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A sequent calculus with procedure calls

Mahfuza Farooque\textsuperscript{1}, Stéphane Lengrand\textsuperscript{1,2}

\textsuperscript{1} CNRS
\textsuperscript{2} Ecole Polytechnique

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Abstract

In this paper, we extend the sequent calculus $\text{LK}_F$ [LM09] into a calculus $\text{LK}(\mathcal{T})$, allowing calls to a decision procedure. We prove cut-elimination of $\text{LK}(\mathcal{T})$.

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1 The sequent calculus $\text{LK}(\mathcal{T})$

The sequent calculus $\text{LK}(\mathcal{T})$ manipulates the formulae of first-order logic, with the specificity that every predicate symbol is classified as either positive or negative, and boolean connectives come in two versions: positive and negative.

**Definition 1 (Formulae)** Literals are predicates (a predicate symbol applied to a list of first-order terms) or negations of predicates. Literals are equipped with the obvious involutive negation, and the negation of a literal $l$ is denoted $\bot l$.

Let $\mathcal{P}$ be the set of literals that are either predicates with positive predicate symbols, or negations of predicates with negative predicate symbols.

Positive formulae $P ::= p \mid A \land^+ B \mid A \lor^+ B \mid \exists x A$

Negative formulae $N ::= p \bot \mid A \land^- B \mid A \lor^- B \mid \forall x A$

Formulae $A, B ::= P \mid N$

where $p$ ranges over $\mathcal{P}$.

**Definition 2 (Negation)** Negation is extended from literals to all formulae:

\[
\begin{array}{c|c|c|c}
(p)^\bot & p^+ & (p^\bot)^\bot \\
(A \land^+ B)^\bot & A^\bot \land B^\bot & (A \land^- B)^\bot & A^\bot \land B^\bot \\
(A \lor^+ B)^\bot & A^\bot \lor B^\bot & (A \lor^- B)^\bot & A^\bot \lor B^\bot \\
(\exists x A)^\bot & \forall x A^\bot & (\forall x A)^\bot & \exists x A^\bot
\end{array}
\]

**Definition 3 (LK($\mathcal{T}$))** The sequent calculus $\text{LK}^p(\mathcal{T})$ manipulates two kinds of sequents:

Focused sequents $\Gamma \vdash \{P\}$

Unfocused sequents $\Gamma \vdash \Delta$

where $\Gamma$ is a multiset of negative formulae and positive literals, $\Delta$ is a multiset of formulae, and $P$ is said to be in the focus of the (focused) sequent. By $\text{lit}(\Gamma)$ we denote the sub-multiset of $\Gamma$ consisting of its literals.

The rules of $\text{LK}^p(\mathcal{T})$, given in Figure 1, are of three kinds: synchronous rules, asynchronous rules, and structural rules. These correspond to three alternating phases in the proof-search process that is described by the rules.

If $S$ is a set of literals, $\mathcal{T}(S)$ is the call to the decision procedure on the conjunction of all literals of $S$. It holds if the procedure returns UNSAT.

2 Admissible rules

**Definition 4 (Assumptions on the procedure)**

We assume that the procedure calls satisfy the following properties:

- **Weakening** If $\mathcal{T}(S)$ then $\mathcal{T}(S, S')$.
- **Contraction** If $\mathcal{T}(S, A, A)$ then $\mathcal{T}(S, A)$.
- **Instantiation** If $\mathcal{T}(S)$ then $\mathcal{T}(\{\forall x \} S)$.
- **Consistency** If $\mathcal{T}(S, p)$ and $\mathcal{T}(S, p^\bot)$ then $\mathcal{T}(S)$.

where $S$ is a set of literals.

