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Algebraic parameter estimation of a biased sinusoidal waveform signal from noisy data

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Abstract: The amplitude, frequency and phase of a biased and noisy sum of two complex exponential sinusoidal signals are estimated via new algebraic techniques providing a robust estimation within a fraction of the signal period. The methods that are popular today do not seem able to achieve such performances. The efficiency of our approach is illustrated by several computer simulations.

Keywords: Parameter identification, differential algebra, sinusoidal waveform.

1. INTRODUCTION

The parameter estimation of a biased sinusoidal signal in a noisy environment is an important issue occurring in many practical engineering problems, *e.g.* the signal demodulation in communications, the regulation of electronic converters power, the circadian rhythm of biological cells and the modal identification for flexible structures (see Trapero et al. [2007b]).

Many different resolution methods have been developed, such as linear or nonlinear regression, subspace methods (Haykin [1991], Roy et al. [1989], Kahn et al. [1992]), the extended Kalman filter (Bittanti et al. [2000]), the notches filter (Regalia et al. [1995]), or alternatively, the use of techniques borrowed from adaptive nonlinear control (Hsu et al. [1999], Mojiri et al. [2004]). However, the robust parameter estimation in a fraction of the time signal, in the presence of noise and of an unknown constant bias, is not yet fully solved. This paper draws its inspiration from the algebraic analysis of Fliess et al. [2003, 2010, 2008], Fliess [2008], Mboup [2009]. In addition to numerical simulations found in these papers, we refer to Neves et al. [2006], Trapero et al. [2007a,b, 2008b] for more very encouraging results in concrete examples.

In his 1795 seminal paper, Riche de Prony studies the parameter estimation of a finite sum of sinusoidal functions (see Riche de Prony [1795], Kahn et al. [1992], Osborne et al. [1995]). In this paper, we are interested in Prony's problem for a two-terms sum of complex sinusoidal functions, meaning that our aim is to estimate the parameters of the signal (see also Neves et al. [2007] for a quite related study)

$$x(t) = \alpha_1 \exp i(\omega_1 t + \phi_1) + \alpha_2 \exp i(\omega_2 t + \phi_2)$$

from the biased and noisy output measure

$$y(t) = x(t) + \beta + \varpi \tag{1}$$

where β is an unknown constant bias and ϖ is a noise [1].

A linear parametric estimation problem may often be formalized as finding a *good* approximation of some vector Θ on the basis of an observed signal that is a linear functional of the *true* signal depending on a set of parameters and a noise corrupting the observation. Here the signal $z(t) = x(t) + \beta$ and Θ are linearly differentially algebraic. Indeed, we have the following differential equation:

$$\ddot{z}(t) - i\,(\omega_1 + \omega_2)\,\dot{z}(t) - \omega_1\omega_2(z(t) - \beta) = 0.$$

Notice that the signal frequencies appear as roots of the characteristic equation of the above relation which is also behind Prony's method and other techniques such as adaptive notch filters: take this equation as the numerator of some transfer function, then the filter zeros align with the signal frequencies. The further applied algebraic operator will take advantage of this remark.

In the operational domain, we obtain

[1] We use here the framework developed in Fliess [2006, 2008]. It is independent of any probabilistic modeling. In this point of view, the noise should be viewed as *fast oscillations* (see Fliess [2006]).

$$s(s-i\omega_1)(s-i\omega_2)Z(s) = s(s-i(\omega_1+\omega_2))z(0) + s\dot{z}(0) - \beta\omega_1\omega_2. \quad (2)$$

Among the unknown parameters, we wish to estimate $\Theta_{\text{est}} := \{\theta_1, \theta_2, \theta_3, \theta_4\}$, but not the bias $\Theta_{\overline{\text{est}}} = \{\theta_5\}$:

$$\theta_1 = -\omega_1\omega_2, \quad \theta_2 = -i(\omega_1+\omega_2), \quad (3)$$

$$\theta_3 = -x(0) = \beta - z(0), \quad \theta_4 = -\dot{z}(0) = -\dot{x}(0), \quad (4)$$

$$\theta_5 = -\beta \quad (5)$$

Notice that the parameters $\alpha_1, \omega_1, \phi_1, \alpha_2, \omega_2, \phi_2$ can be easily deduced from Θ_{est}, see Lemma 15 in the Appendix.

