Representation theorems and fundamental solutions for micropolar solid-fluid mixtures under steady state vibrations

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Abstract: The isothermal theory of binary micropolar solid-fluid mixture is considered in this paper. For the dynamical problem, a Galerkin type representation of the solution is established. Then, a fundamental solution is given for the three dimensional partial differential system which describes the steady vibrations. Also, some basic properties of the fundamental solution and a direct application to the point load problem are presented.

Representation theorems and fundamental solutions for micropolar solid-fluid mixtures under steady state vibrations

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Abstract

The isothermal theory of binary micropolar solid-fluid mixture is considered in this paper. For the dynamical problem, a Galerkin type representation of the solution is established. Then, a fundamental solution is given for the three dimensional partial differential system which describes the steady vibrations. Also, some basic properties of the fundamental solution and a direct application to the point load problem are presented.

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1. Introduction

A microcontinuum media, roughly speaking, is a continuum media whose properties and behaviour are affected by the local motions and deformations of the primitive elements. A special case of microcontinuum media is that of micropolar continua. In the micropolar continuum theory, the rotational degrees of freedom play a central role. Thus, we have six degrees of freedom, instead of three degrees of freedom considered in classical elasticity and fluid mechanics. The number of published works in these fields presently exceeds several hundred papers. A modern presentation, the state-of-art and the intended applications of these theories can be found in the books (Eringen, 1999, 2001). The higher-order or higher-grade continuum theories are necessary to capture size effects in small length scales, where the fundamental assumption of the classical continuum theory is that the (physical, chemical,

mechanical, etc.) properties of a material are uniformly distributed throughout its volume fails (Tekoglu & Onck, 2008). We remind that the theory of micropolar elastic materials has many applications concerning cellular solids (Lakes, 1983, 1986; Mora & Wass, 2000).

On the other hand, many natural or synthetic materials are not pure materials. They are mixtures of two or more co-existence constituents. Sometimes the presence of a constituent can be ignored, if there is a preponderant constituent, but in many situations the local mechanical effects of each ingredient of the mixture cannot be ignored. In classical continuum theory (Rajagopal & Tao, 1995), a mixture is idealized by assuming that every point in the mixture is occupied simultaneously by each constituent. A mixture is thereby envisioned as a superposition of several continuous media.

Taking into account the microstructural motions Twiss & Eringen (1971, 1972) introduce the mixture theory of materials with microstructure. In the last years many papers (Eringen, 2003; Ieşan, 2007, 2009; Chiriţă & Galeş, 2008) got back in discussion the study of mixtures with microstructure. Eringen (2003) has developed a continuum theory for a mixture of a micropolar elastic solid and a micropolar viscous fluid. This theory can be successfully applied to the study of engineering materials, as well as soils, rocks, granular materials, sand and underground water mixtures. Consolidation problems in the building industry, earthquake problems, oil exploration problems and cellular solids can be studied with the help of the mixture theories (de Boer, 2005; Elangovan et al., 2008).

In the framework of the theory developed by Eringen (2003), the existence, uniqueness, continuous data dependence of the solution of the initial-boundary value problem, the asymptotic partition of the energy and the stability problem have been studied by Ghiba (2006, 2007, 2008, 2009).

In various boundary-value problems from continuum mechanics it is important to give a representation of the general solution of the field equations in term of elementary (harmonic, biharmonic etc.) functions and to find the fundamental solution. In the micropolar elasticity, this type of results have been established by Ieşan (1971), Şandru (1966, 1975), Dragoş (1980, 1983). Ramkissoon & Majundar (1976 a,b) obtained the fundamental solution for slow, steady motion of micropolar fluids, in planar and three dimensional space. Olmstead & Majumdar (1983) gave the fundamental solution for a Oseen flow in the two dimensional case. These results have been completed by the study by Shu & Lee (2008).

In the present paper we consider the isothermal theory of a binary homogeneous mixture of an isotropic micropolar elastic solid with an incompressible micropolar viscous fluid. In the second section the basic equations of this theory are presented. In the third section, using the method introduced

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by Moisil (1952), we establish a representation of Galerkin type for the dynamical problem. In the next section we use some Galerkin representations in order to determine the fundamental solutions for the three–dimensional problem governing the motion of a micropolar solid-fluid mixture in the case of steady state vibrations. A direct application to the point load problem is presented in the last section of the paper.

The representations of the solution and the fundamental solutions for micropolar mixtures have not been studied yet. The equations of the theory developed by Eringen (2003) combine the usual linearized system of equations for isotropic micropolar elastic solids with those for incompressible micropolar viscous fluid and become a complex system of partial differential equations. We outline that the fundamental solutions of the static problems and of the steady oscillations problem for other types of mixtures are constructed in (Svanadze, 1993, 1996; Ciarletta, 1995; Ieşan, 1996; de Boer & Svanadze, 2004; Galeş, 2004 a,b; Svanadze & de Boer, 2005). We also note that in the theories of pure materials with microstructure the proposed problems have been studied by Ieşan & Pompei (1995); Ieşan & Nappa (2001), Svanadze et al. (2007), Nappa (2008) and Cialetta et al. (2009).

2. Basic equations

The space under consideration is the Euclidean three–dimensional space. We refer the motion of a continuum to a fixed system of rectangular Cartesian axes Ox_k ($k = 1, 2, 3$). The Latin subscripts, unless otherwise specified, are understood to range over the integers $1, 2, 3$, the Greek subscripts are confined to the range $1, 2$ and superscripts $\sigma = s, f$ denote the micropolar elastic solid and the incompressible micropolar fluid, respectively. Summation over repeated subscripts and other typical conventions for differential operations are implied such as comma followed by a subscript to denote the partial derivative with respect to the corresponding cartesian coordinate.

We suppose that the mixture is chemical inert, and the fluid is incompressible.

In this paper we use the following notations:

- ρ^σ — the density of the σth constituent;

- π^f — the dynamic pressure in the fluid species;

- u_r^σ — the displacement of the σth constituent;

- ϕ_r^σ — the microrotation vector of the σth constituent;

- v_r^σ — the velocity of the σth constituent;

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- $\nu_r^\sigma -$ the microrotation rate of the σth constituent;

- $\lambda^s, \mu^s, k^s, \alpha^s, \beta^s, \gamma^s -$ the micropolar elastic constants for micropolar elastic solid;

- $\mu^f, k^f, \alpha^f, \beta^f, \gamma^f -$ the micropolar fluid viscosities;

- $\xi -$ the momentum generation coefficient due to the velocity difference;

- $\varpi -$ the momentum generation coefficient due to the difference in gyration;

- $F_r^\sigma -$ the body force;

- $L_r^\sigma -$ the body couple;

- $j^\sigma -$ the microinertia density.

The field equations of the linear theory of isothermal micropolar solid–fluid mixtures are (Eringen, 2003)

$$
\begin{aligned}
(\lambda^s + \mu^s)u_{j,rj}^s + (\mu^s + k^s)u_{r,jj}^s + k^s \varepsilon_{rjk}\phi_{k,j}^s - \\
-\xi(\frac{\partial}{\partial t}u_r^s - v_r^f) + F_r^s = \rho^s \frac{\partial^2}{\partial t^2}u_r^s, \\
(\alpha^s + \beta^s)\phi_{j,rj}^s + \gamma^s \phi_{r,jj}^s + k^s(\varepsilon_{rjk}u_{k,j}^s - 2\phi_r^s) - \\
-\varpi(\frac{\partial}{\partial t}\phi_r^s - \nu_r^f) + L_r^s = \rho^s j^s \frac{\partial^2}{\partial t^2}\phi_r^s, \\
-\pi_{,r}^f + (\mu^f + k^f)v_{r,jj}^f + k^f \varepsilon_{rjk}\nu_{k,j}^f + \xi(\frac{\partial}{\partial t}u_r^s - v_r^f) + F_r^f = \rho^f \frac{\partial}{\partial t}v_r^f, \\
(\alpha^f + \beta^f)\nu_{j,rj}^f + \gamma^f \nu_{r,jj}^f + k^f(\varepsilon_{rjk}v_{k,j}^f - 2\nu_r^f) + \\
+\varpi(\frac{\partial}{\partial t}\phi_r^s - \nu_r^f) + L_r^f = \rho^f j^f \frac{\partial}{\partial t}\nu_r^f, \\
v_{r,r}^f = 0
\end{aligned}
\tag{2.1}
$$

where ε_{rjk} is the permutation symbol. It is easy to see that if we ignore the microstructure then the above equations will describe the behaviour of mixtures which have as constituents an isotropic elastic solid and a classical fluid with λ^s, μ^s and μ^f the classical Lamé constants.

