Stability of slopes and subdifferentials with respect to Wijsman convergence

Marc Lassonde

Stability of Slopes and Subdifferentials with respect to Wijsman Convergence

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Dedicated to Jack Warga on occasion of his 80th birthday

Abstract

We show that the slope introduced by DeGiorgi-Marino-Tosques is stable with respect to the variational convergence introduced by Wijsman. Applications to the stability of subdifferentials at critical points and to subdifferential sum rules are derived.

1 Introduction

The notion of slope of a lsc function introduced in [2] has proved to be central in various parts of nonlinear analysis. Study of the stability of the slope with respect to variational perturbations of the function was initiated in our recent work [4]. This note is a further contribution to this topic. In [4], we showed that the slope of *any linear perturbation* of a given function f is stable under ball-affine perturbations of f, where the notion of ball-affine convergence was introduced in [3] as a viable alternative to slice convergence for lsc, non necessarily convex, functions. Here we show instead that the slope of f is stable under Wijsman perturbations of f, a weaker and very classical notion of convergence (see [1, 6]). Our preceding result can be recovered from the following fact: *A sequence $\{f_n\}$ ball-affine converges to f if and only if $\{f_n + x^*\}$ Wijsman converges to $f + x^*$ for every $x^* \in X^*$.* (To our knowledge, this fact is new even for convex functions — when ball-affine reduces to slice.) Applications to the stability of subdifferentials under Wijsman convergence are adapted from [4]. The last part showing the link between the stability results and the subdifferential sum rules is new.

Notation. Except where otherwise stated, X stands for a real Banach space and X^* for its topological dual. All functions are assumed to be extended-real-valued and lower semicontinuous (lsc); $\mathrm{LSC}(X)$ denotes the space of all such functions on X. For f in $\mathrm{LSC}(X)$, we denote by $\operatorname{dom} f := \{x \in X \mid f(x) < \infty\}$ the domain of f and by $\operatorname{epi} f := \{\, (x, \alpha) \mid f(x) \le \alpha \,\}$ the epigraph of f. The indicator of $A \subset X$ is the function $\delta_A : X \to \mathbb{R} \cup \{\infty\}$ defined by $\delta_A(x) := 0$ for $x \in A$, ∞ otherwise. The closed ε-ball centered at point x is written $B_\varepsilon(x)$.

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2 Wijsman Convergence

A sequence of sets $\{F_n\}$ in a metric space (Y, d) is said to *Wijsman converge* to F, *W-converge* to F for short, provided $d(y, F_n) \to d(y, F)$ for every $y \in Y$ [6, 1]. A sequence of lsc functions $\{f_n\}$ on a normed space X is said to *W-converge* to f provided the sequence of their epigraphs $\{\operatorname{epi} f_n\}$ W-converges to $\operatorname{epi} f$ in $X \times \mathbb{R}$ supplied with the norm $\|(x, t)\| := \max\{\|x\|, |t|\}$.

It is known that W-convergence may not be preserved under equivalent renorming of the underlying space X. In fact, Beer proved that a sequence of convex functions $\{f_n\}$ W-convergences to f with respect to any equivalent norm of X iff f_n slice converges to f (see [1]). Similarly, W-convergence may not be preserved under linear perturbations of the functions. In fact:

Proposition 1 *On a normed space X, a sequence of lsc functions $\{f_n\}$ ball-affine converges to f iff any sequence $\{f_n + x^*\}$ with $x^* \in X^*$ W-converges to $f + x^*$.*

Ball-affine convergence was introduced in [3]. It agrees with slice convergence on the space of convex functions.

The following more tractable characterization of W-convergence of functions in terms of the convergence of their local infima is an adaptation of results in [3, 4]:

Theorem 2 *On a normed space X, a sequence of lsc functions $\{f_n\}$ W-converges to f iff for every $x \in X$ and $\lambda \ge 0$ there is a sequence $\{x_n\} \subset X$ such that $x_n \to x$, $f_n(x_n) \to f(x)$ and*

$$\sup_{\delta>0} \inf_{B_{\lambda+\delta}(x)} f \le \sup_{\delta>0} \liminf_n \inf_{B_{\lambda+\delta}(x)} f_n. \qquad (\dagger)$$

Theorem 2 suggests possible localization of the concept of W-convergence 'at a given point x'. Thus, a sequence of lsc functions $\{f_n\}$ will be declared to *W-converge to f at x* provided there is $\{x_n\}$ such that $x_n \to x$, $f_n(x_n) \to f(x)$ and $(\dagger)$ holds at x for any *small* $\lambda \ge 0$. Clearly, (global) W-convergence implies W-convergence at each point, but the converse need not be true.

