Parameter sensitivity of CIR process

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Abstract

We study the differentiability of the CIR process with respect to its parameters. We give a stochastic representation for these derivatives in terms of the paths of V.

1 Introduction

The CIR process is defined as the unique solution of the following stochastic differential equation:

$$dV_t = (a - bV_t)dt + \sigma\sqrt{V_t}dW_t, \quad V_0 = v, \tag{1.1}$$

where $a,\ \sigma,\ v\ \geq 0$ and $b \in \mathbb{R}$ (see [8] for the existence and uniqueness of the solution of the SDE). This process is widely used in finance to model short term interest rate (see [3]) but also used to model stochastic volatility in the Heston stochastic volatility model. The option prices in these models depend in the values of the parameters of CIR process. On the other hand, these parameters are often calibrated to market prices of derivatives, so they tend to change their values regularly. The knowledge of the derivatives of the CIR process with respect to its parameters is therefore crucial for the study the sensitivities of prices in these models.

The most common approach to study the sensitivity of stochastic differential equation with respect to its parameters is to use the Malliavin calculus, especially for the sensitivity with respect to the initial value. The Malliavin derivative gives a stochastic representation of the sensitivity of process with respect to its initial value. We note that the coefficients of (1.1) are neither differentiable in 0 nor globally Lipschitz, so the standard results (see e.g [9],[5]) cannot be used here. Nevertheless, for the special case of CIR process, Alòs and Ewald ([1]) show the existence of Malliavin derivative of the CIR process under assumption ($2a > \sigma^2$). In mathematical finance, the sensitivities of option prices with respect to not only the initial point, but also other parameters, need to be studied very carefully.

In this article, we study the differentiability of the solution of (1.1) with respect to the parameters a, b and σ in $\mathcal{L}_p$ sense (see next section). We show that, under some assumptions, this process is differentiable with respect to these parameters and give a stochastic representation of its derivatives.

2 Differentiability

For technical reasons, we will rather consider the square root of V^v, denoted X^v. Throughout this paper, we assume that

$$2a \geq \sigma^2 \tag{2.1}$$

Under this assumption, we have for any $T, v > 0$, $\mathbb{P}\left(\forall t \in [0,T] : V_t^v > 0\right) = 1$. The process X^v is the unique solution of the following stochastic differential equation

$$dX_t^v = \left(\left(\frac{a}{2} - \frac{\sigma^2}{8} \right) \frac{1}{X_t^v} - \frac{b}{2} X_t^v \right) dt + \frac{\sigma}{2} dW_t, \ X_0^v = \sqrt{v}. \tag{2.2}$$

We start by studying the differentiability of X with respect to the parameter a. We consider here the $\mathcal{L}_p$-differentiability of the function $a \longmapsto X^v(a)$, i.e the existence of a process $\dot{X}_a$ so that

$$\lim_{\epsilon \to 0} \left\| \sup_{s \leq t} \left| \frac{X_s^v(a+\epsilon) - X_s^v(a)}{\epsilon} - \dot{X}_a(s) \right| \right\|_p = 0 \tag{2.3}$$

We have the following result

Proposition 2.1. *Let $b \in \mathbb{R}$ and $\sigma,\ x \geq 0$. For every $a \in]\sigma^2, +\infty[$, let X_a be the unique* *solution of the SDE :*

$$dX_t = \left(\left(\frac{a}{2} - \frac{\sigma^2}{8} \right) \frac{1}{X_t} - \frac{b}{2} X_t \right) dt + \frac{\sigma}{2} dW_t, \; X(0) = x$$

and let $a_0 > \sigma^2$. Then the function $a \longmapsto X_a$ is $\mathcal{L}_p$-differentiable at a_0, for any $1 \leq p \leq \frac{2a_0}{\sigma^2} - 1$ and its derivative $(\dot{X}_a)$ is given by

$$\dot{X}_a(t) = \int_0^t \frac{1}{2X_s} \exp\left(-\frac{b}{2}(t-u) - (\frac{a}{2} - \frac{\sigma^2}{8}) \int_s^t \frac{du}{X_u^2} \right) ds. \tag{2.4}$$

