Persistency of wellposedness of Ventcel's boundary value problem under shape deformations

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M. Dambrine* and D. Kateb †

Abstract

Ventcel boundary conditions are second order differential conditions that appear in asymptotic models. Like Robin boundary conditions, they lead to well-posed variational problems under a sign condition of the coefficient. This is achieved when physical situations are considered. Nevertheless, situations where this condition is violated appeared in several recent works where absorbing boundary conditions or equivalent boundary conditions on rough surface are sought for numerical purposes. The well-posedness of such problems was recently investigated : up to a countable set of parameters, existence and uniqueness of the solution for the Ventcel boundary value problem holds without the sign condition. However, the values to be avoided depend on the domain where the boundary value problem is set. In this work, we address the question of the persistency of the solvability of the boundary value problem under domain deformation.

1 Introduction and statement of the results

Let Ω be a smooth bounded domain of $\mathbb{R}^d$ with $d \geq 2$. Let α and β denote two real numbers and fix $f \in \mathrm{L}^2(\Omega)$. The Ventcel boundary value problem for the Laplace operator reads as follows

$$\left\{ \begin{array}{rcll} -\Delta u &=& f & \text{in } \Omega, \\ \partial_n u + \alpha u + \beta \Delta_\tau u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where Δ_τ stands for the Laplace-Beltrami operator on $\partial\Omega$. The boundary condition appears in asymptotic models for coated structures: the second order term $\beta\Delta_\tau u$ represents surface diffusion on the boundary which models the tangential effects of the diffusion in the coating layer.

Surface and volume diffusion both induce similar effects, therefore the coefficient β is naturally signed. In this case, β is nonpositive and a variational approach is available. Define the bilinear form A and the linear form B by

$$A(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) \mathrm{d}x + \int_{\partial\Omega} \alpha\, u(x)v(x) - \beta\, \nabla_\tau u(x) \cdot \nabla_\tau v(x) \mathrm{d}\sigma(x),$$

$$B(v) = \int_\Omega f(x)v(x)\mathrm{d}x,$$

on the variational space

$$\mathcal{H}(\Omega) = \left\{ u \in \mathrm{H}^1(\Omega),\ u_{|\partial\Omega} \in \mathrm{H}^1(\partial\Omega) \right\}.$$

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Endowed with the norm

$$\|u\|^2_{\mathcal{H}(\Omega)} = \|u\|^2_{\mathrm{H}^1(\Omega)} + \|u\|^2_{\mathrm{H}^1(\partial\Omega)},$$

the space $\mathcal{H}(\Omega)$ is a Hilbert space. Then, the weak formulation of problem (1) is the following:

$$\text{Find } u \in \mathcal{H}(\Omega) \text{ such that for all } v \in \mathcal{H}(\Omega),\ A(u,v) = B(v).$$

When $\beta < 0$ and $\alpha > 0$, the bilinear form A is coercive. The existence and uniqueness of a solution to (1) is a consequence of the Lax-Milgram theorem. A large literature has been devoted to that case of great importance: the condition $\beta < 0$ is generally satisfied in the applications since the pioneering works of Feller and Ventcel, [6, 7, 15, 14]. For the specific case of the Laplace operator, we refer to [1] and [5, 8, 10].

In the case $\beta > 0$, the quadratic form $u \mapsto A(u,u)$ is neither positive, nor negative. Lax-Milgram's theorem does not apply. To the best of our knowledge, the condition $\beta > 0$ appear ed for the first time in a recent work of D. Bresch and V. Milisic [3] on wall laws in fluid mechanics. Marigo and Pideri also found such a boundary condition when studying equivalent boundary conditions for a elastic body damaged on surface [12].

In the recent work [2], a first study of the case $\beta > 0$ has been performed. The main idea is to study the boundary value problem (1) as a nonlocal equation on the boundary. Take as new unknown w the trace of the previous unknown u on $\partial\Omega$. After a lifting, (1) is rewritten as a nonlocal equation on the boundary:

$$\beta\Delta_\tau w + \Lambda w + \alpha w = \varphi, \text{ on } \partial\Omega, \tag{2}$$

where Λ denotes the Dirichlet-to-Neumann map. This equation has a sense in the space $\mathrm{H}^1(\partial\Omega)$. The original unknown u is then recovered by solving a usual Dirichlet boundary value problem. Applying Fredholm alternative to this pseudodifferential equation, Bonnaillie-Noël and her coauthors obtained the following result:

Theorem 1.1 *The operator $P_{\alpha,\beta} = -\beta\Delta_\tau - \Lambda - \alpha\mathrm{Id}$ is an elliptic self-adjoint semi-bounded from below pseudodifferential operator of order 2. Besides, for fixed $\beta > 0$, there exists a sequence $(\alpha_n(\Omega))_{n\in\mathbb{N}}$ growing to infinity such that for any $\phi \in \mathrm{H}^s(\partial\Omega)$ with $s \in \mathbb{R}$, we have*

1. *If $\alpha \notin \{\alpha_n(\Omega)\}$, then equation $-P_{\alpha,\beta}w = \phi$ admits a unique solution in $\mathcal{S}'(\partial\Omega)$ which, in addition, belongs to $\mathrm{H}^{s+2}(\partial\Omega)$;*

2. *If $\alpha \in \{\alpha_n(\Omega)\}$, then there is either no solution or a complete affine finite dimensional space of $\mathrm{H}^{s+2}(\partial\Omega)$ solutions.*

Let us give an illustration of this result: consider the case of the unit ball in dimension three. As proved in the annex, the Ventcell boundary value problem is uniquely solvable if and only if

$$\alpha \notin \Big\{\beta n^2 + (\beta - 1)n,\ n \in N\Big\}. \tag{3}$$

In this work, we consider $\alpha \notin \{\alpha_n(\Omega)\}$, i.e., α is chosen in such a way that the boundary value problem (1) has a unique solution. Deform Ω into another domain close to Ω. We address the following question: does the boundary value Problem (1) also have a unique solution on the perturbed domain?

Assume that Ω is a $\mathcal{C}^2$ domain of $\mathbb{R}^d$, we prove that for small deformations of Ω the Ventcel boundary value problem remains uniquely solvable. Precisely, our result is the following.

Theorem 1.2 *Let Ω be a $\mathcal{C}^3$ domain of $\mathbb{R}^d$ and consider $\alpha \notin \{\alpha_n(\Omega)\}$. Consider a $\mathcal{C}^2$ vector field $\boldsymbol{h}$ and the application $T_{\boldsymbol{h}} : \mathbb{R}^d \to \mathbb{R}^d$ defined by $T_{\boldsymbol{h}} = I_{\mathbb{R}^d} + \boldsymbol{h}$. Then, there exists $\varepsilon_0 > 0$ such that the Ventcel boundary value problem*

$$\left\{ \begin{array}{rcll} -\Delta u & = & f & \text{in } \Omega_{\boldsymbol{h}} = T_{\boldsymbol{h}}(\Omega), \\ \partial_n u + \alpha u + \beta \Delta_\tau u & = & 0 & \text{on } \partial\Omega_{\boldsymbol{h}}. \end{array} \right. \tag{4}$$

is uniquely solvable for all vector fields $\boldsymbol{h}$ satisfying $\|h\|_{\mathcal{C}^2} \leq \varepsilon_0$.