Using the notation (3)-(5), the equation (2) reads as

$$\left(s^3 + \theta_2 s^2 + \theta_1 s\right) Z(s) + (s^2 + \theta_2 s)\theta_3 + \theta_4 s + \left(s^2 + \theta_2 s + \theta_1\right)\theta_5 = 0. \quad (6)$$

From (6) we would like to obtain a system of equations on Θ_{est} and independent of $\Theta_{\overline{\text{est}}}$. We consider the algebraic extensions $\mathbb{C}_\Theta := \mathbb{C}(\Theta)$, $\mathbb{C}_{\Theta_{\text{est}}} := \mathbb{C}(\Theta_{\text{est}})$ and $\mathbb{C}_{\Theta_{\overline{\text{est}}}} := \mathbb{C}(\Theta_{\overline{\text{est}}})$. Moreover, $\mathbb{C}_\Theta[s]$ (respectively $\mathbb{C}_{\Theta_{\text{est}}}[s]$, $\mathbb{C}_{\Theta_{\overline{\text{est}}}}[s]$) denotes the polynomial ring in the variable s with coefficients in $\mathbb{C}_{\Theta_{\text{est}}}$ (respectively in $\mathbb{C}_{\Theta_{\overline{\text{est}}}}$). We obtain the relation:

$$\mathcal{R}\left(s, Z(s), \Theta_{\text{est}}, \Theta_{\overline{\text{est}}}\right) := P(s)\, Z(s) + Q(s) + \overline{Q}(s) = 0 \quad (7)$$

$$\text{with} \quad P(s) = s\, T(s) \text{ and}$$
$$Q(s) = s^2\theta_3 + s(\theta_4 + \theta_2\theta_3) \in \mathbb{C}_{\Theta_{\text{est}}}[s]$$
$$\text{and} \quad \overline{Q}(s) = T(s)\theta_5 \in \mathbb{C}_{\Theta_{\overline{\text{est}}}}[s],$$

where we set $T(s) = s^2 + \theta_2 s + \theta_1 \ \in \mathbb{C}_{\Theta_{\text{est}}}[s]$.

Now, we proceed in three steps:

(1) Algebraic elimination of all terms in $\Theta_{\overline{\text{est}}}$: all differential operators that annihilate $\overline{Q}$ can be generated by a single operator in $\mathbb{C}_{\Theta_{\text{est}}}(s)\left[\frac{d}{ds}\right]$ [2], called a $\overline{Q}$-*minimal annihilator*. These annihilators will be rewritten in a canonical form.

(2) Obtaining a system of equations on Θ_{est}: we apply on $\mathcal{R}$ several differential operators annihilating $\overline{Q}$. Using their canonical forms, choices will be made to obtain a system of equations with good numerical properties (once back in the time domain).

(3) Resolution of the obtained system: we use the inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{s^m}\frac{d^p Z(s)}{ds^p}\right) = \frac{(-1)^p t^{m+p}}{(m-1)!}\int_0^1 w^{m-1,p}(\tau) z(t\tau) d\tau \quad (8)$$

with $w^{m,p}(t) = (1-t)^m t^p, \forall\, p, m \in \mathbb{N}, m \geq 1$ to bring the equations back in the time domain. The integers m, p will be chosen as small as possible so that the resulting estimation is as least as possible sensitive to the noise.

The first point emphasizes the central role played by $\overline{Q}$-minimal annihilators. They are defined in Section 2 where we describe the algebraic structure behind them and the algebraic elimination technique to eliminate $\Theta_{\overline{\text{est}}}$ (subsection 2.1). Minimal annihilators can be rewritten in a canonical form, detailed in subsection 2.1. The parameter estimation is provided in section 3. Section 4 contains convincing numerical experiments that illustrate our techniques and are easily implementable. These experiments are compared to the well-known modified Prony's method.

[2] The polynomial ring in $\frac{d}{ds}$ with coefficients in $\mathbb{C}_{\Theta_{\text{est}}}[s]$

2. AN ALGEBRAIC FRAMEWORK FOR OBTAINING ANNIHILATORS

The algebraic framework is borrowed from Fliess et al. [2010, 2003, 2008], Fliess [2008], Mboup [2009] [3]. More details about the algebraic notions can be found in Dixmier et al. [1974] and McConnell et al. [2000].

