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Modeling and numerical simulation of a nonlinear system of piano strings coupled to a soundboard.

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ABSTRACT

Construction of a physical model for the grand piano implies complex and multidimensional phenomena. We present a model of piano strings coupled to a soundboard, and its numerical approximation. Measurements on piano strings and bridge show phantom partials and a time precursor that both cannot be explained by the linear scalar string model. Because of all the nonlinearities and the couplings, this is not an easy task and this paper will outline the difficulties, and propose efficient solutions.

MODELING THE PIANO

The work presented in this paper is the ongoing subject of the first author’s PhD, which consists in modeling a grand piano. This project is the third collaboration between UME-ENSTA, laboratory specialized in mechanics and musical acoustics, and POEMS-INRIA, laboratory specialized in numerical analysis. In the past, two other PhD works have considered the numerical simulation of a timpani [12] and a guitar [8]. Apart from the fluid/structure coupling, the major difficulty of the timpani modeling was to take into account the nonlinear interaction between the timpani stick and the membrane, while the major difficulty of the guitar was to model the soundboard. For such complex, coupled problems, the priority when performing numerical simulations is the stability of the numerical scheme, which is not an easy issue in a nonlinear context. The energy approach has proven to be very efficient and lead to intuitive numerical schemes, especially in order to treat the different coupling conditions.

The two issues mentioned earlier have still to be considered for the piano modeling, for the interaction between hammer and strings and the modeling of the soundboard. As explained later, we must also consider a nonlinear model for the strings. The object of this paper is to explain the modeling choices of the authors, and to construct a stable numerical scheme that represents a system of hammer, strings and soundboard. Because of all the nonlinearities and the couplings, this is not an easy task and this paper will outline the difficulties, and propose efficient solutions.

A first section will present the string models and how to approximate them, then a second section will suggest how to model the hammer/strings interaction. A description of the soundboard and the models we use is then proposed, and finally we will present the whole coupled problem and its numerical approximation.

STRINGS

A nonlinear string model has been introduced by Morse & Ingard [10], in which the string vibration problem is considered as a nonlinear coupled system referred to as “Geometrically Exact Model” (GEM). Conklin [7] has seen in his measurements on piano spectra that some partials could not be explained with the linear string vibration theory, and Bank & Sujbert [2] have shown that these so called “phantom partials” appeared at frequency values being sum or differences of harmonic partials. Several authors [2, 3] have done numerical simulations using a nonlinear string model coming from Taylor developments of the GEM. These developments are made so that the energy of the string remains positive, giving a stable numerical scheme. We present here the GEM as well as a mathematical justification of the developed models, for small transversal initial data.

The geometrically exact model

We consider an infinitely thin string, parametrized at rest with \( x \in [0, L] \), where \( L \) is the length of the string in meters. We will call \( \mu \) the linear mass of the string, \( A \) the area of its section, \( E \) its Young’s modulus, \( T_0 \) its tension at rest. All along this paper, the notations \( \partial_x \equiv \frac{\partial}{\partial x} \) and \( \partial_t \equiv \frac{\partial}{\partial t} \) will denote the partial derivative along time \( t \) and space \( x \) respectively. Vectorial unknowns will be noted in thick or underlined font unlike the scalar unknowns. The standard nonlinear geometrically exact
model [10] couples the transversal displacement of the string \( u(x,t) \) to the longitudinal displacement \( v(x,t) \):

\[
\begin{align*}
\mu \frac{\partial^2 u}{\partial t^2} & = \partial_x \left[ E A \partial_x u - (E A - T_0) \frac{\partial u}{\partial x} \right] \\
\mu \frac{\partial^2 v}{\partial t^2} & = \partial_x \left[ E A \partial_x v - (E A - T_0) \frac{\partial v}{\partial x} \right]
\end{align*}
\]

This system is obtained by a geometrical description of the motion of a material point of the string, using the dynamic fundamental law on the elementary system subjected only to tension forces. Hooke’s law is used to link the local tension of the string to the relative elongation. We can use a dimensionless system, by introducing

\[
\begin{align*}
\alpha = \frac{E A - T_0}{E A} \in [0,1], \quad T = L \sqrt{\frac{\mu}{E A}} \\
x^* = x/L, \quad u^* = u/L, \quad v^* = v/L, \quad t^* = t/T
\end{align*}
\]

