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Asymptotic analysis of an adhesive joint with mismatch strain

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This paper proposes the study of the equilibrium problem, where two elastic bodies are bonded to a thin elastic film under mismatch strain conditions resulting in a state of residual stress. The asymptotic behavior of the film/adherent system is modeled as the film thickness tends to zero, using a method based on asymptotic expansions and energy minimization procedures. This method yields a family of non-local imperfect interface laws, which define a jump in the displacement and the traction vector fields. The amplitudes of the jumps turn out to be correlated with the state of residual stress and the elastic properties of the materials. As an example, the interface law is calculated at order zero in the case of a pure homogeneous mismatch strain and a thin isotropic film consisting of Blatz-Ko material.

1. Introduction

A thin layer is a fine film of material deposited onto or bonded to another material, called the adherent. The purpose of the thin layer is to endow the surface of the adherent with specific properties. At the same time it takes advantage of the massive properties of the adherent, such as its mechanical strength and/or thermal properties (Park and Park, 1999; Fang and Lo, 2000; Resel, 2003). Thin films are used in microelectronic integrated circuits, magnetic information storage systems, optical coatings, wear resistant coatings, corrosion resistant coatings, etc. (see (Hu and Wang, 2006; Nie et al., 2006; Gan, 2008) and references therein). There is a large number of (physical, chemical, etc.) deposition processes. From a mechanical point of view, a thin film is a layer of material ranging in thickness from just a few fractions of a nanometer (in the case of a monolayer) to several micrometers. It is therefore characterized by at least one small dimension, which is much smaller than the other two.

Over the last few decades, many experiments have been performed and many mathematical models and numerical approaches have been developed to study the mechanical behavior of thin films (Huang and Rosakis, 2005; Mishnaevsky and Gross, 2005; Ngo et al., 2007; Cheng and Lee, 2008; Janssen et al., 2009; Pureza et al., 2009; Steigmann, 2009; Pureza et al., 2010; Xie and Fan, 2010). In particular, it has now been established that there are residual stresses multilayered structures and that their presence plays an important role in the ability of these structures to withstand external loads (Freund and Suresh, 2003). The aim of this study is to present a new method of modeling of thin films in the context of multilayered structures.

Many factors are known to influence the distribution of residual stresses, i.e., the mechanical properties of the films and adherents, mismatches between the thermal expansion of films and those of the adherents, processing parameters such as the deposition rate, film thickness, adherent temperature and chamber pressure during deposition, and film/adherents adhesion characteristics. Here, we focus on the mechanical properties of both the film and the adherents and on the strain mismatch in the film, which can be of various origins (thermal, chemical, etc.). The aim is to apply asymptotic techniques to predict the response of the film/adherent system under conditions where a mismatch strain results in a state of residual stress. In particular, asymptotic techniques are used to calculate an interface law to substitute the thin film and to consider the mismatch strain and the elastic properties of the film/adherent system. Asymptotic techniques have been previously used by the authors to model the behavior of thin films with a similar rigidity to that of the adherents (Lebon and Rizzoni, 2010, 2011a), to study soft thin films (Lebon et al., 2004, 1997) and thin adhesives governed by a non convex energy (Lebon and Rizzoni, 2008), and to analyze imperfect adhesion between adhesive and adherents (Lebon and Zaittouni, 2010).

The approach presented in (Paroni and Rizzoni, 2005) and recalled in Section 2 is used to model the effects of the mismatch strain. This strain is viewed as a finite deformation f_0^ε from the natural configuration of a thin interphase to a stressed

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configuration, which is compatible with that of the adherents and maintained by external constraints. The stressed interphase is then bonded to the adherents, the constraints are released and the film/adherent system deforms. The interphase and the adherents are assumed to be hyperelastic and the deformation of the film/adherent system, f, is assumed to be infinitesimal. The latter assumption appears to be justified when film strain is not changed appreciably as a result of the deformation of the adherents. Within these and further regularizing assumptions, the equilibrium problem is expressed as the minimization of the total energy given by the sum of two terms: the deformation energy of the interphase after the infinitesimal deformation f has been superimposed on the state of strain f_0^ε, and the deformation energy of the adherents undergoing the small strain f. Both f_0^ε and the deformation energy in the film can depend on the thickness ε of the interphase.

In Section 3, the results obtained in (Paroni, 2006) are recalled, which provide suitable conditions for ensuring the existence of at least one minimizer of the total energy. Section 4 deals with the asymptotic analysis. In particular, the asymptotic expansion method is introduced and a minimization strategy is used to obtain interface conditions. The strategy presented in (Lebon and Rizzoni, 2011a) is based on two assumptions: first, we assume the existence of expansions in series of the displacement and residual stress vector fields in terms of the small parameter describing the thickness; next, we assume that we can obtain the fields which are stationary points of the three dimensional energy by finding the stationary points of the energies obtained at the various levels of the expansion. At order zero, this method yields a continuous displacement vector field at the interface and a jump in the traction vector field; the amplitude of the jump is directly related to the residual stress and the film elastic properties (cfr. (50)). At higher orders, non-local imperfect laws defining the jumps in both the displacement and traction vector fields are obtained.

For the sake of completeness, an example is presented in the last section of the paper. The interface law calculated at order zero is restricted to the simple case of a pure homogeneous mismatch strain and an isotropic film consisting of Blatz-Ko material.

2. Three-dimensional energy of a mismatch strained interphase

Let us consider a thin interphase with a reference configuration B_0^ε placed between two bodies $\Omega_\pm^\varepsilon \subset R^3$, as depicted in Fig. 1a. The configuration B_0^ε is a natural state for the interphase, and there is a mismatch strain between the two bodies. We assume there is a deformation f_0^ε from B_0^ε to a configuration $B^\varepsilon \subset R^3$ that is compatible with $\Omega_\pm^\varepsilon$, is assumed to exist. Some constraints are assumed to act on B^ε to maintain the mismatch strain f_0^ε prescribed. The strained interphase is then assumed to be brought into contact with the two bodies and bonded to them (Fig. 1b). After the bonding, the constraints are released and the joint deforms. Let $f^\varepsilon : B^\varepsilon \mapsto R^3$ describe its deformation (Fig. 1c).