Recall that we wish to annihilate

$$\overline{Q} = \left(s^2 + \theta_2 s + \theta_1\right)\theta_5 \in \mathbb{C}_{\Theta_{\overline{\text{est}}}}[s].$$

Since we have a polynomial in s, the natural idea is to look for operator in $\frac{d}{ds}$. It is clear that operators of order [4] greater than 3 will annihilate $\overline{Q}$, *e.g.* $\Pi_1 = (s\frac{d}{ds} - 2) \circ (s\frac{d}{ds} - 1) \circ \left(s\frac{d}{ds}\right)$ and $\Pi_2 = \frac{d^3}{ds^3}$. One question would be whether these annihilators are the same. Another question would be whether there exists a lower order annihilator. Answers are provided by the algebraic structure of the Weyl algebra $\mathbb{C}_\Theta(s)\left[\frac{d}{ds}\right]$.

2.1 Weyl algebra

We consider $\mathbb{K}$ a field of characteristic zero (here $\mathbb{K} = \mathbb{C}$).

Definition 1. Let $k \in \mathbb{N} \setminus \{0\}$. The *Weyl algebra* $\mathsf{A}_k(\mathbb{K})$ is the $\mathbb{K}$-algebra generated by $p_1, q_1, \ldots, p_k, q_k$ satisfying the relations

$$[p_i, q_j] = \delta_{ij},\ [p_i, p_j] = [q_i, q_j] = 0, \forall\ 1 \leq i, j \leq k$$

where $[\cdot, \cdot]$ is the commutator defined by $[u, v] := uv - vu$, $\forall\ u, v \in \mathsf{A}_k(\mathbb{K})$. We will simply write A_k instead of $\mathsf{A}_k(\mathbb{K})$ when we do not need to make explicit the base field.

A well-known fact is that A_k can be realized as the algebra of polynomial differential operators on $\mathbb{K}[s_1, \ldots, s_k]$ with

$$p_i = \frac{\partial}{\partial s_i} \quad \text{and} \quad q_i = s_i \times \cdot\ , \forall\ 1 \leq i \leq k. \quad (9)$$

We can also write

$$\mathsf{A}_k = \mathbb{K}[q_1, \ldots, q_k][p_1, \ldots, p_k] = \mathbb{K}[s_1, \ldots, s_k]\left[\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_k}\right]$$

(notice that we use the same notation for the variable s_i and for the operator *multiplication by* s_i here).

There is a closely related algebra to A_k: it is defined as the set of differential operators on $\mathbb{K}[s_1, \ldots, s_k]$ with coefficients in the rational functions field $\mathbb{K}(s_1, \ldots, s_k)$. We denote it by $\mathsf{B}_k(\mathbb{K})$ and write

$$\mathsf{B}_k(\mathbb{K}) := \mathbb{K}(q_1, \ldots, q_k)[p_1, \ldots, p_k] = \mathbb{K}(s_1, \ldots, s_k)\left[\frac{\partial}{\partial s_1}, \ldots, \frac{\partial}{\partial s_k}\right]$$

Sometimes we will simply write B_k instead of $\mathsf{B}_k(\mathbb{K})$. In the case $k = 1$ for instance, we have

$$\mathsf{A}_1 = \langle p, q \mid pq - qp = 1\rangle = \mathbb{K}[s]\left[\frac{d}{ds}\right] \quad \text{and} \quad \mathsf{B}_1 = \mathbb{K}(s)\left[\frac{d}{ds}\right]$$

[3] Similar tools were also used for numerical differentiation of noisy signal (see Mboup et al. [2009], Liu et al. [2011]) and change-point detection (see Fliess et al. [2010]).

[4] The order of an operator $\Pi \in \mathbb{C}_\Theta(s)\left[\frac{d}{ds}\right]$ is its degree as a polynomial in the variable $\frac{d}{ds}$.

Proposition 2. A basis for A_k is given by $\left\{q^I p^J \mid I, J \in \mathbb{N}^k\right\}$ where $q^I := q_1^{i_1} \dots q_k^{i_k}$ and $p^J := p_1^{j_1} \dots p_k^{j_k}$.

An element $F \in \mathsf{A}_k$ can be written in its *canonical form*,

$$F = \sum_{I,J} \lambda_{IJ} q^I p^J \text{ with } \lambda_{IJ} \in \mathbb{K}.$$

Example 3. In this paper the following identities are useful:

$$q^m p^n = p^n q^m + \sum_{k=1}^{m} \binom{m}{k}\binom{n}{k} k!(-1)^k p^{n-k} q^{m-k}$$

$$p^n q^m = q^m p^n + \sum_{k=1}^{n} \binom{n}{i}\binom{m}{i} i! q^{m-i} p^{n-i}$$

Similarly, an element $F \in \mathsf{B}_k$ can be written as

$$F = \sum_I g_I(s) p^I \quad \text{with} \quad g_I(s) \in \mathbb{K}(s_1, \dots, s_k).$$

The *order* of an element $F \in \mathsf{B}_k$, $F = \sum_I g_I(s) p^I$ is defined as

$$\operatorname{ord}(F) := \max\{[I] \mid g_I(s) \neq 0\}.$$

Notice that the same definition is valid for the Weyl algebra A_k since $\mathsf{A}_k \subset \mathsf{B}_k$.