The new system is, forgetting the starred notations,

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} & = \partial_x \left[ \partial_x u - \alpha \frac{\partial u}{(\partial_x u)^2 + (1 + \partial_x v)^2} \right] \\
\frac{\partial^2 v}{\partial t^2} & = \partial_x \left[ \partial_x v - \alpha \frac{\partial v}{(\partial_x u)^2 + (1 + \partial_x v)^2} \right]
\end{align*}
\]

If we introduce the potential energy

\[
\mathcal{W}_{ex}(u,v) = \frac{1}{2} \alpha u^2 + \frac{1}{2} v^2 - \alpha \left[ \sqrt{u^2 + (1+v)^2} - (1+v) \right]
\]

then the solution of (4) satisfies the preservation of the energy:

\[
\frac{d}{dt} \left\{ \frac{1}{2} \int_0^L \left[ \partial_x u^2 + \frac{1}{2} \int_0^L \partial_x v^2 + \int_0^L \mathcal{W}_{ex}(\partial_x u, \partial_x v) \right] \right\} = 0
\]

**Expanded models and their asymptotic justification**

In the papers of [1–3], a developed model can be found, which has been established with the method presented above, but performing a Taylor development of the square root in (5) and neglecting some terms, in order to keep a positive potential energy. We wanted to give a more precise explanation of the origin of this model, by using standard asymptotic methods on the model (4). We present here the method used and the results obtained, the reader can refer to [4] for calculation details.

We solve, for a small initial amplitude \( \varepsilon \) on the transversal data, and

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \partial_x \left[ \sqrt{\mathcal{W}_{ex}(\partial_x u \partial_x v)} \right] & = 0 \\
\frac{\partial^2 v}{\partial t^2} & = \partial_x \left[ \partial_x v \partial_x u \right]
\end{align*}
\]

We wonder what would be the system of equation on \( u^\varepsilon \) if we neglect all terms in factor of \( \varepsilon^2 \). We write the fourth order Taylor development of (6) and inject it in (7). Then we group all the terms in factor of \( \varepsilon, \varepsilon^2 \) and \( \varepsilon^3 \) respectively, to obtain three different systems on the unknowns \( u_1 \) and \( v_1 \), and we find by calculation that \( v_1 = u_2 = v_1 = 0 \). If we regroup at this point all the terms together, neglecting \( \varepsilon^4 \), we find the following system on the new unknowns \( \tilde{u} = \varepsilon^0 u_1 + \varepsilon^3 u_3 \) and \( \tilde{v} = \varepsilon^0 u_2 \).

\[
\begin{align*}
\frac{\partial^2 \tilde{u}}{\partial t^2} - \partial_x \left[ (1 - \alpha) \frac{\partial \tilde{u}^2}{\partial x} + \alpha \frac{\partial \tilde{v}^2}{\partial x} + \frac{\alpha}{2} \left( \frac{\partial \tilde{u}}{\partial x} \right)^2 \right] & = 0, \\
\frac{\partial^2 \tilde{v}}{\partial t^2} - \partial_x \left[ \frac{\partial \tilde{v}^2}{\partial x} + \frac{\alpha}{2} \left( \frac{\partial \tilde{u}}{\partial x} \right)^2 \right] & = 0
\end{align*}
\]

which corresponds to the Cauchy problem (7) replacing the potential energy \( \mathcal{W}_{ex} \) with

\[
\mathcal{W}_{BS}(u,v) = \frac{1 - \alpha}{2} u^2 + \frac{1}{2} v^2 + \frac{\alpha}{2} u^2 v + \frac{\alpha}{8} u^4
\]

This energy is the one found empirically in [1–3]. We give here a restriction: this model is valid only if the string is excited transversally.

