear that A is LSC at every point of $\operatorname{dom} A$ if and only if for every open set $U \subset Y$, the set $A^{-1}(U)$ is open in X.

Example : Let $T : X \to Y$ be a mapping. Then T is continuous at $x_0 \in X$ if and only if T, considered as a relation $T \subset X \times Y$, is LSC at x_0; T is open if and only if T^{-1}, considered as a relation $T^{-1} \subset Y \times X$, is LSC at every point of $\operatorname{dom} T^{-1} = \operatorname{Im} T$.

Theorem 3 *Let X, Y be Hausdorff LCTVS such that $X \times Y$ verifies* (T+).

(1) *If $h : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is lsc convex, then the marginal function $\varphi : x \in X \mapsto \inf_{y \in Y} h(x, y)$ is continuous on the core of its domain.*

(2) *If $A \subset X \times Y$ is a relation with closed convex graph, then A is LSC on the core of its domain.*

Proof. (1) By Proposition 1 (2), the epi-sum of h and of the indicator function ψ of the closed convex set $\{0\} \times Y$ is continuous on the core of its domain. But $\varphi(x) = (h +_e \psi)(x, 0)$, so φ is continuous at every point x such that $(x, 0) \in \operatorname{cor}(\operatorname{dom}(h +_e \psi))$, that is, at every point of $\operatorname{cor}(\operatorname{dom} \varphi)$.

(2) Let $x_0 \in \operatorname{cor}(\operatorname{dom} A)$ and let $y_0 \in U \cap Ax_0$ where U is open in Y. We must show that x_0 belongs to $\operatorname{int}(A^{-1}(U))$. We may assume that $x_0 = 0$ and $y_0 = 0$. Let $V \subset U$ be a closed convex neighborhood of 0 in Y and let h be the indicator function of the closed convex set $A \cap (X \times V)$. By (1), the marginal function φ is continuous on $\operatorname{cor}(\operatorname{dom} \varphi)$. But $\operatorname{dom} \varphi = A^{-1}(V)$ and it is immediat that 0 belongs to $\operatorname{cor}(A^{-1}(V))$. Whence 0 belongs to $\operatorname{int}(A^{-1}(V)) \subset \operatorname{int}(A^{-1}(U))$. □

When X and Y are Banach spaces, the above theorem is due to Robinson [15]; see also Jameson [7], Ursescu [20], Borwein [3]. For a converse of (2), see Ricceri [14].

Banach Theorems *Let X and Y be Fréchet spaces, $T : X \to Y$ be a linear mapping with closed graph. Then T is continuous and, if it is onto, it is open.*

Proof. By the preceding theorem applied to $T \subset X \times Y$, the *relation* T is LSC on $\operatorname{cor}(\operatorname{dom} T) = X$, which amounts to saying that the *mapping* T

is continuous. By the same theorem applied to $T^{-1} \subset Y \times X$, the relation T^{-1} is LSC on $\operatorname{cor}(\operatorname{dom} T^{-1}) = Y$, hence T is open. □

Epilogue

By combining Proposition 1 with the algebraic theorems of Section 1 we immediatly obtain topological versions of these theorems. For instance:

Theorem 1 – topological version *If X verifies* (T)*, then for any lsc convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that $f(0) \geq 0$ and $0 \in \operatorname{cor}(\operatorname{dom} f)$ the set*

$$S(f) = \{\, x^* \in X^* \mid x^* \leq f \,\}$$

is non-empty and equicontinuous.

Theorem 2 – topological version *If X verifie* (T+)*, then for any lsc convex functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ such that $-g \leq f$ and $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$ the set*

$$S(f, g) = \{\, x^* \in X^* \mid -g \leq x^* + r \leq f \quad \textit{for some } r \in \mathbb{R} \,\}$$

is non-empty and equicontinuous.

Fenchel Theorem – topological version *If X verifies* (T+)*, then for any lsc convex functions $f, g : X \to \mathbb{R} \cup \{+\infty\}$ such that $0 \in \operatorname{cor}(\operatorname{dom} f - \operatorname{dom} g)$ there exists $x' \in X'$ such that*

$$\inf(f + g) = \inf(f - x') + \inf(g + x').$$

For X a Banach space, the above theorem is due to Attouch-Brezis [1] (see also Borwein [4, p. 421]); for X a Fréchet space, it is proved in Rodrigues-Simons [17].

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