Proposition 4. A_k is a domain. Moreover, A_k is simple and Noetherian.

However, A_k is neither a principal right domain, nor a principal left domain.

Proposition 5. For the algebra B_k, one has:

(1) B_k is a domain. Moreover, B_k is simple and Noetherian.
(2) B_1 admits a left division algorithm, that is, if F, $G \in \mathsf{B}_1$, then there exists q, $r \in \mathsf{B}_1$ such that $F = q\, G + r$ and $\operatorname{ord}(r) < \operatorname{ord}(G)$. As a consequence, B_1 is a principal left domain.

Since $\frac{d}{ds}$ is a derivation operator we have:

Proposition 6. (Derivation). For $F \in \mathbb{C}[s]$ and $X(s)$ the Laplace transform of a signal $x(t)$, we have (Leibniz rule):

$$\frac{d^n}{ds^n}(F\ X) = \sum_{k=0}^{n} \binom{n}{k} \frac{d^k F}{ds^k}\ \frac{d^{n-k} X}{ds^{n-k}}$$

2.2 Annihilator

In the sequel, to distinguish whether an annihilator depends on Θ_{est} or not, we work with $\mathsf{B} := \mathsf{B}_1 = \mathbb{C}(s)\left[\frac{d}{ds}\right]$ and $\mathsf{B}_{\Theta_{\text{est}}} := \mathbb{C}_{\Theta_{\text{est}}}(s)\left[\frac{d}{ds}\right]$.

Definition 7. Let $R \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}[s]$. Consider the left ideal:

$$\operatorname{Ann}_{\mathsf{B}}(R) = \{F \in \mathsf{B} \mid F(R) = 0\}.$$

A *R-annihilator w.r.t.* B is an element $\Pi \in \operatorname{Ann}_{\mathsf{B}}(R)$.

Remark 8. Let us note that $\operatorname{Ann}_{\mathsf{B}}(R)$ is a left principal ideal. Thus it is generated by a single generator $\Pi_{\min} \in \mathsf{B}$ called a *minimal R-annihilator w.r.t.* B. So

$$\operatorname{Ann}_{\mathsf{B}}(R) = \mathsf{B}\ \Pi_{\min}.$$

Remark that $\operatorname{Ann}_{\mathsf{B}}(R)$ contains annihilators in finite integral form, *i.e.* operators with coefficients in $\mathbb{C}\left[\frac{1}{s}\right]$.

Remark 9. Remark 8 still holds if we replace B by $\mathsf{B}_{\Theta_{\text{est}}}$ in Definition 7. That means that $\operatorname{Ann}_{\mathsf{B}_{\Theta_{\text{est}}}}(R)$ is generated by a unique generator $\Pi_{\min} \in \mathsf{B}_{\Theta_{\text{est}}}$ (up to multiplication by a polynomial in $\mathbb{C}_{\Theta_{\text{est}}}(s)$) called a *minimal R-annihilator w.r.t.* $\mathsf{B}_{\Theta_{est}}$.

We have the following lemmas:

Lemma 10. Let $Q_n = s^n$, $n \in \mathbb{N}$. A minimal Q_n-annihilator is

$$\Pi_n = s\frac{d}{ds} - n.$$

(unique up to a multiplication by a polynomial in $\mathbb{C}(s)$).

Let us note that for m, $n \in \mathbb{N}$, the operators Π_m and Π_n commute. Thus one can take advantage of the following lemma

Lemma 11. Let $P_1, P_2 \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}[s]$. Let Π_i be a P_i-annihilator $(i = 1, 2)$ such that $\Pi_1\Pi_2 = \Pi_2\Pi_1$. Then $\Pi_1\Pi_2$ is a $(\mu P_1 + \eta P_2)$-annihilator for all $\mu, \eta \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}$.