**Remark 1** For a more accurate development in long time, perturbation methods can be used [11]. The principle is to consider different time scales, leading to more complex expressions of the equations satisfied by the developed solution. See figure 2 for a comparison between classical method and multi-scale method.
If we write the kinetic and potential energies as:

\[
\begin{align*}
\mathcal{T}(\dot{q}) &= \frac{\rho A}{2} (\dot{q}u)^2 + \frac{\rho A}{2} (\dot{q}v)^2 + \frac{\rho l}{2} (\dot{q} \phi)^2 \\
\mathcal{V}(q, \phi) &= \frac{EA}{2} (\partial_u q)^2 + \frac{EA}{2} (\partial_v q)^2 + \frac{E l}{2} (\partial \phi)^2 \\
&\quad - (EA - T_0) \sqrt{(\partial_u q)^2 + (1 + \partial_v q)^2} + \frac{AGk^T}{2} (\partial_u q - \phi)^2
\end{align*}
\]

(11a) (11b)

\[
\begin{align*}
\delta \mathcal{V}(\dot{q}) - \partial_q \delta \mathcal{V}(q, \phi) + \nabla_q \mathcal{V}(q, \phi) = 0
\end{align*}
\]

(13)

which preserves the energy

\[
\mathcal{E}(q, \phi, \dot{q}, \dot{\phi}) = \int_0^L \mathcal{T}(\dot{q}) + \int_0^L \mathcal{V}(q, \phi). \quad (14)
\]

We will now on consider that

\[
\mathcal{T}(\dot{q}) = \sum_{k=1}^K \mathcal{T}_k (\dot{q}_k)^2.
\]

(15)

**Remark 2** In all what follows, we will abusively mingle notations and treat the triplet \((u, v, \phi)\) as \((q_1, q_2, q_3)\) and equally refer to the unknowns with their number in \(q\) or their letter symbol in the triplet. To illustrate this fact, we shall do, in our present case, write that \(\mathcal{T}_1 = \mathcal{T}_2 = \rho A, \mathcal{T}_3 = \rho l\) or \(\mathcal{T}_u = \mathcal{T}_d = \rho A, \mathcal{T}_p = \rho l\).

**Numerical approximation**

We have shown in [5] that in a certain class of energy preserving numerical schemes, it was impossible to construct an explicit scheme unless the original equation is linear. We have shown that the intuitive scheme, that approximate the gradient of \(\mathcal{V}\) with a directional finite difference does not, in general, lead to a preserving scheme. The numerical scheme that we propose here is an implicit, second order accurate in time, unconditionally stable numerical scheme.

In a very general context, we had to introduce the functions

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\begin{align*}
\delta \mathcal{V}(\dot{q}) - \partial_q \delta \mathcal{V}(q, \phi) + \nabla_q \mathcal{V}(q, \phi) = 0
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\]

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\mathcal{T}(\dot{q}) = \sum_{k=1}^K \mathcal{T}_k (\dot{q}_k)^2.
\]

(15)
coming from the interactions. Since the function $\Phi$ is nonlinear, we have to treat the hammer implicitly with the string, which is not a great cost since it is a scalar unknown. The line for the hammer should be:

$$M^{\text{ham}} = \frac{\varepsilon^{n+1} - 2\varepsilon^n + \varepsilon^{n-1}}{\Delta t}$$  

(20)

\[ N \sum_{i=1}^{N} \left[ k_{i}^{\text{ham}} \left( U_{i,h}^{n+1} - \frac{\varepsilon^{n+1}}{2} \right) - \left( U_{i,h}^{n+1} - \frac{\varepsilon^n}{2} \right) - \left( U_{i,h}^{n-1} - \frac{\varepsilon^{n-1}}{2} \right) - \frac{k_{i}^{\text{ham}}}{2\Delta t} \right] \Psi(U_{i,h}^{n+1} - \frac{\varepsilon^{n+1}}{2}) + \frac{k_{i}^{\text{ham}}}{2\Delta t} \Psi(U_{i,h}^{n-1} - \frac{\varepsilon^{n-1}}{2}) - \frac{k_{i}^{\text{ham}}}{2\Delta t} \Psi(U_{i,h}^{n-1} - \frac{\varepsilon^n}{2}) - \frac{k_{i}^{\text{ham}}}{2\Delta t} \Psi(U_{i,h}^{n+1} - \frac{\varepsilon^n}{2}) \right) \]

