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Finite-sample distribution of regression quantiles

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Abstract

The finite sample distributions of the regression quantile and of the extreme regression quantile are derived for a broad class of distributions of the model errors, even for the non-i.i.d case. The distributions are analogous to the corresponding distributions in the location model; this again confirms that the regression quantile is a straightforward extension of the sample quantile. As an application, the tail behavior of the regression quantile is studied.

Key words: order statistic, regression quantile, extreme regression quantile, regression rank scores, score function, tail behavior

2008 MSC: 62E15, 62J05, 62G30, 62H10

1. Introduction

Consider the linear regression model

$$Y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + e_i, \quad i = 1, \dots, n \quad (1)$$

with an unknown parameter $\boldsymbol{\beta}$, where $\mathbf{x}_i \in \mathbb{R}^p$, $x_{i1} = 1$, $i = 1, \dots, n$ and where $e_1, \dots, e_n$ are i.i.d. errors with distribution function F and density f . We assume that the matrix $\mathbf{X}$ with the rows $\mathbf{x}_1^\top, \dots, \mathbf{x}_n^\top$ is of rank p and that $f(z)$ is absolutely continuous and positive for $z \in (a, b)$ where $a = \inf\{z : F(z) > 0\}$ and $b = \sup\{z : F(z) < 1\}$.

The regression α -quantile ($0 < \alpha < 1$) of Koenker and Bassett (1978) is any vector $\hat{\boldsymbol{\beta}}_n(\alpha)$ minimizing the criterion

$$\sum_{i=1}^n \rho_\alpha(Y_i - \mathbf{x}_i^\top \mathbf{b}), \quad \mathbf{b} \in \mathbb{R}^p \quad (2)$$

where $\rho_\alpha(z) = |z|\{\alpha I[z > 0] + (1 - \alpha)I[z < 0]\}$. Dual to $\hat{\boldsymbol{\beta}}_n(\alpha)$ in the linear programming sense is the vector $\hat{\mathbf{a}}_n(\alpha) = (\hat{a}_{n1}(\alpha), \dots, \hat{a}_{nn}(\alpha))^\top$ of the regression rank scores, defined as a solution of the linear programming problem

$$\sum_{i=1}^n \hat{a}_{ni}(\alpha) Y_i = \max$$

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$$\begin{aligned} \text{under } \sum_{i=1}^n \hat{a}_{ni}(\alpha) &= n(1 - \alpha) \\ \sum_{i=1}^n x_{ij} \hat{a}_{ni}(\alpha) &= (1 - \alpha) \sum_{i=1}^n x_{ij}, \quad j = 2, \dots, p \\ 0 \leq \hat{a}_{ni}(\alpha) &\leq 1, \quad i = 1, \dots, n. \end{aligned} \quad (3)$$

Because of the restrictions in (3), the $\hat{a}_{ni}$ are unchanged if Y_i is replaced with $e_i = Y_i - \mathbf{x}_i^\top \boldsymbol{\beta}$, $i = 1, \dots, n$. As mutually dual, $\hat{\mathbf{a}}_n(\alpha)$ and $\hat{\boldsymbol{\beta}}_n(\alpha)$ are in the following relation:

$$\hat{a}_{ni}(\alpha) = \begin{cases} 1 & \dots & Y_i > \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha) \\ 0 & \dots & Y_i < \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha) \end{cases}, \quad i = 1, \dots, n, \quad (4)$$

while $0 < \hat{a}_{ni}(\alpha) < 1$ for $Y_i = \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)$, $i = 1, \dots, n$; the latter correspond to the optimal basis of p vectors among $\mathbf{x}_1, \dots, \mathbf{x}_n$. Moreover, the identity

$$\sum_{i=1}^n \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha) = - \sum_{i=1}^n Y_i \frac{d\hat{a}_i(\alpha)}{d\alpha} \quad (5)$$

holds at the point α of continuity of $\hat{\boldsymbol{\beta}}(\cdot)$ (see. e.g., Hallin and Jurečková (1999)). The derivative of $\hat{a}_i(\alpha)$ is nonzero only if $Y_i = \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}_n(\alpha)$.