This result is useful for numerical simulation when computations are made on a close but distinct domain: it ensures that the boundary value problem remains solvable on the approximating domain. It is also required to prove that the solution of (1) is differentiable with respect to the shape. Such a differentiability result is interesting for optimization purposes. In the particular case of the sphere, this theorem implies the existence of a neighborhood of the sphere in which the Ventcel boundary value problem is uniquely solvable.

Our strategy to prove Theorem 1.2 is the following. First, we transport the boundary value problem (4) defined on Ω_h onto the fix domain Ω. We obtain a new boundary value problem written now on $\partial\Omega$, this modified problem is not of the type of (1). Differential operators are modified by the change of variable. However, the key is the following: if the geometric deformation is sufficiently small, then the new operators are perturbations of the original one and we use a perturbation argument around the configuration on Ω.

The main difficulties are directly connected with the transport of Ω. First, the transport on $\partial\Omega$ of the Laplace-Beltrami operator on $\partial\Omega_{\boldsymbol{h}}$ has to be derived and linked to the Laplace-Beltrami operator on $\partial\Omega$. Second, orthogonality is not preserved in the transport and once transported the Dirichlet-to-Neumann map on $\partial\Omega_{\boldsymbol{h}}$ is not a usual Dirichlet-to-Neumann map on $\partial\Omega$.
The paper is organized as follows. After introducing some definitions and notations, we present in section 3 preliminary results on the transport for the Laplace-Beltrami operator.After, we consider the transport of the Dirichlet-to-Neumann map. The last section is devoted to the proof of Theorem 1.2.

Let us introduce some notations. We denote by H^s the usual Lebesgue and Sobolev spaces. The transpose and the determinant of a matrix A is denoted respectively by A^* and $\det(A)$. The space of real-valued square matrices of size N is denoted by $\mathcal{M}_{N\times N}$. We denote in bold the vectorial quantities: for example, $\boldsymbol{n}$ and $\boldsymbol{n}_{\boldsymbol{h}}$ stand respectively for the unit normal vector field on $\partial\Omega$ and $\partial\Omega_{\boldsymbol{h}}$. The notation $\mathbf{a}\cdot\mathbf{b}$ stands for the Euclidian scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$. The tangential differential operators will be denoted by the subscript τ: for functions φ and $\boldsymbol{V}$ defined in a neighborhood of $\partial\Omega$. We recall that

- $\nabla_\tau \varphi := \nabla\varphi - \partial_{\boldsymbol{n}}\varphi\, \boldsymbol{n}$ is the tangential gradient of the scalar function φ. As usual, we have set $\partial_{\boldsymbol{n}}\varphi = \nabla\varphi \cdot \boldsymbol{n}$,
- $\mathrm{div}_\tau\, \boldsymbol{V} := \mathrm{div}\, \boldsymbol{V} - (D\boldsymbol{V}\boldsymbol{n})\, \boldsymbol{n}$ is the tangential divergence of the vector field $\boldsymbol{V}$,
- $\Delta_\tau \varphi := \mathrm{div}_\tau\, (\nabla_\tau \varphi)$ is the Laplace-Beltrami operator on $\partial\Omega$.

These quantities are only defined on the boundary $\partial\Omega$. As the deformed domain $\Omega_{\boldsymbol{h}}$ depends on a parameter $\boldsymbol{h}$, the operators related to $\partial\Omega_{\boldsymbol{h}}$ also depend on $\boldsymbol{h}$, and are denoted by $\nabla_{\tau,\boldsymbol{h}}$, $\mathrm{div}_{\tau,\boldsymbol{h}}$ and $\Delta_{\tau,\boldsymbol{h}}$. Through the paper, we will use the notation $DT_{\boldsymbol{h}}$ for the Jacobian matrix of the transformation $T_{\boldsymbol{h}}$. Here, $DT_{\boldsymbol{h}} = I + D\boldsymbol{h}$ since $T_{\boldsymbol{h}} = I_{\mathbb{R}^d} + \boldsymbol{h}$.

2 Boundary perturbations and transported problem

In this section, $\boldsymbol{h}$ is fixed so that $T_{\boldsymbol{h}}$ is a diffeomorphism from $\mathbb{R}^d$ into itself. We aim to transport Equation (2) set on the perturbed domain's boundary $\partial\Omega_{\boldsymbol{h}}$ into a new equation set on the boundary $\partial\Omega$ of the reference domain. Therefore we have to compute the transport of both the Laplace-Beltrami operator and the Dirichlet-to-Neuman map under the deformation $T_{\boldsymbol{h}}$.

Let fix some notations: we set

$$\omega_{\boldsymbol{h}}(x) = \det(DT_{\boldsymbol{h}})(x)\|(DT_{\boldsymbol{h}}(x)^*)^{-1}\boldsymbol{n}(x)\|.$$

The density $\omega_{\boldsymbol{h}}$ is the surface jacobian. It can be easily checked that $\omega_{\boldsymbol{h}}$ is a smooth function of $\boldsymbol{h}$ satisfying

$$\omega_{|\boldsymbol{h}=\boldsymbol{0}}(x) = 1 \text{ and } (D\omega_{\boldsymbol{h}})_{|\boldsymbol{h}=\boldsymbol{0}}.\boldsymbol{\xi} = \mathrm{div}_\tau\, \boldsymbol{\xi}. \tag{5}$$

We also set

$$A_{\boldsymbol{h}}(x) = (DT_{\boldsymbol{h}}(x))^{-1}(DT_{\boldsymbol{h}}(x)^*)^{-1} \text{ and } C_h(x) = \omega_{\boldsymbol{h}}(x)A_{\boldsymbol{h}}(x).$$

We first prove a useful technical lemma.