In this paper we suppose that the dissipation potential and the internal energy density (Eringen, 2003) are positive definite. This is true if and only if

$$
\begin{aligned}
3\lambda^s + 2\mu^s + k^s > 0, \quad 2\mu^\sigma + k^\sigma > 0, \quad k^\sigma > 0, \quad \varpi > 0, \quad \xi > 0, \\
3\alpha^\sigma + \beta^\sigma + \gamma^\sigma > 0, \quad \gamma^\sigma + \beta^\sigma > 0, \quad \gamma^\sigma - \beta^\sigma > 0, \quad (\sigma = s, f).
\end{aligned}
\tag{2.2}
$$

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3. Complete solutions of the field equations

In this section we establish a Galerkin type representation of the solution of the field equations.

Let us introduce the differential operators:

$$Q_1^s(\Delta) = (\mu^s + k^s)\Delta - \xi\frac{\partial}{\partial t} - \rho^s\frac{\partial^2}{\partial t^2}, \quad Q_2^s(\Delta) = \gamma^s\Delta - 2k^s - \varpi\frac{\partial}{\partial t} - \rho^s j^s\frac{\partial^2}{\partial t^2},$$

$$Q_3^s(\Delta) = (\lambda^s + \mu^s)\Delta + Q_1^s, \quad Q_4^s(\Delta) = (\alpha^s + \beta^s)\Delta + Q_2^s,$$

$$Q_5^s(\Delta) = (k^s)^2\Delta + Q_1^s Q_2^s,$$

$$Q_1^f(\Delta) = (\mu^f + k^f)\Delta - \xi - \rho^f\frac{\partial}{\partial t}, \quad Q_2^f(\Delta) = \gamma^f\Delta - 2k^f - \varpi - \rho^f j^f\frac{\partial}{\partial t},$$

$$Q_3^f(\Delta) = (\alpha^f + \beta^f)\Delta + Q_2^f, \quad Q_4^f(\Delta) = (k^f)^2\Delta + Q_1^f Q_2^f,$$

$$P_1(\Delta) = \varpi^2\frac{\partial}{\partial t} - Q_2^f Q_2^s, \quad P_2(\Delta) = \xi^2\frac{\partial}{\partial t} - Q_1^f Q_1^s, \tag{3.1}$$

where $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \dfrac{\partial^2}{\partial x_3^2}$ is the Laplace operator.

In the above quantities and in the following, if $Q_n,\ n = 1, 2, ..., m$ are differential operators and G is, for example, a $C^\infty(\mathbb{R})$ function, we use the notation

$$Q_1 Q_2 ... Q_m G \equiv Q_1(Q_2(...(Q_m(G))...)).$$

With the help of these operators, we also define the following differential operators:

$$D_1(\Delta) = -\varpi^2\frac{\partial}{\partial t} + Q_3^f Q_4^s,$$

$$D_2(\Delta) = Q_5^s Q_4^f + 2\xi\varpi k^s k^f\Delta\frac{\partial}{\partial t} - (\varpi^2 Q_1^s Q_1^f + \xi^2 Q_2^s Q_2^f)\frac{\partial}{\partial t} + \xi^2\varpi^2\frac{\partial^2}{\partial t^2},$$

$$D_3(\Delta) = D_2 - (Q_4^f Q_2^s - \frac{\partial}{\partial t}\varpi^2 Q_1^f)Q_3^s, \tag{3.2}$$

$$D_4(\Delta) = Q_4^f Q_3^f[(k^s)^2 - (\alpha^s + \beta^s)Q_1^s] + 2\xi\varpi k^s k^f Q_3^f\frac{\partial}{\partial t} +$$
$$+\varpi^2 Q_1^s[(k^f)^2 - (\alpha^f + \beta^f)Q_1^f]\frac{\partial}{\partial t} + \xi^2(\alpha^s + \beta^s)Q_2^f Q_3^f\frac{\partial}{\partial t} + (\alpha^f + \beta^f)\xi^2\varpi^2\frac{\partial^2}{\partial t^2},$$

$$D_5(\Delta) = Q_5\varpi(k^f)^2 + \xi k^s k^f(\varpi^2\frac{\partial}{\partial t} + Q_3^f Q_4^s) + \varpi\{[(\alpha^s + \beta^s)Q_2^f +$$
$$+(\alpha^f + \beta^f)Q_2^s + (\alpha^f + \beta^f)(\alpha^s + \beta^s)\Delta]P_2 + (k^s)^2 Q_1^f Q_2^f\},$$

$$D_6(\Delta) = Q_5^s Q_4^s[(k^f)^2 - (\alpha^f + \beta^f)Q_1^f] + 2\xi\varpi k^s k^f Q_4^s\frac{\partial}{\partial t} +$$
$$+\varpi^2 Q_1^f[(k^s)^2 - (\alpha^s + \beta^s)Q_1^s]\frac{\partial}{\partial t} + \xi^2(\alpha^f + \beta^f)Q_2^s Q_4^s\frac{\partial}{\partial t} + \xi^2\varpi^2(\alpha^s + \beta^s)\frac{\partial^2}{\partial t^2}.$$

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Using the method introduced by Moisil (1952) we construct a representation of Galerkin type for the solution of the dynamical problem.

Theorem 3.1. *Let*

$$
\begin{aligned}
u_r^s &= \Delta Q_3^s D_1(-\varpi^2\frac{\partial}{\partial t}Q_1^f + Q_4^f Q_2^s)G_r^s + D_1 D_3 G_{j,rj}^s + \\
&\quad +\varepsilon_{rjk}\Delta Q_3^s D_1(k^s Q_4^f + \xi\varpi k^f\frac{\partial}{\partial t})H_{j,k}^s + \\
&\quad +\Delta Q_3^s D_1(\varpi k^s k^f \Delta + \xi P_1)G_r^f - Q_3^s D_1(\varpi k^s k^f \Delta + \xi P_1)G_{j,rj}^f - \\
&\quad -\varepsilon_{ijk}\Delta Q_3^s D_1(\varpi k^s Q_1^f + \xi k^f Q_2^s)H_{j,k}^f - \xi P_{,r},
\end{aligned}
$$

$$
\begin{aligned}
\phi_r^s &= \varepsilon_{rjk}\Delta Q_3^s D_1(k^s Q_4^f + \xi\varpi k^f\frac{\partial}{\partial t})G_{j,k}^s + \\
&\quad +\Delta Q_3^s D_1[(k^f)^2\Delta Q_1^s - P_2 Q_2^f]H_r^s + \Delta Q_3^s D_4 H_{j,rj}^s - \\
&\quad -\varepsilon_{rjk}D_1\Delta Q_3^s(\varpi k^f Q_1^s + \xi k^s Q_2^f)G_{j,k}^f + \\
&\quad +\Delta Q_3^s D_1(\xi k^s k^f \Delta + \varpi P_2)H_r^f - \Delta Q_3^s D_5 H_{j,rj}^f,
\end{aligned}
$$