3 Stability of Slopes

The *slope* of $f : X \to \mathbb{R} \cup \{\infty\}$ at $x \in \operatorname{dom} f$, introduced in [2], is defined by

$$\operatorname{slope} f(x) := \limsup_{\substack{y \to x \\ y \neq x}} \frac{(f(x) - f(y))^+}{\|x - y\|},$$

where $\alpha^+ = \max(0, \alpha)$ for $\alpha \in \mathbb{R} \cup \{\infty\}$. A point x with $\operatorname{slope} f(x) = 0$ is called a *critical point* of f.

Theorem 3 *If f_n W-converges to f at $x \in \operatorname{dom} f$, then there is a sequence $\{x_n\} \subset X$ such that $x_n \to x$, $f_n(x_n) \to f(x)$, and*

$$\limsup_n \operatorname{slope} f_n(x_n) \leq \operatorname{slope} f(x).$$

Corollary 3.1 *Let x be a critical point of f. If f_n W-converges to f at x, then there is a sequence $\{x_n\}$ such that $x_n \to x$, $f_n(x_n) \to f(x)$ and* $\operatorname{slope} f_n(x_n) \to 0$.

Corollary 3.2 *Assume that f is bounded from below. If f_n W-converges to f, then there is a sequence $\{x_n\}$ such that $f_n(x_n) \to \inf f(X)$ and* $\operatorname{slope} f_n(x_n) \to 0$.

4 Stability of Subdifferentials

Let ∂ be any operator which associates a subset $\partial f(x)$ of X^* to any Banach space X, any $f \in \mathrm{LSC}(X)$ and any $x \in X$ in such a way that:
(A0) If f attains a finite local minimum at x, then $0 \in \partial f(x)$;
(A1) If f is convex near x, then $\partial f(x)$ is the subdifferential of convex analysis;
(A2) If $F : X \times X \to \mathbb{R} \cup \{\infty\}$ is given by $F(x, y) = f(x) + g(y)$, then $\partial F(x, y) \subset \partial f(x) \times \partial g(y)$.
We write $\partial f := \{(x, x^*) \in X \times X^* \mid x^* \in \partial f(x)\}$ for the graph of ∂f.

Say that X is *∂-appropriate* provided

$$\left\{ \begin{array}{l} \forall g \text{ lsc}, \forall \varphi \text{ convex continuous, if } x \text{ is local minimum} \\ \text{of } g + \varphi, \text{ then there are } (y_n, y_n^*) \subset \partial g,\ (z_n, z_n^*) \subset \partial \varphi \\ \text{such that: } y_n, z_n \to x,\ g(y_n) \to g(x),\ y_n^* + z_n^* \to 0. \end{array} \right.$$

Examples. 1. X is a Hilbert space, ∂ is the proximal subdifferential;
2. X is an Asplund space, ∂ is the canonical Fréchet subdifferential;
3. X is a separable Banach space, ∂ is the Hadamard subdifferential;
4. X is any Banach space, ∂ is the subdifferential of Clarke, of Michel-Penot, or of Ioffe.

Theorem 4 *Assume that X is ∂-appropriate. Let x be a critical point of f. If f_n W-converges to f at x, then there is a sequence $\{(x_n, x_n^*)\} \subset \partial f_n$ such that $x_n \to x$, $f_n(x_n) \to f(x)$ and $x_n^* \to 0$.*

5 Subdifferential Sum Rules

Say that $\{g_1, \ldots, g_k\}$ is *decouplable* at $x \in X$ provided for any small $\lambda \geq 0$,

$$\sup_{\delta > 0} \inf_{B_{\lambda+\delta}(x)} \sum g_i \leq$$

$$\sup_{\delta > 0} \inf \left\{ \sum g_i(x_i) \mid x_i \in B_\lambda(x), \|x_i - x_j\| \leq \delta \right\}. \quad (\ddagger)$$

Examples. 1. At least one of the functions has compact lower level sets near x;
2. All but at most one of the functions are uniformly continuous near x.

Let $F : (x_1, \ldots, x_k) \in X^k \mapsto \sum g_i(x_i)$ and let Y be the diagonal of X^k. Then the sequence $f_n := F + nd_Y$ W-converges to $f := F + \delta_Y$ at $(x, \ldots, x)$. Indeed, (†) reduces to (‡)!

Theorem 5 [5] *Assume that X^k is ∂-appropriate. Let x be a critical point of $\sum g_i$. If $\{g_1, \ldots, g_k\}$ is decouplable at x, then there are sequences $\{(x_{i,n}, x_{i,n}^*)\} \subset \partial g_i$ such that $x_{i,n} \to x$, $g_i(x_{i,n}) \to g_i(x)$, $\sum x_{i,n}^* \to 0$ and* $\operatorname{diam}(x_{1,n}, \ldots, x_{k,n}) \|x_{i,n}^*\| \to 0$.

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