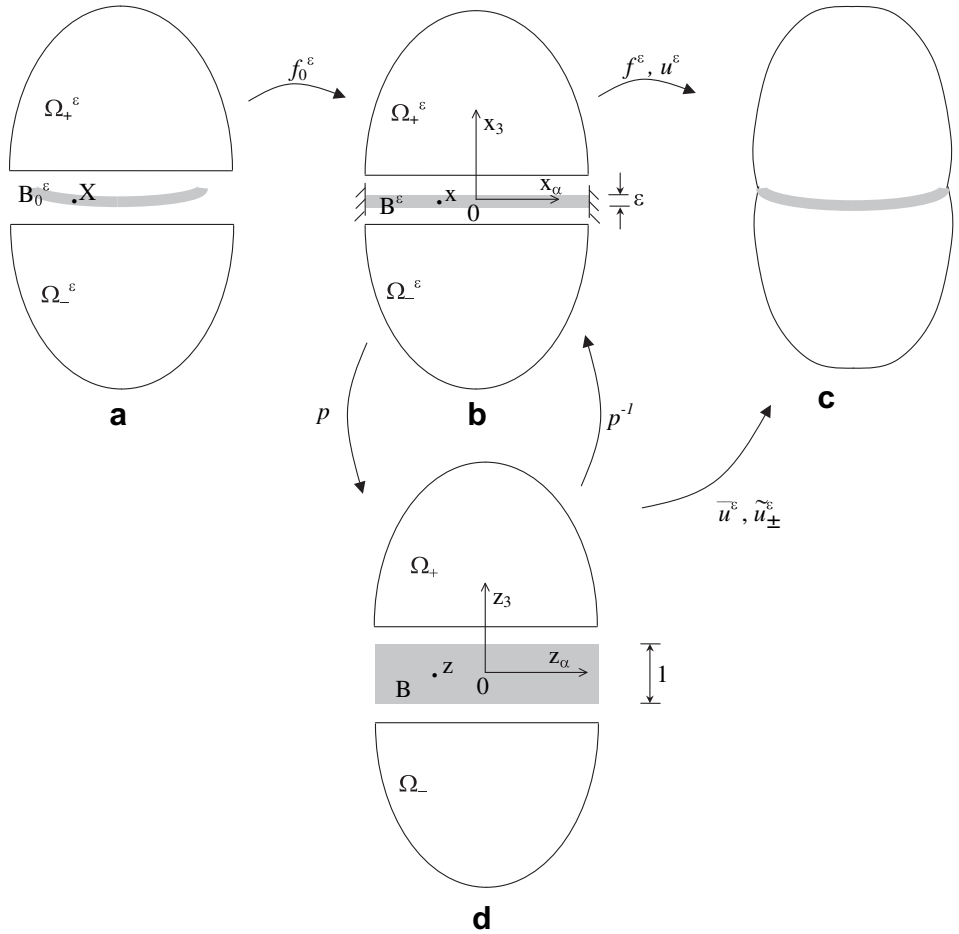


Fig. 1. (a) natural state of a thin interphase having a mismatch strain between two bodies; (b) the strained configuration of the thin interphase just before it is bonded to the two bodies; (c) the final deformed configuration of the joint; (d) the rescaled configuration of the interphase.

Let X and x indicate points in B_0^ε and in B^ε, respectively. The deformation f^ε is assumed to be infinitesimal, i.e.

$$f^\varepsilon(x) = x + u^\varepsilon(x) = f_0^\varepsilon(X) + u^\varepsilon(f_0^\varepsilon(X)) \tag{1}$$

where $u^\varepsilon: B^\varepsilon \mapsto R^3$ is a small displacement superimposed on the state of strain f_0^ε. The interphase is assumed to be elastic, and the strain energy density per unit reference volume dX is written as

$$w^\varepsilon(\nabla f^\varepsilon(X)) = w^\varepsilon(\nabla f_0^\varepsilon(X) + \nabla u^\varepsilon(x)\nabla f_0^\varepsilon(X)). \tag{2}$$

To simplify the notation we write F^ε, F_0^ε instead of ∇f^ε, ∇f_0^ε, and we omit the variables X, x. Take the expansion

$$\begin{aligned} w^\varepsilon(F^\varepsilon) =& w^\varepsilon(F_0^\varepsilon) + w_F^\varepsilon(F_0^\varepsilon)\cdot\nabla u^\varepsilon F_0^\varepsilon + \frac{1}{2} w_{F\,F}^\varepsilon(F_0^\varepsilon)(\nabla u^\varepsilon F_0^\varepsilon)\cdot\nabla u^\varepsilon F_0^\varepsilon \\ &+ o\Big(|\nabla u^\varepsilon|^2\Big), \end{aligned} \tag{3}$$

where the index F denotes differentiation and the dot indicates the scalar product between tensors. Without any loss of generality, we set $w^\varepsilon(F_0^\varepsilon) = 0$. The displacement gradient ∇u^ε is assumed to be infinitesimal, so that all the terms of an order higher than one in ∇u^ε can be neglected. All the subsequent formulae are valid only up to an error which tends to zero, $|\nabla u^\varepsilon|^2$. Using the tensor product $(A\boxtimes B)H = A\,H\,B^T$ (Del Piero, 1979), and introducing the tensors

$$T_0^\varepsilon := (\det F_0^\varepsilon)^{-1} w_F^\varepsilon(F_0^\varepsilon) F_0^{\varepsilon T} \tag{4}$$

$$b_0^\varepsilon := (\det F_0^\varepsilon)^{-1} (I\boxtimes F_0^\varepsilon) w_{F\,F}^\varepsilon(F_0^\varepsilon)\Big(I\boxtimes F_0^{\varepsilon T}\Big) \tag{5}$$

defining the Cauchy stress tensor and a fourth order elastic tensor, we obtain

$$w^\varepsilon(F^\varepsilon) = \det F_0^\varepsilon\Big(T_0^\varepsilon\cdot\nabla u^\varepsilon + \frac{1}{2} b_0^\varepsilon(\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\Big). \tag{6}$$

After integrating over the domain B^ε, we obtain the total deformation energy of the interphase subjected to the displacement u^ε:

$$\int_{B^\varepsilon}\Big(T_0^\varepsilon\cdot\nabla u^\varepsilon + \frac{1}{2} b_0^\varepsilon(\nabla u^\varepsilon)\cdot\nabla u^\varepsilon\Big)dx. \tag{7}$$

In (Del Piero, 1979; Del Piero and Rizzoni, 2008) it is proved that if w^ε is frame-indifferent, then b_0^ε can be decomposed into

$$b_0^\varepsilon = c_0^\varepsilon + I\boxtimes T_0^\varepsilon, \tag{8}$$

where I is the second-order identity tensor and c_0^ε is a tensor with two minor symmetries. Using these symmetries, the deformation energy of the interphase takes the form

$$\int_{B^\varepsilon}\Big(T_0^\varepsilon\cdot e(u^\varepsilon) + \frac{1}{2}\nabla u^\varepsilon T_0^\varepsilon\cdot\nabla u^\varepsilon + \frac{1}{2} c_0^\varepsilon(e(u^\varepsilon))\cdot e(u^\varepsilon)\Big)dx, \tag{9}$$

where $e(u^\varepsilon) = 1/2(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$ is the symmetric part of ∇u^ε.

3. Statement of the three dimensional problem

Let $S\subset R^2$ be an open bounded set with a Lipschitz boundary. In the rest of the paper, we take B^ε to be restricted to the cylindrical domain with a constant cross-section S and a constant small thickness $\varepsilon \ll 1$. Let $S_\pm^\varepsilon$ denote the flat interfaces between the interphase and the two bodies $\Omega_\pm^\varepsilon$ and let $\Omega^\varepsilon = \Omega_\pm^\varepsilon \cup S_\pm^\varepsilon \cup B^\varepsilon$ denote the composite comprising the interphase and the two bodies. Let (O, x_1, x_2, x_3) denote an orthogonal frame with its origin at the center of the interphase midplane and with the x_3-axis perpendicular to the interfaces $S_\pm^\varepsilon$ (Fig. 1b).