$$
\begin{aligned}
v_r^f &= \frac{\partial}{\partial t}\Delta Q_3^s D_1(\varpi k^s k^f \Delta + \xi P_1)G_r^s - \frac{\partial}{\partial t}Q_3^s D_1(\varpi k^s k^f \Delta + \xi P_1)G_{j,rj}^s - \\
&\quad -\varepsilon_{rjk}\frac{\partial}{\partial t}\Delta Q_3^s D_1(\varpi k^f Q_1^s + \xi k^s Q_2^f)H_{j,k}^s + \\
&\quad +\Delta Q_3^s D_1(-\varpi^2\frac{\partial}{\partial t}Q_1^s + Q_5^s Q_2^f)G_r^f - Q_3^s D_1(-\varpi^2\frac{\partial}{\partial t}Q_1^s + Q_5^s Q_2^f)G_{j,rj}^f + \\
&\quad +\varepsilon_{rjk}\Delta Q_3^s D_1(\xi\varpi k^s\frac{\partial}{\partial t} + k^f Q_5^s)H_{j,k}^f + Q_3^s P_{,r},
\end{aligned}
$$

$$
\begin{aligned}
\nu_r^f &= -\varepsilon_{rjk}\Delta\frac{\partial}{\partial t}Q_3^s D_1(\varpi k^s Q_1^f + \xi k^f Q_2^s)G_{j,k}^s + \\
&\quad +\frac{\partial}{\partial t}\Delta Q_3^s D_1(\xi k^s k^f \Delta + \varpi P_2)H_r^s - \frac{\partial}{\partial t}\Delta Q_3^s D_5 H_{j,rj}^s + \\
&\quad +\varepsilon_{rjk}\Delta Q_3^s D_1(\xi\varpi k^s\frac{\partial}{\partial t} + k^f Q_5^s)G_{j,k}^f + \\
&\quad +\Delta Q_3^s D_1[(k^s)^2\Delta Q_1^f - P_2 Q_2^s]H_r^f + \Delta Q_3^s D_6 H_{j,rj}^f,
\end{aligned}
$$

$$
\pi^f = \xi\frac{\partial}{\partial t}D_1 D_2 G_{r,r}^s - Q_3^s D_1 D_2 G_{r,r}^f - (\xi^2\frac{\partial}{\partial t} - Q_1^f Q_3^s)P,
$$

(3.3)

where $G_r^s, G_r^f, H_r^s, F_r^f$ *and* P *satisfy*

$$\Delta Q_3^s D_1 D_2 G_r^\sigma = -F_r^\sigma, \ \ \Delta Q_3^s D_1 D_2 H_r^\sigma = -L_r^\sigma, \ \ \Delta Q_3^s P = 0 \ (\sigma = s, f). \quad (3.4)$$

Then u_r^s, v_r^f, ϕ_r^s, ν_r^f *and* π^f *satisfy the equations* (2.1).

Proof. It is easy to see that for every $\mathbf{A} \in C^3(\mathbb{R}^3) \times C^3(\mathbb{R}^3) \times C^3(\mathbb{R}^3)$ we have

$$\varepsilon_{rjk}A_{j,ki} = 0, \ \ \varepsilon_{jmn}A_{m,nrj} = 0, \ \ \varepsilon_{rjk}A_{m,kmj} = 0, \quad (3.5)$$

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and

$$\varepsilon_{krj}\varepsilon_{kmn} = \delta_{rm}\delta_{jn} - \delta_{rn}\delta_{jm}. \tag{3.6}$$

To prove that v_r^f satisfies the incompressibility condition $(2.1)_5$, we use the relations (3.4), (3.6) and the relations (3.5) for $\mathbf{A} = \mathbf{H}^s$ and $\mathbf{A} = \mathbf{H}^f$.

By a direct substitution of the relations (3.3) in (2.1) and using the relations (3.4), (3.5), (3.6) and the identities

$$Q_3^s D_4 - \varpi\frac{\partial}{\partial t}D_5 = D_1\{k^s(k^sQ_4^f + \xi\varpi k^f\frac{\partial}{\partial t}) - (\alpha^s + \beta^s)[(k^f)^2\Delta Q_1^s - P_2Q_2^f]\},$$
$$Q_3^s D_5 - \varpi D_6 = D_1[k^s(\varpi k^sQ_1^f + \xi k^fQ_2^s) + (\alpha^s + \beta^s)(\xi k^fk^s\Delta + \varpi P_2)],$$
$$Q_3^f D_5 - \varpi D_4 = D_1[(\alpha^f + \beta^f)(\xi k^fk^s\Delta + \varpi P_1) + k^f(\varpi k^fQ_1^s + \xi k^sQ_2^f)],$$
$$Q_3^f D_6 - \varpi\frac{\partial}{\partial t}D_5 = D_1\{k^f(\xi\varpi k^s\frac{\partial}{\partial t} + k^fQ_5^s) - (\alpha^f + \beta^f)[(k^s)^2\Delta Q_1^f - P_2Q_2^s]\},$$

$$\varepsilon_{ijk}P_{,kj} = 0, \tag{3.7}$$

we have that u_r^s, v_r^f, ϕ_r^s, ν_r^f and π^f are solutions of the basic system of equations (2.1), and the proof is complete.

4. Fundamental solution for steady state vibrations

In this section we use the representation described in the previous section in order to determine the fundamental solution of equations of motion for the case of steady vibrations. We suppose that

$$F_r^\sigma = \mathrm{Re}\Big[F_r^{*\sigma}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\Big], \quad L_r^\sigma = \mathrm{Re}\Big[L_r^{*\sigma}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\Big],$$
$$u_r^\sigma = \mathrm{Re}\Big[u_r^{*\sigma}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\Big], \quad \phi_r^\sigma = \mathrm{Re}\Big[\phi_r^{*\sigma}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\Big], \tag{4.1}$$
$$\pi^f = \mathrm{Re}\Big[\pi^{*f}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\Big], \quad (\sigma = s, f),$$

where i is the imaginary unit and $\omega > 0$ is the frequency of the vibration.

Regarding the propagation of plane harmonic waves in micropolar materials we have to say that the micropolar effects become important in high-frequency and short wave-length regions of waves (Eringen, 1999). In the theory of micropolar mixtures introduced by Eringen (2003) this aspect has been studied by Singh & Tomar (2006). In the present paper we consider that the frequency is high enough for the solution for steady state vibrations to exist.

Let us introduce the differential operators:

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$$Q_1^{*s}(\Delta) = (\mu^s + k^s)\Delta + i\omega\xi - \rho^s\omega^2, \quad Q_2^{*s}(\Delta) = \gamma^s\Delta - 2k^s + i\omega\varpi - \rho^s\omega^2 j^s,$$

$$Q_3^{*s}(\Delta) = (\lambda^s + \mu^s)\Delta + Q_1^{*s}, \quad Q_4^{*s}(\Delta) = (\alpha^s + \beta^s)\Delta + Q_2^{*s},$$

$$Q_5^{*s}(\Delta) = (k^s)^2\Delta + Q_1^{*s}Q_2^{*s}, \quad Q_1^{*f}(\Delta) = (\mu^f + k^f)\Delta - \xi + i\rho^f\omega,$$

$$Q_2^{*f}(\Delta) = \gamma^f\Delta - 2k^f - \varpi + i\rho^f\omega j^f, \quad Q_3^{*f}(\Delta) = (\alpha^f + \beta^f)\Delta + Q_2^{*f},$$

$$Q_4^{*f}(\Delta) = (k^f)^2\Delta + Q_1^{*f}Q_2^{*f}, \, P_1^*(\Delta) = -i\omega\varpi^2 - Q_2^{*f}Q_2^{*s},$$

$$P_2^*(\Delta) = -i\omega\xi^2 - Q_1^{*f}Q_1^{*s}, \quad D_1^*(\Delta) = i\omega\varpi^2 + Q_3^{*f}Q_4^{*s}, \qquad (4.2)$$

