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ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA AND OTHER RELATED VARIETIES.

JEAN-YVES CHARBONNEL AND MOUCHIRA ZAITER

Abstract. The nilpotent cone of a reductive Lie algebra has a desingularization given by the cotangent bundle of the flag variety. Analogously, the nullcone of a cartesian power of the algebra has a desingularization given by a vector bundle over the flag variety. As for the nullcone, the subvariety of elements whose components are in a same Borel subalgebra, has a desingularization given by a vector bundle over the flag variety. In this note, we study geometrical properties of these varieties. For the study of the commuting variety, the analogous variety to the flag variety is the closure in the Grassmannian of the set of Cartan subalgebras. So some properties of this variety are given. In particular, it is smooth in codimension 1. We introduce the generalized isospectral commuting varieties and give some properties. Furthermore, desingularizations of these varieties are given by fiber bundles over a desingularization of the closure in the grassmannian of the set of Cartan subalgebras contained in a given Borel subalgebra.

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1. Introduction

In this note, the base field $\mathbb{k}$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, $\ell$ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and $G$ is its adjoint group. As usual, $\mathfrak{b}$ denotes a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$, and $B$ the normalizer of $\mathfrak{b}$ in $G$.

1.1. Main results. Let $\mathcal{B}^{(k)}$ be the subset of elements $(x_1, \ldots, x_k)$ of $\mathfrak{g}^k$ such that $x_1, \ldots, x_k$ are in a same Borel subalgebra of $\mathfrak{g}$. This subset of $\mathfrak{g}^k$ is closed and contains two interesting subsets: the nullcone of $\mathfrak{g}^k$ denoted by $\mathcal{N}^{(k)}$ and the generalized commuting variety of $\mathfrak{g}$ that is the closure in $\mathfrak{g}^k$ of the subset of elements $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k$ are in a same Cartan subalgebra of $\mathfrak{g}$. We denote it by $\mathcal{C}^{(k)}$. According to [Mu65, Ch.2, §1, Theorem], for $(x_1, \ldots, x_k)$ in $\mathcal{B}^{(k)}$, $(x_1, \ldots, x_k)$ is in $\mathcal{N}^{(k)}$ if and only if $x_1, \ldots, x_k$ are nilpotent. According to a Richardson Theorem [Ri79], $\mathcal{C}^{(2)}$ is the commuting variety of $\mathfrak{g}$.

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There is a natural projective morphism $G \times_b b^k \rightarrow \mathcal{B}^{(k)}$. For $k = 1$, this morphism is not birational but for $k \geq 2$, it is birational (see Lemma 2.2 and Lemma 2.4). Furthermore, denoting by $\mathcal{X}$ the subvariety of elements $(x, y)$ of $g \times h$ such that $y$ is in the closure of the orbit of $x$ under $G$, the morphism

$$G \times h \rightarrow \mathcal{X}, \quad (g, x) \mapsto (g.x, x)$$

with $\mathcal{X}$ the projection of $x$ onto $h$ defines through the quotient a projective and birational morphism $G \times_h b \rightarrow \mathcal{X}$ and $g$ is the categorical quotient of $\mathcal{X}$ under the action of $W(R)$ on the factor $h$, with $W(R)$ the Weyl group of $g$. For $k \geq 2$, the inverse image of $\mathcal{B}^{(k)}$ by the canonical projection from $\mathcal{X}^k$ to $g^k$ is not irreducible but the canonical action of $W(R)^k$ on $\mathcal{X}^k$ induces a simply transitive action on the set of its irreducible components. Setting:

$$\mathcal{B}^{(k)}_x := \{(g(x_1), x_1), \ldots, (g(x_k), x_k) \mid (g, x_1, \ldots, x_k) \in G \times b^k\},$$

$\mathcal{B}^{(k)}_x$ is an irreducible component of the inverse image of $\mathcal{B}^{(k)}$ in $\mathcal{X}^k$ (see Corollary 2.8) and we have a commutative diagram

$$\begin{array}{ccc}
G \times_B b^k & \rightarrow & \mathcal{B}^{(k)}_x \\
\gamma & & \downarrow \eta \\
\mathcal{B}^{(k)} & \rightarrow & \mathcal{B}^{(k)}
\end{array}$$

with $\eta$ the restriction to $\mathcal{B}^{(k)}_x$ of the canonical projection $\pi$ from $\mathcal{X}^k$ to $g^k$. The first main theorem of this note is the following theorem:

**Theorem 1.1.** (i) The variety $N^{(k)}$ is normal if and only if so is $\mathcal{B}^{(k)}_x$.

(ii) The variety $N^{(k)}$ is Cohen-Macaulay if and only if so is $\mathcal{B}^{(k)}_x$.

(iii) The variety $N^{(k)}$ has rational singularities if and only if it is Cohen-Macaulay.

(iv) The variety $\mathcal{B}^{(k)}_x$ has rational singularities if and only if it is Cohen-Macaulay.

(v) The algebra $k[\mathcal{B}^{(k)}_x]$ is a free extension of $k[\mathcal{B}^{(k)}]^G$ which identifies with $S(b^k)$.

(vi) The algebra $k[\mathcal{B}^{(k)}]^G$ identifies with $S(b^k)^{W(R)}$ with respect to the diagonal action of $W(R)$ in $b^k$.

(vii) The ideal $k[\mathcal{B}^{(k)}]^G$ of $k[\mathcal{B}^{(k)}]$ is strictly contained in the ideal of definition of $N^{(k)}$ in $k[\mathcal{B}^{(k)}]$.