Proof: Let X^ϵ be the unique solution of the stochastic differential equation

$$dX_t^\epsilon = \left(\left(\frac{a+\epsilon}{2} - \frac{\sigma^2}{8} \right) \frac{1}{X_t^\epsilon} - \frac{b}{2} X_t^\epsilon \right) dt + \frac{\sigma}{2} dW_t, \; X_0^\epsilon = \sqrt{v}.$$

For $\epsilon > 0$, define $R_0^\epsilon(t) := X_t^\epsilon - X_t$. We can easily see that R_0^ϵ is given by

$$R_0^\epsilon(t) = \epsilon U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1} \frac{1}{2X_s} ds,$$

where

$$U^\epsilon = \exp\left(-\int_0^t \alpha_s^\epsilon ds \right), \quad \text{with} \;\; \alpha_t^\epsilon = \left(\frac{a+\epsilon}{2} - \frac{\sigma^2}{8} \right) \frac{1}{X_s^\epsilon X_s} + \frac{b}{2}.$$

We have, using the fact that for any $s \leq t$, $e^{-\int_s^t \alpha_u^\epsilon du} \leq e^{-bt/2} \vee 1$ a.s,

$$\frac{|R_0^\epsilon(t)|}{\epsilon} \;\; \leq \;\; \frac{t(e^{-bt/2} \vee 1)}{2} \sup_{s \leq t} \frac{1}{X_s^v}.$$

On the other hand, we have, using Lemma 2.3.2 of [4] ,

$$\forall p < 2\left(\frac{2a}{\sigma^2} - 1 \right), \;\; \mathbb{E}\left[\sup_{s \leq t} \frac{1}{X_s^p} \right] < +\infty. \tag{2.5}$$

In particular, we have for any $p \in \left[1, 2\left(\frac{2a}{\sigma^2} - 1\right)\right[$,

$$\|R_0^\epsilon\|_p \leq C\epsilon.$$

Let's now set

$$\dot{X}_a(t) := \lim_{\epsilon\to 0}\frac{R_0^\epsilon}{\epsilon}(t) = U_t^0\int_0^t (U_s^0)^{-1}\frac{1}{2X_s}ds.$$

We have $\left\|\dot{X}_a\right\|_p \leq C$. Furthermore, $\dot{X}_a$ is solution of the stochastic differential equation:

$$d\dot{X}_a(t) = -\left(\left(\frac{a}{2}-\frac{\sigma^2}{8}\right)\frac{1}{X_t^2}+\frac{b}{2}\right)\dot{X}_a(t)dt + \frac{1}{2X_t}dt.$$

Let $R_1^\epsilon(t) = X_t^\epsilon - X_t - \epsilon\dot{X}_a(t)$. The process R_1^ϵ is a solution of the stochastic differential equation

$$dR_1^\epsilon(t) = \left(-\alpha_t^\epsilon R_1^\epsilon(t) - \epsilon\dot{X}_a(t)\left(\alpha_t^\epsilon - \left[(\frac{a}{2}-\frac{\sigma^2}{8})\frac{1}{X_t^2}+\frac{b}{2}\right)\right]\right)dt.$$

On the other hand, we have

$$\alpha_t^\epsilon - \left((\frac{a}{2}-\frac{\sigma^2}{8})\frac{1}{X_t^2}+\frac{b}{2}\right) = -\left(\frac{\alpha_t^\epsilon}{X_t}-\frac{b}{2X_t}\right)R_0^\epsilon(t) + \frac{\epsilon}{2X_t^2}.$$