**Lemma 1 (Admissibility of weakening and contraction)**

The following rules are admissible in $\text{LK}(\mathcal{T})$.

\[
\begin{array}{c|c|c|c|c|c}
\Gamma \vdash [B] & \Gamma \vdash \Delta \\
\hline
\Gamma, A \vdash [B] & \Gamma, A \vdash \Delta \\
\Gamma, A, A \vdash [B] & \Gamma, A, A \vdash \Delta \\
\Gamma, A \vdash [B] & \Gamma, A \vdash \Delta
\end{array}
\]

**Proof:** By induction on the derivation of the premiss. □
Synchronous rules
\[
\begin{align*}
\Gamma \vdash [A] & \quad \Gamma \vdash [B] \\
\Gamma \vdash [A \land B] \\
\Gamma \vdash [A_1 \lor A_2] & \quad \Gamma \vdash [\exists x A] \\
\Gamma, p \vdash [p] & \quad \mathcal{T}(\text{lit}(\Gamma), p^+) \\
\Gamma \vdash [N] \\
\end{align*}
\]

\(p\) positive literal

Asynchronous rules
\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma \vdash B, \Delta \\
\Gamma \vdash A \land \neg B, \Delta & \quad \Gamma \vdash A_1, A_2, \Delta \\
\Gamma \vdash A \lor \neg A_2, \Delta & \quad \Gamma \vdash (\forall x A), \Delta \\
\Gamma, A^+ \vdash \Delta & \quad \Gamma \vdash A, \Delta \\
\Gamma \vdash A, \Delta \\
\end{align*}
\]

\(A\) positive or literal

Structural rules
\[
\begin{align*}
\Gamma, P^+ \vdash [P] \\
\Gamma, P^+ \vdash P & \quad \mathcal{T}(\text{lit}(\Gamma)) \\
\Gamma \vdash - \\
\end{align*}
\]

Lemma 2 (Admissibility of instantiation) The following rules are admissible in \(\text{LK}(\mathcal{T})\).
\[
\begin{align*}
\Gamma \vdash [B] & \quad \Gamma \vdash \Delta \\
\{x\} \Gamma \vdash \{x\} [B] & \quad \{x\} \Gamma \vdash \{x\} \Delta \\
\end{align*}
\]

Proof: By induction on the derivation of the premiss. \(\square\)

3 Invertibility of the asynchronous phase

Lemma 3 (Invertibility of asynchronous rules)
All asynchronous rules are invertible in \(\text{LK}(\mathcal{T})\).

Proof: By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of \(A \land \neg B\): by case analysis on the last rule actually used
  \[
  \begin{align*}
  \Gamma \vdash A \land \neg B, C, \Delta' & \quad \Gamma \vdash A \land \neg B, D, \Delta' \\
  \Gamma \vdash A \land \neg B, C \land D, \Delta' & \quad \Gamma \vdash A \land \neg B, C \lor D, \Delta' \\
  \Gamma \vdash A \land \neg B, C \lor D, \Delta' \\
  \end{align*}
  \]
  By induction hypothesis we get
  \[
  \begin{align*}
  \Gamma \vdash A, C, \Delta' & \quad \Gamma \vdash A, D, \Delta' \\
  \Gamma \vdash B, C, \Delta' & \quad \Gamma \vdash B, D, \Delta' \\
  \end{align*}
  \]

- Inversion of \(A \lor \neg B\), \(C \lor D\), \(x \notin \text{FV}(\Gamma, \Delta')\)
  \[
  \begin{align*}
  \Gamma \vdash A \lor \neg B, C, \Delta' \\
  \Gamma \vdash A \lor \neg B, C \lor D, \Delta' \\
  \Gamma \vdash A \lor \neg B, C \land D, \Delta' \\
  \end{align*}
  \]
  By induction hypothesis we get
  \[
  \begin{align*}
  \Gamma \vdash A, C, \Delta' & \quad \Gamma \vdash A, D, \Delta' \\
  \Gamma \vdash B, C, \Delta' & \quad \Gamma \vdash B, D, \Delta' \\
  \end{align*}
  \]
By induction hypothesis we get
\[ \Gamma \vdash A, C, \Delta' \quad x \notin \text{FV}(\Gamma, \Delta', A) \] and
\[ \Gamma \vdash B, C, \Delta' \quad x \notin \text{FV}(\Gamma, \Delta', B) \]
\[ \Gamma, C \vdash A \land B, \Delta' \] \quad C positive or literal

\begin{align*}
\Gamma &\vdash A \land B, \Delta' \\
&\quad \text{By induction hypothesis we get} \\
&\Gamma, C \vdash A, \Delta' \\
&\quad \text{C positive or literal} \\
&\Gamma \vdash B, \Delta' \\
&\quad \text{C positive or literal} \\
\end{align*}