Lemma 2.1 *Let Ω be a $\mathcal{C}^3$ smooth bounded domain of $\mathbb{R}^d$ and let Ψ be a function in $\mathcal{C}^2(\partial\Omega)$, then there is a extension ψ of Ψ in $\mathcal{C}^2(\mathbb{R}^d)$ such that $\boldsymbol{n}\cdot\nabla\psi = 0$ on $\partial\Omega$.*

Proof of Lemma 2.1:
By assumption, $\partial\Omega$ is a compact $\mathcal{C}^2$ manifold, then its cut locus ρ is non negative. Any $x \in \mathbb{R}^d$ such that the distance $d(x, \partial\Omega)$ from x to the boundary $\partial\Omega$ is strictly less than ρ has a unique orthogonal projection on $\partial\Omega$ denoted by $p_{\partial\Omega}(x)$. Let $d_{\partial\Omega}$ be the signed distance function to $\partial\Omega$ and fix $\chi : \mathbb{R} \to \mathbb{R}_+$ a $\mathcal{C}^\infty$ cutoff function such that:

$$\chi(t) = 1 \text{ if } |t| < \rho/3 \text{ and } \chi(t) = 0 \text{ if } |t| > \rho/2.$$

The function $\psi(x) = \chi\circ d_{\partial\Omega}(x)\Psi\circ p_{\partial\Omega}(x)$ satisfies the requirements since the functions $x \mapsto d_{\partial\Omega}(x)$ and $x \mapsto p_{\partial\Omega}(x)$ are respectively $\mathcal{C}^3$ and $\mathcal{C}^2$ in the tubular neighborhood of $\partial\Omega$ of radius ρ. This is a consequence of the implicit functions theorem, see for example Theorems 4-2 & 4-3 in Chapter 5 of [4] for precise statements and detailled proofs. ■

To describe the transport of the Laplace-Beltrami operator Δ_τ, we first tackle the transport on the manifold $\partial\Omega$ of the tangential gradient of a function defined on $\partial\Omega_{\boldsymbol{h}}$.

Transport of the tangential gradient. Let y be a point on $\partial\Omega_{\boldsymbol{h}}$ and $x := T_{\boldsymbol{h}}^{-1}(y)$ be the corresponding point on $\partial\Omega$. Given $\varphi \in C^2(\mathbb{R}^d)$, we give an explicit formula of $(\nabla_{\tau,\boldsymbol{h}}\varphi)\circ T_{\boldsymbol{h}}^{-1}$, that is the back transport on $\partial\Omega$ of the tangential gradient of a quantity defined on $\partial\Omega_{\boldsymbol{h}}$. By definition, the tangential gradient on $\partial\Omega_{\boldsymbol{h}}$ is

$$\nabla_{\tau,\boldsymbol{h}}\varphi(y) = \nabla\varphi(y) - \boldsymbol{n}_{\boldsymbol{h}}(y)\cdot\nabla\varphi(y)\,\boldsymbol{n}_{\boldsymbol{h}}(y)$$

for all $y \in \partial\Omega_{\boldsymbol{h}}$, that is to say for all $x \in \partial\Omega$

$$(\nabla_{\tau,\boldsymbol{h}}\varphi)\circ T_{\boldsymbol{h}}(x) = (\nabla\varphi)\circ T_{\boldsymbol{h}}(x) - \boldsymbol{n}_{\boldsymbol{h}}\circ T_{\boldsymbol{h}}(x)\cdot(\nabla\varphi)\circ T_{\boldsymbol{h}}(x)\;\boldsymbol{n}_{\boldsymbol{h}}\circ T_{\boldsymbol{h}}(x).$$

Let $\varphi^b = \varphi\circ T_h$ be the back transport of φ on $\partial\Omega$. By the chain rule, one has

$$(\nabla\varphi)\circ T_{\boldsymbol{h}}(x) = (DT_{\boldsymbol{h}}(x)^*)^{-1}(\nabla(\varphi\circ T_{\boldsymbol{h}})(x)) \text{ that is to say } \nabla\varphi(y) = (DT_{\boldsymbol{h}}(x)^*)^{-1}\nabla\varphi^b(x).$$

Let us point out that the normal fields are linked with each other through the relation

$$\boldsymbol{n_h}(y) = \boldsymbol{n_h} \circ T_{\boldsymbol{h}}(x) = \frac{(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)}{\|(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\|}.$$

Plugging the expression of the transported fields in the equation defining the tangential gradient, we obtain the expression of the transported tangential gradient at a point of $\partial\Omega$ as

$$(\nabla_{\tau,\boldsymbol{h}}\varphi) \circ T_{\boldsymbol{h}}(x) = \; (DT^*_{\boldsymbol{h}})^{-1}(x)\nabla\varphi^b(x) - \frac{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\nabla\varphi^b(x)}{\|(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\|^2}\,(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x). \tag{6}$$

Note that if $\nabla\varphi.\boldsymbol{n_h} = 0$, then this expression simplifies since

$$(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\cdot(DT^*_{\boldsymbol{h}})^{-1}(x)\nabla\varphi^b(x) = \|(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\|\boldsymbol{n_h}(y)\cdot\nabla\varphi(y).$$

To summarize, we have proved the following lemma.

Lemma 2.2 *For a function $\phi \in \mathcal{C}^2(\partial\Omega_{\boldsymbol{h}})$, we consider any extension $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ such that $\nabla\varphi.\boldsymbol{n_h} = 0$. Then we have*

$$(\nabla_{\tau,\boldsymbol{h}}\phi) \circ T_{\boldsymbol{h}}(x) = \; (DT^*_{\boldsymbol{h}})^{-1}(x)\nabla\varphi^b(x).$$

The transport of the Laplace-Beltrami operator. Let us also recall from [9] how to integrate by parts on $\partial\Omega$ the boundary of a domain Ω. Let φ be a function in $H^2(\mathbb{R}^d)$ and $\boldsymbol{V}$ be a vector field supposed to be sufficiently regular, then

$$\int_{\partial\Omega}(\nabla\varphi(x)\cdot\boldsymbol{V}(x) + \varphi(x)\;\mathrm{div}_\tau\,\boldsymbol{V}(x))\mathrm{d}\sigma(x) = \int_{\partial\Omega}\left(\frac{\partial\varphi}{\partial n}(x) + H(x)\varphi(x)\right)V_n(x)\mathrm{d}\sigma(x). \tag{7}$$

where H denotes the mean curvature of $\partial\Omega$. We now are in position to derive the expression of the transported Laplace-Beltrami operator for a function defined only on $\partial\Omega_{\boldsymbol{h}}$.

Lemma 2.3 *For all functions $\phi \in \mathrm{H}^2(\partial\Omega_{\boldsymbol{h}})$, it holds*

$$\Delta_{\tau,\boldsymbol{h}}\phi(y) = \frac{1}{\omega(\boldsymbol{h})(x)}\mathrm{div}_\tau\left(C_{\boldsymbol{h}}(x)\nabla_\tau\varphi^b(x)\right)\;\text{ on } \partial\Omega. \tag{8}$$

Proof of Lemma 2.3:
Fix ϕ in $\mathrm{H}^2(\partial\Omega_{\boldsymbol{h}})$ and an extension φ such that $\nabla\varphi\cdot\boldsymbol{n_h} = 0$. We seek to compute the quantity $\aleph$ such that for any $\psi \in \mathrm{H}^2(\partial\Omega_{\boldsymbol{h}})$, it holds

$$\int_{\partial\Omega_{\boldsymbol{h}}}\Delta_{\tau,\boldsymbol{h}}\varphi(y)\;\psi(y)\mathrm{d}\sigma(y) \;=\int_{\partial\Omega}\aleph(x)\psi^b(x)\omega_{\boldsymbol{h}}(x)\mathrm{d}\sigma(x).$$

Here, the exponent b denotes the back transport on $\partial\Omega$. To compute $\aleph$, we use the variational characterization of the Laplace-Beltrami operator. We fix a test function ψ in $\mathrm{H}^2(\partial\Omega_{\boldsymbol{h}})$ and an extension Ψ such that $\nabla\Psi\cdot\boldsymbol{n_h} = 0$.