In the location model, where $\mathbf{X} = \mathbf{1}_n$, the regression α -quantile reduces to the $[n\alpha]$ -order statistic $Y_{n:[n\alpha]}$, and the vector of α -regression rank scores reduces to the Hájek's rank scores which can be expressed as follows:

$$a_{ni}^*(R_i, \alpha) = \begin{cases} 0 & \dots & R_i < n\alpha \\ R_i - n\alpha & \dots & (R_i - 1)/n < \alpha \leq R_i/n \\ 1 & \dots & \alpha \leq (R_i - 1)/n \end{cases} \quad i = 1, \dots, n \quad (6)$$

where R_i is the rank of Y_i among $Y_1, \dots, Y_n$. Rank scores $a_n^*(R_i, \alpha)$ were first used by Hájek (1965) as a starting point for the construction of the nonlinear rank tests. We see that the identity (5) is also true in the location model, where $\frac{d\hat{a}_i(\alpha)}{d\alpha} \neq 0$ only for i such that $n\alpha < R_i < n\alpha + 1$, $i = 1, \dots, n$.

In the location model we have i.i.d. observations $Y_1, \dots, Y_n$ with the ranks $R_1, \dots, R_n$, and the vector of ranks is independent of the vector of order statistics $\mathbf{Y}_{(\cdot)} = (Y_{n:1}, \dots, Y_{n:n})$, $Y_{n:1} \leq \dots \leq Y_{n:n}$. The k -th order statistic $Y_{n:k}$ has the density

$$\frac{n!}{(k-1)!(n-k)!} (F(y))^{k-1} (1 - F(y))^{n-k} f(y),$$

while $P\{R_1 = r_1, \dots, R_n = r_n\} = \frac{1}{n!}$ for any permutation $r_1, \dots, r_n$ of numbers $1, 2, \dots, n$.

Generally, if the vector $Y_1, \dots, Y_n$ has density $p(y_1, \dots, y_n)$, the ranks and order statistics can be mutually dependent; their joint distribution can be written in the following way:

$$P(\mathbf{Y}_{(\cdot)} \in B, R_1 = r_1, \dots, R_n = r_n) = \int_B \dots \int p(y_{n:r_1}, \dots, y_{n:r_n}) dy_{n:1} \dots dy_{n:n}.$$

for any Borel subset of the ordered quadrant of $\mathbb{R}^n$ and for any permutation $r_1, \dots, r_n$.

The asymptotic distribution of regression quantiles and of regression rank scores under $n \rightarrow \infty$ was studied by more authors, under various conditions on f and on the regressors [let us mention Koenker and Bassett (1978), Ruppert and Carroll (1980), Gutenbrunner and Jurečková (1992), Gutenbrunner et al. (1993), among others]. The asymptotic properties of autoregression quantiles and the rank scores was studied by Koul and Saleh (1995). However, the proofs of the asymptotic distributions available in the literature impose various restrictions on the tails of the distribution of errors and on the regression matrix, even though the numerical evidence demonstrates that e.g. the regression rank scores tests work under quite general conditions. For instance, the asymptotic distribution of the regression rank scores criterion was proved only when either the tails of the basic distribution are lighter than those of the t -distribution with 4 degrees of freedom or when the score-generating function $\varphi : (0, 1) \mapsto \mathbb{R}^1$ is zero outside $(\varepsilon, 1 - \varepsilon) \subset (0, 1)$ (cf. e.g. Gutenbrunner and Jurečková (1992), Gutenbrunner et al. (1993)). Moreover, the asymptotic distribution, which is typically normal, does not provide the full information on the behavior of $\hat{\beta}(\alpha)$, and it can stretch its true behavior under heavy-tailed f .