It follows that R_1^ϵ can be written as

$$R_1^\epsilon(t) = U_t^\epsilon\int_0^t (U_s^\epsilon)^{-1}\left(\dot{X}_a(t)\left(\epsilon\left(\frac{\alpha_t^\epsilon}{X_t}-\frac{b}{2X_t}\right)R_0^\epsilon(t) - \frac{\epsilon^2}{2X_t^2}\right)\right)ds,$$

Using (2.5) and the fact that for any $s\leq t$, we have $e^{-\int_s^t\alpha_u^\epsilon du} \leq 1\vee e^{-bt/2}$ and $\int_0^t \alpha_s^\epsilon\, e^{-\int_s^t\alpha_u^\epsilon du}ds = 1 - e^{-\int_0^t\alpha_u^\epsilon du}$, we get

$$\forall 1\leq p < \frac{2a}{\sigma^2}-1,\quad \|R_1^\epsilon\|_p \leq C\epsilon^2. \qquad \square$$

The differentiability with respect to b is obtained in the same. The proof of the next Proposition is almost identical to Proposition 2.1.

Proposition 2.2. *Let $x, a, \sigma \geq 0$ so that $4a > 3\sigma^2$. For every $b\in\mathbb{R}$, let X_b be the unique solution of the SDE : $dX_t = \left(\left(\frac{a}{2}-\frac{\sigma^2}{8}\right)\frac{1}{X_t}-\frac{b}{2}X_t\right)dt + \frac{\sigma}{2}dW_t,\ X_0 = x$ and let $b_0\in\mathbb{R}$. The function $b\longmapsto X_b$ is $\mathcal{L}_p$-differentiable at b_0, for any $1\leq p < 2(\frac{2a}{\sigma^2}-1)$ and its derivative $\dot{X}_b$ is given by*

$$\dot{X}_b(t) = -\int_0^t \frac{X_s}{2}\exp\left(-\frac{b}{2}(t-u) - (\frac{a}{2}-\frac{\sigma^2}{8})\int_s^t\frac{du}{X_u^2}\right)ds \tag{2.6}$$

We now consider the differentiability of X with respect to the parameter σ. We propose an extension of the result of Benhamou et al (cf. [2]) who show that $\sigma \longmapsto X$ is $\mathcal{C}^2$ in neighborhood of 0. We will show that this function is $\mathcal{C}^1$ in $[0, \sqrt{a}[$ and C^∞ around 0.

Proposition 2.3. *For any $\sigma \in [0, \sqrt{a}[$, the function $\sigma \longmapsto X$ is $\mathcal{C}^1$ at σ in $\mathcal{L}_p$-sense, for every $p \in [1, \frac{2a}{\sigma^2} - 1[$ and its derivative is the unique solution of the SDE :*

$$d\dot{X}_\sigma(t) = \left(-\frac{\sigma}{4X_t} - \left(\frac{a}{2} - \frac{\sigma^2}{8}\right)\frac{\dot{X}_\sigma(t)}{X_t} - \frac{b}{2}\dot{X}_\sigma(t)\right)dt + \frac{1}{2}dW_t. \tag{2.7}$$

Proof: Let X^ϵ be the unique solution of the SDE :

$$dX_t^\epsilon = \left(\left(\frac{a}{2} - \frac{(\sigma+\epsilon)^2}{8}\right)\frac{1}{X_t^\epsilon} - \frac{b}{2}X_t^\epsilon\right)dt + \frac{\sigma+\epsilon}{2}dW_t,\ X_0^\epsilon = \sqrt{v}.$$