- Inversion of \( A \lor \neg B \)
\[ \Gamma \vdash A \lor \neg B, C, \Delta' \quad \Gamma \vdash A \lor \neg B, D, \Delta' \]
\[ \Gamma \vdash A \lor \neg B, C \land D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, B, C, \Delta' \quad \Gamma \vdash A, B, D, \Delta' \]
\[ \Gamma \vdash A \lor \neg B, C \land D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, B, C, D, \Delta' \]
\[ \Gamma \vdash A \lor \neg B, C \lor D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, B, C, D, \Delta' \]
\[ \Gamma \vdash A \lor \neg B, (\forall x C), \Delta' \quad x \notin \text{FV}(\Gamma, \Delta') \]
\[ \Gamma \vdash A, B, (\forall x C), \Delta' \quad x \notin \text{FV}(\Gamma, \Delta') \]
\[ \Gamma, C \vdash A \lor \neg B, \Delta' \] \quad \text{C positive or literal}
\[ \Gamma \vdash A \lor \neg B, C, \Delta' \] \quad \text{C positive or literal}

- Inversion of \( \forall x A \)
\[ \Gamma \vdash (\forall x A), C, \Delta' \quad \Gamma \vdash (\forall x A), D, \Delta' \]
\[ \Gamma \vdash (\forall x A), C \lor D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, C, \Delta' \quad x \notin \text{FV}(\Gamma, \Delta') \] and
\[ \Gamma \vdash A, D, \Delta' \]
\[ \Gamma \vdash (\forall x A), C \lor D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, C, D, \Delta' \]
\[ \Gamma \vdash (\forall x A), C \lor D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, C, \Delta' \quad x \notin \text{FV}(\Gamma, \Delta') \]
\[ \Gamma \vdash A, (\forall x C), \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma \vdash A, C, \Delta' \quad x \notin \text{FV}(\Gamma, \Delta') \]
\[ \Gamma \vdash (\forall x A), \Delta' \] \quad \text{C positive or literal}
\[ \Gamma \vdash (\forall x A), C, \Delta' \] \quad \text{C positive or literal}

- Inversion of literals and positive formulae \((A)\)
\[ \Gamma \vdash A, C, \Delta' \quad \Gamma \vdash A, D, \Delta' \]
\[ \Gamma \vdash A \lor \neg D, \Delta' \] \quad \text{By induction hypothesis we get}
\[ \Gamma, A \lor \neg D \vdash C, \Delta' \quad \Gamma, A \lor \neg D \vdash D, \Delta' \]
Theorem 4 (cut-elimination) The following rules are admissible in $LK(T)$.

\[
\begin{array}{c}
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma \vdash \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash [B]} \text{cut}_2
\end{array}
\]

Proof: By simultaneous induction on the derivation of the right premiss.
We reduce $\text{cut}_1$ by case analysis on the last rule used to prove the right premiss.

\[
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1
\]

reduces to

\[
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1
\]

reduces to

\[
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash B, \Delta} \text{cut}_1
\]

We have $T(\text{lit}(\Gamma), p^+, B^+)$ as we assume the procedure to satisfy weakening.

If $P^+ \in (\Gamma, p)$,

\[
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash [P]} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash [P]} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash [P]} \text{cut}_1 \\
\frac{T(\text{lit}(\Gamma), p^+)}{\Gamma, p \vdash [P]} \text{cut}_1
\]

as $P^+ \in (\Gamma)$.
Proof: By simultaneous induction on the following lexicographical measure:

\[ T(\text{lit}(\Gamma), p) \] reduces to \[ T(\text{lit}(\Gamma)) \]

using the assumption of consistency.

