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A note on marginal and conditional independence

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SUMMARY

Some statistical models imply that two random vectors are marginally independent as well as being conditionally independent with respect to another random vector. When the joint distributions of the vectors is normal, Canonical Correlation Analysis may lead to relevant simplifications of the dependence structure. A similar application can be found in elliptical models, where linear independence does not imply statistical independence. Implications for Bayes analysis of the general linear model are discussed.

Some key words: Bayes linear analysis; Canonical Correlation Analysis; Elliptical distributions; Sylvester law of nullity; Unrelated parameters.

1. INTRODUCTION

Data modelling and interpretation are greatly simplified when independence relations can be assumed or derived. In order to achieve the simplest structure from the least of

constraints, it is necessary to derive all implications of the assumed independence relations. Properties in Dawid (1979) are very useful to this purpose and can be regarded as axioms for a logical system. Unfortunately, no finite set of axioms can completely characterize conditional independence (Studeny, 1992). Other implications of conditional independence relations require more probabilistic structure than that embodied in Dawid's axioms, as it happens when dealing with marginally independent random vectors X and Y which are also conditionally independent with respect to a random vector Z (Dawid, 1998). We shall write $X \perp Y$ and $X \perp Y|Z$ to denote the former and the latter independence relation, respectively.

Statistical implications of $X \perp Y$ and $X \perp Y|Z$ were first investigated within the framework of three-way tables. Yule (1903) showed that $X \perp Y$ and $X \perp Y|Z$ imply either $(Z, X) \perp Y$ or $X \perp (Y, Z)$, when X, Y, Z are dichotomous random variables. The result was independently rediscovered by Simpson (1951) while considering situations where $X \perp Y$ may be a reasonable hypothesis while $X \perp Y|Z$ is irrelevant, and vice versa. Birch (1963) generalized the result to $r \times s \times 2$ tables, but considered doubtful whether there were any practical situation in which both $X \perp Y$ and $X \perp Y|Z$ might be expected to hold. According to Darroch (1962), a relevant example of such a situation was given by perfect tables, where $X \perp Y$ holds if and only if $X \perp Y|Z$ does.

Marginal and conditional independence also occur in Bayesian analysis, when the distribution of the sample Y is indexed by the parameter $\Theta = (A, \Psi)$, with a prior distribution on it. Subparameters A and Ψ are said to be unrelated (Basu, 1977) if they are independent in the prior distribution ($A \perp \Psi$) as well as in the posterior one ($A \perp \Psi|Y$). Dawid (1980) introduced the strong identifiability condition and showed its usefulness in deriving further independence relations when a statistic A is marginally ancillary for the

parameter A (i.e. $A \perp\!\!\!\perp A$), as well as being specific ancillary for it (i.e. $A \perp\!\!\!\perp A | \mathcal{P}$). Basu & Pereira (1983) also assumed strong identifiability, when dealing with the case of a statistic T , ancillary for Θ (i.e. $T \perp\!\!\!\perp \Theta$), and a statistic V , sufficient for Θ (leading to $T \perp\!\!\!\perp \Theta | V$).

Statistical analysis in graphical models sometimes focuses on the conditional distribution of Y_S given Y_C , before and after marginalization over Y_M , where Y_S , Y_C and Y_M are the random vectors associated to the sets of nodes S , C and M , respectively. In particular, it may be of interest to know whether independence relations between Y_S and Y_C , like $Y_S \perp\!\!\!\perp Y_C | Y_M$, also holds after marginalization over Y_M , i.e. $Y_S \perp\!\!\!\perp Y_C$. Cox & Wermuth (1996, section 8.5) deal with the issue using summary graphs and collisionless paths. Wermuth & Cox (2004) state conditions preventing simultaneous occurrence of $Y_S \perp\!\!\!\perp Y_C | Y_M$ and $Y_S \perp\!\!\!\perp Y_C$. Cox & Wermuth (1993) discuss the issues with respect to chordless four-cycles in the covariance and in the concentration matrix.