Let $u^\varepsilon : \Omega^\varepsilon \mapsto R^3$ be the displacement vector field from Ω^ε, i.e., the displacement undergone by the adherents after being bonded to the strained interphase. For the continuity of the displacement field across the surfaces $S_\pm^\varepsilon$, it is necessary that

$$[u^\varepsilon]_\pm = 0 \quad \text{on } S_\pm^\varepsilon \tag{10}$$

where

$$[u^\varepsilon]_\pm := u^\varepsilon\Big(x_1, x_2, \Big(\pm\frac{\varepsilon}{2}\Big)^+\Big) - u^\varepsilon\Big(x_1, x_2, \Big(\pm\frac{\varepsilon}{2}\Big)^-\Big) \tag{11}$$

stands for the jumps in the displacement across $S_\pm^\varepsilon$. In (11), $u^\varepsilon(x_1, x_2, (\varepsilon/2)^+)$ (resp. $u^\varepsilon(x_1, x_2, (\varepsilon/2)^-)$ indicates the limit of $u^\varepsilon(x_1, x_2, x_3)$ as x_3 tends to $(\varepsilon/2)$ and $x_3 \geq (\varepsilon/2)$ (resp. $x_3 \leq (\varepsilon/2)$).

The adherents are assumed to be homogeneous and linear elastic with elasticity tensors $a_\pm$. A body force density, ϕ, is applied to Ω^ε and a surface force density, γ, to $\Gamma_\gamma \subset \partial\Omega^\varepsilon$. Homogeneous boundary conditions are prescribed on $\Gamma_u = \partial\Omega^\varepsilon/(\Gamma_\gamma/(\partial\Omega^\varepsilon \cap \partial B^\varepsilon))$:

$$u^\varepsilon = 0 \quad \text{on } \Gamma_u. \tag{12}$$

We also make the following assumptions:

$$H1)\quad \begin{cases} T_0^\varepsilon, c_0^\varepsilon \in L^\infty(B^\varepsilon), a_\pm \in L^\infty(\Omega_\pm^\varepsilon), \\ (a_\pm)_{ijkl} = (a_\pm)_{klij} = (a_\pm)_{jilk} = (a_\pm)_{ijlk}, \\ (c_0^\varepsilon)_{ijkl} = (c_0^\varepsilon)_{klij} = (c_0^\varepsilon)_{jilk} = (c_0^\varepsilon)_{ijlk}, \\ \exists \eta_\pm, \eta_0^\varepsilon > 0 : a_\pm(e)\cdot e \geq \eta_\pm e\cdot e, \\ c_0^\varepsilon(x)(e)\cdot e \geq \eta_0^\varepsilon e\cdot e \quad \forall e \in R^9, e = e^T, x \in B^\varepsilon, \end{cases}$$

$$H2)\quad \exists \varepsilon_0 : B^\varepsilon \cap (\Gamma_\gamma \cup \mathrm{supp}(\phi)) = \varnothing, \quad \forall \varepsilon < \varepsilon_0,$$

$$H3)\quad \phi \in (L^2(\Omega)), \gamma \in (L^2(\Gamma_\gamma)).$$

Assumption (H1) concerns the usual symmetry properties and positive definiteness hypothesis of the elasticity tensors. Assumption (H2) means that Γ_γ is located far from the interphase. In (H3), the fields of the external forces are taken to have sufficient regularity to ensure the existence of equilibrium configurations.

The equilibrium configurations of the composite body are the minimizers of the total energy

$$\begin{aligned} E^\varepsilon(u^\varepsilon) =& \int_{\Omega_\pm^\varepsilon}\Big(\frac{1}{2} a_\pm(e(u^\varepsilon))\cdot e(u^\varepsilon) - \phi\cdot u^\varepsilon\Big)dx - \int_{\Gamma_\gamma}\gamma\cdot u^\varepsilon ds_x \\ &+ \int_{B^\varepsilon}\Big(T_0^\varepsilon\cdot e(u^\varepsilon) + \frac{1}{2}\nabla u^\varepsilon T_0^\varepsilon\cdot\nabla u^\varepsilon + \frac{1}{2} c_0^\varepsilon(e(u^\varepsilon))\cdot e(u^\varepsilon)\Big)dx \end{aligned} \tag{13}$$

in the space of kinematically admissible displacements

$$V^\varepsilon = \{u \in H(\Omega^\varepsilon) : u = 0 \quad \text{on } \Gamma_u\}. \tag{14}$$

Here $H(\Omega^\varepsilon)$ is the space of vector-valued functions in the set Ω^ε which are continuous and differentiable as many times as necessary.

Based on a result proved in Paroni (2006, Thm. 3.2), the coercivity of the energy E^ε (and hence the existence result, via the Lax-Milgram theorem) holds if

$$\eta_0^\varepsilon > c_k\frac{\tau_0^\varepsilon}{\varepsilon^2}, \tag{15}$$

where c_k is a constant which is independent of ε and τ_0^ε is the essential sup of the minimum eigenvalue of T_0^ε

$$\tau_0^\varepsilon := |\mathrm{essinf}_{x\in B^\varepsilon}(\min_{a\in R^3}\{T_0^\varepsilon(x)a\cdot a\})|. \tag{16}$$

As pointed out in Paroni (2006), condition (16) requires "the compression due to the residual stress" not be too large. Note that if the stress tensor T_0^ε is non-negative definite, then τ_0^ε will vanish and, in view of (15), coercivity simply follows from the positive definiteness of η_0^ε.

4. Asymptotic analysis

In this Section, the asymptotic expansion method is used to obtain the interface conditions giving the effects of the interphase on the mechanical behavior of the film/adherent system Ω^ε (Lebon and Rizzoni, 2010, 2011a,b). In order to reformulate the equilibrium problem in a domain which is independent of ε, the following change of variables is made:

$$(\hat{z}, z_3) = p(\hat{x}, x_3) = \begin{cases} (\hat{x}, x_3\,\varepsilon^{-1}) & (\hat{x}, x_3)\in B^\varepsilon, \\ \left(\hat{x}, x_3 \mp \dfrac{\varepsilon}{2} \pm \dfrac{1}{2}\right) & (\hat{x}, x_3)\in \Omega_\pm^\varepsilon, \end{cases} \tag{17}$$

where $\hat{x} = (x_1, x_2), \hat{z} = (z_1, z_2)$. With this change of variables, B^ε is rescaled by a factor ε^{-1} along the interphase thickness and the bodies $\Omega_\pm^\varepsilon$ are translated $\pm 1/2(1-\varepsilon)$ in the same direction, as depicted in Fig. 1d. In the new coordinate system, the interphase occupies the domain

$$B = \left\{(z_1, z_2, z_3)\in R^3 : (z_1, z_2) \in S, |z_3| < \frac{1}{2}\right\}, \tag{18}$$

and the adherents occupy the domains $\Omega_\pm = \Omega_\pm^\varepsilon \pm 1/2(1-\varepsilon)e_3$, where e_3 denotes the unit vector along the z_3-axis. Let $S_\pm = \{(\hat{z}, z_3)\in R^3 : \hat{z}\in S, z_3 = \pm 1/2\}$ denote the interfaces between the interphase and the adherents after rescaling, and let $\Omega = \Omega_+ \cup \Omega_- \cup B \cup S_+ \cup S_-$ denote the configuration of the composite body after the change of variables. Lastly, let $\Gamma_u^\pm$ and $\Gamma_\gamma^\pm$ denote the translations of Γ_u and Γ_γ, respectively. We take

$$\tilde{u}_\pm^\varepsilon(\hat{z}, z_3) := \left(u^\varepsilon \circ p^{-1}\right)(\hat{z}, z_3), \quad (\hat{z}, z_3)\in \Omega_\pm, \tag{19}$$

to denote the displacement in the adherents adjacent to the rescaled interphase, and

$$\overline{u}^\varepsilon(\hat{z}, z_3) := \left(u^\varepsilon \circ p^{-1}\right)(\hat{z}, z_3), \quad (\hat{z}, z_3)\in B, \tag{20}$$

to denote the displacement in the rescaled interphase. Continuity of the displacements at the interfaces $S_\pm^\varepsilon$ entails that

$$\tilde{u}_\pm^\varepsilon\left(\hat{z}, \pm\frac{1}{2}\right) = \overline{u}^\varepsilon\left(\hat{z}, \pm\frac{1}{2}\right), \quad \hat{z}\in S. \tag{21}$$