By the variational definition of the Laplace-Beltrami operator, we get

$$\begin{aligned}\int_{\partial\Omega_{\boldsymbol{h}}}\Delta_{\tau,\boldsymbol{h}}\varphi(y)\;\psi_(y)\mathrm{d}\sigma(y) \;&= -\int_{\partial\Omega_{\boldsymbol{h}}}\nabla_{\tau,\boldsymbol{h}}\varphi(y)\cdot\nabla_{\tau,\boldsymbol{h}}\psi_{\boldsymbol{h}}(y)\mathrm{d}\sigma(y)\\ &= -\int_{\partial\Omega}(DT_{\boldsymbol{h}}(x)^*)^{-1}\nabla\varphi^b(x)\cdot(DT_{\boldsymbol{h}}(x)^*)^{-1}\nabla\Psi^b(x)\;\omega_{\boldsymbol{h}}(x)\mathrm{d}\sigma(x)\text{ (by Lemma 2.2)}\\ &= -\int_{\partial\Omega}C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\nabla\Psi^b(x)\mathrm{d}\sigma(x).\end{aligned}$$

Integrating (7) by parts, we obtain that

$$\int_{\partial\Omega} C_{\boldsymbol{h}}(x)\nabla\varphi^b(x).\nabla\Psi^b(x)\mathrm{d}\sigma(x) = \int_{\partial\Omega}\Big(\partial_n\Psi^b(x)+H(x)\Psi^b(x)\Big)\, C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)\mathrm{d}\sigma(x) \\ - \int_{\partial\Omega}\Psi^b\mathrm{div}_\tau\,\Big(C_{\boldsymbol{h}}(x)\nabla_\tau\varphi^b(x)\Big)\,\mathrm{d}\sigma(x).$$

Let us simplify the formula by eliminating the first term of the right hand side: the symmetry of $C_{\boldsymbol{h}}$ enables to get

$$C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x) = \nabla\varphi^b(x)\cdot C_{\boldsymbol{h}}(x)\boldsymbol{n}(x) = 0$$

Thus, we have obtained the announced expression for the transported Laplace-Beltrami operator. ∎

In the application we are interested in, the function ϕ is defined as the trace of a Sobolev function φ defined in $\Omega_{\boldsymbol{h}}$. The choice of a specific extension (such as in the proof of Lemma 2.3) cannot be done since φ is already given. Thus, Lemma 2.3 cannot be applied in our case and we need to adapt it to a function defined on $\Omega_{\boldsymbol{h}}$. Before we state our result, we need to introduce the operator $\mathcal{L}(\boldsymbol{h})$ defined by

$$\mathcal{L}(\boldsymbol{h})\left[\varphi\circ T_{\boldsymbol{h}}\right](x) \\ =\frac{1}{\omega_{\boldsymbol{h}}(x)}\,\mathrm{div}_\tau\left\{C_{\boldsymbol{h}}(x)\nabla_\tau\left[\varphi\circ T_{\boldsymbol{h}}\right](x) - \frac{C_{\boldsymbol{h}}(x)\nabla\left[\varphi\circ T_{\boldsymbol{h}}\right](x).\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x).\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\right\} \tag{9}$$

for $\varphi\in \mathrm{H}^{5/2}(\Omega_{\boldsymbol{h}})$. We have

Lemma 2.4 *The identity*

$$\left[\Delta_{\tau,\boldsymbol{h}}\phi\right]\circ T_{\boldsymbol{h}} = \mathcal{L}(\boldsymbol{h})\left[\varphi\circ T_{\boldsymbol{h}}\right] \tag{10}$$

holds for all functions φ belonging to $\mathrm{H}^{5/2}(\Omega_{\boldsymbol{h}})$.

Proof of Lemma 2.4:
Let $\varphi\in\mathrm{H}^{5/2}(\Omega_{\boldsymbol{h}})$, its trace also denoted by φ belongs to $\mathrm{H}^2(\partial\Omega_{\boldsymbol{h}})$. We follow the proof of Lemma 2.3: we fix a test function $\psi\in\mathrm{H}^2(\partial\Omega_{\boldsymbol{h}})$ and an extension Ψ such that $\nabla\Psi.\boldsymbol{n}_{\boldsymbol{h}}=0$.

By the variational definition of the Laplace-Beltrami operator, we get

$$\int_{\partial\Omega_{\boldsymbol{h}}}\Delta_{\tau,\boldsymbol{h}}\varphi(y)\;\psi_{\boldsymbol{h}}(y)\mathrm{d}\sigma(y) \;\; = -\int_{\partial\Omega_{\boldsymbol{h}}}\nabla_{\tau,\boldsymbol{h}}\varphi(y).\nabla_{\tau,\boldsymbol{h}}\psi_{\boldsymbol{h}}(y)\mathrm{d}\sigma(y).$$

By the change of variables $y=T_{\boldsymbol{h}}(x)$, we get integrals defined on the fixed boundary $\partial\Omega$. Thanks to the chain rule (see formula (6)), we obtain

$$\int_{\partial\Omega}\left(\left(\Delta_{\tau,\boldsymbol{h}}\varphi\right)\circ T_{\boldsymbol{h}}\right)(x)\Psi^b(x)\;\omega_h(x)\mathrm{d}\sigma(x) \;\; = \\ = -\int_{\partial\Omega}\left(DT_{\boldsymbol{h}}(x)^*\right)^{-1}\nabla\varphi^b(x)\cdot\left(DT_{\boldsymbol{h}}(x)^*\right)^{-1}\nabla\Psi^b(x)\;\omega_{\boldsymbol{h}}(x)\mathrm{d}\sigma(x) \\ + \int_{\partial\Omega}\frac{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x).\nabla\varphi^b(x)}{\|(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\|^2}\left(DT^*_{\boldsymbol{h}}\right)^{-1}(x)\boldsymbol{n}(x)\cdot\left(DT_{\boldsymbol{h}}(x)^*\right)^{-1}\nabla\Psi^b(x)\;\omega_{\boldsymbol{h}}(x)\mathrm{d}\sigma(x)$$

Since $\nabla\Psi(y)\cdot \boldsymbol{n_h}(y) = 0$: we have

$$\frac{(DT_{\boldsymbol{h}}^*)^{-1}(x)\boldsymbol{n}(x)}{\|(DT_{\boldsymbol{h}}^*)^{-1}(x)\boldsymbol{n}(x)\|}\cdot (DT_{\boldsymbol{h}}(x)^*)^{-1}\nabla\Psi^b(x) = \nabla\Psi(y)\cdot\boldsymbol{n_h}(y) = 0$$

hence

$$\int_{\partial\Omega} \left((\Delta_{\tau,\boldsymbol{h}}\varphi)\circ T_{\boldsymbol{h}}\right)(x)\Psi^b(x)\ \omega_h(x)\mathrm{d}\sigma(x) = -\int_{\partial\Omega} C_{\boldsymbol{h}}(x)\nabla\varphi^b(x).\nabla\Psi^b(x)\mathrm{d}\sigma(x).$$