(21)

and the following contribution should be added to the $u$-line of each string, for any test function $\Psi$:

$$K_{i}^{\text{ham}} \left[ \Psi(U_{i,h}^{n+1} - \frac{\varepsilon^{n+1}}{2}) - \Psi(U_{i,h}^{n+1} - \frac{\varepsilon^n}{2}) - \frac{k_{i}^{\text{ham}}}{2\Delta t} \Psi(U_{i,h}^{n-1} - \frac{\varepsilon^{n-1}}{2}) \right]$$

So, the main purpose of the ribs where $U_{i,h}$ is the vector of unknowns linked to the degrees of freedom for the variable $u_{i}$, $V_{i,h}$ for $v_{i}$ and $\Phi_{i,h}$ for $\phi_{i}$. This scheme preserves the energy

$$\varepsilon_{h,s}^{n+1/2} = \sum_{i=1}^{N} \left[ \frac{1}{2} \int_{0}^{L} \sum_{j=1}^{N} \left( \frac{\partial q_{i,j}^{n+1} - \partial q_{i,j}^{n}}{\Delta t} \right) + M^{\text{ham}} \left( \frac{\varepsilon^{n+1} - 2\varepsilon^n + \varepsilon^{n-1}}{\Delta t} \right) \right] + N \sum_{i=1}^{N} \left[ \frac{1}{2} \int_{0}^{L} \frac{\partial}{\partial x} \left( \frac{\partial q_{i,j}^{n+1} - \partial q_{i,j}^{n}}{\Delta t} \right) + \frac{k_{i}^{\text{ham}}}{2} \Psi(U_{i,h}^{n+1} - \frac{\varepsilon^{n+1}}{2}) + \frac{k_{i}^{\text{ham}}}{2} \Psi(U_{i,h}^{n-1} - \frac{\varepsilon^{n-1}}{2}) \right]$$

(22)

\section*{SOUNDBOARD}

\subsection*{Geometry}

The piano soundboard is a wooden plate (mostly in spruce) with a variable shape depending on the piano. It is stiffened by ribs, which are glued perpendicularly to the fibers of the wood, and a bridge (see figure 3). The main purpose of the ribs is to restore a certain isotropy in a fundamentally orthotropic material: wood. This objective is achieved for the first modes by ribs, which are glued perpendicularly to the fibers of the wood, and a bridge. The two models that we will consider are Kirchhoff Love and Mindlin Reissner models. They both model a stiff plate, and can take into account orthotropy, variations in the thickness or the material. The Kirchhoff Love model can be derived from the Mindlin Reissner model when making an additional assumption: the straight sections remain orthogonal to the neutral fiber (excluding the ribs and the bridges).

These two plate models can be grouped into the following general frame, with $A$ and $B$ two selfadjoint operators. We seek a displacement field $u : \omega \times \mathbb{R}^{+} \rightarrow \mathbb{R}$ and an angle field $\theta : \omega \times \mathbb{R}^{+} \rightarrow \mathbb{R}^{2}$ such that:

\begin{align}
& c_{\theta} \frac{\partial^{2} \theta}{\partial t^{2}} + A \theta + C u = 0 \quad \text{(23a)} \\
& c_{u} \frac{\partial^{2} u}{\partial t^{2}} + B u + C^{\theta} \theta = f \chi_{\omega}(x,y) \quad \text{(23b)}
\end{align}

where $\chi_{\omega}$ is a repartition function which distributes the force $f(t)$ over the plate.

<table>
<thead>
<tr>
<th>Mindlin Reissner</th>
<th>Kirchhoff Love</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{\theta}$</td>
<td>$\rho \delta^{3/2}$</td>
</tr>
<tr>
<td>$c_{u}$</td>
<td>$\rho \delta$</td>
</tr>
<tr>
<td>$A \theta$</td>
<td>$-\left(\delta^{3/2}/12\right) \text{Div}(C \varepsilon(\theta)) + \delta G \theta$</td>
</tr>
<tr>
<td>$B u$</td>
<td>$-\delta \text{div}(G \varepsilon(u))$</td>
</tr>
<tr>
<td>$C u$</td>
<td>$\delta G \varepsilon(u)$</td>
</tr>
<tr>
<td>$C^{\theta} \theta$</td>
<td>$-\delta G \text{div} \theta$</td>
</tr>
</tbody>
</table>