Let set $R_0^\epsilon(t) = X_t^\epsilon - X_t$. In particular, R_0^ϵ solves the stochastic differential equation:

$$\begin{aligned} dR_0^\epsilon(t) &= \left(\left(\frac{a}{2} - \frac{(\sigma+\epsilon)^2}{8}\right)\frac{1}{X_t^\epsilon} - \frac{b}{2}X_t^\epsilon - \left(\frac{a}{2} - \frac{\sigma^2}{8}\right)\frac{1}{X_t} + \frac{b}{2}X_t\right)dt + \frac{\epsilon}{2}dW_t \\ &= \left(-\left[\left(\frac{a}{2} - \frac{(\sigma+\epsilon)^2}{8}\right)\frac{1}{X_s^\epsilon X_s} + \frac{b}{2}\right]R_0^\epsilon(t) - \frac{2\epsilon\sigma+\epsilon^2}{8X_t}\right)dt + \frac{\epsilon}{2}dW_t. \end{aligned}$$

It follows that R_0^ϵ can be written as

$$R_0^\epsilon(t) = U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1}\left(-\frac{2\epsilon\sigma+\epsilon^2}{8X_s}ds + \frac{\epsilon}{2}dW_s\right),$$

where U^ϵ is given by

$$U_t^\epsilon = \exp\left(-\int_0^t \alpha_s^\epsilon ds\right) \tag{2.8}$$

and

$$\alpha_s^\epsilon = \left(\frac{a}{2} - \frac{(\sigma+\epsilon)^2}{8}\right)\frac{1}{X_s^\epsilon X_s} + \frac{b}{2}. \tag{2.9}$$

Applying the Itô formula to the product $(U_t^\epsilon)^{-1}W_t$, we have

$$R_0^\epsilon(t) = -\frac{2\epsilon\sigma+\epsilon^2}{8}U_t^\epsilon\int_0^t (U_s^\epsilon)^{-1}\frac{ds}{X_s} + \frac{\epsilon}{2}W_t + U_t^\epsilon\int_0^t W_s d(U^\epsilon)_s^{-1}.$$

On the other hand, using the fact that $\alpha_t^\epsilon \geq b/2, a.s$, we know that for any $s \leq t$, we have $0 \leq U_t^\epsilon (U_s^\epsilon)^{-1} \leq 1 \vee e^{-bt/2}$, $a.s.$ It follows that

$$\begin{aligned}
|R_0^\epsilon(t)| &\leq c(t) \int_0^t \frac{ds}{X_s} + \frac{\epsilon}{2} \left(\sup_{s\leq t} W_s + \sup_{s\leq t} W_s (1 - U_t^\epsilon) U_t^\epsilon \right) \\
&\leq c(t) \sup_{s\leq t} \frac{1}{X_s} + \epsilon \sup_{s\leq t} W_s (1 + U_t^\epsilon) U_t^\epsilon .
\end{aligned}$$

Using (2.5), we have, for any $1 \leq p < 2\left(\frac{2a}{\sigma^2} - 1\right)$,

$$\|R_0^\epsilon\|_p \leq C\epsilon. \tag{2.10}$$

Let's now set

$$\dot{X}_\sigma(t) := U_t^0 \int_0^t (U_s^0)^{-1} \left(-\frac{\sigma}{4X_s} ds + \frac{1}{2} dW_s \right).$$

We have $\left\| \dot{X}_\sigma \right\|_p \leq C$. Furthermore, we can easily see that $\dot{X}_\sigma$ is solution to the stochastic differential equation:

$$d\dot{X}_\sigma(t) = -\left(\left(\frac{a}{2} - \frac{\sigma^2}{8} \right) \frac{1}{X_t^2} + \frac{b}{2} \right) \dot{X}_\sigma(t) dt - \frac{\sigma}{4X_t} dt + \frac{1}{2} dW_t.$$

Set $R_1^\epsilon(t) = X_t^\epsilon - X_t - \epsilon \dot{X}_\sigma(t)$. The process R_1^ϵ solves the stochastic differential equation:

$$dR_1^\epsilon(t) = \left(-\alpha_t^\epsilon R_1^\epsilon(t) - \epsilon \dot{X}_\sigma(t) \left(\alpha_t^\epsilon - \left[(\frac{a}{2} - \frac{\sigma^2}{8}) \frac{1}{X_t^2} + \frac{b}{2} \right) \right] - \frac{\epsilon^2}{8X_t} \right) dt.$$