We reduce cut₂ again by case analysis on the last rule used to prove the right premiss.

\[ \Gamma, p \vdash [B] \quad \Gamma, p \vdash [C] \]

\[ T(\text{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [B \land C] \]

reduces to

\[ T(\text{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [B] \quad \Gamma, p \vdash [C] \]

\[ \Gamma \vdash [B \land C] \]

Finally,\[ \Gamma, p \vdash N \]

\[ T(\text{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [N] \]

\[ \Gamma \vdash [N] \]

If \( p' \in \Gamma, p \),

\[ T(\text{lit}(\Gamma), p^{\perp}) \quad \Gamma, p \vdash [p'] \]

\[ \Gamma \vdash [p'] \]

\[ \Gamma, p \vdash [p'] \quad \Gamma \vdash [p'] \]

\[ \Gamma \vdash [p'] \]

since weakening gives \( T(\text{lit}(\Gamma), p^{\perp}, p^{\perp}) \) and consistency then gives \( T(\text{lit}(\Gamma), p^{\perp}) \).

\[ \square \]

**Theorem 5 (cut₁, cut₄ and cut₅)** The following rules are admissible in LK(T).

\[ \Gamma \vdash [A] \quad \Gamma \vdash A^{\perp}, \Delta \]

\[ \Gamma \vdash \Delta \]

\[ \Gamma, N \vdash \Delta \]

\[ \Gamma \vdash [B] \]

**Proof:** By simultaneous induction on the following lexicographical measure:

- the size of the cut-formula \( (A \text{ or } N) \)
- the fact that the cut-formula \( (A \text{ or } N) \) is positive or negative (if of equal size, a positive formula is considered smaller than a negative formula)
- the height of the derivation of the right premiss

6
Weakenings and contractions (as they are admissible in the system) are implicitly used throughout this proof.

In order to eliminate \texttt{cut}_3, we analyse which rule is used to prove the left premiss. We then use invertibility of the negative phase so that the last rule used in the right premiss is its dual one.

- \[\Gamma \vdash [A] \quad \Gamma \vdash [B]\]
- \[\Gamma \vdash [A \land B] \quad \Gamma \vdash A^+, B^+, \Delta\] \texttt{cut}_3

We will describe below how \texttt{cut}_4 is reduced.

- \[\Gamma \vdash \exists x A\]
- \[\Gamma \vdash A^+, \Delta\] \texttt{cut}_3

In order to eliminate \texttt{cut}_5, we analyse which rule is used to prove the right premiss.

- \[\Gamma \vdash [N] \quad \Gamma \vdash [N^+], \Delta\] \texttt{cut}_3

if \(N\) is not an literal (hence, it is not passed on to the procedure).

- \[\Gamma \vdash \{\forall x A\}\]
- \[\Gamma \vdash A^+, \Delta\] \texttt{cut}_3

if \(p^+\) is an literal passed on to the procedure.

- \[\Gamma \vdash [N] \quad \Gamma \vdash [N^+]\] \texttt{cut}_3

\[\cdots\]
\[\Gamma \vdash N \quad \Gamma, N \vdash \Lambda \quad \Gamma, N \vdash C, \Delta \]
\[\Gamma \vdash B, \Delta \quad \Gamma \vdash N \quad \Gamma, N \vdash B \wedge C, \Delta \]
\[\Gamma \vdash B, \Delta \quad \Gamma \vdash N \quad \Gamma, N \vdash C, \Delta \]
\[\Gamma \vdash B \wedge C, \Delta \]

reduces to
\[\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta \quad \Gamma \vdash B, \Delta \quad \Gamma \vdash N \quad \Gamma, N \vdash C, \Delta \quad \Gamma \vdash C, \Delta \]
\[\Gamma \vdash B \wedge C, \Delta \]

\[\Gamma, N \vdash B, C, \Delta \]
\[\Gamma \vdash B \wedge C, \Delta \]
\[\Gamma \vdash N \quad \Gamma, N \vdash B \lor C, \Delta \quad \Gamma \vdash N \quad \Gamma, N \vdash B, \Delta \]
\[\Gamma \vdash \forall x B, \Delta \]
\[\Gamma, N, B \downarrow \vdash \Delta \]
\[\Gamma \vdash N \quad \Gamma, N \vdash B, \Delta \]
\[\Gamma \vdash B, \Delta \]

using weakening, and if \(B\) is positive or a negative literal.