We expect that the finite-sample distribution of the α -regression quantile reminds the distribution of the sample quantile in the location model. Koenker and Bassett (1978) provided one possible form of the finite-sample distribution of $\hat{\beta}(\alpha)$ (see also Koenker (2005)). However, these authors themselves do not find their form very tractable and adaptable to practical statistical inference. A possible form of the finite sample distribution, similar to the distribution of the sample quantile, is given in the present paper. Using the regression rank scores, dual to regression quantiles, we first derive the joint distribution of $(\hat{\mathbf{a}}_n(\alpha), \hat{\beta}(\alpha))$ in the linear regression model (1). Unlike in Koenker and Bassett (1978), our main tool is using the score function of $(\hat{\mathbf{a}}_n(\alpha), \hat{\beta}(\alpha))$ and the fact that the score function of a statistic $S = S(\mathbf{Y})$ is equal to the conditional expectation of the score function of $\mathbf{Y}$ under given S , provided it exists (see, e.g., Kagan et al. (1972) and Jurečková and Milhaud (2003)). We also benefit from the fact that the regression quantiles and rank scores are mutually dual optimal solutions of a linear programming problem. Because $0 \leq \hat{a}_i(\alpha) \leq 1$, $i = 1, \dots, n$, the vector $\hat{\mathbf{a}}(\alpha)$ can be interpreted as a test function, and we can use the generalized Neyman-Pearson lemma.

2. Distribution of α -regression quantile under i.i.d. errors

Consider the linear regression model (1) with deterministic regression matrix $\mathbf{X}$ with the first column equal to $\mathbf{1}_n$. Consider first the situation that the errors $e_1, \dots, e_n$ are i.i.d. with distribution function F and density f . We assume that the matrix $\mathbf{X}$ with the rows $\mathbf{x}_1^\top, \dots, \mathbf{x}_n^\top$ has the rank p and that $f(z)$ is absolutely continuous and positive for $z \in (a, b)$ where $a = \inf\{z : F(x) > 0\}$ and $b = \sup\{z : F(z) < 1\}$. The first step is to derive the score function of $(\hat{\mathbf{a}}(\alpha), \hat{\beta}(\alpha))$.

Generally, the score function of the vector $Y_1, \dots, Y_n$ with density $p(y_1, \dots, y_n, \boldsymbol{\theta}) = p(\mathbf{y}, \boldsymbol{\theta})$, $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subset \mathbb{R}^p$, is the vector function

$$\left(\frac{\partial \ln p(\mathbf{y}, \boldsymbol{\theta})}{\partial \theta_j}, j = 1, \dots, p \right)^\top.$$

The score function is in the 1:1 correspondence with the probability distribution, and it is one of its main characteristics. Using the score functions was recommended by Hampel (1973), Field and Hampel (1982), Field and Ronchetti (1990), among others. A natural argument for the use of the score function is that it is linear for the normal distribution, which is considered as a unit in the family of probability distributions.

Fix α and consider the set

$$\begin{aligned} \mathbf{A}_n(\alpha) = & \left\{ \mathbf{a} : 0 \leq a_i \leq 1, i = 1, \dots, n, \sum_{i=1}^n \mathbf{x}_i a_i I[a_i > 0] = (1 - \alpha) \sum_{i=1}^n \mathbf{x}_i \right. \\ & \text{and } 0 < a_{i_j} < 1, j = 1, \dots, p \text{ for } 1 \leq i_1, \dots, i_p \leq n, \\ & \left. \text{such that } \mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_p} \text{ is a basis of } \mathbb{R}^p \right\}. \end{aligned} \quad (7)$$

The score function of the vector $(\hat{\mathbf{a}}(\alpha), \hat{\boldsymbol{\beta}}(\alpha))$ follows.

Lemma 2.1. *Under the conditions stated above, the score function of $(\hat{\mathbf{a}}(\alpha), \hat{\boldsymbol{\beta}}(\alpha))$ has the form*

$$\begin{aligned} \mathbf{r}(\mathbf{a}, \mathbf{b}, \boldsymbol{\beta}) = & - \sum_{i=1}^n \mathbf{x}_i \left\{ f(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \left(\frac{I[a_i = 0]}{F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} - \frac{I[a_i = 1]}{1 - F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} \right) \right. \\ & \left. + I[0 < a_i < 1] \frac{f'(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])}{f(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} \right\}, \quad \mathbf{a} \in \mathbf{A}_n(\alpha), \mathbf{b} \in \mathbb{R}^p. \end{aligned} \quad (8)$$

Proof. The score function of model (1) is the vector function

$$\begin{aligned} & \left(\frac{\partial \ln \prod_{i=1}^p f(y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}{\partial \beta_j}, j = 1, \dots, p \right)^\top \\ & = \left(\frac{\sum_{i=1}^n \partial \ln f(y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}{\partial \beta_j}, j = 1, \dots, p \right)^\top = - \sum_{i=1}^n \mathbf{x}_i \frac{f'(y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}{f(y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}. \end{aligned} \quad (9)$$