$$D_2^*(\Delta) = Q_5^{*s}Q_4^{*f} - 2i\omega\xi\varpi k^s k^f\Delta + i\omega(\varpi^2 Q_1^{*s}Q_1^{*f} + \xi^2 Q_2^{*s}Q_2^{*f}) + \xi^2\omega^2\varpi^2,$$

$$D_3^*(\Delta) = D_2^* - (Q_4^{*f}Q_2^{*s} + i\omega\varpi^2 Q_1^{*s})Q_3^{*s},$$

$$D_4^*(\Delta) = Q_4^{*f}Q_3^{*f}[(k^s)^2 - (\alpha^s + \beta^s)Q_1^{*s}] - 2i\omega\xi\varpi k^s k^f Q_3^{*f} -$$

$$-i\omega\varpi^2 Q_1^{*s}[(k^f)^2 - (\alpha^f + \beta^f)Q_1^{*f}] - i\omega\xi^2(\alpha^s + \beta^s)Q_2^{*f}Q_3^{*f} + (\alpha^f + \beta^f)\xi^2\omega^2\varpi^2,$$

$$D_5^*(\Delta) = Q_5^*\varpi(k^f)^2 + \xi k^s k^f(-i\omega\varpi^2 + Q_3^{*f}Q_4^{*s}) + \varpi\{[(\alpha^s + \beta^s)Q_2^{*f} +$$

$$+ (\alpha^f + \beta^f)Q_2^{*s} + (\alpha^f + \beta^f)(\alpha^s + \beta^s)\Delta]P_2^* + (k^s)^2 Q_1^{*f}Q_2^{*f}\},$$

$$D_6^*(\Delta) = Q_5^{*s}Q_4^{*s}[(k^f)^2 - (\alpha^f + \beta^f)Q_1^{*f}] - 2i\omega\xi\varpi k^s k^f Q_4^{*s} -$$

$$-i\omega\varpi^2 Q_1^{*f}[(k^s)^2 - (\alpha^s + \beta^s)Q_1^{*s}] - i\omega\xi^2(\alpha^f + \beta^f)Q_2^{*s}Q_4^{*s} + \omega^2\xi^2\varpi^2(\alpha^s + \beta^s).$$

We introduce the differential matrix operator

$$\mathbf{D}\left(\frac{\partial}{\partial\mathbf{x}}\right) = \left\|D_{mn}\left(\frac{\partial}{\partial\mathbf{x}}\right)\right\|_{13\times13}, \qquad (4.3)$$

where

$$D_{rj} = Q_1^{*s}\delta_{rj} + (\lambda^s + \mu^s)\frac{\partial^2}{\partial x_r \partial x_j},$$

$$D_{r;3+j} = -D_{3+j;r} = k^s\varepsilon_{rkj}\frac{\partial}{\partial x_k},$$

$$D_{r;j+6} = D_{j+6;r} = -i\omega\xi,$$

$$D_{r;j+9} = D_{j+9;r} = D_{r;13} = D_{13;r} = D_{r+3;j+6} = D_{j+6;r+3} = 0,$$

$$D_{r+3;13} = D_{13;r+3} = D_{r+9;13} = D_{13;r+9} = D_{13;13} = 0,$$

$$D_{r+3;j+3} = Q_2^{*s}\delta_{rj} + (\alpha^s + \beta^s)\frac{\partial^2}{\partial x_r \partial x_j},$$

$$D_{r+3;j+9} = D_{j+9;r+3} = -i\omega\varpi,$$

$$D_{r+6;j+6} = -i\omega Q_1^{*f}\delta_{rj},$$

$$D_{r+6;j+9} = -D_{j+9;r+6} = -i\omega k^f\varepsilon_{rkj}\frac{\partial}{\partial x_k},$$

$$D_{r+9;j+9} = -i\omega Q_2^{*f}\delta_{rj} - i\omega(\alpha^f + \beta^f)\frac{\partial^2}{\partial x_r \partial x_j},$$

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$$D_{r+6;13} = -\frac{1}{\chi}D_{13;r+6} = -\frac{\partial}{\partial x_r},$$

$$\chi = (\lambda^s + 2\mu^s + k^s)^{-1}\prod_{\sigma=s,f}[(\alpha^\sigma + \beta^\sigma + \gamma^\sigma)\gamma^\sigma(k^\sigma + \mu^\sigma)]^{-1}.$$

If we introduce the thirteen–dimensional vectors $\mathbf{U} = (\mathbf{u}^{*s}, \boldsymbol{\phi}^{*s}, \mathbf{u}^{*f}, \boldsymbol{\phi}^{*f}, \pi^{*f})$ and $\mathbf{F} = (\mathbf{F}^{*s}, \mathbf{L}^{*s}, \mathbf{F}^{*f}, \mathbf{L}^{*f}, 0)$, it is easy to see that the basic system of equations which describes the behaviour of the amplitudes $\mathbf{U}$ of the steady vibrations can be written in the form

$$\mathbf{D}\left(\frac{\partial}{\partial \mathbf{x}}\right)\mathbf{U} = -\mathbf{F}. \tag{4.4}$$

Definition 4.1. Let be $\mathbf{y} \in \mathbf{E}^3$. A fundamental solution of the system (4.4) is a matrix $\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y}; \omega) = \|\Gamma_{rj}\|_{13\times13}$ which satisfies the condition (Hörmander, 1964)

$$\mathbf{D}\left(\frac{\partial}{\partial \mathbf{x}}\right)\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y}; \omega) = -\delta(\mathbf{x} - \mathbf{y})\mathbf{I}, \quad \mathbf{x} \in \mathbf{E}^3, \tag{4.5}$$

where $\delta(\cdot)$ is the Dirac delta and $\mathbf{I} = \|\delta_{rj}\|_{13\times13}$ is the unit matrix.

According to the general theory of the fundamental solutions of the differential operators (Kythe, 1996), we have to say that a fundamental solution is unique up to a matrix which has as columns solutions of the homogeneous system

$$\mathbf{D}\left(\frac{\partial}{\partial \mathbf{x}}\right)\mathbf{U} = 0. \tag{4.6}$$

As a consequence of the Theorem 3.1 we obtain the following result:

Theorem 4.1. Let

$$
\begin{aligned}
u_r^{*s} &= \Delta Q_3^{*s}D_1^*(\mathrm{i}\omega\varpi^2 Q_1^{*f} + Q_4^{*f}Q_2^{*s})G_r^{*s} + D_1^*D_3^*G_{j,rj}^{*s} + \\
&\quad +\varepsilon_{rjk}\Delta Q_3^{*s}D_1^*(k^s Q_4^{*f} - \mathrm{i}\omega\xi\varpi k^f)H_{j,k}^{*s} + \\
&\quad +\Delta Q_3^{*s}D_1^*(\varpi k^s k^f \Delta + \xi P_1^*)G_r^{*f} - Q_3^{*s}D_1^*(\varpi k^s k^f \Delta + \xi P_1^*)G_{j,rj}^{*f} - \\
&\quad -\varepsilon_{rjk}\Delta Q_3^{*s}D_1^*(\varpi k^s Q_1^{*f} + \xi k^f Q_2^{*s})H_{j,k}^{*f} - \xi P_{,r}^*, \\
\phi_r^{*s} &= \varepsilon_{rjk}\Delta Q_3^{*s}D_1^*(k^s Q_4^{*f} - \mathrm{i}\omega\xi\varpi k^f)G_{j,k}^{*s} + \\
&\quad +\Delta Q_3^{*s}D_1^*[(k^f)^2\Delta Q_1^{*s} - P_2^*Q_2^{*f}]H_r^{*s} + \Delta Q_3^{*s}D_4^*H_{j,rj}^{*s} - \\
&\quad -\varepsilon_{rjk}D_1^*\Delta Q_3^{*s}(\varpi k^f Q_1^{*s} + \xi k^s Q_2^{*f})G_{j,k}^{*f} + \\
&\quad +\Delta Q_3^{*s}D_1^*(\xi k^s k^f \Delta + \varpi P_2^*)H_r^{*f} - \Delta Q_3^{*s}D_5^*H_{j,rj}^{*f},
\end{aligned}
$$