Using this Lemma and considering $\overline{Q}$ in (7), we obtain a minimal $\overline{Q}$-annihilator w.r.t. B where $\overline{Q} = \left(s^2 + \theta_2 s + \theta_1\right)\theta_5 \in \mathbb{C}_{\overline{\Theta_{\text{est}}}}[s]$:

$$\Pi_{\min} = \left(s\frac{d}{ds} - 2\right) \circ \left(s\frac{d}{ds} - 1\right) \circ \left(s\frac{d}{ds}\right). \tag{10}$$

The identities in Example 3 give:

$$\Pi_{\min} = s^3 \frac{d^3}{ds^3}.$$

Lemma 12. Let $R \in \mathbb{C}_{\Theta_{\text{est}}}[s]$. Then a minimal R-annihilator w.r.t $\mathsf{B}_{\Theta_{\text{est}}}$ is

$$\Pi_{\min} = R\frac{d}{ds} - \frac{dR}{ds}.$$

In our example, by Lemma 12 we obtain that the $\overline{Q}$-annihilator w.r.t. $\mathsf{B}_{\Theta_{\text{est}}}$ is:

$$\Pi_{\min}^{\Theta_{\text{est}}} = \left(s^2 + \theta_2 s + \theta_1\right)\frac{d}{ds} - (2s + \theta_2).$$

3. PARAMETER ESTIMATION

Our estimation problem is equivalent to find a family of annihilators $(\Pi_i)_{i=1}^r$ in $\mathbb{C}(s)\left[\frac{d}{ds}\right]$ that applied on (7) provides a set of equations enabling the computation of Θ_{est} in the time domain. This family will be generated by a minimal $\overline{Q}$-annihilator.

Formula (8) justifies the use of finite-integral form annihilators. Moreover, these operators should be of minimal degree in $\frac{d}{ds}$ to minimize noise sensitivity. Lastly, the obtained system of equations should be well-balanced to provide *good* numerical estimation.

Recall that the minimal $\overline{Q}$-annihilator $\Pi_{\min}$ is a generator of the ideal $\operatorname{Ann}_{\mathsf{B}}(\overline{Q})$, so a general $\overline{Q}$-annihilator will be of the form:

$$\Pi = \left(\sum_{i=0}^{\ell} g_i(s)\frac{d^i}{ds^i}\right) \circ \Pi_{\min} \tag{11}$$

where $g_i(s) \in \mathbb{C}(s), \forall\ i = 0, \dots, \ell$.

3.1 Estimation of θ_1 and θ_2

Applying $\Pi_{\min} = s^3 \frac{d^3}{ds^3}$, the minimal $\overline{Q}$-annihilator w.r.t. B, on relation (7) gives

$$\Pi_{\min}(P(s)\ Z(s)) = P_3(s)\frac{d^3Z(s)}{ds^3} + P_2(s)\frac{d^2Z(s)}{ds^2} + P_1(s)\frac{dZ(s)}{ds} + P_0(s)Z(s)$$

where $P_0(s) = 6s^3$, $P_2(s) = 9s^5 + 6s^4\theta_2 + 3s^3\theta_1$, $P_1(s) = 18s^4 + 6s^3\theta_2$, $P_3(s) = s^6 + s^5\theta_2 + s^4\theta_1$, and $\Pi_{\min}(Q(s)) = \Pi_{\min}\left(\overline{Q}(s)\right) = 0$.
That provides the following algebraic relation

$$P_3(s)\frac{d^3Z(s)}{ds^3} + P_2(s)\frac{d^2Z(s)}{ds^2} + P_1(s)\frac{dZ(s)}{ds} + P_0(s)Z(s) = 0.$$

We obtain a single equation in θ_1 and θ_2. To linearly identify these two parameters, we need two independent equations. However, the following result show that this is not possible in the operational domain (see the appendix for a proof):

Theorem 13. There do not exist two $\overline{Q}$-annihilators w.r.t B leading to two independent equations in θ_1 and θ_2.

Remark 14. Let us note that for a similar parameter identification problem of a single sinusoid, it is indeed possible to find two independent equations in the operational domain (see Ushirobira et al. [2011]).

Therefore we will use such a construction in the time domain. For this, since $\overline{Q}$-annihilators are of the form (11), to get two equations we select $\ell = 1$. That leads to the following 4^{th}-order annihilator written in the canonical form:

$$\Pi = g_0(s)\frac{d^3}{ds^3} + g_1(s)\frac{d^4}{ds^4}, \tag{12}$$

where $g_0(s), g_1(s) \in \mathbb{C}(s)$. The choices of $g_0(s) = 1$, $g_1(s) = 0$ and then $g_0(s) = 0$, $g_1(s) = 1$, give two equations in the operational domain leading to the following system in the time domain:

$$\begin{pmatrix} -\frac{1}{6}\ t\ I_1 & -I_2 \\ \frac{1}{6}\ t^2\ I_3 & -tI_4 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{t}I_5 \\ I_6 \end{pmatrix}$$

where

$I_1 = \int_0^1 \left(6w^{2,3}(\tau) - 3w^{2,2}(\tau)\right) z(t\tau)d\tau$

$I_2 = \int_0^1 \left(5w^{1,3}(\tau) - 5w^{1,2}(\tau) + w^{1,1}(\tau)\right) z(t\tau)d\tau$