We focus on the right side of the equation : we have to make some computations in order to get a formula depending only on Ψ^b. Integrating (7) by parts, we obtain

$$\int_{\partial\Omega} C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\nabla\Psi^b(x)\mathrm{d}\sigma(x) = \int_{\partial\Omega}\Big(\partial_n\Psi^b(x) + H(x)\Psi^b(x)\Big)\, C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)\mathrm{d}\sigma(x)$$
$$-\int_{\partial\Omega}\Psi^b\mathrm{div}_\tau\,\Big(C_{\boldsymbol{h}}(x)\nabla_\tau\varphi^b(x)\Big)\,\mathrm{d}\sigma(x). \quad (11)$$

It remains to deal with the term involving $\partial_n\Psi^b(x)$. From the property $\nabla\Psi\cdot\boldsymbol{n_h} = 0$ satisfied by the extended test function, we back transport it on $\partial\Omega_{\boldsymbol{h}}$ and decompose $\nabla\Psi^b$:

$$0 = A_{\boldsymbol{h}}(x)\nabla\Psi^b(x)\cdot\boldsymbol{n}(x) = A_{\boldsymbol{h}}(x)\left(\nabla_\tau\Psi^b(x) + \partial_n\Psi^b(x)\boldsymbol{n}(x)\right)\cdot\boldsymbol{n}(x).$$

Hence, it comes that

$$\partial_n\Psi^b = -\frac{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\nabla_\tau\Psi^b(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}.$$

Then we inject this expression in the right hand side of (11) and integrate by parts:

$$\int_{\partial\Omega}\partial_n\Psi^b(x)C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)\mathrm{d}\sigma(x)$$
$$= \int_{\partial\Omega} -\frac{C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\nabla_\tau\Psi^b(x)\mathrm{d}\sigma(x)$$
$$= \int_{\partial\Omega}\left[\mathrm{div}_\tau\left\{\frac{C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\right\}\Psi^b(x) - H(x)C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)\right]\mathrm{d}\sigma(x).$$

Gathering the terms, we obtain

$$\int_{\partial\Omega}\left((\Delta_{\tau,\boldsymbol{h}}\varphi)\circ T_{\boldsymbol{h}}\right)(x)\ \omega_h(x)\ \Psi^b(x)\mathrm{d}\sigma(x) =$$
$$\int_{\partial\Omega}\mathrm{div}_\tau\left\{C_{\boldsymbol{h}}(x)\nabla_\tau\varphi^b(x) - \frac{C_{\boldsymbol{h}}(x)\nabla\varphi^b(x)\cdot\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\right\}\Psi^b(x)\mathrm{d}\sigma(x).$$

This achieves the proof. ■

Transport of the Dirichlet-to-Neuman map. Let us consider the Dirichlet-to-Neumann operator defined on its natural space $\Lambda_{\boldsymbol{h}} : \mathrm{H}^{1/2}(\partial\Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$. It maps a function ϕ in $\mathrm{H}^{1/2}(\partial\Omega_{\boldsymbol{h}})$ onto the normal derivative of its harmonic expansion in $\Omega_{\boldsymbol{h}}$, i.e., $\Lambda_{\boldsymbol{h}}(\phi) = \partial_{\boldsymbol{n_h}}u$, where u solves the boundary value problem:

$$\left\{\begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega_{\boldsymbol{h}}, \\ u & = & \phi & \text{on } \partial\Omega_{\boldsymbol{h}}. \end{array}\right. \quad (12)$$

To compute the quantity $\aleph$ such that $\aleph(\phi^b) = [\Lambda_{\boldsymbol{h}}(\phi)] \circ T_{\boldsymbol{h}}$, we back transport the boundary value problem (12) on the domain Ω. Setting $u_{\boldsymbol{h}} = u \circ T_{\boldsymbol{h}}$, we check from the variational formulation, that the function $v_{\boldsymbol{h}}$ is the unique solution of the transported boundary value problem:

$$\left\{ \begin{array}{rcll} -\mathrm{div}\,(\tilde{A}_{\boldsymbol{h}} \nabla u_{\boldsymbol{h}}) & = & 0 & \text{in } \Omega, \\ u_{\boldsymbol{h}} & = & \phi^b & \text{on } \partial\Omega, \end{array} \right. \tag{13}$$

where $\tilde{A}_{\boldsymbol{h}} = \det(T_{\boldsymbol{h}}) A_{\boldsymbol{h}}$. Hence, we get formally

$$\begin{aligned} \Lambda_{\boldsymbol{h}}(\phi)(y) = \nabla u(y) \cdot \boldsymbol{n}_h(y) = (DT_{\boldsymbol{h}}(x)^*)^{-1} \nabla u_{\boldsymbol{h}}(x) \cdot \frac{(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)}{\|(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\|} \\ = \frac{1}{\|(DT^*_{\boldsymbol{h}})^{-1}(x)\boldsymbol{n}(x)\|} A_{\boldsymbol{h}}(x)\boldsymbol{n}(x) \cdot \nabla u_{\boldsymbol{h}}(x). \end{aligned}$$

Here again, we can give a sense to the conormal derivative $A_{\boldsymbol{h}}\boldsymbol{n}.\nabla v$ thanks to the boundary value problem (13). This quantity is defined in a weak sense as the previous Dirichlet-to-Neumann operator Λ_h. To be more precise, we have the following result

Lemma 2.5 *For $\phi \in \mathrm{H}^{1/2}(\partial\Omega)$, we define $\mathcal{D}_{\boldsymbol{h}}\phi$ as the element of $\mathrm{H}^{-1/2}(\partial\Omega)$ such that*

$$f \in \mathrm{H}^{1/2}(\partial\Omega) \mapsto \langle \mathcal{D}_{\boldsymbol{h}}\phi, f\rangle_{\mathrm{H}^{-1/2}(\partial\Omega)\times\mathrm{H}^{1/2}(\partial\Omega)} := \int_\Omega \tilde{A}_{\boldsymbol{h}}(x) \nabla u_{\boldsymbol{h}}(x) \cdot \nabla E(f)(x) \mathrm{d}x,$$

where E is a continuous extension operator from $\mathrm{H}^{-1/2}(\partial\Omega)$ to $\mathrm{H}^1(\Omega)$. Then, for all functions $\varphi \in \mathrm{H}^{1/2}(\Omega_{\boldsymbol{h}})$, it holds

$$\Lambda_{\boldsymbol{h}}\varphi = \mathcal{D}_{\boldsymbol{h}}\left[\varphi \circ T_h\right]. \tag{14}$$

Now, we consider the restriction of the operator $\mathcal{D}_{\boldsymbol{h}}$ to $\mathrm{H}^1(\partial\Omega)$. It will still be denoted by $\Lambda_{\boldsymbol{h}}$ and is now to be considered as a linear continuous operator in $\mathcal{L}(\mathrm{H}^1(\partial\Omega), \mathrm{H}^{-1}(\partial\Omega))$. We can now state the main result of this section.