In econometric theory, independence often models Granger noncausality or one of its variants (Granger, 1969; 1980; 1988). Both $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Y | Z$ might hold when spurious and indirect causal relationships are identified by adding new variables to the information set or removing from it already included variables, an approach proposed by Hsiao (1982) and later developed by Eichler (2009). Unfortunately, independence properties often lead to inferential difficulties, especially within complex, nonparametric models (Dahlhaus & Eichler, 2003). Linear Granger causality avoids the problem by focusing on linear dependencies (Hosoya, 1977; Florens & Mouchart, 1985; Dahlhaus, 2000; Eichler, 2007). As a direct consequence, focus shifts from marginal and conditional independence to ordinary uncorrelation and partial uncorrelation, respectively.

If $Z=(Z_1, Z_2)$ and $(X, Z_1) \perp\!\!\!\perp (Y, Z_2)$, then both $X \perp\!\!\!\perp Y$, $X \perp\!\!\!\perp Y|Z$ hold, as a trivial application of axioms in Dawid (1979). On the other hand, $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Y|Z$ do not imply that there exists a one-to-one function of Z into (Z_1, Z_2) such that $(X, Z_1) \perp\!\!\!\perp (Y, Z_2)$ holds (a simple counterexample can be found in Dawid, 1975). A natural question is then whether $X \perp\!\!\!\perp Y$, $X \perp\!\!\!\perp Y|Z$ imply $(X, Z_1) \perp\!\!\!\perp (Y, Z_2)$, at least in relevant statistical models. The paper approaches the problem using correlations and partial correlations, in order to apply well-known results of linear algebra. Independence and conditional independence easily follow when dealing with special families of distributions, i.e. the normal ones. Other kind of independencies are also considered. The paper is structured as follows: § 2 contains the main results; § 3 shows their relations with Canonical Correlation Analysis; § 4 applies them to Bayes analysis of the general linear model; § 5 discusses the results. Proof of Proposition 1 is in the Appendix.

2. MAIN RESULTS

The statement “ Y is irrelevant to X once Z is given” can be formalized in several ways , other than $X \perp\!\!\!\perp Y|Z$ (Dawid, 1998), and uncorrelation is a natural choice. We shall write $X \perp\!\!\!\perp Y$ to denote two uncorrelated random vectors and $X \perp\!\!\!\perp Y||Z$ to denote that residuals of linear regression of X and Y on Z are uncorrelated (it is implicitly assumed that all necessary moments exist). The following proposition shows an implication of their joint occurrence.

Proposition 1: *Let Y_a, Y_b and Y_c be three random vectors satisfying $Y_a \perp\!\!\!\perp Y_b$, $Y_a \perp\!\!\!\perp Y_b||Y_c$.*

Then $(Y_a, Y_A) \perp\!\!\!\perp (Y_b, Y_B)$, where (Y_A, Y_B) is a one-to-one linear function of Y_c .

The relation $(Y_a, Y_A) \perp (Y_b, Y_B)$ implies $(Y_a, HY_A) \perp (Y_b, KY_B)$, where H and K are two square, full rank matrices of appropriate dimensions. In particular, if $\text{cov}(Y_b, Y_c)$ is a null matrix, $Y_A=Y_c$, Y_B is a constant and $(Y_a, Y_c) \perp Y_b$.

In the normal case, (partial) uncorrelation implies (conditional) independence. The following corollary relies on this very convenient property.

Corollary 1: *Let Y_a , Y_b and Y_c be three random vectors whose joint distribution is multivariate normal and such that $Y_a \perp Y_b$, $Y_a \perp Y_b | Y_c$. Then $(Y_a, Y_A) \perp (Y_b, Y_B)$, where (Y_A, Y_B) is a one-to-one linear function of Y_c .*