$I_3 = \int_0^1 \left(7w^{2,4}(\tau) - 4w^{2,3}(\tau)\right) z(t\tau)d\tau$

$I_4 = \int_0^1 \left(-7w^{1,4}(\tau) + 8w^{1,3}(\tau) - 2w^{1,2}(\tau)\right) z(t\tau)d\tau$

$I_5 = \int_0^1 \left(w^{3,0}(\tau) - 9w^{2,1}(\tau) + 9w^{1,2}(\tau) - w^{0,3}(\tau)\right) z(t\tau)d\tau$

$I_6 = \int_0^1 \left(-4w^{3,1}(\tau) + 18w^{2,2}(\tau) - 12w^{1,3}(\tau) + w^{0,4}(\tau)\right) z(t\tau)d\tau$

The expressions for θ_1 and θ_2 are thus obtained:

$$\theta_1 = -\frac{6}{t^2}\frac{(-I_4I_5 + I_6I_2)}{I_4I_1 + I_3I_2}$$

$$\theta_2 = \frac{1}{t}\frac{I_1I_6 - I_3I_5}{I_4I_1 + I_2I_3}$$

3.2 Estimation of θ_3 and θ_4

Using annihilators generated by the minimal $\overline{Q}$-annihilator w.r.t. $\mathsf{B} := \mathbb{C}(s)\left[\frac{d}{ds}\right]$ given by (10), we could linearly identify the parameters θ_1 and θ_2. These annihilators do not depend on the parameters to be found. Now, one can show that it is not possible to identify linearly the remaining parameters θ_3 and θ_4, so we will use nonlinear equations in θ_1 and θ_2. So, let us consider the $\overline{Q}$-annihilator w.r.t. $\mathsf{B}_{\Theta_{\text{est}}}$:

$$\Pi_{\min}^{\Theta_{\text{est}}} = \left(s^2 + \theta_2 s + \theta_1\right)\frac{d}{ds} - (2s + \theta_2).$$

Applying it on (7) gives:

$$\Pi_{\min}^{\Theta_{\text{est}}}(P(s)\ Z(s)) = T(s)^2\left(s\frac{dZ(s)}{ds} + Z(s)\right)$$
$$\Pi_{\min}^{\Theta_{\text{est}}}(Q(s)) = -\theta_4 s^2 + 2\theta_1\theta_3 s + \theta_1(\theta_4 + \theta_2\theta_3)$$
$$\Pi_{\min}^{\Theta_{\text{est}}}(\overline{Q}(s)) = 0$$

where $T(s) = s^2 + \theta_2 s + \theta_1 \in \mathbb{C}[s]$. That gives the following algebraic relation:

$$T(s)^2\left(Z(s) + s\frac{dZ(s)}{ds}\right) + (\theta_1 - s^2)\theta_4 + \theta_1(2s + \theta_2)\theta_3 = 0.$$

We obtain a single equation in θ_3 and θ_4. As in the previous subsection, we use (11) with $\ell = 1$, where $\Pi_{\min}$ is replaced by $\Pi_{\min}^{\Theta_{\text{est}}}$. That leads to the following 4^{th}-order annihilator written in the canonical form:

$$\Pi = g_0(s)\Pi_{\min}^{\Theta_{\text{est}}} + g_1(s)\left(T(s)\frac{d^2}{ds^2} - 2\right), \tag{13}$$

where $g_1(s), g_2(s) \in \mathbb{C}(s)$. Selecting $g_0(s) = 1, g_1(s) = 0$ and then $g_0(s) = 0, g_1(s) = 1$, we obtain two equations:

$$\begin{pmatrix} \theta_1(2s + \theta_2) & (\theta_1 - s^2) \\ 2\theta_1 & -2s \end{pmatrix} \begin{pmatrix} \theta_3 \\ \theta_4 \end{pmatrix} = -\begin{pmatrix} \mathcal{B}_3 \\ \mathcal{B}_4 \end{pmatrix}, \tag{14}$$

with

$$\mathcal{B}_3 = T(s)^2\left(Z(s) + s\frac{dZ(s)}{ds}\right) \tag{15}$$

$$\mathcal{B}_4 = sT(s)^2\frac{d^2Z(s)}{ds^2} + T(s)(6s^2 + 4s\theta_2 + 2\theta_1)\frac{dZ(s)}{ds} + T(s)(4s + 2\theta_2)Z(s) \tag{16}$$

The above matrix can be easily written in a diagonal form and the expressions for θ_3 and θ_4 are thus obtained. Notice that they depend on θ_1 and θ_2.