\section*{Numerical approximation}

We proceed to the semi discretization of the plate problem, the problem becomes to seek $(\theta_{p}, u_{p}) : [0,T] \rightarrow (\Theta_{p}^{\text{disc}}, \mathcal{Y}_{p}^{\text{disc}})$ such that $\forall \theta^{*} \in \Theta_{p}, \forall u^{*}_{p} \in \mathcal{Y}_{p}$

\begin{align}
& \int_{0}^{T} \int_{\omega} c_{\theta} \frac{\partial^{2} \theta}{\partial t^{2}} \theta + \int_{\omega} A \theta \theta^{*} + \int_{\omega} (C u) \cdot \theta^{*} = 0, \quad \text{(24a)} \\
& \int_{0}^{T} \int_{\omega} c_{u} \frac{\partial^{2} u}{\partial t^{2}} u + \int_{\omega} B u \theta^{*} + \int_{\omega} (C^{\theta} \theta) u^{*}_{p} = f \int_{\omega} \chi_{\omega}(x,y) u^{*}_{p}.
\end{align}
which is equivalent to

\[
\frac{\partial^2}{\partial t^2} M_h^U \Lambda_{h,\text{mod}} + R_h \Lambda_{h,\text{EF}} = \left( f_h, 0 \right),
\]

where \( \Lambda_{h,\text{mod}} = \left( M_h^U \right)^{1/2} \Lambda_{h,\text{mod}} \). We diagonalize the real symmetric matrix \( R_h \) in a \( M_h \)-orthogonal basis. Let \( \Lambda_{h,\text{mod}} \) be the basis of eigenvectors. Then, there exist a diagonal matrix \( D_h \) and a matrix \( P_h \) orthogonal for the scalar product \( \langle h, \cdot \rangle \) such that

\[
\begin{align*}
\mathbb{I} \cdot R_h \cdot P_h \cdot \mathbb{I} = D_h \\
\Lambda_{h,\text{mod}} = P_h \cdot \Lambda_{h,\text{mod}} \cdot P_h^{-1} = P_h \cdot \Lambda_{h,\text{mod}} \cdot P_h^{-1}
\end{align*}
\]

The problem is now constituted of decoupled ODEs with second member and initial values: \( \forall t \in \left[ t_{n-1/2}, t_{n+1/2} \right] \)

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} \Lambda_{h,\text{mod}} + D_h \Lambda_{h,\text{mod}} &= f_h \mathbb{I} \\
\Lambda_{h,\text{mod}}(t = t_{n-1/2}) &= \Lambda_{h,\text{mod},n-1/2} \\
\frac{\partial}{\partial t} \Lambda_{h,\text{mod}}(t = t_{n-1/2}) &= \frac{\partial}{\partial t} \Lambda_{h,\text{mod},n-1/2}.
\end{align*}
\]

Which we can solve exactly, using the technique of [8]. We introduce the operators \( \mathcal{M}_h(U_0, U_1) \), which gives the solution, after \( \Delta t \) time, to the homogeneous problem with initial values \( U_0 \) and \( U_1 \), and \( \mathbb{I} \cdot \Lambda_{h,\Delta t}(F) \) which gives the solution to the problem between \( t_{n-1/2} \) and \( t_{n+1/2} \) with second member \( F \) (see (34c)). Out of linearity, we can write that

\[
\Lambda_{h,\text{mod}}(t = t_{n-1/2}) = \mathcal{M}_h(\Lambda_{h,\text{mod},n-1/2}, \frac{\partial}{\partial t} \Lambda_{h,\text{mod},n-1/2}) + \mathcal{M}_h(0, \frac{\partial}{\partial t} \Lambda_{h,\text{mod},n-1/2}).
\]