$$
\begin{aligned}
u_r^{*f} &= \Delta Q_3^{*s} D_1^*(\varpi k^s k^f \Delta + \xi P_1^*) G_r^{*s} - Q_3^{*s} D_1^*(\varpi k^s k^f \Delta + \xi P_1^*) G_{j,rj}^{*s} - \\
&\quad -\varepsilon_{rjk} \Delta Q_3^{*s} D_1^*(\varpi k^f Q_1^{*s} + \xi k^s Q_2^{*f}) H_{j,k}^{*s} + \\
&\quad +\frac{\mathrm{i}}{\omega} \Delta Q_3^{*s} D_1^*(\mathrm{i}\omega\varpi^2 Q_1^{*s} + Q_5^{*s} Q_2^{*f}) G_r^{*f} - \frac{\mathrm{i}}{\omega} Q_3^{*s} D_1^*(\mathrm{i}\omega\varpi^2 Q_1^{*s} + Q_5^{*s} Q_2^{*f}) G_{j,rj}^{*f} + \\
&\quad +\frac{\mathrm{i}}{\omega}\varepsilon_{rjk} \Delta Q_3^{*s} D_1(-\mathrm{i}\omega\xi\varpi k^s + k^f Q_5^{*s}) H_{j,k}^{*f} + \frac{\mathrm{i}}{\omega} Q_3^{*s} P_{,r}^*, \\
\phi_r^{*f} &= -\varepsilon_{rjk} \Delta Q_3^{*s} D_1^*(\varpi k^s Q_1^{*f} + \xi k^f Q_2^{*s}) G_{j,k}^{*s} + \\
&\quad +\Delta Q_3^{*s} D_1^*(\xi k^s k^f \Delta + \varpi P_2^*) H_r^{*s} - \Delta Q_3^{*s} D_5^* H_{j,rj}^{*s} + \\
&\quad +\frac{\mathrm{i}}{\omega}\varepsilon_{rjk} \Delta Q_3^{*s} D_1^*(-\mathrm{i}\omega\xi\varpi k^s + k^f Q_5^{*s}) G_{j,k}^{*f} + \\
&\quad +\frac{\mathrm{i}}{\omega} \Delta Q_3^{*s} D_1[(k^s)^2 \Delta Q_1^{*f} - P_2^* Q_2^{*s}] H_r^{*f} + \frac{\mathrm{i}}{\omega} \Delta Q_3^{*s} D_6^* H_{j,rj}^{*f},
\end{aligned}
$$

$$
\pi^{*f} = -\mathrm{i}\omega\xi \ D_1^* D_2^* G_{r,r}^{*s} - Q_3^{*s} D_1^* D_2^* G_{r,r}^{*f} + (\mathrm{i}\xi^2\omega + Q_1^{*f} Q_3^{*f}) P^*, \tag{4.7}
$$

where $G_r^{*s}, G_r^{*f}, H_r^{*s}, F_r^{*f}$ and P^* satisfy

$$
\Delta Q_3^{*s} D_1^* D_2^* G_r^{*\sigma} = -F_r^{*\sigma}, \ \Delta Q_3^{*s} D_1^* D_2^* H_r^{*\sigma} = -L_r^{*\sigma}, \ \Delta Q_3^{*s} P^* = 0 \ (\sigma = s, f). \tag{4.8}
$$

Then u_r^{*s}, u_r^{*f}, ϕ_r^{*s}, ϕ_r^{*f} and π^{*f} satisfy the equations (4.4).

We denote by k_n^2, $n = 1, 2$ and respectively, by k_m^2, $m = 3, 4, 5, 6$ the roots of the equations

$$
D_1^*(-k) = 0, \ D_2^*(-k) = 0. \tag{4.9}
$$

It is convenient to write

$$
Q_3^{*s}(\Delta) = (\lambda^s + 2\mu^s + k^s)(\Delta + k_7^2), \tag{4.10}
$$

where k_7 is the complex number defined by

$$
k_7^2 = -\frac{1}{\lambda^s + 2\mu^s + k^s}(\rho^s\omega^2 - \mathrm{i}\omega\xi). \tag{4.11}
$$

We assume that $k_n^2 \neq k_m^2$, for $n \neq m$, $n, m = 1, 2, ..., 7$, and we choose the complex number k_n such that $\mathrm{Im}[k_n] \geq 0$, for $n = 1, 2, ..., 7$.

With the help of these quantities, we can rewrite the equations (4.8) in the following form

$$
\Delta \prod_{n=1}^{7}(\Delta + k_n^2) G_r^{*\sigma} = -\chi F_r^{*\sigma}, \ \ \Delta \prod_{n=1}^{7}(\Delta + k_n^2) H_r^{*\sigma} = -\chi L_r^{*\sigma},
$$
$$
\Delta(\Delta + k_7^2) P^* = 0 \ (\sigma = s, f). \tag{4.12}
$$

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Proposition 4.1. *Assume that $F_r^{*s} = \delta_{rk}\delta(\mathbf{x} - \mathbf{y})$, $L_r^{*s} = 0$, $F_r^{*f} = 0$ and $L_r^{*f} = 0$. Then, the equations (4.12) have the solution $G_r^{*s} = \delta_{rk}E(\mathbf{x}, \mathbf{y}; \omega)$, $H_r^{*s} = 0, G_r^{*f} = 0$, $H_r^{*f} = 0, P^* = 0$, where*

$$
E(\mathbf{x}, \mathbf{y}; \omega) = \sum_{n=1}^{7} c_n E_n(\mathbf{x}, \mathbf{y}; \omega), \quad E_n = \frac{\chi}{4\pi\varrho k_n^2}(1 - e^{ik_n\varrho}),
$$

$$
c_n^{-1} = \prod_{m=1, m \neq n}^{7}(k_n^2 - k_m^2) \quad n = 1, 2, ..., 7, \quad \varrho^2 = (x_r - y_r)(x_r - y_r).
$$
(4.13)

Proof. First of all, we remark that

$$
\Delta(\Delta + k_n^2)E_n = -\chi\delta(\mathbf{x} - \mathbf{y}).
$$
(4.14)

Taking into account the relations

$$
\sum_{n=1}^{7} c_n = 0, \sum_{n=m}^{7} c_n \prod_{l=1}^{m-1}(k_l^2 - k_n^2) = 0 \quad \text{for} \quad m = 2, 3, ..., 6
$$

$$
(\Delta + k_n^2)E_m = \chi\delta(\mathbf{x} - \mathbf{y}) + (k_n^2 - k_m^2)E_m \quad \text{for} \quad n, m = 1, 2, ..., 7,
$$
(4.15)

and the method presented in the paper (Svanadze, 1996), we have

$$
\Delta\prod_{n=1}^{7}(\Delta + k_n^2)E(\mathbf{x}, \mathbf{y}; \omega) = -\chi\delta(\mathbf{x} - \mathbf{y}),
$$
(4.16)

and the proof is complete.