Unfortunately, Corollary 1 may not hold when the density of the data $Y=(Y_1, \dots, Y_p)^T$ is elliptical, i.e. it depends on the outcome $y=(y_1, \dots, y_p)^T$ only through the quadratic form $(y-\xi)^T \Omega^{-1} (y-\xi)$, $\xi \in \mathbb{R}^p$, $\Omega = \{\omega_{ij}\}$, $\Omega^{-1} = \{\omega^{ij}\} \in \mathbb{R}^p \times \mathbb{R}^p$. Components of an elliptical, nonnormal vector are always dependent, so that another formalization of the statement “ Y is irrelevant to X once Z is given” is called for. We shall write $X \perp Y$ and $X \perp Y | Z$ to denote that, almost surely, $E(X|Y)=E(X)$ and $E(X|Y,Z)=E(X|Z)$, respectively (it is implicitly assumed that all necessary moments exist). When $\omega_{ij}=0$ and first (second) moments exist, $Y_i \perp Y_j$ ($Y_i \perp Y_j$). Similarly, if $\omega^{ij}=0$ and first (second) moments exist, $Y_i \perp Y_j | Y_{\setminus ij}$ ($Y_i \perp Y_j | Y_{\setminus ij}$), where $Y_{\setminus ij}$ is the vector of all components of Y but Y_i and Y_j . Given constraints on matrices Ω and Ω^{-1} lead to relevant simplifications of the dependence structure, as shown in the following corollary.

Corollary 2: *Let Y_a , Y_b and Y_c be three random vectors whose joint distribution is elliptical and such that $Y_a \perp Y_b$, $Y_a \perp Y_b | Y_c$. Then $(Y_a, Y_A) \perp (Y_b, Y_B)$, where (Y_A, Y_B) is a one-to-one linear function of Y_c .*

The proof easily follows from the expectation $E(X|Y)$ being linear in Y , when the joint distribution of X and Y is elliptical and $E(X)$ exists finite.

3. CANONICAL CORRELATION ANALYSIS

Linear transformations leading to Y_A and Y_B can be obtained through basic results in Canonical Correlation Analysis. However, unlike Canonical Correlation Analysis, here interest focuses on linear functions minimizing, rather than maximizing, squared correlations. Moreover, standard Canonical Correlation Analysis does not impose any structure on the covariance matrix, whereas here some structure is required in order to satisfy assumptions of Proposition 1. The procedure for obtaining Y_A and Y_B can be outlined as follows. Let

$$U_1 = \begin{pmatrix} Y_a \\ Y_A \end{pmatrix} \in \mathfrak{R}^h \quad U_2 = \begin{pmatrix} Y_b \\ Y_B \end{pmatrix} \in \mathfrak{R}^k \quad X_1 = \begin{pmatrix} Y_a \\ Y_c \end{pmatrix} \in \mathfrak{R}^p \quad X_2 = \begin{pmatrix} Y_b \\ Y_c \end{pmatrix} \in \mathfrak{R}^q.$$

Without loss of generality we can assume that $p \geq q$ and that Y_a, Y_b, Y_c are linearly independent. Consider now the following matrices

$$\text{var} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \quad R = \Sigma_{11}^{-1/2} \Sigma_{12} \Sigma_{22}^{-1/2} = A \begin{pmatrix} D \\ O \end{pmatrix} B^T,$$

where the columns $a_1, \dots, a_p$ ($b_1, \dots, b_p$) of A (B) are the eigenvectors of RR^T ($R^T R$). The matrix $D = \text{diag}(\lambda_1, \dots, \lambda_q)$ is a diagonal matrix with $\lambda_1 \geq \dots \geq \lambda_q \geq 0$.

Proposition 1 and basic results in Canonical Correlation Analysis imply existence of two random vectors $Z = (Z_1, \dots, Z_h)^T$ and $W = (W_1, \dots, W_h)^T$, where

$$Z_i = \alpha_i^T \Sigma_{11}^{-1/2} X_1 \quad \alpha_i \in \{a_1, \dots, a_p\}; \quad W_j = \beta_j^T \Sigma_{22}^{-1/2} X_2 \quad \beta_j \in \{b_1, \dots, b_q\},$$

satisfying

$$V(Z_i) = V(W_j) = 1 \quad \text{cov}(Z_i, W_j) = 0 \quad i = 1, \dots, h \quad j = 1, \dots, k.$$

The constraint $\text{cov}(Z, W) = O_{hk}$ implies that $Z(W)$ is linear one-to-one transformation of $U_1(U_2)$, i.e. $Z = A_1 U_1$ ($W = A_2 U_2$), so that $U_1 = A_1^{-1} Z$ ($U_2 = A_2^{-1} W$).