Proposition 2.6 *Let w be a function on $\partial\Omega_{\boldsymbol{h}}$. One has*

$$\beta\Delta_\tau w + \Lambda w + \alpha w = \varphi \text{ on } \partial\Omega_{\boldsymbol{h}} \tag{15}$$

if and only if $w_{\boldsymbol{h}} = w \circ T_{\boldsymbol{h}}$ its back transport on $\partial\Omega$ satisfies

$$\beta\mathcal{L}_{\boldsymbol{h}} w_{\boldsymbol{h}} + \mathcal{D}_{\boldsymbol{h}} w_{\boldsymbol{h}} + \alpha w_{\boldsymbol{h}} = \varphi \circ T_h \text{ on } \partial\Omega. \tag{16}$$

3 The transported problem seen as a perturbation.

Let us introduce the operator $L_{\boldsymbol{h}}$ defined on $\mathrm{H}^1(\partial\Omega)$ with values in $\mathrm{H}^{-1}(\partial\Omega)$:

$$L_{\boldsymbol{h}} w = \beta\mathcal{L}_{\boldsymbol{h}} w + \mathcal{D}_{\boldsymbol{h}} w + \alpha w.$$

Our goal is to prove that $L_{\boldsymbol{h}}$ is a perturbation of $L_{\boldsymbol{0}}$. To that end, we now want to express that the operator $\mathcal{L}_{\boldsymbol{h}}$ (resp. $\mathcal{D}_{\boldsymbol{h}}$) is a perturbation of the Laplace-Beltrami operator Δ_τ (resp. of the Dirichlet-to-Neumann map Λ). The persistency of the existence and uniqueness result under shape deformation is deduced from these two results.

Study of $\mathcal{L}_{\boldsymbol{h}}$

Lemma 3.1 *There exists a constant $C > 0$ such that*

$$\|\mathcal{L}_{\boldsymbol{h}} - \Delta_\tau\|_{\mathcal{L}(\mathrm{H}^1(\partial\Omega),\mathrm{H}^{-1}(\partial\Omega))} \leq C\,\|h\|_{\mathcal{C}^2}$$

holds for all $\boldsymbol{h} \in C^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ with $\|\boldsymbol{h}\|_{\mathcal{C}^2}$ sufficiently small.

Proof of Lemma 3.1:
Let $\phi \in H^1(\partial\Omega)$, we write

$$\mathcal{L}_{\boldsymbol{h}}\varphi = \mathcal{L}_{\boldsymbol{h},1}\varphi + \mathcal{L}_{\boldsymbol{h},2}\varphi + \mathcal{L}_{\boldsymbol{h},3}\varphi + \mathcal{L}_{\boldsymbol{h},4}\varphi,$$

where

$$\mathcal{L}_{\boldsymbol{h},1}\varphi := \frac{1-\omega_{\boldsymbol{h}}}{\omega_h}\mathrm{div}_\tau\,\{C_{\boldsymbol{h}}(x)\nabla_\tau\varphi\}\,, \tag{17}$$

$$\mathcal{L}_{\boldsymbol{h},2}\varphi := \mathrm{div}_\tau\,\{(C_{\boldsymbol{h}}(x)-I)\nabla_\tau\varphi\}\,, \tag{18}$$

$$\mathcal{L}_{\boldsymbol{h},3}\varphi := \mathrm{div}_\tau\,\{\nabla_\tau\varphi\} = \Delta_\tau\varphi, \tag{19}$$

and

$$\mathcal{L}_{\boldsymbol{h},4}\varphi := -\frac{1}{\omega_{\boldsymbol{h}}(x)}\mathrm{div}_\tau\left\{\frac{C_{\boldsymbol{h}}(x)\nabla_\tau\varphi(x)\cdot\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\right\}. \tag{20}$$

Hence

$$(\mathcal{L}_{\boldsymbol{h}} - \Delta_\tau)\varphi = \mathcal{L}_{\boldsymbol{h},1}\varphi + \mathcal{L}_{\boldsymbol{h},2}\varphi + \mathcal{L}_{\boldsymbol{h},4}\varphi.$$

We point out that if $T_{\boldsymbol{h}}$ is a small perturbation of the identity in the norm of $W^{1,\infty}$, then $DT_{\boldsymbol{h}}$ and $DT_{\boldsymbol{h}}^{-1}$ belong to $L^\infty(\partial\Omega,\mathcal{M}_{N\times N})$. It follows that $C_{\boldsymbol{h}}$ belongs also to $L^\infty(\partial\Omega,\mathcal{M}_{N\times N})$.
Let $\psi \in H^1(\partial\Omega)$. After integration by parts, we get

$$\int_{\partial\Omega}\mathcal{L}_{\boldsymbol{h},1}\varphi\;\psi\;\mathrm{d}\sigma = -\int_{\partial\Omega}C_h\nabla_\tau\varphi\cdot\nabla_\tau\left(\frac{1-\omega_{\boldsymbol{h}}}{\omega_h}\psi\right)\;\mathrm{d}\sigma;$$

hence there exists a strictly positive constant $C > 0$ such that when $\|h\|$ is small:

$$\begin{aligned}\left|\int_{\partial\Omega}\mathcal{L}_{\boldsymbol{h},1}\varphi\;\psi\;\mathrm{d}\sigma\right| &\leq \left\|\frac{1}{\omega_h}-1\right\|_{W^{1,\infty}}\|C_h\|_{L^\infty(\partial\Omega,\mathcal{M}_{N\times N})}\|\varphi\|_{H^1(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}\\ &\leq C\,\|\mathrm{div}_\tau\,h\|_{W^{1,\infty}(\partial\Omega)}\,\|\varphi\|_{H^1(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}\\ &\leq C\,\|h\|_2\,\|\varphi\|_{H^1(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}.\end{aligned}$$

Here, we used the fact that the derivative of the surface jacobian is the tangential divergence as stated in (5).