Regarding the inequalities (4), the conditional distribution of Y_i , given $[\hat{a}_i(\alpha) = 0 \Leftrightarrow Y_i < \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha)]$ and $[\hat{\boldsymbol{\beta}}(\alpha) = \mathbf{b}]$ has the density

$$\frac{f(y - \mathbf{x}_i^\top \boldsymbol{\beta})}{F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} I[\hat{a}_{ni}(\alpha) = 0].$$

Similarly, the conditional density of Y_i given $[\hat{a}_i(\alpha) = 0 \Leftrightarrow Y_i > \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha)]$ and $[\hat{\boldsymbol{\beta}}(\alpha) = \mathbf{b}]$ is

$$\frac{f(y - \mathbf{x}_i^\top \boldsymbol{\beta})}{1 - F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} I[\hat{a}_{ni}(\alpha) = 1].$$

This further implies for $i = 1, \dots, n$

$$\begin{aligned} & \mathbb{E}_{\boldsymbol{\beta}} \left\{ \left(- \frac{f'(Y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}{f(Y_i - \mathbf{x}_i^\top \boldsymbol{\beta})} \right) \middle| \hat{\boldsymbol{\beta}}_n(\alpha) = \mathbf{b}, \hat{a}_{ni}(\alpha) = 0 \right\} \\ & = \frac{1}{F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} \int_{-\infty}^{\mathbf{x}_i^\top \mathbf{b}} \left(- \frac{f'(y - \mathbf{x}_i^\top \boldsymbol{\beta})}{f(y - \mathbf{x}_i^\top \boldsymbol{\beta})} \right) f(y - \mathbf{x}_i^\top \boldsymbol{\beta}) dy = \frac{-f(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])}{F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}])} \end{aligned} \quad (10)$$

and similarly,

$$\begin{aligned} & \mathbb{E}_{\beta} \left\{ \left(-\frac{f'(Y_i - \mathbf{x}_i^{\top} \beta)}{f(Y_i - \mathbf{x}_i^{\top} \beta)} \right) \middle| \hat{\beta}_n(\alpha) = \mathbf{b}, \hat{a}_{ni}(\alpha) = 1 \right\} \\ &= \frac{1}{1 - F(\mathbf{x}_i^{\top} [\mathbf{b} - \beta])} \int_{\mathbf{x}_i^{\top} \mathbf{b}}^{\infty} \left(-\frac{f'(y - \mathbf{x}_i^{\top} \beta)}{f(y - \mathbf{x}_i^{\top} \beta)} \right) f(y - \mathbf{x}_i^{\top} \beta) dy = \frac{f(\mathbf{x}_i^{\top} [\mathbf{b} - \beta])}{1 - F(\mathbf{x}_i^{\top} [\mathbf{b} - \beta])}. \end{aligned} \quad (11)$$

Moreover,

$$\mathbb{E}_{\beta} \left\{ \left(-\frac{f'(Y_i - \mathbf{x}_i^{\top} \beta)}{f(Y_i - \mathbf{x}_i^{\top} \beta)} \right) \middle| \hat{\beta}_n(\alpha) = \mathbf{b}, 0 < \hat{a}_{ni}(\alpha) < 1 \right\} = -\frac{f'(\mathbf{x}_i^{\top} [\mathbf{b} - \beta])}{f(\mathbf{x}_i^{\top} [\mathbf{b} - \beta])}, \quad (12)$$

The score function of $(\hat{\mathbf{a}}_n(\alpha), \hat{\beta}_n(\alpha))$ follows from (10), (11), (12). $\square$

Integrating (8) over β , we obtain the joint distribution of the α -regression rank scores $\hat{\mathbf{a}}_n(\alpha)$ and of the α -regression quantile $\hat{\beta}_n(\alpha)$:

Lemma 2.2. *Under the above conditions, the joint distribution of the α -regression rank score and α -regression quantile has the form*

$$\begin{aligned} & P_{\beta} \{ \hat{\beta}_n(\alpha) \in \mathbf{B}, \hat{\mathbf{a}}_n(\alpha) = \mathbf{a} \} \\ &= C \int_{\mathbf{B}} \prod_{i=1}^n \left[\left(F(\mathbf{x}_i^{\top} [\mathbf{b} - \beta]) \right)^{I[a_i=0]} \left(1 - F(\mathbf{x}_i^{\top} [\mathbf{b} - \beta]) \right)^{I[a_i=1]} \left(f(\mathbf{x}_i^{\top} [\mathbf{b} - \beta]) \right)^{I[0 < a_i < 1]} \right] d\mathbf{b}, \end{aligned} \quad (13)$$

$\mathbf{a} \in \mathbf{A}_n(\alpha)$, $\mathbf{B} \in \mathcal{B}^p$, $C > 0$ is a normalizing constant.

Proof. Denote $\mathcal{H} = \left\{ \left(\sum_{i=1}^n x_{i1} a_i, \sum_{i=1}^n x_{i2} a_i, \dots, \sum_{i=1}^n x_{ip} a_i \right)^{\top} : 0 \leq a_i \leq 1, i = 1, \dots, n \right\}$. Then $\mathcal{H}$ is closed and convex. Let $\mathbf{a} \in \mathbf{A}_n(\alpha)$, $0 < a_{ij} < 1$ for $j = 1, \dots, p$, and let $\mathbf{x}_1^{\top}, \dots, \mathbf{x}_p^{\top}$ be linearly independent rows of matrix $\mathbf{X}$. Fix $\mathbf{b} \in \mathbb{R}^p$, $\mathbf{b} \neq \mathbf{0}$, and choose $y_1, \dots, y_n$ so that

$$\begin{aligned} y_{ij} &= \mathbf{x}_j^{\top} \mathbf{b}, \quad j = 1, \dots, p \\ y_i &> \mathbf{x}_i^{\top} \mathbf{b} \quad \text{if } a_i = 1, \\ y_i &< \mathbf{x}_i^{\top} \mathbf{b} \quad \text{if } a_i = 0, \quad i = 1, \dots, n. \end{aligned}$$

Then $\mathbf{a}$ maximizes $\sum_{i=1}^n y_i a_i$ under $\mathbf{a} \in \mathbf{A}_n(\alpha)$, and $\mathbf{b}$ is the α -regression quantile pertaining to $(y_1, \dots, y_n)$. On the other hand, let $\mathbf{a}$ solve (3). Because $(1 - \alpha) \sum_{i=1}^n \mathbf{x}_i$ is an inner point of $\mathcal{H}$, there exists the vector $\mathbf{b}$ of Lagrange multipliers such that

$$\begin{aligned} y_i &> \mathbf{x}_i^{\top} \mathbf{b} \quad \text{if } a_i = 1, \\ y_i &< \mathbf{x}_i^{\top} \mathbf{b} \quad \text{if } a_i = 0, \quad i = 1, \dots, n, \end{aligned}$$

the remaining a_i corresponding to $y_i = \mathbf{x}_i^{\top} \mathbf{b}$ and the values of $\mathbf{b}$ are determined so that $\sum_{i=1}^n \mathbf{x}_i a_i = (1 - \alpha) \sum_{i=1}^n \mathbf{x}_i$ (cf. the Neyman-Pearson lemma [Lemma 3 in (18), Chapter 3]). $\square$

Lemmas 2.1 and 2.2 finally give the density of the α -regression quantile.

Theorem 2.3. Consider the linear regression model (1) with deterministic regression matrix $\mathbf{X}$ of the rank p , with the first column equal to $\mathbf{1}_n$. Assume that the errors $e_1, \dots, e_n$ are i.i.d. with absolutely continuous distribution function F and with density f , absolutely continuous and positive for $z \in (a, b)$ where $a = \inf\{z : F(z) > 0\}$ and $b = \sup\{z : F(z) < 1\}$. Then the α -regression quantile $\hat{\beta}(\alpha)$, $0 < \alpha < 1$ has the density

$$g(\mathbf{b}; \alpha) = \sum_{\mathbf{a} \in \mathbf{A}_n(\alpha)} \prod_{i=1}^n \left[\left(F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I[a_i=0]} \left(1 - F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I[a_i=1]} \left(f(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I[0 < a_i < 1]} \right],$$