Note also that in view of the change of variables, we can write

$$u^\varepsilon\left(\hat{x}, \pm\frac{\varepsilon}{2}\right) = \tilde{u}_\pm^\varepsilon\left(\hat{z}, \pm\frac{1}{2}\right), \quad \hat{x}, \hat{z}\in S. \tag{22}$$

$$u^\varepsilon\left(\hat{x}, \pm\frac{\varepsilon}{2}\right) = \overline{u}_\pm^\varepsilon\left(z_1, z_2, \pm\frac{1}{2}\right), \quad \hat{x}, \hat{z}\in S. \tag{23}$$

Let $\tilde{\phi} := \phi \circ p^{-1}$ and $\tilde{\gamma} := \gamma \circ p^{-1}$ denote the rescaled external forces and let $\tilde{T}_0^\varepsilon$, $\tilde{c}_0^\varepsilon$ denote the rescaled Cauchy stress tensor and the rescaled elastic tensor, respectively.

We rewrite the assumption (H2) as follows:

$$H2') \quad B \cap \left(\Gamma_\gamma^\pm \cup \mathrm{supp}\left(\tilde{\phi}\right)\right) = \varnothing. \tag{24}$$

To rewrite the energy in an appropriate form, we introduce the notation

$$\begin{aligned} &\left(t_0^{\varepsilon,i}\right)_j := \left(\tilde{T}_0^\varepsilon\right)_{ij}, \quad i,j = 1,2,3, \\ &\left(K_0^{\varepsilon,ij}\right)_{kl} := \left(\tilde{c}_0^\varepsilon\right)_{kilj} + \left(\tilde{T}_0^\varepsilon\right)_{ij}\delta_{kl}, \end{aligned} \tag{25}$$

where δ_{kl} is the Kronecker delta. In view of the symmetry of $\tilde{c}_0^\varepsilon$ and of $\tilde{T}_0^\varepsilon$, we have the property

$$\left(K_0^{\varepsilon,ij}\right)^T = K_0^{\varepsilon,ji}. \tag{26}$$

After changing the variables and using the definitions (25) and (26), we express the energy (13) over the fixed domain Ω:

$$\begin{aligned} E^\varepsilon(u^\varepsilon) = &\int_{\Omega_\pm} \left(\frac{1}{2}a_\pm\left(e\left(\tilde{u}_\pm^\varepsilon\right)\right)\cdot e\left(\tilde{u}_\pm^\varepsilon\right) - \tilde{\phi}\cdot\tilde{u}_\pm^\varepsilon\right)dz \\ &- \int_{\Gamma_\gamma^\pm} \tilde{\gamma}\cdot\tilde{u}_\pm^\varepsilon ds_z + \int_B \left(\varepsilon t_0^{\varepsilon,\alpha}\cdot\overline{u}_{,\alpha}^\varepsilon + t_0^{\varepsilon,3}\cdot\overline{u}_{,3}^\varepsilon\right)dz \\ &+ \int_B \left(\frac{\varepsilon}{2}K_0^{\varepsilon,\alpha\beta}\left(\overline{u}_{,\alpha}^\varepsilon\right)\cdot\overline{u}_{,\beta}^\varepsilon + K_0^{\varepsilon,\alpha 3}\left(\overline{u}_{,\alpha}^\varepsilon\right)\cdot\overline{u}_{,3}^\varepsilon\right. \\ &\left. + \frac{1}{2\varepsilon}K_0^{\varepsilon,33}\left(\overline{u}_{,3}^\varepsilon\right)\cdot\overline{u}_{,3}^\varepsilon\right)dz =: \mathcal{E}^\varepsilon\left(\overline{u}^\varepsilon, \tilde{u}_\pm^\varepsilon\right) \end{aligned} \tag{27}$$

The standard summation convention over repeated indices is used throughout this paper and the indexes α, β range from 1 to 2. With the assumptions (H1) ÷ (H3) and (H2′), we minimize the rescaled energy (27) in the following set of displacements:

$$\begin{aligned} V = &\left\{\left(\overline{u}^\varepsilon, \tilde{u}_\pm^\varepsilon\right)\in H(B)\times H(\Omega_+)\times H(\Omega_-) : \right. \\ &\left. \tilde{u}_\pm^\varepsilon = \overline{u}^\varepsilon \text{ on } S_\pm, \tilde{u}_\pm^\varepsilon = 0 \text{ on } \Gamma_u^\pm\right\}. \end{aligned} \tag{28}$$

Since we are looking for the behavior of the minimizers of (13) when the interphase thickness ε is small, we assume that the minimizing displacements can be expressed as the sum of the series

$$\tilde{u}_\pm^\varepsilon = \tilde{u}_\pm^0 + \varepsilon\tilde{u}_\pm^1 + \varepsilon^2\tilde{u}_\pm^2 + o\left(\varepsilon^2\right), \tag{29}$$

$$\overline{u}^\varepsilon = \overline{u}^0 + \varepsilon\overline{u}^1 + \varepsilon^2\overline{u}^2 + o\left(\varepsilon^2\right), \tag{30}$$

where the displacement vectors $\tilde{u}_\pm^0, \tilde{u}_\pm^1, \ldots$ are independent of ε. We also assume that the Cauchy stress $\tilde{T}_0^\varepsilon$ and the elastic tensor $\tilde{c}_0^\varepsilon$ can be expressed as the sum of the series:

$$\begin{aligned} \tilde{T}_0^\varepsilon &= \frac{1}{\varepsilon}T_0^{-1} + T_0^0 + \varepsilon T_0^1 + \varepsilon^2 T_0^2 + o\left(\varepsilon^2\right), \\ \tilde{c}_0^\varepsilon &= \frac{1}{\varepsilon}c_0^{-1} + c_0^0 + \varepsilon c_0^1 + \varepsilon^2 c_0^2 + o\left(\varepsilon^2\right), \end{aligned} \tag{31}$$

and set

$$\left(K_0^{I,ij}\right)_{kl} := \left(c_0^I\right)_{kilj} + \left(T_0^I\right)_{ij}\delta_{kl}, \quad \left(t_0^{I,i}\right)_j := \left(T_0^I\right)_{ij}, \tag{32}$$

where $I = -1, 0, 1, \ldots$ and $i, j = 1, 2, 3$. We then substitute the expansions (29) and (30) into (27) to obtain