We denote by $(u_r^{*s(k)}, \phi_r^{*s(k)}, u_r^{*f(k)}, \phi_r^{*f(k)})$ the amplitudes of displacements caused by the concentrated loads $F_r^{*s} = \delta_{rk}\delta(\mathbf{x} - \mathbf{y})$, $L_r^{*s} = 0$, $F_r^{*f} = 0$ and $L_r^{*f} = 0$. In view of the relations (4.7) we get

$$
u_r^{*s(k)} = \Delta Q_3^{*s}D_1^*(i\omega\varpi^2 Q_1^{*f} + Q_4^{*f}Q_2^{*s})\delta_{rk}E + D_1^*D_3^*E_{,rk},
$$

$$
\phi_r^{*s(k)} = \varepsilon_{rkl}\Delta Q_3^{*s}D_1^*(k^s Q_4^{*f} - i\omega\xi\varpi k^f)E_{,l},
$$

$$
u_r^{*f(k)} = \Delta Q_3^{*s}D_1^*(\varpi k^s k^f \Delta + \xi P_1^*)\delta_{rk}E - Q_3^{*s}D_1^*(\varpi k^s k^f\Delta + \xi P_1^*)E_{,rk},
$$
(4.17)

$$
\phi_r^{*f(k)} = -\varepsilon_{rkl}\Delta Q_3^{*s}D_1^*(\varpi k^s Q_1^{*f} + \xi k^f Q_2^{*s})E_{,l},
$$

$$
\pi^{*f(k)} = -i\omega\xi \, D_1^*D_2^*E_{,k}.
$$

Corresponding to the concentrated loads, $F_r^{*s} = 0$, $L_r^{*s} = \delta_{rk}\delta(\mathbf{x} - \mathbf{y})$, $F_r^{*f} = 0$ and $L_r^{*f} = 0$, we have the following amplitude of displacement,

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denoted by $(u_r^{*s(3+k)},\ \phi_r^{*s(3+k)},\ u_r^{*f(3+k)}, \phi_r^{*f(3+k)})$,

$$u_r^{*s(3+k)} = \varepsilon_{rkl}\Delta Q_3^{*s}D_1^*(k^s Q_4^{*f} - \mathrm{i}\omega\xi\varpi k^f)E_{,l},$$
$$\phi_r^{*s(3+k)} = \Delta Q_3^{*s}D_1^*[(k^f)^2\Delta Q_1^{*s} - P_2^*Q_2^{*f}]\delta_{rk}E + \Delta Q_3^{*s}D_4^*E_{,rk},$$
$$u_r^{*f(3+k)} = -\varepsilon_{rkl}\Delta Q_3^{*s}D_1^*(\varpi k^f Q_1^{*s} + \xi k^s Q_2^{*f})E_{,l}, \qquad (4.18)$$
$$\phi_r^{*f(3+k)} = \Delta Q_3^{*s}D_1^*(\xi k^s k^f \Delta + \varpi P_2^*)\delta_{rk}E - \Delta Q_3^{*s}D_5^*E_{,rk},$$
$$\pi^{*f(3+k)} = 0.$$

If $F_r^{*s} = 0$, $L_r^{*s} = 0$, $F_r^{*f} = \delta_{rk}\delta(\mathbf{x}-\mathbf{y})$ and $L_r^{*f} = 0$, then the corresponding displacement vectors denoted by $(u_r^{*s(6+k)},\ \phi_r^{*s(6+k)},\ u_r^{*f(6+k)}, \phi_r^{*f(6+k)})$ are

$$u_r^{*s(6+k)} = \Delta Q_3^{*s}D_1^*(\varpi k^s k^f \Delta + \xi P_1^*)\delta_{rk}E - Q_3^{*s}D_1^*(\varpi k^s k^f \Delta + \xi P_1^*)E_{,rk},$$
$$\phi_r^{*s(6+k)} = -\varepsilon_{rkl}D_1^*\Delta Q_3^{*s}(\varpi k^f Q_1^{*s} + \xi k^s Q_2^{*f})E_{,l}$$
$$u_r^{*f(6+k)} = \frac{\mathrm{i}}{\omega}\Delta Q_3^{*s}D_1^*(\mathrm{i}\omega\varpi^2 Q_1^{*s} + Q_5^{*s}Q_2^{*f})\delta_{rk}E-$$
$$\qquad -\frac{\mathrm{i}}{\omega}Q_3^{*s}D_1^*(\mathrm{i}\omega\varpi^2 Q_1^{*s} + Q_5^{*s}Q_2^{*f})E_{,rk},$$
$$\phi_r^{*f(6+k)} = \frac{\mathrm{i}}{\omega}\varepsilon_{rkl}\Delta Q_3^{*s}D_1^*(-\mathrm{i}\omega\xi\varpi k^s + k^f Q_5^{*s})E_{,l},$$
$$\pi^{*f(6+k)} = -Q_3^{*s}D_1^*D_2^*E_{,k}.$$

$$(4.19)$$

Finally, if $F_r^{*s} = 0$, $L_r^{*s} = 0$, $F_r^{*f} = 0$ and $L_r^{*f} = \delta_{rk}\delta(\mathbf{x}-\mathbf{y})$, then we have for the displacement vectors $(u_r^{*s(9+k)},\ \phi_r^{*s(9+k)},\ u_r^{*f(9+k)}, \phi_r^{*f(9+k)})$ the expressions

$$u_r^{*s(9+k)} = -\varepsilon_{rkl}\Delta Q_3^{*s}D_1^*(\varpi k^s Q_1^{*f} + \xi k^f Q_2^{*s})E_{,l},$$
$$\phi_r^{*s(9+k)} = \Delta Q_3^{*s}D_1^*(\xi k^s k^f \Delta + \varpi P_2^*)\delta_{rk}E - \Delta Q_3^{*s}D_5^*E_{,rk}, \qquad (4.20)$$
$$u_r^{*f(9+k)} = \frac{\mathrm{i}}{\omega}\varepsilon_{rkl}\Delta Q_3^{*s}D_1(-\mathrm{i}\omega\xi\varpi k^s + k^f Q_5^{*s})E_{,l},$$

$$\phi_r^{*f(9+k)} = \frac{\mathrm{i}}{\omega}\Delta Q_3^{*s}D_1[(k^s)^2\Delta Q_1^{*f} - P_2^*Q_2^{*s}]\delta_{rk}E + \frac{\mathrm{i}}{\omega}\Delta Q_3^{*s}D_6^*E_{,rk},$$
$$\pi^{*f(9+k)} = 0.$$

From the above discussion we can conclude:

Theorem 4.1. *The matrix* $\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y}; \omega)$ *defined by*

$$\Gamma_{r;k} = u_r^{*s(k)}, \Gamma_{3+r;k} = \phi_r^{*s(k)}, \ \Gamma_{6+r;k} = u_r^{*f(k)}, \ \Gamma_{9+r;k} = \phi_r^{*f(k)},$$
$$\Gamma_{r;3+k} = u_r^{*s(3+k)}, \Gamma_{3+r;3+k} = \phi_r^{*s(3+k)}, \Gamma_{6+r;3+k} = u_r^{*f(3+k)},$$
$$\Gamma_{9+r;3+k} = \phi_r^{*f(3+k)}, \Gamma_{r;6+k}(\mathbf{x}, \omega) = u_r^{*s(6+k)}, \Gamma_{3+r;6+k} = \phi_r^{*s(6+k)},$$
$$\Gamma_{6+r;6+k} = u_r^{*f(6+k)}, \Gamma_{9+r;6+k} = \phi_r^{*f(6+k)}, \Gamma_{r;9+k}(\mathbf{x}, \omega) = u_r^{*s(9+k)}, \qquad (4.21)$$
$$\Gamma_{3+r;9+k} = \phi_r^{*s(9+k)}, \Gamma_{6+r;9+k} = u_r^{*f(9+k)}, \Gamma_{9+r;9+k} = \phi_r^{*f(9+k)},$$
$$-\Gamma_{k;13} = \Gamma_{13;k} = \pi^{*f(k)}, -\Gamma_{6+k;13} = \Gamma_{13;6+k} = \pi^{*f(6+k)},$$
$$\Gamma_{3+k;13} = \Gamma_{13;3+k} = \Gamma_{9+k;13} = \Gamma_{13;9+k} = \Gamma_{13;13} = 0$$

is a fundamental solution of the system (4.4).