$$\mathbf{b} \in \mathbb{R}^p. \quad (14)$$

3. Remarks and application to the tail behavior of regression quantiles

In the location model with $\mathbf{X} = \mathbf{1}_n$, density (14) reduces to the density of the order statistic $Y_{n:k}$ with $k = \lceil n\alpha \rceil$. Indeed, then $\mathbf{X}_n \equiv \mathbf{1}_n$ and the regression rank scores reduce to the *Hájek scores* (6). Then $\hat{a}_{ni}(\alpha) \in (0, 1)$ iff the rank of Y_i equals $R_i = \lceil n\alpha \rceil$, what happens just for one among $Y_1, \dots, Y_n$ with probability 1, and $\hat{a}_{nk}(\alpha) = 0$ or 1 according to whether R_k is greater or less than $\lceil n\alpha \rceil$, respectively.

Even in the location model the moments of order statistics must be calculated numerically. In spite of that, the BLUE estimators of Lloyd (1952) [see also Sarhan and Greenberg (1962), David (1981)] are considered useful in various contexts. For instance, the Shapiro-Wilk (1965) test is a surprisingly good intuitive test of normality based on the ratio of the MLE and BLUE estimators of σ under normal f . Sen et al. (2003) extended the test to the model with the nuisance linear regression, where it also works well.

With the above methods we can derive the joint distribution of two regression quantiles, $\hat{\beta}(\alpha_1), \hat{\beta}(\alpha_2)$. Their expectations and covariances are sufficient just under $\boldsymbol{\beta} = \mathbf{0}$, because of the affine equivariance of regression quantiles and of the invariance of regression rank scores.

Even if the finite sample distribution is more complex than the asymptotic one, and the finite sample moments are calculated via the numerical integration, we cannot limit ourselves to the asymptotics, which can stretch the truth and is sometimes informative only for a very large number of observations.

As an illustration, we apply the methods of Section 2 to the study of the finite sample tail behavior of regression quantiles. Typically, the quantiles and other estimators inherit their tail behavior from the parent distribution, though their asymptotic (normal) distribution is light-tailed. Following He et al. (1990), consider the following finite-sample measure of the right-tail performance of a regression estimator $\mathbf{T}_n$ in model (1) (assume $0 < F < 1$, $f > 0$, for the brevity)

$$B(\mathbf{T}_n, \gamma) = \frac{-\ln P_\beta \left(\max_{1 \leq i \leq n} \left\{ \mathbf{x}_i^\top (\mathbf{T}_n - \boldsymbol{\beta}) \right\} > \gamma \right)}{-\ln(1 - F(\gamma))}, \quad \gamma > 0. \quad (15)$$

The probability in (15) tends to 0 as $\gamma \rightarrow \infty$, but the rate of this convergence depends on the tail of F . Then the bounds of (15) under finite n and $\gamma \rightarrow \infty$ are of interest, because they show how faster is the convergence $P_\beta \left(\max_{1 \leq i \leq n} \left\{ \mathbf{x}_i^\top (\mathbf{T}_n - \boldsymbol{\beta}) \right\} > \gamma \right) \rightarrow 0$ than $1 - F(\gamma) \rightarrow 0$ as $\gamma \rightarrow \infty$.

Put $\boldsymbol{\beta} = \mathbf{0}$, without loss of generality, and assume for simplicity that $n\alpha = k > 0$, an integer, and $k + p < n$. Regarding (3) and (4), we can write

$$P_0 \left(\max_i \left\{ \mathbf{x}_i^\top \hat{\beta}(\alpha) \right\} \leq \gamma \right) \leq P_0 \left(\text{at least } k + 1 \text{ among } Y\text{'s are } \leq \gamma \right)$$

hence

$$\begin{aligned} P_0\left(\max_i \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha) > \gamma\right) &\geq P_0(Y_{n:k+1} > \gamma) = \frac{n!}{k!(n-k-1)!} \int_{F(\gamma)}^1 u^k (1-u)^{n-k-1} du \\ &\geq \frac{n!}{k!(n-k-1)!} (F(\gamma))^k \int_0^{1-F(\gamma)} v^{n-k-1} dv = \frac{n!}{k!(n-k)!} (F(\gamma))^k (1-F(\gamma))^{n-k}. \end{aligned}$$