$$\begin{aligned} \mathcal{E}^\varepsilon\left(\overline{u}^\varepsilon, \tilde{u}_\pm^\varepsilon\right) = &\frac{1}{\varepsilon^2}\mathcal{E}^{-2}(\overline{u}^0) + \frac{1}{\varepsilon}\mathcal{E}^{-1}(\overline{u}^0, \overline{u}^1) + \mathcal{E}^0\left(\overline{u}^0, \overline{u}^1, \tilde{u}_\pm^0\right) \\ &+ \varepsilon\mathcal{E}^1\left(\overline{u}^0, \ldots, \overline{u}^3, \tilde{u}_\pm^0, \tilde{u}_\pm^1\right) \\ &+ \varepsilon^2\mathcal{E}^2\left(\overline{u}^0, \ldots, \overline{u}^4, \tilde{u}^0, \ldots, \tilde{u}^2\right) + o\left(\varepsilon^2\right), \end{aligned} \tag{33}$$

where the energies $\mathcal{E}^I, I=-2,-1,0,1,2$ are independent of ε and are defined as follows:

$$\mathcal{E}^{-2}(\overline{u}^0) := \int_B \frac{1}{2}\left(K_0^{-1,33}\left(\overline{u}^0_{,3}\right)\cdot\overline{u}^0_{,3}\right)dz, \tag{34}$$

$$\mathcal{E}^{-1}(\overline{u}^0,\overline{u}^1) := \int_B \left(t_0^{-1,3}\cdot\overline{u}^0_{,3} + K_0^{-1,\alpha 3}\left(\overline{u}^0_{,\alpha}\right)\cdot\overline{u}^0_{,3} + \frac{1}{2}\sum_{I=-1}^{0}\sum_{J=0}^{-I} K_0^{I,33}\left(\overline{u}^{(-I-J)}_{,3}\right)\cdot\overline{u}^J_{,3}\right)dz, \tag{35}$$

$$\begin{aligned}\mathcal{E}^0\left(\overline{u}^0,\overline{u}^1,\overline{u}^2,\tilde{u}^0_\pm\right) := {} & \int_{\Omega_\pm}\left(\frac{1}{2}a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)\cdot e\left(\tilde{u}^0_\pm\right) - \tilde{\phi}\cdot\tilde{u}^0_\pm\right)dz - \int_{\tilde{\Gamma}_\gamma^\pm}\tilde{\gamma}\cdot\tilde{u}^0_\pm ds_z + \int_B t_0^{-1,\alpha}\cdot\overline{u}^0_{,\alpha} + \frac{1}{2}K_0^{-1,\alpha\beta}\left(\overline{u}^0_{,\alpha}\right)\cdot\overline{u}^0_{,\beta} + \sum_{I=-1}^{0} t_0^{I,3}\cdot\overline{u}^{(-I)}_3\Bigg)dz \\ & + \int_B\left(\sum_{I=-1}^{0}\sum_{J=0}^{-I}K_0^{I,\alpha 3}\left(\overline{u}^{(-I-J)}_{,\alpha}\right)\cdot\overline{u}^J_{,3} + \frac{1}{2}\sum_{I=-1}^{1}\sum_{J=0}^{1-I}K_0^{I,33}\left(\overline{u}^{(1-I-J)}_{,3}\right)\cdot\overline{u}^J_{,3}\right)dz,\end{aligned} \tag{36}$$

$$\begin{aligned}\mathcal{E}^1\left(\overline{u}^0,\overline{u}^1,\overline{u}^2,\overline{u}^3,\tilde{u}^0_\pm,\tilde{u}^1_\pm\right) := {} & \int_{\Omega_\pm}\left(a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)\cdot e\left(\tilde{u}^1_\pm\right) - \tilde{\phi}\cdot\tilde{u}^1_\pm\right)dz - \int_{\tilde{\Gamma}_\gamma^\pm}\tilde{\gamma}\cdot\tilde{u}^1_\pm ds_z + \int_B\Bigg(\sum_{I=-1}^{0} t_0^{I,\alpha}\cdot\overline{u}^{(-I)}_\alpha + \sum_{I=-1}^{1} t_0^{I,3}\cdot\overline{u}^{(1-I)}_3 \\ & + \frac{1}{2}\sum_{I=-1}^{0}\sum_{J=0}^{-I}K_0^{I,\alpha\beta}\left(\overline{u}^{(-I-J)}_{,\alpha}\right)\cdot\overline{u}^J_{,\beta}\Bigg)dz + \int_B\left(\sum_{I=-1}^{1}\sum_{J=0}^{1-I}K_0^{I,\alpha 3}\left(\overline{u}^{(1-I-J)}_{,\alpha}\right)\cdot\overline{u}^J_{,3} + \frac{1}{2}\sum_{I=-1}^{2}\sum_{J=0}^{2-I}K_0^{I,33}\left(\overline{u}^{(2-I-J)}_{,3}\right)\cdot\overline{u}^J_{,3}\right)dz,\end{aligned} \tag{37}$$

$$\begin{aligned}\mathcal{E}^2\left(\overline{u}^0,\overline{u}^1,\overline{u}^2,\overline{u}^3,\overline{u}^4,\tilde{u}^0_\pm,\tilde{u}^1_\pm,\tilde{u}^2_\pm\right) := {} & \int_{\Omega_\pm}\left(a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)\cdot e\left(\tilde{u}^2_\pm\right) - \tilde{\phi}\cdot\tilde{u}^2_\pm\right)dz - \int_{\tilde{\Gamma}_\gamma^\pm}\tilde{\gamma}\cdot\tilde{u}^2_\pm ds_z + \int_{\Omega_\pm}\frac{1}{2}a_\pm\left(e\left(\tilde{u}^1_\pm\right)\right)\cdot e\left(\tilde{u}^1_\pm\right)dz \\ & + \int_B\sum_{I=-1}^{1} t_0^{I,\alpha}\cdot\overline{u}^{(1-I)}_\alpha + \sum_{I=-1}^{2} t_0^{I,3}\cdot\overline{u}^{(2-I)}_3\Bigg)dz + \int_B\Bigg(\frac{1}{2}\sum_{I=-1}^{1}\sum_{J=0}^{1-I}K_0^{I,\alpha\beta}\left(\overline{u}^{(1-I-J)}_{,\alpha}\right)\cdot\overline{u}^J_{,\beta} \\ & + \sum_{I=-1}^{2}\sum_{J=0}^{2-I}K_0^{I,\alpha 3}\left(\overline{u}^{(2-I-J)}_{,\alpha}\right)\cdot\overline{u}^J_{,3}\Bigg)dz + \int_B\left(\frac{1}{2}\sum_{I=-1}^{3}\sum_{J=0}^{3-I}K_0^{I,33}\left(\overline{u}^{(3-I-J)}_{,3}\right)\cdot\overline{u}^J_{,3}\right)dz.\end{aligned} \tag{38}$$

Let us now consider the minimization of each of these energies. The function class in which we look for the solution of each energy minimization problem is the natural class of displacements with finite energy; kinematic constraints are added whenever they follow from the minimization of the energies at lower orders. In the following analysis, we also make the simplifying assumption

$$t_0^{-1,3} = 0 \tag{39}$$

i.e., at the lowest order, plane stress state is assumed to exist.