4. SIMULATIONS

Figure 1 shows the simulation results for the estimation of the parameters θ_1 and θ_2 vs the estimation time. The modified Prony's method (PM) is used as a reference. Each point is obtained by averaging the results over 200 trials. The first plot displays the real part of a sample realization of the noisy signal (1), with $N = 1024$ samples and SNR = 20 dB). The corresponding value of the constant bias is $\beta = 7.9$. The results in the second and third plots show that the estimation is unbiased and insensitive to the constant bias β for the presented method (solid line curve). For the same experiment, the results obtained with the modified Prony's method display a bias in the estimates (dot-dashed line curve). Note that even by

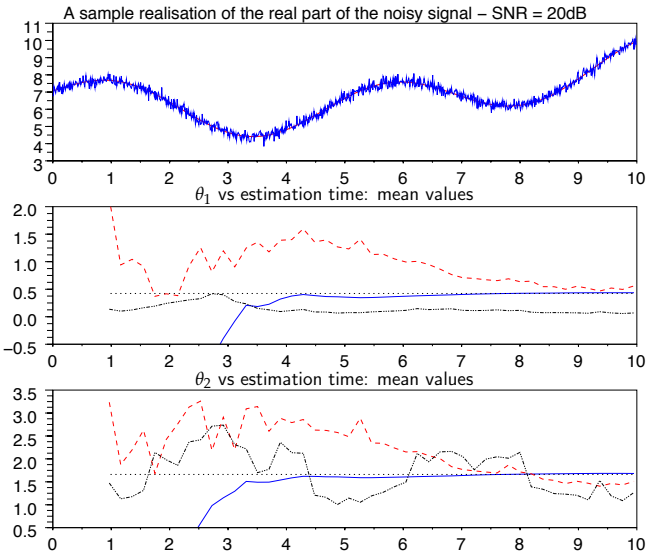

Fig. 1. Comparison with the modified Prony's method: sample mean

setting $\beta = 0$ with the Modified Prony's method (dashed line curve), the results of the proposed method still show more robustness to the noise corruption. This is confirmed by the corresponding variances which are displayed below

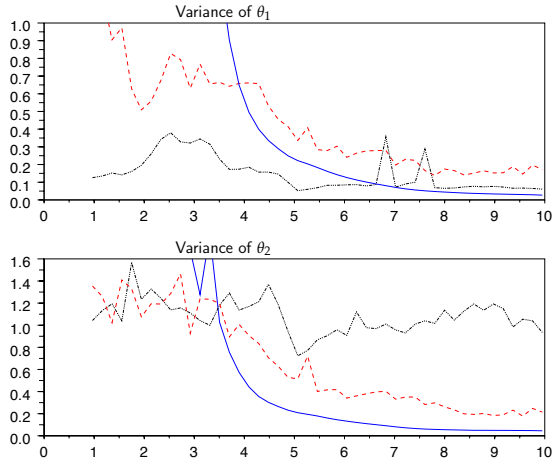

Fig. 2. Comparison with the modified Prony's method: sample variance

5. CONCLUSION

The algebraic method we used in this paper proved to be very efficient in this parameter estimation problem of a sum of two sinusoidal waveform signal. Indeed, the two triplets (amplitude, frequency, phase) could be easily identified using the minimal annihilators proposed here.

In the case of a single sinusoidal wave, the results are also convincing (see Ushirobira et al. [2011]).

The perspective of a extension of this algebraic method to a sum of several sinusoidal waveform signal is challenging, but hopefully very likely to be succeeded. A positive result in the three-sinusoid case has already been obtained.

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Appendix A. FINDING THE ORIGINAL PARAMETERS

Lemma 15. The parameters $\alpha_1, \omega_1, \phi_1, \alpha_2, \omega_2, \phi_2$ can be obtained from Θ_{est}.