We focus now on $\mathcal{L}_{\boldsymbol{h},2}$. A straightforward calculation shows that $\boldsymbol{h} \in W^{1,\infty} \mapsto (DT_h)^{-1} \in L^\infty(\partial\Omega,\mathcal{M}_{N\times N})$ is $\mathcal{C}^\infty$ when $\|\boldsymbol{h}\|$ is sufficiently small (see [9], p 184). Hence, there exists a strictly positive constant $C > 0$ such

$$\begin{aligned}\left|\int_{\partial\Omega}\mathcal{L}_{\boldsymbol{h},2}\varphi\;\psi\;\mathrm{d}\sigma\right| &\leq \|C_h - I\|_{L^\infty(\partial\Omega,\mathcal{M}_{N\times N})}\,\|\nabla_\tau\varphi\|_{L^2(\partial\Omega)}\,\|\nabla_\tau\psi\|_{L^2(\partial\Omega)}\\ &\leq C\,|\omega_{\boldsymbol{h}}-1|_{L^\infty(\partial\Omega)}\,\|\nabla_\tau\varphi\|_{L^2(\partial\Omega)}\,\|\nabla_\tau\psi\|_{L^2(\partial\Omega)}\\ &\leq C\,\|\mathrm{div}_\tau\,h\|_{W^{1,\infty}(\partial\Omega)}\,\|\varphi\|_{H^1(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}\\ &\leq C\,\|h\|_2\,\|\varphi\|_{H^1(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}.\end{aligned}$$

Concerning $\mathcal{L}_{\boldsymbol{h},4}\varphi$, we use the fact that $\boldsymbol{n}.\nabla_\tau\phi = 0$ and rewrite $\mathcal{L}_{\boldsymbol{h},4}\varphi$ as

$$\mathcal{L}_{\boldsymbol{h},4}\varphi = -\frac{1}{\omega_{\boldsymbol{h}}(x)}\,\mathrm{div}_\tau\left\{\frac{(C_{\boldsymbol{h}}(x)-I)\nabla_\tau\varphi(x)\cdot\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\right\}. \tag{21}$$

We then obtain

$$\int_{\partial\Omega}\mathcal{L}_{\boldsymbol{h},4}\varphi\ \psi\ \mathrm{d}\sigma \quad = \quad \int_{\partial\omega}\frac{(C_{\boldsymbol{h}}(x)-I)\nabla_\tau\varphi(x)\cdot\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\nabla_\tau\Big(\frac{1}{\omega_{\boldsymbol{h}}(x)}\psi\Big)\ \mathrm{d}\sigma,$$

and following the same arguments as before, we get

$$\begin{aligned}\left|\int_{\partial\Omega}\mathcal{L}_{\boldsymbol{h},2}\varphi\ \psi\ \mathrm{d}\sigma\right| \quad &= \quad \left|\int_{\partial\Omega}\frac{(C_{\boldsymbol{h}}(x)-I)\nabla_\tau\varphi(x).\boldsymbol{n}(x)}{A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\boldsymbol{n}(x)}A_{\boldsymbol{h}}(x)\boldsymbol{n}(x)\cdot\nabla_\tau\Big(\frac{1}{\omega_{\boldsymbol{h}}(x)}\psi\Big)\ \mathrm{d}\sigma\right|\\ &\le \quad C\,\|(C_{\boldsymbol{h}}-I\|_{L^\infty(\partial\Omega,\mathcal{M}_{N\times N})}\,\|\nabla_\tau\varphi\|_{L^2(\partial\Omega)}\,\|\psi\|_{H^1(\partial\Omega)}\\ &\le \quad C\,\|(C_{\boldsymbol{h}}-I\|_{L^\infty(\partial\Omega,\mathcal{M}_{N\times N})}\,\|\varphi\|_{H^1(\partial\Omega)}\,\|\psi\|_{H^1(\partial\Omega)}\\ &\le \quad C\,\|h\|_2\,\|\varphi\|_{H^1(\partial\Omega)}\|\psi\|_{H^1(\partial\Omega)}.\end{aligned}$$

■

Study of $\mathcal{D}_{\boldsymbol{h}}$ Our aim is to prove the following result stating that the perturbed D i richlet-to-Neumann map depends continuously on the perturbation on the natural space for the Laplace-Beltrami operator .

Lemma 3.2 *There is a modulus of continuity ω_2 such that for all $\boldsymbol{h}$ with $\|\boldsymbol{h}\|_{\mathcal{C}^2} < 1$, one has*

$$\|\mathcal{D}_{\boldsymbol{h}} - \Lambda\|_{\mathcal{L}(\mathrm{H}^1(\partial\Omega),\mathrm{H}^{-1}(\partial\Omega))} \le \omega_2(\|\boldsymbol{h}\|_{\mathcal{C}^2}).$$

Proof of Lemma 3.2:
Take ϕ and f two functions in $\mathrm{H}^1(\partial\Omega)$. Let us notice (see [11]) that $\mathrm{H}^1(\partial\Omega)$ is the trace space of $\mathrm{H}^{3/2}(\Omega)$ and that the normal derivative of an element of $\mathrm{H}^{3/2}(\Omega)$ is not defined in general except under regularity properties of its Laplacian. Hence, the Dirichlet-to-Neumann map is still defined by the weak formulation and we will prove the estimation first in the norm of $\mathcal{L}(\mathrm{H}^{1/2}(\partial\Omega),\mathrm{H}^{-1/2}(\partial\Omega))$. The passage to $\mathcal{L}(\mathrm{H}^1(\partial\Omega),\mathrm{H}^{-1}(\partial\Omega))$ is then a consequence of the hierarchy of the norms.

By definitions of the operators $\mathcal{D}_{\boldsymbol{h}}$ and Λ, one has

$$\begin{aligned}\langle(\mathcal{D}_{\boldsymbol{h}}-\Lambda)\phi, f\rangle_{\mathrm{H}^{-1/2}(\partial\Omega)\times\mathrm{H}^{1/2}(\partial\Omega)} &= \int_\Omega (A_{\boldsymbol{h}}(x)\nabla u_h(x) - \nabla u(x))\cdot\nabla E_{\boldsymbol{h}}(f)(x)\ \mathrm{d}x\\ &= \int_\Omega [(A_{\boldsymbol{h}}-I)(x)\nabla u_h(x) + \nabla(u_{\boldsymbol{h}}(x)-u(x))]\cdot\nabla E_{\boldsymbol{h}}(f)(x)\ \mathrm{d}x,\end{aligned}$$

where $u_{\boldsymbol{h}}$ and u are the respective solutions of the boundary value problem

$$\left\{\begin{array}{rcll}-\mathrm{div}\,(A_{\boldsymbol{h}}\nabla u_{\boldsymbol{h}}) &=& 0 & \text{in } \Omega,\\ u_{\boldsymbol{h}} &=& \phi & \text{on } \partial\Omega.\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll}-\Delta u &=& 0 & \text{in } \Omega,\\ u &=& \phi & \text{on } \partial\Omega.\end{array}\right.$$

It remains to estimate the variations $A_{\boldsymbol{h}} - I = A_{\boldsymbol{h}} - A_{\boldsymbol{0}}$ and $u_h - u = u_{\boldsymbol{h}} - u_{\boldsymbol{0}}$.