5. Basic properties of the matrix $\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y}; \omega)$

In this section we point out some basic properties of the fundamental solution constructed in the previous section. These basic properties of fundamental matrix are useful if we want to apply the potential method for the framework theory.

Let us first note that

Proposition 5.1. *The fundamental matrix* $\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y}; \omega)$ *is so that*

(i) $\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y}; \omega) = \boldsymbol{\Gamma}^T(\mathbf{y}, \mathbf{x}; \omega)$;
(ii) *If* $\mathbf{x} \neq \mathbf{y}$, *then each column* $\boldsymbol{\Gamma}^{(m)}(\mathbf{x}, \mathbf{y}; \omega), (m = 1, 2, ..., 13)$ *of the matrix* $\boldsymbol{\Gamma}$ *satisfies at* $\mathbf{x}$ *the homogeneous system*

$$\mathbf{D}\left(\frac{\partial}{\partial \mathbf{x}}\right) \boldsymbol{\Gamma}^{(m)}(\mathbf{x}, \mathbf{y}; \omega) = 0. \qquad (5.1)$$

Lemma 5.1. *The function* E *has the following properties:*

(i) $\dfrac{\partial E}{\partial x_r^{s_1} \partial x_l^{s_2} \partial x_k^{s_3}} = O(\varrho) \ (\varrho \to 0)$, *for all even* $s \leq 11$;

(ii) $\dfrac{\partial E}{\partial x_r^{s_1} \partial x_l^{s_2} \partial x_k^{s_3}} = \text{const} + O(\varrho^2) \ (\varrho \to 0)$, *for all odd* $s \leq 11$;

(iii) $\dfrac{\partial E}{\partial x_r^{s_1} \partial x_l^{s_2} \partial x_k^{s_3}} = O(\varrho^{13-s}) \ (\varrho \to 0)$, *for all* $s \geq 12$;

where $s_1, s_2, s_3 \in \mathbb{N}^*$ *and* $s = s_1 + s_2 + s_3$.

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Proof. It is easy to see that in the neighbourhood of $\mathbf{y}$, we have

$$\frac{1 - e^{\mathrm{i}k_n\varrho}}{k_n^2\varrho} = -\frac{\mathrm{i}}{k_n} + \sum_{m=0}^{\infty} \frac{(\mathrm{i}k_n)^m}{(m+2)!}\varrho^{m+1}. \tag{5.2}$$

On the other hand, we can deduce that

$$\sum_{n=1}^{7} c_n k_n^{2p} = 0, \text{ for } p = 0, 1, ..., 5 \text{ and } \sum_{n=1}^{7} c_n k_n^{12} = 1. \tag{5.3}$$

Thus, we obtain

$$E = \frac{\chi}{4\pi}\left[-\mathrm{i}\sum_{n=1}^{7}\frac{c_n}{k_n} + \sum_{n=1}^{7}\sum_{m=1}^{6}\frac{c_n(\mathrm{i}k_n)^{2m-1}}{(2m+1)!}\varrho^{2m} + \frac{1}{14!}\varrho^{13} + \sum_{n=1}^{7}\sum_{m=13}^{\infty}\frac{c_n(\mathrm{i}k_n)^m}{(m+2)!}\varrho^{m+1}\right]. \tag{5.4}$$

Using this relation we obtain the conclusions of lemma.

Let us introduce the matrix $\mathbf{\Pi}$ defined by

$$\Pi_{r;k} = \frac{1}{4\pi}\left(\frac{1}{\mu^s + k^s} - \frac{1}{2a^s}\right)\delta_{rk}\frac{1}{\varrho} + \frac{1}{8\pi a^s}\frac{x_r x_k}{\varrho^3},$$

$$\Pi_{3+r;k} = \Pi_{6+r;k} = \Pi_{9+r;k} = \Pi_{13;k} = 0,$$

$$\Pi_{3+r;3+k} = \frac{1}{4\pi}\left(\frac{1}{\gamma^s} - \frac{1}{2b^s}\right)\delta_{rk}\frac{1}{\varrho} + \frac{1}{8\pi b^s}\frac{x_r x_k}{\varrho^3},$$

$$\Pi_{r;3+k} = \Pi_{6+r;3+k} = \Pi_{9+r;3+k} = \Pi_{13;3+k} = 0,$$

$$\Pi_{6+r;6+k} = \frac{\mathrm{i}}{8\pi\omega(\mu^f + k^f)}\left(\delta_{rk}\frac{1}{\varrho} + \frac{x_r x_k}{\varrho^3}\right), \tag{5.5}$$

$$\Pi_{r;6+k} = \Pi_{3+r;6+k} = \Pi_{9+r;6+k} = 0, \ \Pi_{13;6+k} = -\Pi_{6+k;13} = -\frac{1}{4\pi}\frac{x_k}{\varrho^3},$$

$$\Pi_{9+r;9+k} = \frac{\mathrm{i}}{4\pi\omega}\left(\frac{1}{\gamma^s} - \frac{1}{2b^s}\right)\delta_{rk}\frac{1}{\varrho} + \frac{\mathrm{i}}{8\pi\omega b^s}\frac{x_r x_k}{\varrho^3},$$

$$\Pi_{r;9+k} = \Pi_{3+r;9+k} = \Pi_{6+r;9+k} = \Pi_{13;3+k} = 0,$$

$$\Pi_{k;13} = \Pi_{3+k;13} = \Pi_{9+k;13} = \Pi_{13;13} = 0,$$

where

$$a^s = \frac{(\mu^s + k^s)(\lambda^s + 2\mu^s + k^s)}{\lambda^s + \mu^s}$$

$$b^\sigma = \frac{\gamma^\sigma(\alpha^\sigma + \beta^\sigma + \gamma^\sigma)}{\alpha^\sigma + \beta^\sigma} \tag{5.6}$$

We can observe that

$$\mathbf{\Pi}(\mathbf{x}, \mathbf{y}; \omega) = \mathbf{\Pi}(\mathbf{y}, \mathbf{x}; \omega), \ \ \mathbf{\Pi}(\mathbf{x}, \mathbf{y}; \omega) = \mathbf{\Pi}^T(\mathbf{x}, \mathbf{y}; \omega). \tag{5.7}$$

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We denote by $\mathbf{\Pi}^{(m)}(\mathbf{x}, \mathbf{y}; \omega)$, $(m = 1, 2, ..., 13)$ the columns of the matrix $\mathbf{\Pi}(\mathbf{x}, \mathbf{y}; \omega)$.

In view of Lemma 5.1 we have

Proposition 5.2. *The differences*

$$\boldsymbol{G}^{(m)}(\mathbf{x}, \mathbf{y}; \omega) = \mathbf{\Gamma}^{(m)}(\mathbf{x}, \mathbf{y}; \omega) - \mathbf{\Pi}^{(m)}(\mathbf{x}, \mathbf{y}; \omega), \qquad (5.8)$$

remain bounded when $\mathbf{x} = \mathbf{y}$ *and the first derivations of these differences have only a pole of the first order for* $\mathbf{x} = \mathbf{y}$.

A thirteen–dimensional vector $\mathbf{U} = (\mathbf{u}^{*s}, \boldsymbol{\phi}^{*s}, \mathbf{u}^{*f}, \boldsymbol{\phi}^{*f}, \pi^{*f})$ defined on $\mathbb{R}^3$, is called regular if $\mathbf{u}^{*s}, \boldsymbol{\phi}^{*s}, \mathbf{u}^{*f}, \boldsymbol{\phi}^{*f} \in \mathbf{C}^2(\mathbb{R}^3)$, $\pi^f \in C^1(\mathbb{R}^3)$ and $u_{j,r}^{*s}, \phi_{j,r}^{*s}, u_{j,r}^{*f}$, $\phi_{j,r}^{*f}, \pi^f \in L^2(\Sigma(\mathbf{0}, R))$ for every $R > 0$, and satisfies the asymptotic relations of the type

$$u_r^{*\sigma} = O(\varrho^{-1}), \quad u_{r,l}^{*\sigma} = o(\varrho^{-1}), \quad \phi_r^{*\sigma} = o(\varrho^{-1}), \quad \phi_{r,l}^{*\sigma} = O(\varrho^{-1}), \qquad (5.9)$$

where $\Sigma(\mathbf{0}, R)$ is the sphere with its center at $\mathbf{0}$ and radius R.