The energy preserved by this method, if \( f = 0 \), is

\[
\begin{align*}
\varepsilon_{p, n+1/2} &= \frac{1}{2} \| \frac{\partial}{\partial t} \Lambda_{h,\text{mod},n+1/2} \|^2 + \frac{1}{2} \| \Lambda_{h,\text{mod},n+1/2} \|^2 \\
&= \frac{1}{2} \| \frac{\partial}{\partial t} \Lambda_{h,\text{mod}}(t = t_{n+1/2}) \|^2 + \frac{1}{2} \| \Lambda_{h,\text{mod}}(t = t_{n+1/2}) \|^2.
\end{align*}
\]

**Remark 4** An interesting possibility is to solve the problem only on the first eigen modes of the soundboard, which concentrates most energy, by considering a truncated (hence rectangular) matrix \( \Lambda_{h,\text{mod}} \). This approach justifies the (expensive) diagonalization of the matrix \( M_h \) since it enforces us to use a physically more relevant basis of approximation with less degrees of freedom.
The standard choice of \( \mathcal{E}, \mathcal{Y}, \Phi, \Theta_p, \Psi_p \) is to set
\[
\left\{ \begin{array}{l}
\mathcal{E} = \mathcal{E}^H([0,L]), \quad \mathcal{Y} = \mathcal{Y}^H([0,L]) \\
\Phi_p = H^1([0,L]), \\
\Theta_p = H^1(\Omega)^2, \quad \Psi_p = H^1(\Omega)
\end{array} \right.
\]
But in this case, the expression \( \mathbf{v} \cdot \mathbf{n} \mathcal{E}_c(\mathbf{q}_i, \partial \mathbf{q}_i)(x = L,t) \) does not make sense anymore. We choose to introduce new unknowns, Lagrange multipliers, \( \mathcal{F}^{\text{disc}} = [0,T] \rightarrow \mathbb{R}^{Nc} \), which will be implicitly determined by the equation (34a).

**Semi discretization in space**

We choose, to approximate the string spaces \( \{ \mathcal{E}_c, \mathcal{Y}_c, \Phi_c \} \) (which are very close to \( \mathcal{E}^H([0,L]) \)), Lagrange finite elements \( \mathbb{P}_k \), with, associated to the degree of freedom \( j \), the basis function \( \phi_j \). We introduce
\[
(M^\text{disc}_h)_{i,j} = \int_0^L \phi_i \phi_j \, \text{d}x, \quad \text{and} \quad \alpha_j = \phi_j(L).
\]
As previously, we call \( U_{i,h} \) the vector of coordinates of \( u_i \) in the basis \( (\phi_j) \), \( V_{i,h} \) the coordinates of \( v_i \), and \( \Phi_{i,h} \) the coordinates of \( \phi_i \).