We have reduced all cases of \(\text{cut}_4\); we now reduce the cases for \(\text{cut}_5\) (again, by case analysis on the last rule used to prove the right premise).
\[\Gamma, N \vdash [B] \quad \Gamma, N \vdash [C] \quad \Gamma \vdash N \quad \Gamma, N \vdash [B \wedge C] \quad \Gamma \vdash N \quad \Gamma, N \vdash [B] \quad \Gamma \vdash N \quad \Gamma, N \vdash [C] \]
\[\Gamma \vdash [B \wedge C] \]
\[\Gamma \vdash N \quad \Gamma, N \vdash [B_1] \quad \Gamma \vdash [B] \]
\[\Gamma \vdash N \quad \Gamma, N \vdash [B_1 \lor B_2] \quad \Gamma \vdash [B_1 \lor B_2] \quad \Gamma \vdash N \quad \Gamma, N \vdash [\{x\} B] \quad \Gamma \vdash N \quad \Gamma, N \vdash [\{x\} B] \]
\[\Gamma \vdash [\exists x B] \quad \Gamma \vdash [\exists x B] \quad \Gamma \vdash N \quad \Gamma, N \vdash N' \quad \Gamma \vdash N' \quad \Gamma \vdash [N'] \quad \Gamma \vdash [N'] \]
\[\Gamma \vdash N \quad \Gamma, N \vdash [p] \quad \Gamma \vdash [p] \quad \Gamma \vdash N \quad \Gamma, N \vdash [\text{lit}(\Gamma), p^+] \]
\[\Gamma \vdash [p] \quad \Gamma \vdash [p] \]

\(\square\)
Theorem 6 (cut₆, cut₇, cut₈, and cut₉) The following rules are admissible in LK(T).

\[
\begin{align*}
\Gamma \vdash N, \Delta \quad & \quad \Gamma, N \vdash \Delta & \quad \text{cut₆} \\
\hline
\Gamma \vdash \Delta & \quad \Gamma \vdash A, \Delta \quad & \quad \Gamma \vdash A^\perp, \Delta & \quad \text{cut₇} \\
\hline
\Gamma \vdash \Delta & \quad \Gamma, l \vdash \Delta & \quad \text{cut₈} \\
\hline
\Gamma, l, l^\perp \vdash \Delta & \quad \Gamma, l_1, \ldots, l_n \vdash \Delta & \quad \Gamma, (l^\perp_1 \lor \cdots \lor l^\perp_n) \vdash \Delta & \quad \text{cut₉}
\end{align*}
\]

Proof: cut₆ is proved admissible by induction on the multiset \(\Delta\): the base case is the admissibility of cut₄, and the other cases just require the inversion of the connectives in \(\Delta\).

For cut₇, we can assume without loss of generality (swapping \(A\) and \(A^\perp\)) that \(A\) is negative. Applying inversion on \(\Gamma \vdash A^\perp, \Delta\) gives a proof of \(\Gamma, A \vdash \Delta\), and cut₇ is then obtained by cut₆:

\[
\Gamma \vdash A, \Delta \quad \Gamma, A \vdash \Delta & \quad \text{cut₆}
\]

cut₈ is obtained as follows:

\[
\Gamma, l^\perp \vdash \Delta \quad \Gamma \vdash l, \Delta & \quad \text{cut₇}
\]

cut₉ is obtained as follows:

\[
\Gamma, (l^\perp_1 \lor \cdots \lor l^\perp_n) \vdash \Delta & \quad \text{cut₇}
\]

5 Conclusion

It is worth noting that an instance of such a theory is the theory where \(T(S)\) holds if and only if there is a literal \(p \in S\) such that \(p^\perp \in S\).

We proved the admissibility of cut₄ and cut₉ as they are used to simulate the DPLL(T) procedure [NOT06] as the proof-search mechanism of LK(T).

Further work will consist in using the cut-admissibility results to:

- show that changing the polarities of the connectives and predicates that are present in a sequent, does not change the provability of that sequent in LK(T);
- prove the completeness of LK(T) with respect to the standard notion of provability in first-order logic, working in a particular theory \(T\) for which we have a (sound and complete) decision procedure;
- show how the DPLL(T) procedure can be simulated in LK(T) (with backtracking as well as with backjumping and lemma learning).

References