As in (Galeş, 2004 b), we can prove that the system of equations (4.4) has a unique solution in the class of regular vectors.

Let remark that in view of the above proprieties and because we choose the complex numbers k_n, $n = 1, 2, ..., 7$ such that $\text{Im}[k_n] \geq 0$, we can conclude that all the columns of the matrix $\mathbf{\Gamma}(\mathbf{x}, \mathbf{y}; \omega)$ are regular vectors. Thus, the matrix $\mathbf{\Gamma}(\mathbf{x}, \mathbf{y}; \omega)$ is the unique fundamental solution, up to a rearrangement of the columns, for which the columns are regular vectors.

6. Solution for point load problem in cylindrical coordinates

We consider an infinite micropolar solid-fluid mixture and a point $\mathbf{y}$ in the mixture. A concentrated force $\mathbf{F}^{*s}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})\mathbf{m}$ is applied to the mixture, where $\mathbf{m}$ is an unit vector. Based on the general solution described in Section 4, we give the solution of the problem corresponding to this point force.

We choose a system of the Cartesian axes such that the origin O is in the point $\mathbf{y}$ and the direction of Ox_3 is given by the unit vector $\mathbf{m}$. In the Cartesian coordinates (x_1, x_2, x_3) we have $\mathbf{F}^{*s}(\mathbf{x}) = \delta(\mathbf{x})\mathbf{e}_3$.

Using (6.10) we find that the displacement of the solid is given by

$$\mathbf{u}^* = \Delta Q_3^{*s} D_1^*(\mathrm{i}\omega\varpi^2 Q_1^{*f} + Q_4^{*f} Q_2^{*s})E\mathbf{e}_3 + D_1^* D_3^* \mathbf{grad}E_{,3}, \qquad (6.10)$$

where $\mathbf{grad} = \dfrac{\partial}{\partial x_k}\mathbf{e}_k$ is the gradient operator and $\mathbf{e}_k$ are the unit vectors of the Cartesian axes.

In the cylindrical coordinates (r, θ, z), because E is independent by θ, the components of the displacement of the solids are

$$u_r^{*s} = D_1^* D_3^* \frac{zr}{\varrho^2} \left(\frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial E}{\partial \varrho} \right),$$

$$u_\theta^{*s} = 0,$$

$$u_z^{*s} = \Delta Q_3^{*s} D_1^* (i\omega\varpi^2 Q_1^{*f} + Q_4^{*f} Q_2^{*s}) E + D_1^* D_3^* \frac{1}{\varrho^2} \left(\frac{\partial^2 E}{\partial \varrho^2} z^2 + \frac{r^2}{\varrho} \frac{\partial E}{\partial \varrho} \right),$$

(6.11)

where we use that $\varrho^2 = r^2 + z^2$.

Similarly, we find

$$u_r^{*f} = -Q_3^{*s} D_1^* (\varpi k^s k^f \Delta + \xi P_1^*) \frac{zr}{\varrho^2} \left(\frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial E}{\partial \varrho} \right),$$

$$u_\theta^{*f} = 0,$$

(6.12)

$$u_z^{*f} = \Delta Q_3^{*s} D_1^* (\varpi k^s k^f \Delta + \xi P_1^*) E - Q_3^{*s} D_1^* (\varpi k^s k^f \Delta + \xi P_1^*) \frac{1}{\varrho^2} \left(\frac{\partial^2 E}{\partial \varrho^2} z^2 + \frac{r^2}{\varrho} \frac{\partial E}{\partial \varrho} \right),$$

and

$$\pi^{*f} = -i\omega\xi \ D_1^* D_2^* \frac{\partial E}{\partial z}.$$

(6.13)

On the other hand, the microrotations are

$$\boldsymbol{\phi}^{*s} = -\Delta Q_3^{*s} D_1^* (k^s Q_4^{*f} - i\omega\xi\varpi k^f) \mathbf{curl}(0, 0, E),$$

$$\boldsymbol{\phi}^{*f} = \Delta Q_3^{*s} D_1^* (\varpi k^s Q_1^{*f} + \xi k^f Q_2^{*s}) \mathbf{curl}(0, 0, E),$$

(6.14)

and thus, we have

$$\phi_r^{*s} = \phi_z^{*s} = 0,$$

$$\phi_\theta^{*s} = \Delta Q_3^{*s} D_1^* (k^s Q_4^{*f} - i\omega\xi\varpi k^f) \frac{r}{\varrho} \frac{\partial E}{\partial \varrho},$$

$$\phi_r^{*f} = \phi_z^{*f} = 0,$$

$$\phi_\theta^{*f} = -\Delta Q_3^{*s} D_1^* (\varpi k^s Q_1^{*f} + \xi k^f Q_2^{*s}) \frac{r}{\varrho} \frac{\partial E}{\partial \varrho}$$

(6.15)

and the solution of the point force problem is complete.

Let now consider that the concentrated couple $\mathbf{L}^{*s}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})\mathbf{m}$ is applied to the mixture.

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As above, we will find that the corresponding solution is

$$u_r^{*s} = u_z^{*s} = 0,$$

$$u_\theta^{*s} = \Delta Q_3^{*s} D_1^* (k^s Q_4^{*f} - \mathrm{i}\omega\xi\varpi k^f)\frac{r}{\varrho}\frac{\partial E}{\partial \varrho},$$

$$\phi_r^{*s} = \Delta Q_3^{*s} D_4^* \frac{zr}{\varrho^2}\left(\frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho}\frac{\partial E}{\partial \varrho}\right),$$

$$\phi_\theta^{*s} = 0,$$

$$\phi_z^{*s} = \Delta Q_3^{*s} D_1^*[(k^f)^2\Delta Q_1^{*s} - P_2^* Q_2^{*f}]E + \Delta Q_3^{*s} D_4^* \frac{1}{\varrho^2}\left(\frac{\partial^2 E}{\partial \varrho^2}z^2 + \frac{r^2}{\varrho}\frac{\partial E}{\partial \varrho}\right),$$

$$u_r^{*f} = u_z^{*f} = 0,$$

$$u_\theta^{*f} = -\Delta Q_3^{*s} D_1^* (\varpi k^f Q_1^{*s} + \xi k^s Q_2^{*f})\frac{r}{\varrho}\frac{\partial E}{\partial \varrho},$$

$$\phi_r^{*f} = \Delta Q_3^{*s} D_5^* \frac{zr}{\varrho^2}\left(\frac{\partial^2 E}{\partial \varrho^2} - \frac{1}{\varrho}\frac{\partial E}{\partial \varrho}\right),$$

$$\phi_\theta^{*s} = 0,$$

$$\phi_z^{*s} = \Delta Q_3^{*s} D_1^* (\xi k^s k^f \Delta + \varpi P_2^*)E - \Delta Q_3^{*s} D_5^* \frac{1}{\varrho^2}\left(\frac{\partial^2 E}{\partial \varrho^2}z^2 + \frac{r^2}{\varrho}\frac{\partial E}{\partial \varrho}\right),$$

$$\pi^{*f} = 0.$$

$$(6.16)$$

Similarly we can find the solutions which correspond to the concentrated loads $\mathbf{F}^{*f}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})\mathbf{m}$ and $\mathbf{L}^{*f}(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{y})\mathbf{m}$ and the problem is solved.

Acknowledgment

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