As a numerical example, let Y_1, Y_2, Y_3, Y_4 be four standardized random variables whose variance and concentration matrix are

$$\begin{pmatrix} 1 & 0 & 0.12 & 0.16 \\ 0 & 1 & 0.48 & 0.36 \\ 0.12 & 0.48 & 1 & 0.96 \\ 0.16 & 0.36 & 0.96 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1.042 & 0 & 0.447 & -0.595 \\ 0 & 1.563 & -2.679 & 2.009 \\ 0.447 & -2.679 & 17.538 & -15.944 \\ -0.595 & 2.009 & -15.944 & 15.679 \end{pmatrix},$$

respectively. It follows that $Y_1 \perp Y_2$ and $Y_1 \perp Y_2 \mid (Y_3, Y_4)$, but no other independence relation is apparent from the above matrices. Proposition 1 shows that the above covariance structure can be simplified by choosing an appropriate linear transformation of (Y_3, Y_4) , while the method described in this section leads to $(Y_1, 3Y_3 + 4Y_4) \perp (Y_2, 4Y_3 + 3Y_4)$.

4. BAYESIAN LINEAR MODEL

Bayesian analysis of the normal linear model provides a statistical application of results in § 2. Let

$$Y | \beta_1, \beta_2 \sim N(X_1 \beta_1 + X_2 \beta_2, \sigma^2 I_n), \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \sim N(m, \sigma^2 V),$$

where $Y \in \mathbb{R}^n$ is a vector of observations, $X_1 \in \mathbb{R}^n \times \mathbb{R}^h$ and $X_2 \in \mathbb{R}^n \times \mathbb{R}^k$ are matrices of known coefficients, $\beta_1 \in \mathbb{R}^h$ and $\beta_2 \in \mathbb{R}^k$ are the parameters (O'Hagan, 1994, § 9). The matrix V , the vector m and the scalar σ are assumed to be known (the latter assumption will be relaxed later in this section). Also, let the parameters β_1 and β_2 be unrelated

(Basu, 1977), i.e. $\beta_1 \perp\!\!\!\perp \beta_2$ and $\beta_1 \perp\!\!\!\perp \beta_2 | Y$. As a direct consequence of Corollary 1, there exists a one-to-one linear function $(A_1 Y, A_2 Y)$ of Y satisfying $(\beta_1, A_1 Y) \perp\!\!\!\perp (\beta_2, A_2 Y)$. Axioms in Dawid (1979) lead to $Y \perp\!\!\!\perp \beta_2 | (\beta_1, A_2 Y)$, $A_1 Y \perp\!\!\!\perp \beta_2$ and $Y \perp\!\!\!\perp \beta_1 | (\beta_2, A_1 Y)$ and $A_2 Y \perp\!\!\!\perp \beta_1$. Equivalently, we can say that $A_1 Y$ and $A_2 Y$ are partially sufficient for β_1 and β_2 , respectively (Basu, 1977).

In the general case, the parameter σ is assumed to be the realization of a random variable, so that:

$$Y | \beta_1, \beta_2, \sigma \sim N(X_1 \beta_1 + X_2 \beta_2, \sigma^2 I_n) \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \Big| \sigma \sim N(m, \sigma^2 V) \quad \sigma \sim F.$$

After marginalization over σ , the joint distribution of β_1, β_2, Y is elliptical (as all scale mixtures of centered normal distributions are) and β_1 and β_2 , are dependent, as well as being dependent conditionally on Y . If appropriate moments exist, however, $\beta_1 \perp\!\!\!\perp \beta_2$ and $\beta_1 \perp\!\!\!\perp \beta_2 | Y$. Hence Corollary 2 implies that there exists a one-to-one linear function $(A_1 Y, A_2 Y)$ of Y satisfying $(\beta_1, A_1 Y) \perp\!\!\!\perp (\beta_2, A_2 Y)$. The statistics $A_1 Y$ and $A_2 Y$ are no longer partially sufficient. However, there is no way that a linear estimate of β_1 (β_2), based on statistics $A_1 Y$ ($A_2 Y$) can be improved by using the full data Y , under mean squared loss. This property is useful in Bayes linear analysis, where observational data are combined with prior judgements using expectation rather than probability as the primitive expression for the judgements themselves. Moreover, there is also a close similarity between the same property and Bayes linear sufficiency (Goldstein & O'Hagan, 1996),

a weakening of the general notion of probabilistic sufficiency when only second-order prior specifications are made.