The semi discrete problem is to find
\[
\dot{\xi} : [0,T] \rightarrow \mathbb{R},
\]
\[
(U_{i,h}, V_{i,h}, \Phi_{i,h}) : [0,T] \rightarrow \mathcal{F}_c^{\text{disc}} \times \mathcal{Y}_c^{\text{disc}} \times \Phi_c^{\text{disc}},
\]
\[
(\Theta_{p,h}, U_{p,h}) : [0,T] \rightarrow (\Theta_c^{\text{disc}}, \Psi_c^{\text{disc}}),
\]
\[
F_i^{\text{disc}} : [0,T] \rightarrow \mathbb{R}^{Nc},
\]
such that
\[
(M^\text{disc}_h)_{i,j} \ddot{\xi}_i(t) = - \sum_j F_i^{\text{disc}}(t) \quad (40a)
\]
\[
F_i^{\text{disc}}(t) = K^\text{disc} \dot{\Phi}_i (\sum_p U_{i,h,p} \phi_p) (t) - \dot{\xi}(t) + R_i^{\text{disc}} \frac{d}{dt} \Phi_i (\sum_p U_{i,h,p} \phi_p) (t) - \xi(t)) \quad (40b)
\]
\[
\left\{ \begin{array}{l}
\mathcal{T}_0 \partial_\alpha^2 (M^\text{disc}_h U_{i,h}) + \int_0^L \int_0^L \partial_\alpha \mathcal{E}_c(\mathbf{q}_i, \partial \mathbf{q}_i) \cdot \partial_\alpha \phi_j + \int_0^L \partial_\alpha \mathcal{Y}_c(\mathbf{q}_i, \partial \mathbf{q}_i) \cdot \partial_\alpha \phi_j = F_i^{\text{disc}}(t) \alpha_j \quad \forall i, \forall j.
\end{array} \right. \quad (40c)
\]
\[
\left\{ \begin{array}{l}
\partial_\alpha^2 (M^\text{disc}_h \Phi_{i,h}) + \int_0^L \int_0^L \partial_\alpha \mathcal{E}_c(\mathbf{q}_i, \partial \mathbf{q}_i) \cdot \partial_\alpha \Phi_j + \int_0^L \partial_\alpha \mathcal{Y}_c(\mathbf{q}_i, \partial \mathbf{q}_i) \cdot \partial_\alpha \Phi_j = J_h \quad \forall i, \forall j.
\end{array} \right. \quad (40d)
\]
\[
\left\{ \begin{array}{l}
\partial_\alpha^2 (M^\text{disc}_h \Theta_{p,h}) + A_h \Theta_{p,h} + C_h U_{p,h} = 0,
\end{array} \right. \quad (40e)
\]
\[
\left\{ \begin{array}{l}
\partial_\alpha^2 (M^\text{disc}_h U_{p,h}) + B_h U_{p,h} + C_h \Theta_{p,h} = - \sum_j F_i^{\text{disc}}(t) J_h
\end{array} \right.
\]
\[
(U_{i,h} \cdot \alpha \left| U_{i,h} \alpha \right.) \cdot Y = U_{p,h} J_h
\]
\[
\nabla^N_{\alpha,0} \mathcal{U} = \mathcal{U}_{i,h}(\mathbf{q}_i, \partial \mathbf{q}_i) (x = L, t)
\]
\[
\text{Figure 5: Schematic of the bridge condition.}
\]

**Time discretization**

The introduction of the Lagrange multiplier is a standard tool often used in linear problems where arithmetic combinations of linear sub-problems can lead to eliminate all unknowns but the Lagrange multiplier, leading to an equation on this multiplier only. It is then possible to determine it, and to treat all sub-problems independently. Here, the nonlinearity of the string problem forbids us to do the same. We will still be able to decouple the plate problem from the hammer/strings/Lagrange multiplier problem, which is rather good since the Lagrange multiplier (\( N_c \) scalar unknowns) does not represent a major cost compared to the string. The idea is to notice that we can write the solution of (40d) in the eigenvectors basis as the superposition of elementary solutions (see (31)).

The coupling constraint that we have to deal with is written to obtain an energy preservation, using a discrete equivalent of the time derivative of (34a):
\[
\frac{U_{i,h}^{n+1} - U_{i,h}^{n-1}}{2 \Delta t} \cdot \mathbf{a} = \frac{A_{i,h}^{n+1/2} - A_{i,h}^{n-1/2}}{\Delta t} \cdot \mathbf{a} + \int_{\tilde{\mathcal{p}}_{h}} A_{i,h} \left( J_h \right) \quad (41)
\]
which can be seen as the following equation, linear in all the \( F_i^{\text{disc}} \) and the \( U_{i,h} \) and \( V_{i,h} \):
\[
\left\{ \begin{array}{l}
\frac{U_{i,h}^{n+1} - U_{i,h}^{n-1}}{2 \Delta t} + \frac{\mathbf{a}}{\Delta t} = \mathbf{a} \quad \forall \mathbf{a} \quad \forall \mathbf{a}.
\end{array} \right. \quad (42)
\]
In this equation, we can evaluate beforehand the quantities \( \mathcal{J}_{\mathcal{T}_{i,h}}(A_{i,h}^{n+1/2} \partial A_{i,h}^{n-1/2}) + \mathcal{J}_{\mathcal{T}_{i,h}}(\tilde{\mathcal{p}}_{h} J_h) \) without knowing the values of the hammer, strings and Lagrange multiplier unknowns. Then, the much smaller sub-problem [hammer, strings, coupling equation] can be solved using a fully implicit scheme. The strings and hammer are treated as in the previous dedicated section, and the coupling is treated as mentioned above in (42).
The fully discrete problem can be written, centered around time $\tau^i$:

$$M^{ex} \frac{z^{n+1} - 2z^n + z^{n-1}}{\Delta t^2} - \sum_{i=1}^{N} K_i \left[ \Psi (\frac{U^{n+1}_{i,h} - \xi^{n+1}}{\Delta t}) - \Psi (\frac{U^{n-1}_{i,h} - \xi^{n-1}}{\Delta t}) \right] = 0$$

$$R_i \left[ \Phi (\frac{U^{n+1}_{i,h} - z^{n+1}}{\Delta t}) - \Phi (\frac{U^{n-1}_{i,h} - z^{n-1}}{\Delta t}) \right] = 0$$

(43a)

$$\left\{ \begin{array}{l}
\frac{1}{2} f \int_{\Omega} \left( M^{h}_{i,h} \frac{V^{n+1}_{i,h} - 2V^n_{i,h} + V^{n-1}_{i,h}}{\Delta t^2} \right) \, dV_i + \\
\frac{1}{2} f \int_{\Omega} \sum_{\sigma \in \Sigma} \zeta(\sigma) \left[ \delta_{\sigma} \Phi (\frac{U^{n+1}_{i,h} - \xi^{n+1}}{\Delta t}) - \delta_{\sigma} \Phi (\frac{U^{n-1}_{i,h} - \xi^{n-1}}{\Delta t}) \right] \, dV_i \\
\frac{1}{2} f \int_{\Omega} \sum_{\sigma \in \Sigma} \zeta(\sigma) \left[ \delta_{\sigma} \Phi (\frac{U^{n+1}_{i,h} - z^{n+1}}{\Delta t}) - \delta_{\sigma} \Phi (\frac{U^{n-1}_{i,h} - z^{n-1}}{\Delta t}) \right] \, dV_i \\
\frac{1}{2} f \int_{\Omega} \sum_{\sigma \in \Sigma} \zeta(\sigma) \left[ \delta_{\sigma} \Phi (\frac{U^{n+1}_{i,h} - \xi^{n+1}}{\Delta t}) - \delta_{\sigma} \Phi (\frac{U^{n-1}_{i,h} - \xi^{n-1}}{\Delta t}) \right] \, dV_i \\
\frac{1}{2} f \int_{\Omega} \sum_{\sigma \in \Sigma} \zeta(\sigma) \left[ \delta_{\sigma} \Phi (\frac{U^{n+1}_{i,h} - z^{n+1}}{\Delta t}) - \delta_{\sigma} \Phi (\frac{U^{n-1}_{i,h} - z^{n-1}}{\Delta t}) \right] \, dV_i \\
\end{array} \right\} = 0$$

(43b)

(43c)

$$\left\{ \begin{array}{l}
\dot{\Lambda}^h(t) + \Lambda^h(t) = 0, \\
\dot{\Lambda}^h(t) = 0,
\end{array} \right\}$$

(43d)

$$\left\{ \begin{array}{l}
\dot{\Lambda}^h(t) = 0,
\end{array} \right\}$$

(43e)
At each time step, this system can be solved in three steps:

- Exact resolution of (43c) (needing only the knowledge of $\lambda_{n+1/2}\mod n$ and $\partial_t \lambda_{n+1/2}\mod n$);
- Implicit resolution of the system \{(43a), (43b), (43d)} via Newton iterations on the strings, hammer and Lagrange multipliers unknowns;
- Adjustment of the soundboard unknowns with (43e).

The soundboard unknowns can finally be calculated in the finite element basis thanks to (29).

PERSPECTIVES

This paper has presented an ongoing work on the modeling of the grand piano. Innovating models, coupling energy preserving conditions and numerical schemes have been presented, as well as the precise method to solve the proposed equations. The paper is divided into pedagogical parts, treating the discretization of each sub system respectively, coming finally to the model of the grand piano. Damping can be considered, in return for some modifications in the numerical approximation.

The total discrete energy must decrease in accordance with the continuous energy decay. Numerical results and comparison with real measurements will be presented in the oral session. The first perspective of this work is to extend the model to the radiated sound. Accurate modeling of bridges and ribs is another goal that should help in a better understanding of the function of the soundboard.

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