On the other hand, we can easily see that

$$\alpha_t^\epsilon - \left((\frac{a}{2} - \frac{\sigma^2}{8}) \frac{1}{X_t^2} + \frac{b}{2} \right) = -\left(\frac{\alpha_t^\epsilon}{X_t} - \frac{b}{2X_t} \right) R_0^\epsilon(t) - \frac{2\epsilon\sigma + \epsilon^2}{8X_t^2}.$$

It follows that R_1^ϵ can be written as

$$R_1^\epsilon(t) = U_t^\epsilon \int_0^t (U_s^\epsilon)^{-1} \left(-\frac{\epsilon^2}{8X_s} ds + \epsilon \dot{X}_\sigma(s) \left(\left(\frac{\alpha_s^\epsilon}{X_s} - \frac{b}{2X_s} \right) R_0^\epsilon(s) + \frac{2\epsilon\sigma + \epsilon^2}{8X_s^2} \right) \right) ds.$$

We have

$$
\begin{aligned}
|R_1^\epsilon(t)| &\leq \int_0^t U_t^\epsilon (U_s^\epsilon)^{-1} \left(\frac{\epsilon^2}{8X_s} ds + \epsilon |\dot{X}_\sigma(s)| \left(\left(\frac{\alpha_t^\epsilon}{X_s} + \frac{b}{2X_s} \right) |R_0^\epsilon(s)| + \frac{2\epsilon\sigma + \epsilon^2}{8X_s^2} \right) \right) ds \\
&\leq \int_0^t U_t^\epsilon (U_s^\epsilon)^{-1} \left(\frac{\epsilon^2}{8X_s} ds + \epsilon |\dot{X}_\sigma(t)| \left(\frac{b}{2X_s} |R_0^\epsilon(s)| + \frac{2\epsilon\sigma + \epsilon^2}{8X_s^2} \right) \right) ds + \\
&\quad \epsilon \int_0^t U_t^\epsilon (U_s^\epsilon)^{-1} \frac{\alpha_t^\epsilon}{X_s} |\dot{X}_\sigma(s)| |R_0^\epsilon(s)| ds \\
&\leq c(t) \int_0^t \left(\frac{\epsilon^2}{8X_s} ds + \epsilon |\dot{X}_\sigma(t)| \left(\frac{b}{2X_s} |R_0^\epsilon(s)| + \frac{2\epsilon\sigma + \epsilon^2}{8X_s^2} \right) \right) ds \\
&\quad + \epsilon c_2(t) \sup_{s \leq t} \left(\frac{|\dot{X}_\sigma(s)| |R_0^\epsilon(s)|}{X_s} \right).
\end{aligned}
$$

Finally, using (2.5), we have, for any $1 \leq p < \left(\frac{2a}{\sigma^2} - 1\right)$,

$$
\|R_1^\epsilon\|_p \leq C\epsilon^2. \quad \square
$$

Proposition 2.4. *Under the assumptions of Propositions 2.3, 2.1, 2.2, the solution of the SDE* (1.1) *is differentiable with respect to the parameters a, b and σ. Its derivatives, denoted by $\dot{V}_a$, $\dot{V}_b$ and $\dot{V}_\sigma$ respectively, are given as*