5. DISCUSSION

Implications of marginal and conditional independence simplify the dependence structure of linear systems with independent components, which include the multivariate normal distribution as a special case. When components are dependent but conditional expectations are linear in the conditioning variables, other simplifications of the dependence structure can be achieved.

Results in the paper rely on Sylvester law of nullity and singular value decomposition, providing another example of tools from linear algebra applied to the statistical analysis of independence. The prominent role of linear algebra in conditional independence theory is well established, to the point that it is not uncommon to devote first sections in papers on graphical models to review of known results in matrix form (Cox & Wermuth, 1993; Wermuth & Cox, 2004). Moreover, linear algebra can benefit from results obtained within a statistical framework, as remarked by Rao (2006). As an example, let the vector spaces V_1, V_2, V_3 and write $V_1 \perp V_2, V_1 \perp V_2 \parallel V_3$ to denote that that vector spaces V_1 and V_2 are orthogonal to each other and that projections of V_1 and V_2 onto the vector space V_3 are orthogonal to each other, too. Minor modifications to Proposition 1 and its proof can be made in order to derive implications of the orthogonality relations $V_1 \perp V_2, V_1 \perp V_2 \parallel V_3$.

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APPENDIX: PROOF OF PROPOSITION 1

We shall denote with n_a , n_b and n_c the dimensions of Y_a , Y_b and Y_c , respectively. Let Y be the vector obtained by stacking Y_a , Y_b and Y_c on top of each other. Without loss of generality we can assume that the covariance Σ of Y is positive definite, so that its inverse Σ^{-1} exists. Moreover, let

$$\text{cov}(Y) = \text{cov} \begin{pmatrix} Y_a \\ Y_b \\ Y_c \end{pmatrix} = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} & \Sigma_{ac} \\ \Sigma_{ba} & \Sigma_{bb} & \Sigma_{bc} \\ \Sigma_{ca} & \Sigma_{cb} & \Sigma_{cc} \end{pmatrix}, \quad \text{con}(Y) = \text{con} \begin{pmatrix} Y_a \\ Y_b \\ Y_c \end{pmatrix} = \begin{pmatrix} \Sigma^{aa} & \Sigma^{ab} & \Sigma^{ac} \\ \Sigma^{ba} & \Sigma^{bb} & \Sigma^{bc} \\ \Sigma^{ca} & \Sigma^{cb} & \Sigma^{cc} \end{pmatrix}.$$

By definition $\Sigma\Sigma^I$ is an identity matrix, so that

$$\Sigma_{aa}\Sigma^{ab} + \Sigma_{ab}\Sigma^{bb} + \Sigma_{ac}\Sigma^{cb} = O_{ab}$$

where O_{ab} is a $n_a \times n_b$ matrix of zeros. Assumptions $Y_a \perp Y_b$ and $Y_a \perp Y_b || Y_c$ imply that Σ_{ab} and Σ^{ab} are null matrices, so that $\Sigma_{ac}\Sigma^{cb} = O_{ab}$. By Sylvester law of nullity, there exists a non-singular $n_c \times n_c$ matrix M such that the last k columns of $\Sigma_{ac}M$ and the first $n_c - k$ rows of $M^1 \Sigma^{cb}$ are null vectors, where k is an integer such that $0 \leq k \leq n_c$. Equivalently, there exists a linear transformation $MY_c = (Y_{\bullet}^T, Y_{*}^T)^T$ such that