4.1. Minimization of $\mathcal{E}^{-2}$

Since $\tilde{c}_0^\varepsilon$ is a positive definite tensor and $t_0^{-1,3}$ is assumed to vanish, the tensor $K_0^{-1,33}$ is also positive definite. The energy $\mathcal{E}^{-2}$ is therefore non negative and the minimizers are independent of z_3:

$$\overline{u}^0_{i,3} = 0, \quad i=1,2,3, \quad a.e.\ \text{in } B. \tag{40}$$

In view of the continuity conditions (21), (22) and (23), we also have

$$\tilde{u}^0_+\left(\hat{z},+\frac{1}{2}\right) = \tilde{u}^0_-\left(\hat{z},-\frac{1}{2}\right), \quad \hat{z}\in S, \tag{41}$$

$$u^0\left(\hat{x},0^+\right) = u^0\left(\hat{x},0^-\right), \quad \hat{x}\in S. \tag{42}$$

4.2. Minimization of $\mathcal{E}^{-1}$

Based on (40), we obtain that the energy $\mathcal{E}^{-1}$ vanishes.

4.3. Minimization of $\mathcal{E}^0$

In view of (39) and (40), the energy $\mathcal{E}^0$ simplifies as follows:

$$\mathcal{E}^0\left(\overline{u}^0,\overline{u}^1,\overline{u}^2,\tilde{u}^0_\pm\right) = \int\limits_{\Omega_\pm}\left(\frac{1}{2}a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)\cdot e\left(\tilde{u}^0_\pm\right)-\tilde{\phi}\cdot\tilde{u}^0_\pm\right)dz-\int\limits_{\tilde{\Gamma}^\pm_\gamma}\tilde{\gamma}\cdot\tilde{u}^0_\pm ds_z+\int\limits_B\left(t_0^{-1,\alpha}\cdot\overline{u}^0_{,\alpha}+\frac{1}{2}K_0^{-1,\alpha\beta}\left(\overline{u}^0_{,\alpha}\right)\cdot\overline{u}^0_{,\beta}\right.$$
$$\left.+K_0^{-1,\alpha 3}\left(\overline{u}^0_{,\alpha}\right)\cdot\overline{u}^1_{,3}+\frac{1}{2}K_0^{-1,33}\left(\overline{u}^1_{,3}\right)\cdot\overline{u}^1_{,3}\right)dz \tag{43}$$

and becomes independent of $\overline{u}^2$. We seek the minimizers of this energy in the class of displacements

$$V_0 = \left\{\left(\overline{u}^0,\overline{u}^1,\tilde{u}^0_\pm\right)\in H(B)\times H(B)\times H(\Omega_\pm):\tilde{u}^0_\pm\left(\hat{z},\pm\frac{1}{2}\right)\right.$$
$$\left.=\overline{u}^0\left(\hat{z},\pm\frac{1}{2}\right),\ \hat{z}\in S,\ \overline{u}^0_{,3}(z)=0,\ z\in B,\ \tilde{u}^0_\pm(z)=0,\ z\in\tilde{\Gamma}^\pm_u\right\}. \tag{44}$$

By minimizing (43) with respect to $\overline{u}^1$, we obtain

$$\overline{u}^1_{,3} = -\left(K_0^{-1,33}\right)^{-1}K_0^{-1,\alpha 3}\left(\overline{u}^0_{,\alpha}\right), \tag{45}$$

which, after integration with respect to z_3, gives

$$[\overline{u}^1] = -<\left(K_0^{-1,33}\right)^{-1}K_0^{-1,\alpha 3}>\left(\overline{u}^0_{,\alpha}\right). \tag{46}$$

Here $<\cdot>$ denotes the average across the thickness of B. After substituting (46) back into the first variation of (43), it turns out that the minimizers of (43) have to satisfy the equilibrium equations

$$\mathrm{div}\left(a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)\right)+\tilde{\phi} = 0 \quad \text{in } \Omega_\pm \tag{47}$$

$$a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)n = \tilde{\gamma} \quad \text{on } \tilde{\Gamma}_\gamma, \tag{48}$$

$$a_\pm\left(e\left(\tilde{u}^0_\pm\right)\right)n = 0 \quad \text{on } \partial\Omega_\pm/\tilde{\Gamma}^\pm_\gamma, \tag{49}$$

and the following jump condition

$$\left[\tilde{\sigma}^0\right]n = -\left(<t_0^{-1,\alpha}>+\hat{K}_0^{-1,\beta\,\alpha}\left(\overline{u}^0_{,\beta}\right)\right)_{,\alpha} \tag{50}$$

where $[\tilde{\sigma}^0]:=\tilde{\sigma}^0_+-\tilde{\sigma}^0_-$, $\tilde{\sigma}^0_\pm:=a_\pm(e(\tilde{u}^0_\pm))$ and

$$\hat{K}_0^{-1,\beta\,\alpha}:=<K_0^{-1,\beta\,\alpha}>-<\left(K_0^{-1,\alpha 3}\right)^T\left(K_0^{-1,33}\right)^{-1}K_0^{-1,\beta 3}>. \tag{51}$$

Condition (50) states that the jump in the traction vector across the rescaled interphase $\overline{B}$ does not vanish. Note that conditions (41) and (50) are mixed transmission conditions involving both the displacement vector at order zero and its derivatives and the lower order terms in the expansions of $\tilde{T}_o$ and $\tilde{c}^\varepsilon_0$.

4.4. Minimization of $\mathcal{E}^1$

In view of (39), (40), (45), (47)–(50), the energy $\mathcal{E}^1$ turns out to depend only on $\overline{u}^0$ and can be therefore $\mathcal{E}^1$ regarded as a constant term.

4.5. Minimization of $\mathcal{E}^2$

In view of (39), (40), (45), (47)–(50), the energy $\mathcal{E}^2$ simplifies into the sum of a term depending only on $\overline{u}_0$, plus the energy

$$\mathcal{E}^{2'}\left(\overline{u}^1,\overline{u}^2,\tilde{u}^1_\pm\right):=\int\limits_{\Omega_\pm}\left(\frac{1}{2}a_\pm\left(e\left(\tilde{u}^1_\pm\right)\right)\cdot e\left(\tilde{u}^1_\pm\right)\right)dz$$
$$+\int\limits_B\left(t_0^{0,\alpha}+\tilde{K}^{0,\beta\alpha}\left(\overline{u}^0_{,\beta}\right)+\frac{1}{2}K_0^{-1,\beta\alpha}\left(\overline{u}^1_{,\beta}\right)\right)\cdot\overline{u}^1_{,\alpha}dz$$
$$+\int\limits_B\left(t_0^{0,3}+\tilde{K}^{0,\alpha 3}\left(\overline{u}^0_{,\alpha}\right)+K_0^{-1,\alpha 3}\left(\overline{u}^1_{,\alpha}\right)\right.$$
$$\left.+\frac{1}{2}K_0^{-1,33}\left(\overline{u}^2_{,3}\right)\right)\cdot\overline{u}^2_{,3}dz, \tag{52}$$

where

$$\tilde{K}^{0,\beta\alpha}:=K_0^{0,\beta\alpha}-K_0^{0,3\alpha}\left(K_0^{-1,33^{-1}}\right)K_0^{-1,\beta 3}, \tag{53}$$

$$\tilde{K}^{0,\alpha 3}:=K_0^{0,\alpha 3}-K_0^{0,33}\left(K_0^{-1,33^{-1}}\right)K_0^{-1,\alpha 3}. \tag{54}$$

The energy term depending only on $\overline{u}_0$ can be taken to be a constant term because $\overline{u}_0$ is completely determined by the minimization of the energy $\mathcal{E}^0$ (cfr. (41), (47)–(50)).