To that end, the key arguments are the two following continuity results. First, the application $\boldsymbol{h}\mapsto A_{\boldsymbol{h}}$ is continuous from $\mathrm{W}^{2,\infty}(\mathbb{R}^N)$ in $\mathrm{L}^\infty(\mathbb{R}^N)$. Second, we claim that there exists a modulus of continuity ω such that

$$\|u_{\boldsymbol{h}} - u_{\boldsymbol{0}}\|_{\mathrm{H}^1(\Omega)} \le \omega(\|h\|_{\mathcal{C}^2}).$$

Let us sketch the proof of that classic claim by *reductio ad absurdum.* We define

$$\omega(\delta) = \sup_{\|\boldsymbol{h}\|\leq\delta} \|u_{\boldsymbol{h}} - u_{\mathbf{0}}\|_{\mathrm{H}^1(\Omega)}.$$

We assume by contradiction that $\lim_{\delta\to 0}\omega(\delta) \neq 0$. Then, there exists a sequence $\boldsymbol{h}_n$ such that $\|\boldsymbol{h}_n\| \leq 1/n$ and $\|u_{\boldsymbol{h}_n} - u_{\mathbf{0}}\|_{\mathrm{H}^1(\Omega)} \geq \alpha > 0$ for all $n \in \mathbb{N}$. Using the classical elliptic estimates, we check that the sequence $u_{\boldsymbol{h}_n}$ is bounded in $\mathrm{H}^{3/2}(\Omega)$. Then, by the compact imbedding of $\mathrm{H}^{3/2}(\Omega)$ into $\mathrm{H}^1(\Omega)$, this sequence has to converge, up to an extraction, to a limit u_∞ in $\mathrm{H}^1(\Omega)$. Noting that

$$\mathrm{div}\,(A_{\boldsymbol{h}_n}\nabla u_{\boldsymbol{h}_n}) - \Delta u_\infty = \mathrm{div}\,([A_{\boldsymbol{h}_n} - I]\nabla u_{\boldsymbol{h}_n}) - \Delta[u_{\boldsymbol{h}_n} - u_\infty],$$

and passing to the limit $n \to +\infty$, we check that u_∞ satisfies

$$\left\{\begin{array}{rcll} -\Delta u & = & 0 & \text{in } \Omega, \\ u & = & \phi & \text{on } \partial\Omega. \end{array}\right.$$

By uniqueness of the solution of this boundary value problem, $u_{\mathbf{0}} = u_\infty$ that contredicts $\|u_{\boldsymbol{h}_n} - u_{\mathbf{0}}\|_{\mathrm{H}^1(\Omega)} \geq \alpha > 0$ for all $n \in \mathbb{N}$. ■

Conclusion Gathering Lemmas 3.1 and 3.2, we have shown that there is a modulus of continuity ω such that

$$\|L_{\boldsymbol{h}} - L_{\mathbf{0}}\|_{\mathcal{L}(\mathrm{H}^1(\partial\Omega),\mathrm{H}^{-1}(\partial\Omega))} \leq \omega(\|\boldsymbol{h}\|).$$

We are now in position to prove our main result Theorem 1.2.

Proof of Theorem 1.2.:
We place ourselves under the assumptions and notations of Theorem 1.2. For a given deformation field $\boldsymbol{h}$, we have shown in Proposition 2.6 that the boundary value problem (4) is equivalent to the nonlocal equation (16). To prove that this equation has a unique solution, it suffices to prove that $L_{\boldsymbol{h}}$ is invertible.

Now, we remark that $L_{\mathbf{0}}$ being invertible by assumption, we can write

$$L_{\boldsymbol{h}} = L_{\mathbf{0}}[I + L_{\mathbf{0}}^{-1}(L_{\boldsymbol{h}} - L_{\mathbf{0}})].$$

Then, for $\|\boldsymbol{h}\|$ small enough, the operator $L_{\boldsymbol{h}}$ is invertible and its inverse can be written in terms of the Neumann's series

$$L_{\boldsymbol{h}}^{-1} = \sum_{n=0}^{\infty}(I - L_{\mathbf{0}}^{-1}L_{\boldsymbol{h}})^n L_{\mathbf{0}}^{-1},$$

expression that provides a uniform bound for the norm of the inverse. ■

A Case of the ball: justification of (3).

We denote by r the radius which is the distance from the point to the origin, by θ and φ the Euler angles. Since u is harmonic, we write it as a sum of spherical harmonics

$$u(r,\varphi,\theta) = \sum_{l=0}^{+\infty} r^l \sum_{m=-l}^{m} u_l^m Y_l^m(\varphi,\theta),$$

where the spherical harmonics (Y_l^m) are defined as

$$Y_l^m(\varphi,\theta) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)}} P_l^m(\cos\theta) e^{im\varphi}, \tag{22}$$

where $P_l^m(\cos\theta)$ are the sequence of the associated Legendre functions. We recall [13] that both the Laplace-Beltrami operator Δ_τ and the Dirichlet-to-Neuman operator Λ are diagonal on the basis of spherical harmonics, i.e., one has

$$\begin{cases} \Delta_\tau Y_l^m &= -l(l+1)Y_l^m \\ \Lambda Y_l^m &= lY_l^m. \end{cases}$$

To solve the Ventcel boundary value problem, we begin first to expand $\phi \in L^2(S)$ into a series of spherical harmonic functions

$$\phi(\varphi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_l^m Y_l^m(\varphi,\theta),$$

where

$$\phi_l^m = \int_S \phi(\varphi,\theta) Y_l^m(\varphi,\theta)\ dS.$$

In the next step, we have to solve the decoupled projected equations for the Fourier coefficient u_l^m of u : it comes obviously that

$$\Big[-\beta l(l+1) + l + \alpha\Big] u_l^m = \phi_l^m, \quad l \in \mathbb{N}, \quad -l \le m \le l.$$

Since each projected equation should have a unique solution in order to obtain a unique solution to the original problem, we obtain that

$$-\beta l(l+1) + l + \alpha \neq 0, \quad \forall l \in \mathbb{N}$$

or equivalently (3).

Concerning the regularity of the found solution, we recall the caracterisation of Sobolev spaces on the sphere in terms of Fourier's coefficients: for an arbitrary $f \in \mathcal{D}'(S)$ we have when $t \in \mathbb{R}$

$$f \in H^t(S) \Leftrightarrow \|f\|^2_{H^t(S)} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1)^{2t} |f_l^m|^2.$$

Hence, for the admissible values of α, Fourier coefficients u_l^m of the solution u satisfy

$$\mid u_l^m \mid \le \frac{C}{l^2} \mid \phi_l^m \mid .$$

We obtain for $\phi \in H^s(S)$:

$$\begin{aligned} \|u\|^2_{H^{s+2}(S)} &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1)^{2s+4} |u_l^m|^2 \le C\Big(\sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{(l+1)^{2s+4}}{l^2} |\phi_l^m|^2 + \frac{1}{|\alpha|^2} |\phi_0^0|^2 \Big) \\ &\le C' \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (l+1)^{2s} |\phi_l^m|^{2s} = C' \|\phi\|^2_{H^s(S)}. \end{aligned}$$

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