$$
\begin{aligned}
\dot{V}_a(t) &= \sqrt{V}_t \int_0^t \frac{1}{\sqrt{V}_s} \exp\left(-\frac{b}{2}(t-u) - (\frac{a}{2} - \frac{\sigma^2}{8}) \int_s^t \frac{du}{V_u} \right) ds, \\
\dot{V}_b(t) &= -\sqrt{V}_t \int_0^t \sqrt{V}_s \exp\left(-\frac{b}{2}(t-u) - (\frac{a}{2} - \frac{\sigma^2}{8}) \int_s^t \frac{du}{V_u} \right) ds, \\
\dot{V}_\sigma(t) &= \frac{2}{\sigma} V_t - \frac{2}{\sigma} \sqrt{V_t} \left(\sqrt{v} e^{-\frac{b}{2}t - (\frac{a}{2} - \frac{\sigma^2}{8}) \int_0^t \frac{dr}{V_r}} + a \int_0^t \frac{e^{-\frac{b}{2}(t-u) - (\frac{a}{2} - \frac{\sigma^2}{8}) \int_u^t \frac{dr}{V_r}}}{\sqrt{V_u}} du \right) \quad (2.11)
\end{aligned}
$$

Proof: As $V_t = X_t^2$, V is differentiable with respect to the parameters a, b and σ under the assumptions of Propositions 2.3, 2.1, 2.2. The derivatives $\dot{V}_\sigma$ is given as solution of the SDE :

$$
d\dot{V}_\sigma(t) = -b\dot{V}_\sigma(t)dt + \sqrt{V_t} dW_t^2 + \sigma \frac{\dot{V}_\sigma(t)}{2\sqrt{V_t}} dW_t, \quad \dot{V}_\sigma(0) = 0.
$$

One can see that the process $Z_t := \dot{V}_\sigma(t) - \frac{2}{\sigma}V_t$ is solution of the SDE :

$$dZ_t = \left(-\frac{2a}{\sigma} - bZ_t\right) dt + \sigma\frac{Z_t}{2\sqrt{V_t}}dW_t^2, \quad Z_0 = -\frac{2}{\sigma}x.$$

On the other hand, applying Itô formula to the process ZV^α, for $\alpha \in \mathbb{R}^*$, we have

$$d(ZV^\alpha)(t) = (-\frac{2a}{\sigma}V_t^\alpha - b(1+\alpha)Z_tV_t^\alpha + (\alpha a + \frac{\alpha^2}{2}\sigma^2)Z_tV^{\alpha-1})dt + (\alpha + \frac{1}{2})ZV^{\alpha-\frac{1}{2}}dW_t^2.$$

It follows that, for $\alpha = -\frac{1}{2}$, the process $Y = ZV^{-\frac{1}{2}}$, Y has finite variation and is given as solution of

$$dY_t = \left(-\frac{2a}{\sigma}V_t^{-\frac{1}{2}} - \frac{b}{2}Y_t - (\frac{a}{2} - \frac{\sigma^2}{8})\frac{Y_t}{V_t}\right) dt \quad , Y_0 = -\frac{2}{\eta}\sqrt{v}.$$

We can easily solve this equation, we get

$$Y_t := \frac{V_\sigma(t) - \frac{2}{\sigma}V_t}{\sqrt{V_t}} = -\frac{2}{\sigma}\sqrt{v}e^{-\gamma_t} - \frac{2a}{\sigma}\int_0^t \frac{e^{-(\gamma_t-\gamma_u)}}{\sqrt{V_u}}du, \quad a.s,$$

where

$$\gamma_t := \frac{b}{2}t + (\frac{a}{2} - \frac{\sigma^2}{8})\int_0^t \frac{dr}{V_r}. \tag{2.12}$$

Thus

$$\dot{V}_\sigma(t) = \frac{2}{\sigma}V_t - \frac{2}{\sigma}\sqrt{V_t}\left(\sqrt{v}e^{-\frac{b}{2}t-(\frac{a}{2}-\frac{\sigma^2}{8})\int_0^t\frac{dr}{V_r}} + a\int_0^t \frac{e^{-\frac{b}{2}(t-u)-(\frac{a}{2}-\frac{\sigma^2}{8})\int_u^t\frac{dr}{V_r}}}{\sqrt{V_u}}du\right), \quad a.s. \square$$

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