It implies

$$\overline{\lim}_{\gamma \rightarrow \infty} \frac{-\ln P_0\left(\max_i \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha) > \gamma\right)}{-\ln(1-F(\gamma))} \leq n-k.$$

Similarly,

$$P_0\left(\max_i \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha) \leq \gamma\right) \geq P_0(Y_{n:k+p} \leq \gamma),$$

hence

$$\begin{aligned} P_0\left(\max_i \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha) > \gamma\right) &\leq \frac{n!}{(k+p-1)!(n-k-p)!} \int_{F(\gamma)}^1 u^{k+p-1} (1-u)^{n-k-p} du \\ \frac{n!}{(k+p-1)!(n-k-p)!} (1-F(\gamma))^{n-k-p} (1-(F(\gamma))^{k+p}) &\leq \frac{n!(k+p-1)}{(k+p)!(n-k-p)!} (1-F(\gamma))^{n-k-p+1} \end{aligned}$$

and this implies

$$\underline{\lim}_{\gamma \rightarrow \infty} \frac{-\ln P_0\left(\max_i \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(\alpha) > \gamma\right)}{-\ln(1-F(\gamma))} \geq n-k-p+1$$

Summarizing, we conclude

$$n-k-p+1 \leq \underline{\lim}_{\gamma \rightarrow \infty} B(\hat{\boldsymbol{\beta}}(\alpha), \gamma) \leq \overline{\lim}_{\gamma \rightarrow \infty} B(\hat{\boldsymbol{\beta}}(\alpha), \gamma) \leq n-k \quad (16)$$

what is consistent with the tail behavior of the order statistic $Y_{n:k}$ in the location model (where $p = 1$).

4. Distribution of the extreme regression quantile

After a slight modification, the above method applies to the extreme regression quantiles, corresponding to $\alpha = 0, 1$. The *maximal regression quantile* $\hat{\boldsymbol{\beta}}(1)$ is a solution of the minimization problem:

$$\min_{\mathbf{b} \in \mathbb{R}^p} \sum_{i=1}^n (Y_i - \mathbf{x}_i^\top \mathbf{b})^+ \quad (17)$$

where $z^+ = \max(z, 0)$ denotes the positive part of z . The minimization (17) can be alternatively described as any solution to the linear program:

$$\min_{\mathbf{b} \in \mathbb{R}^p} \sum_{i=1}^n \{\mathbf{x}_i^\top \mathbf{b}\} \quad \text{s.t.} \quad Y_i \leq \mathbf{x}_i^\top \mathbf{b}, \quad i = 1, \dots, n. \quad (18)$$

This statistic was studied by Smith (1994) who derived its asymptotic distribution under heavy-tailed distribution F and under some conditions on the $\mathbf{x}_i$. Portnoy and Jurečková (1999), Knight

(2001), Chernozhukov (2005), Jurečková (2007) (among others), derived various forms of the asymptotic distributions of the extreme regression quantiles under various conditions. As in the extreme value theory, the asymptotic distribution of the extreme regression quantile depends on the domain of attraction of the model errors, and the results in the literature are proved under various additional conditions on f and on $\mathbf{X}$.

If $(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_p})^\top = \tilde{\mathbf{x}}$ is the optimal basis of the problem (18), then $Y_{i_j} = \mathbf{x}_{i_j}^\top \hat{\boldsymbol{\beta}}(1)$ for $j = 1, \dots, p$ and $Y_i < \mathbf{x}_i^\top \hat{\boldsymbol{\beta}}(1)$ for the remaining components. Denote the set

$$C_n = \left\{ \mathbf{c} = (c_1, \dots, c_n)^\top : c_{i_1}, \dots, c_{i_p} = 1 \text{ for } 1 \leq i_1 < \dots < i_p \leq n \right. \\ \left. \text{such that } (\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_p}) \text{ is a basis of } \mathbb{R}^p, \text{ and } c_i = 0 \text{ otherwise.} \right\} \quad (19)$$

Proceeding analogously as in Lemmas 2.1, 2.2 and Theorem 2.3, we obtain the density of the extreme regression quantile:

Theorem 4.1. *Under the conditions of Theorem 2.3, the extreme regression quantile $\hat{\boldsymbol{\beta}}(1)$ has the density*

$$g(\mathbf{b}; 1) = \sum_{\mathbf{c} \in C_n} \prod_{i=1}^n \left[\left(F(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I_{[c_i=0]}} \left(f(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I_{[c_i=1]}} \right], \quad \mathbf{b} \in \mathbb{R}^p. \quad (20)$$

Proof. The optimal solution $\hat{\boldsymbol{\beta}}(1) = \mathbf{b}$ of (18) determines the optimal basis $(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_p})$ and $Y_{i_j} = \mathbf{x}_{i_j}^\top \mathbf{b}$, $j = 1, \dots, p$. For the remaining i we have the conditional probability

$$P(Y_i < y \mid \hat{\boldsymbol{\beta}}(1) = \mathbf{b}) = \frac{F(y - \mathbf{x}_i^\top \boldsymbol{\beta})}{F(\mathbf{x}_i^\top (\mathbf{b} - \boldsymbol{\beta}))} I[Y_i < \mathbf{x}_i^\top \mathbf{b}].$$

Following the steps in the proofs of Lemmas 2.1, 2.2 and Theorem 2.3, we arrive at (20). In the location model with $\mathbf{X} = \mathbf{1}_n$, density (20) reduces to $n(F(x))^{n-1} f(x)$, the density of the maximal order statistic. $\square$

5. Distribution of α -regression quantile under non-i.i.d. errors

The distribution of the regression quantile can be obtained even when the errors e_i are non-identically distributed, but still independent. Consider the model (1) with the change that the error e_i has a distribution function F_i with density f_i , hence Y_i has distribution function $F_i(y - \mathbf{x}_i^\top \boldsymbol{\beta})$, $i = 1, \dots, n$. The distribution functions $F_1, \dots, F_n$ are generally different, but all are assumed to be absolutely continuous and strictly increasing on $\mathbb{R}^1$ with absolutely continuous densities. A special case is the heteroscedastic model

$$Y_i = \mathbf{x}_i^\top \boldsymbol{\beta} + \sigma_i e_i$$

with $e_1, \dots, e_n$ i.i.d. with distribution function F , i.e. $F_i(z) = F\left(\frac{z}{\sigma_i}\right)$, $i = 1, \dots, n$.

The score function of the random vector $(Y_1, \dots, Y_n)$ in the non-i.i.d. model is

$$-\sum_{i=1}^n \mathbf{x}_i \frac{f'_i(y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}{f_i(y_i - \mathbf{x}_i^\top \boldsymbol{\beta})}.$$

The definitions (2) and (3) of the regression quantile and of the regression rank scores are unchanged, and so are their algebraic properties. So, following the steps of proofs of Lemmas 2.1 and 2.2, we shall arrive in the following form of distribution of the regression quantile in this situation:

Theorem 5.1. *Consider the linear regression model (1) with the deterministic regression matrix $\mathbf{X}$ of the rank p , with the first column equal to $\mathbf{1}_n$. Assume that the errors $e_1, \dots, e_n$ are independent with absolutely continuous and strictly increasing distribution functions F_i , $0 < F_i(x) < 1$, $x \in \mathbb{R}$, which have absolutely continuous densities $f_i > 0$, $i = 1, \dots, n$. Then the α -regression quantile $\hat{\beta}^*(\alpha)$, $0 < \alpha < 1$ has the density*

$$g^*(\mathbf{b}; \alpha) = \sum_{\mathbf{a} \in \mathbf{A}_n(\alpha)} \prod_{i=1}^n \left[\left(F_i(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I[a_i=0]} \left(1 - F_i(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I[a_i=1]} \left(f_i(\mathbf{x}_i^\top [\mathbf{b} - \boldsymbol{\beta}]) \right)^{I[0 < a_i < 1]} \right],$$

$$\mathbf{b} \in \mathbb{R}^p. \quad (21)$$

The distribution of α -regression quantile in the heteroscedastic model we obtain as a corollary.

Concluding remarks

The finite sample distributions of the α -regression quantile and of the extreme regression quantile are analogous to the corresponding distributions in the location model; this again confirms that the regression quantile is an extension of the sample quantile. Moreover, these distributions are true for a broad class of underlying distributions of errors, even in the non-i.i.d. case, while the asymptotic distributions of regression quantiles were derived only under various restrictive conditions.

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