The energy $\mathcal{E}^{2'}$ can be simplified by noting that in view of (46), the vector field $\overline{u}^1$ can be written in the form:

$$\overline{u}^1(\hat{z},z_3) = [\overline{u}^1](\hat{z})z_3+\frac{1}{2}S\left(\tilde{u}^1\right)(\hat{z}) \tag{55}$$

where $S(\tilde{u}^1)(\hat{z}):=\tilde{u}^1(\hat{z},(1/2)^+)+\tilde{u}^1(\hat{z},(-1/2)^-)$. Substituting (55) and (46) into the expression for $\mathcal{E}^{2'}$ and integrating with respect to z_3, we eliminate the dependence on $\overline{u}^1$ and reduce the minimization problem of $\mathcal{E}^{2'}$ to the minimization of the following energy

$$\mathcal{E}^{2'}\left(\overline{u}^2,\tilde{u}^1_\pm\right):=\int\limits_{\Omega_\pm}\left(\frac{1}{2}a_\pm\left(e\left(\tilde{u}^1_\pm\right)\right)\cdot e\left(\tilde{u}^1_\pm\right)\right)dz+\int\limits_S\left(<\left(t_0^{0,\alpha}+\tilde{K}^{0,\beta\alpha}\left(\overline{u}^0_{,\beta}\right)\right)>\cdot\left(\frac{1}{2}S\left(\tilde{u}^1\right)_{,\alpha}\right)\right)ds_z$$
$$-\int\limits_S\left(<\left(z_3K_0^{-1,\beta\alpha}\right)>\left(\left(K_0^{-1,33}\right)^{-1}K_0^{-1,\gamma 3}\left(\overline{u}^0_{,\gamma}\right)\right)_{,\beta}\cdot\left(\frac{1}{2}S\left(\tilde{u}^1\right)_{,\alpha}\right)\right)ds_z+\int\limits_S\left(<K_0^{-1,\beta\alpha}>\left(\frac{1}{2}S\left(\tilde{u}^1\right)_{,\beta}\right)\cdot\left(\frac{1}{2}S\left(\tilde{u}^1\right)_{,\alpha}\right)\right.$$
$$\left.+<\left(K_0^{-1,3\alpha}\right)>\left(\overline{u}^2_{,3}\right)\cdot\frac{1}{2}S\left(\tilde{u}^1\right)_{,\alpha}\right)ds_z+\int\limits_B\left(t_0^{0,3}+\tilde{K}^{0,\alpha 3}\left(\overline{u}^0_{,\alpha}\right)+\frac{1}{2}K_0^{-1,\alpha 3}S\left(\tilde{u}^1\right)_{,\alpha}\right)\cdot\overline{u}^2_{,3}dz$$
$$-\int\limits_B\left(K_0^{-1,\alpha 3}\left(<\left(K_0^{-1,33^{-1}}\right)K_0^{-1,\gamma 3}>\left(\overline{u}^0_{,\gamma}\right)\right)_{,\alpha}z_3\right)\cdot\overline{u}^2_{,3}dz. \tag{56}$$

The energy $\mathcal{E}^{2''}$ is minimized in the following class of displacements:

$$V_1 = \left\{ \left(\overline{u}^2, \tilde{u}^1_{\pm} \right) \in H(B) \times H(\Omega_{\pm}) : \tilde{u}^1_{\pm}(z) = 0, \quad z \in \tilde{\Gamma}^{\pm}_u \right\}. \tag{57}$$

After minimizing with respect to $\overline{u}^2$, we obtain

$$\begin{aligned} \overline{u}^2_{,3} = & -\left(K_0^{-1,33}\right)^{-1} \left(t_0^{0,3} + \tilde{K}^{0,\alpha 3}\left(\overline{u}^0_{,\alpha}\right) + \frac{1}{2} K_0^{-1,\alpha 3} S\left(\tilde{u}^1\right)_{,\alpha} \right. \\ & \left. - K_0^{-1,\alpha 3}\left(< \left(K_0^{-1,33^{-1}}\right) K_0^{-1,\gamma 3} > \left(\overline{u}^0_{,\gamma}\right) \right)_{,\alpha} z_3 \right). \end{aligned} \tag{58}$$

The remaining Euler-Lagrange equations give the equilibrium equations

$$\operatorname{div}\left(a_{\pm}\left(e\left(\tilde{u}^1_{\pm} \right) \right) \right) = 0 \quad \text{in } \Omega_{\pm}, \tag{59}$$

$$a_{\pm}\left(e\left(\tilde{u}^1_{\pm} \right) \right) n = 0 \quad \text{on } \partial\Omega_{\pm}, \tag{60}$$

and the following jump condition

$$\begin{aligned} \left[\tilde{\sigma}^1\right] e_3 = & \left(< \left(t_0^{0,\alpha} + \tilde{K}^{0,\beta\alpha}\left(\overline{u}^0_{,\beta}\right) \right) > - 2 < \left(K_0^{-1,\beta\alpha}\right) > \left(S\left(\tilde{u}^1\right)_{,\beta} \right) \right)_{,\alpha} \\ & - \left(< \left(z_3 K_0^{-1,\beta\alpha} \right) > \left(\left(K_0^{-1,33}\right)^{-1} K_0^{-1,\gamma 3}\left(\overline{u}^0_{,\gamma}\right) \right)_{,\beta} \right)_{,\alpha} \\ & + \left(< \left(K_0^{-1,3\alpha}\right) > \left(\left(K_0^{-1,33}\right)^{-1} \left(t_0^{0,3} + \tilde{K}^{0,\alpha 3}\left(\overline{u}^0_{,\alpha}\right) \right. \right. \right. \\ & \left. \left. \left. + K_0^{-1,\alpha 3}\left(\left(K_0^{-1,33^{-1}}\right) K_0^{-1,\gamma 3}\left(\overline{u}^0_{,\gamma}\right) \right)_{,\alpha} \right) \right) \right)_{,\alpha} \end{aligned} \tag{61}$$

where $[\tilde{\sigma}^1] := \tilde{\sigma}^1_+ - \tilde{\sigma}^1_-$ and $\tilde{\sigma}^1_{\pm} := a_{\pm}(e(\tilde{u}^1_{\pm}))$.