$$\text{cov} \begin{pmatrix} Y_a \\ Y_b \\ Y_{\bullet} \\ Y_{*} \end{pmatrix} = \begin{pmatrix} \Sigma_{aa} & O_{ab} & \Sigma_{a\bullet} & O_{a*} \\ O_{ba} & \Sigma_{bb} & \Sigma_{b\bullet} & \Sigma_{b*} \\ \Sigma_{\bullet a} & \Sigma_{\bullet b} & \Sigma_{\bullet\bullet} & \Sigma_{\bullet*} \\ O_{*a} & \Sigma_{*b} & \Sigma_{*\bullet} & \Sigma_{**} \end{pmatrix}, \quad \text{con} \begin{pmatrix} Y_a \\ Y_b \\ Y_{\bullet} \\ Y_{*} \end{pmatrix} = \begin{pmatrix} \Sigma^{aa} & O^{ab} & \Sigma^{a\bullet} & \Sigma^{a*} \\ O^{ba} & \Sigma^{bb} & O^{b\bullet} & \Sigma^{b*} \\ \Sigma^{\bullet a} & O^{\bullet b} & \Sigma^{\bullet\bullet} & \Sigma^{\bullet*} \\ \Sigma^{*a} & \Sigma^{*b} & \Sigma^{*\bullet} & \Sigma^{**} \end{pmatrix}.$$

Consider now the linear transformation

$$\begin{pmatrix} Y_A \\ Y_B \end{pmatrix} = \begin{pmatrix} I_{\bullet} & -\Sigma_{\bullet\bullet}\Sigma_{**}^{-1} \\ O_{\bullet*} & I_* \end{pmatrix} \begin{pmatrix} y_{\bullet} \\ y_* \end{pmatrix}.$$

The covariance and concentration matrices of Y_a, Y_b, Y_A and Y_B are

$$\text{cov} \begin{pmatrix} Y_a \\ Y_b \\ Y_A \\ Y_B \end{pmatrix} = \begin{pmatrix} \Sigma_{aa} & O_{ab} & \Sigma_{aA} & O_{aB} \\ O_{ba} & \Sigma_{bb} & \Sigma_{bA} & \Sigma_{bB} \\ \Sigma_{Aa} & \Sigma_{Ab} & \Sigma_{AA} & O_{AB} \\ O_{Ba} & \Sigma_{Bb} & O_{BA} & \Sigma_{BB} \end{pmatrix}, \quad \text{con} \begin{pmatrix} Y_a \\ Y_b \\ Y_A \\ Y_B \end{pmatrix} = \begin{pmatrix} \Sigma^{aa} & O^{ab} & \Sigma^{aA} & \Sigma^{aB} \\ O^{ba} & \Sigma^{bb} & O^{bA} & \Sigma^{bB} \\ \Sigma^{Aa} & O^{Ab} & \Sigma^{AA} & \Sigma^{AB} \\ \Sigma^{Ba} & \Sigma^{Bb} & \Sigma^{BA} & \Sigma^{BB} \end{pmatrix}.$$

Since the two matrices are inverses, the following equality holds:

$$\Sigma_{Aa} O^{ab} + \Sigma_{Ab} \Sigma^{bb} + \Sigma_{AA} O^{Ab} + O_{AB} \Sigma^{Bb} = O_{ab}$$

By assumption Σ is a positive definite matrix, so that Σ^{bb} is a positive definite matrix and Σ_{Ab} is a null matrix. By rearranging elements in the above vectors and matrices we obtain

$$\text{cov} \begin{pmatrix} Y_a \\ Y_A \\ Y_b \\ Y_B \end{pmatrix} = \begin{pmatrix} \Sigma_{aa} & \Sigma_{aA} & O_{ab} & O_{aB} \\ \Sigma_{Aa} & \Sigma_{AA} & O_{Ab} & O_{AB} \\ O_{ba} & O_{bA} & \Sigma_{bb} & \Sigma_{bB} \\ O_{Ba} & O_{BA} & \Sigma_{Bb} & \Sigma_{BB} \end{pmatrix}$$

Hence $(Y_a, Y_A) \perp (Y_b, Y_B)$. □

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