Proof. From θ_1 and θ_2 one can easily deduce ω_1 and ω_2 since $i\omega_1$ and $i\omega_2$ are the roots of the polynomial $\omega^2 + \theta_2\omega + \theta_1 = 0$. From θ_3 and θ_4 one obtains $x(0) = -\theta_3 = \alpha_1 \exp i\phi_1 + \alpha_2 \exp i\phi_2$ and $\dot{x}(0) = -\theta_4 = i(\omega_1\alpha_1 \exp i\phi_1 + \omega_2\alpha_2 \exp i\phi_2)$ and then easily deduce the remaining parameters $(\alpha_1, \phi_1, \alpha_2, \phi_2)$:

$$\begin{pmatrix} -\theta_3 \\ -\theta_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i\omega_1 & i\omega_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \exp i\phi_1 \\ \alpha_2 \exp i\phi_2 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 \exp i\phi_1 \\ \alpha_2 \exp i\phi_2 \end{pmatrix} = \frac{1}{\omega_2 - \omega_1} \begin{pmatrix} -i\theta_4 - \omega_2\theta_3 \\ i\theta_4 + \omega_1\theta_3 \end{pmatrix}$$

Taking absolute values and arguments and ordering the frequencies $\omega_2 > \omega_1$:

$$\alpha_1 = \frac{\sqrt{\left(\Im(\theta_4) - \omega_2\Re(\theta_3)\right)^2 + \left(\Re(\theta_4) + \omega_2\Im(\theta_3)\right)^2}}{\omega_2 - \omega_1} \quad \text{(A.1)}$$

$$\tan(\phi_1) = -\frac{\Re(\theta_4) + \omega_2\Im(\theta_3)}{\Im(\theta_4) - \omega_2\Re(\theta_3)} \quad \text{(A.2)}$$

$$\alpha_2 = \frac{\sqrt{\left(-\Im(\theta_4) + \omega_1\Re(\theta_3)\right)^2 + \left(\Re(\theta_4) + \omega_1\Im(\theta_3)\right)^2}}{\omega_2 - \omega_1} \quad \text{(A.3)}$$

$$\tan(\phi_2) = -\frac{\Re(\theta_4) + \omega_1\Im(\theta_3)}{-\Im(\theta_4) + \omega_1\Re(\theta_3)} \quad \text{(A.4)}$$

Appendix B. PROOF OF THEOREM 13

A $\overline{Q}$-annihilator is of the form (11), that is

$$\Pi = \sum_{i=0}^{\ell} g_i(s) \frac{d^{i+3}}{ds^{i+3}}, \quad \text{(B.1)}$$

where $g_i(s) \in \mathbb{C}_{\Theta_{\text{est}}}(s)$, $\forall\ 1 \leq i \leq \ell$. We apply such annihilator on relation $\Re$ (7). Since $\Pi(Q(s)) = \Pi(\overline{Q}) = 0$, we compute $\Pi(P(s)\ Z(s))$. Using proposition 6, one obtains:

$$\frac{d^{i+3}}{ds^{i+3}} \left(P(s)\ Z(s)\right) = \sum_{k=0}^{4} \binom{i+3}{k} P^{(k)}(s) Z^{(i+3-k)}(s)$$

since $P \in \mathbb{C}[s]$ is of degree 3. Moreover: $P^{(0)}(s) = s^3 + \theta_2 s^2 + \theta_1 s$, $P^{(1)}(s) = 3s^2 + 2\theta_2 s + \theta_1$, $P^{(2)}(s) = 6s + 2\theta_2$, and $P^{(2)}(s) = 6$. Thus using notations (9), Π given by (B.1) applied on relation $\Re$ (7) reads in the Weyl algebra framework as:

$$\sum_{i=0}^{\ell} g_i(q) \left(A_i(p,q)\theta_1 + B_i(p,q)\theta_2 + C_i(p,q)\right) = 0 \quad \text{(B.2)}$$

where $A_i(p,q) = (qp + a_i)\, p^{i+2}$,

$$B_i(p,q) = \left(q^2p^2 + 2a_iqp + b_i\right) p^{i+1},$$

$$C_i(p,q) = \left(q^3p^3 + 3a_iq^2p^2 + 3b_iqp + c_i\right) p^i,$$

with $a_i = (i+3), b_i = a_i(i+2), c_i = b_i(i+1)$, $\forall 0 \leq i \leq \ell$. Denote by $L_i := [A_i(p,q)B_i(p,q)C_i(p,q)]$, then for $i > j$ we have by induction on $(j-i)$ and by example 3:

$$L_j = p^{(j-i)} L_i \quad \text{(B.3)}$$

and that completes the proof of theorem 13.