5. An example of the interface law in the case of a thin isotropic interphase

In this section, the general result obtained (50) will be applied to the simple special case of a mismatch strain which is homogeneous and independent of ε and an isotropic interphase with deformation energy scaling like ε^{-1}. In particular, the strained configuration is taken to be B^{ε} related to B^{ε}_0 by a pure homogeneous strain with the deformation gradient

$$F_0 = \sum_{i=1}^{3} \lambda_i (e_i \otimes e_i), \tag{62}$$

where e_1, e_2, e_3 are the directions parallel to axes 1, 2 and 3, respectively, and λ_i, $i = 1, 2, 3$, are constant stretches which are independent of ε. As an example, we take a film of Blatz-Ko material (Holzapfel, 2000)

$$w^{\varepsilon}(F) = \alpha^{\varepsilon}\left(\frac{1}{2}(F \cdot F - 3) + \gamma^{-1}\left((\det F)^{-1} - 1 \right) \right), \tag{63}$$

where

$$\alpha^{\varepsilon} = \frac{A}{\varepsilon} \tag{64}$$

and A, γ are positive constants. In Del Piero and Rizzoni (2008) it was established that the Cauchy stress and the elastic tensor c^{ε}_0 (see (8)) can be written as

$$T^{\varepsilon}_0(F_0) = \frac{A}{\varepsilon}(\det F_0)^{-1}\left(F_0 F_0^T - (\det F_0)^{-\gamma} I \right), \tag{65}$$

$$c^{\varepsilon}_0(F_0) = \frac{A}{\varepsilon}(\det F_0)^{-1-\gamma}(2\mathbb{S} + \gamma I \otimes I), \tag{66}$$

where $\mathbb{S}$ is the symmetry mapping on the set of all second-order tensors (Del Piero, 1979). For F_0 as in (62), we obtain

$$T^{\varepsilon}_0 = \frac{A}{\varepsilon\Delta} \sum_{i=1}^{3} \left(\lambda_i^2 - \Delta^{-\gamma} \right)(e_i \otimes e_i), \tag{67}$$

$$c^{\varepsilon}_0 = \frac{A}{\varepsilon \Delta^{1+\gamma}}(2\mathbb{S} + \gamma I \otimes I,) \tag{68}$$

where $\Delta := \lambda_1\lambda_2\lambda_3$. We take the stretches λ_1, λ_2, λ_3 such that

$$\lambda_3 = \Delta^{-\gamma}, \tag{69}$$

$$\lambda_1\lambda_2\lambda_3 > 0, \tag{70}$$

$$\lambda_2 > \max\left\{ \lambda_1^{-\frac{2(1+\gamma)}{\gamma}}, \lambda_1^{-\frac{\gamma}{2(1+\gamma)}} \right\} = \begin{cases} \lambda_1^{-\frac{2(1+\gamma)}{\gamma}}, & \gamma \le 2, \\ \lambda_1^{-\frac{\gamma}{2(1+\gamma)}}, & \gamma > 2. \end{cases} \tag{71}$$

With this choice, the condition (39) is satisfied and the Cauchy stress T^{ε}_0 turns out to be non-negative definite. As discussed in Section 3, the existence of energy minimizers of (13) is therefore guaranteed, provided that the smallest eigenvalue of c^{ε}_0

$$\eta^{\varepsilon}_0 = \frac{2A}{\varepsilon \Delta^{1+\gamma}} \tag{72}$$

is positive. The positivity of η^{ε}_0 follows from (70) and from the positivity of A.

Let us now look at the form taken by (50). From (25),

$$K_0^{-1,\alpha\alpha} = \frac{A}{\Delta^{1+\gamma}}\left(\gamma e_\alpha \otimes e_\alpha + \left(1 + \lambda_\alpha^2 \Delta^{\gamma} \right) I \right), \quad \alpha = 1, 2, \tag{73}$$

$$K_0^{-1,33} = \frac{A}{\Delta^{1+\gamma}}(\gamma e_3 \otimes e_3 + 2I), \tag{74}$$

$$K_0^{-1,ij} = \frac{A}{\Delta^{1+\gamma}}(2e_i \otimes e_j + \gamma e_j \otimes e_i), \quad i \neq j, i, j = 1, 2, 3, \tag{75}$$

and from (51),

$$\hat{K}_0^{-1,11} = \frac{A}{\Delta^{1+\gamma}}\left(\left(\frac{2+3\gamma}{2+\gamma} \right) e_1 \otimes e_1 + e_2 \otimes e_2 - e_3 \otimes e_3 + \lambda_1^2 \Delta^{\gamma} I \right), \tag{76}$$

$$\hat{K}_0^{-1,12} = \frac{A}{\Delta^{1+\gamma}}\left(2e_1 \otimes e_2 + \left(\frac{2\gamma}{2+\gamma} \right) e_2 \otimes e_1 \right), \tag{77}$$

$$\hat{K}_0^{-1,21} = \frac{A}{\Delta^{1+\gamma}}\left(\left(\frac{2\gamma}{2+\gamma} \right) e_1 \otimes e_2 + 2e_2 \otimes e_1 \right), \tag{78}$$

$$\hat{K}_0^{-1,22} = \frac{A}{\Delta^{1+\gamma}}\left(\left(\frac{2+3\gamma}{2+\gamma} \right) e_2 \otimes e_2 + e_1 \otimes e_1 - e_3 \otimes e_3 + \lambda_2^2 \Delta^{\gamma} I \right). \tag{79}$$

The interface law (50) therefore takes the form

$$\left[\tilde{\sigma}^0_{13}\right] = -\frac{A}{\Delta^{1+\gamma}}\left(\left(\frac{2+3\gamma}{2+\gamma}+\lambda_1^2\Delta^\gamma\right)\overline{u}^0_{1,11}+4\left(\frac{1+\gamma}{2+\gamma}\right)\overline{u}^0_{2,12} + \left(1+\lambda_2^2\Delta^\gamma\right)\overline{u}^0_{1,22}\right), \tag{80}$$

$$\left[\tilde{\sigma}^0_{23}\right] = -\frac{A}{\Delta^{1+\gamma}}\left(\left(1+\lambda_1^2\Delta^\gamma\right)\overline{u}^0_{2,11}+4\left(\frac{1+\gamma}{2+\gamma}\right)\overline{u}^0_{1,12} + \left(\frac{2+3\gamma}{2+\gamma}+\lambda_2^2\Delta^\gamma\right)\overline{u}^0_{2,22}\right), \tag{81}$$

$$\left[\tilde{\sigma}^0_{33}\right] = -\frac{A}{\Delta^{1+\gamma}}\left(\left(-1+\lambda_1^2\Delta^\gamma\right)\overline{u}^0_{3,11}+\left(-1+\lambda_2^2\Delta^\gamma\right)\overline{u}^0_{3,22}\right). \tag{82}$$

As mentioned above, these relations along with the continuity condition (41) give imperfect non-local interface laws for the mechanical behavior of a thin isotropic interphase with a mismatch strain between the adherents. In (80)–(82) we note the presence of the tangential derivatives of $\overline{u}^0$, showing non-local character of these relationships. We also note that in (80)–(82) the derivatives of the residual stress terms $\langle t_0^{-1,\alpha}\rangle$, $\alpha = 1, 2$, arising from the Cauchy stress T_0^{-1}, vanish because they are constant terms. The dependence on the mismatch strain, and therefore on the residual stress, is recovered however via the elasticity tensors (51) (cfr. (32)).

6. Conclusion

In this paper, an asymptotic method is applied to obtain a mathematical model describing the mechanical behavior of a composite consisting of two elastic adherents separated by a thin interphase with a mismatch strain.

At the lowest orders, the model is based on the equilibrium Eqs. (47)–(49) and (59), (60), and four non-local imperfect laws (41), (50), (55), (61), which define a jump in the displacements and the traction vector fields in terms of the elastic properties of the interfaces and the mismatch strain.

In future studies, it is proposed to test the validity of this model, which was applied here to the simple case involving a thin isotropic interphase consisting of Blatz-Ko material and a pure homogeneous mismatch strain, and the results obtained will be compared with those obtained using more classical approaches and with experimental data.

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