According to K. Vilonen and T. Xue [VX15], $N^{(k)}$ and $\mathcal{B}^{(k)}_x$ are not normal in general. In the study of the generalized commuting variety, the closure in $\text{Gr}_r(g)$ of the orbit of $h$ under the action of $G$ plays an analogous role to the flag variety. Denoting by $X$ the closure in $\text{Gr}_r(b)$ of the orbit of $h$ under $B$. $G.X$ is the closure of the orbit of $G.h$ and we have the following second main result:

**Theorem 1.2.** Let $X'$ be the set of centralizers of regular elements of $h$ whose semisimple components is regular or subregular.

(i) All element of $X$ is a commutative algebraic subalgebra of $g$.

(ii) For $x$ in $g$, the set of elements of $G.X$ containing $x$ has dimension at most $\dim g^r - \ell$.

(iii) The sets $X \setminus B.h$ and $G.X \setminus G.h$ are equidimensional of dimension $n - 1$ and $2n - 1$ respectively.

(iv) The sets $X'$ and $G.X'$ are smooth big open subsets of $X$ and $G.X$ respectively.

This is a main result with respect to the generalized commuting varieties as it will be shown in the next two notes. We recall that an element of $g$ is subregular if its centralizer in $g$ has dimension
Let \( x_{0,k} \) be the closure in \( B^k \) of \( B.b^k \) and let \( \Gamma \) be a desingularization of \( X \) in the category of \( B \)-varieties. Let \( \mathcal{E}_0 \) be the tautological bundle over \( X \) and set:

\[
\mathcal{E}_s := \mathcal{E}_0 \times_X \Gamma, \quad \mathcal{E}_s^{(k)} := \mathcal{E}_s \times_{\Gamma} \cdots \times_{\Gamma} \mathcal{E}_s.
\]

Then \( \mathcal{E}_s^{(k)} \) is a desingularization of \( x_{0,k} \). Set: \( \mathcal{E}_x^{(k)} := \eta^{-1}(\mathcal{E}_s^{(k)}) \). The following theorem is the third main result of this note:

**Theorem 1.3.** The variety \( \mathcal{E}_x^{(k)} \) is irreducible and \( G \times_B \mathcal{E}_s^{(k)} \) is a desingularization of \( \mathcal{E}_x^{(k)} \).

It will be proved in a next note that the normalizations of \( x_{0,k}, \mathcal{E}_x^{(k)} \) and \( \mathcal{E}_x^{(k)} \) are Gorenstein with rational singularities. As a matter of fact, as a consequence, \( x_{0,k} \) is normal.

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### 1.2. Notations

- An algebraic variety is a reduced scheme over \( k \) of finite type.
- For \( V \) a vector space, its dual is denoted by \( V^* \) and the augmentation ideal of its symmetric algebra \( S(V) \) is denoted by \( S_+(V) \). For \( A \) a graded algebra over \( \mathbb{N} \), \( A_+ \) is the ideal generated by the homogeneous elements of positive degree.
- All topological terms refer to the Zariski topology. If \( Y \) is a subset of a topological space \( X \), denote by \( \overline{Y} \) the closure of \( Y \) in \( X \). For \( Y \) an open subset of the algebraic variety \( X \), \( Y \) is called a **big open subset** if the codimension of \( X \setminus Y \) in \( X \) is at least 2. For \( Y \) a closed subset of an algebraic variety \( X \), its dimension is the biggest dimension of its irreducible components and its codimension in \( X \) is the smallest codimension in \( X \) of its irreducible components. For \( X \) an algebraic variety, \( \mathcal{O}_X \) is its structural sheaf, \( k[X] \) is the algebra of regular functions on \( X \) and \( \mathcal{O}(X) \) is the field of rational functions on \( X \) when \( X \) is irreducible.
- For \( X \) an algebraic variety and for \( M \) a sheaf on \( X \), \( \Gamma(V, M) \) is the space of local sections of \( M \) over the open subset \( V \) of \( X \). For \( i \) a nonnegative integer, \( H^i(X, M) \) is the \( i \)-th group of cohomology of \( M \). For example, \( H^0(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) \).

**Lemma 1.4.** [EGAII, Corollaire 5.4.3] Let \( X \) be an irreducible affine algebraic variety and let \( Y \) be a desingularization of \( X \). Then \( H^0(Y, \mathcal{O}_Y) \) is the integral closure of \( k[X] \) in its fraction field.

- For \( K \) a group and for \( E \) a set with a group action of \( K \), \( E^K \) is the set of invariant elements of \( E \) under \( K \). The following lemma is straightforward and will be used in the proof of Corollary 2.23.

**Lemma 1.5.** Let \( A \) be an algebra generated by the subalgebras \( A_1 \) and \( A_2 \). Let \( K \) be a group acting on \( A_2 \). Suppose that the following conditions are verified:

1. \( A_1 \cap A_2 \) is contained in \( A_2^K \).
2. \( A \) is a free \( A_2 \)-module having a basis contained in \( A_1 \).
3. \( A_1 \) is a free \( A_1 \cap A_2 \)-module having the same basis.

Then there exists a unique group action of \( K \) on the algebra \( A \) extending the action of \( K \) on \( A_2 \) and fixing all the elements of \( A_1 \). Moreover, if \( A_1 \cap A_2 = A_2^K \) then \( A^K = A_1 \).

- For \( E \) a finite set, its cardinality is denoted by \( |E| \). For \( E \) a vector space and for \( x = (x_1, \ldots, x_k) \) in \( E^k \), \( E_x \) is the subspace of \( E \) generated by \( x_1, \ldots, x_k \). Moreover, there is a canonical action of
GLₜ(𝕜) in Eₜ given by:

\[(a_{i,j}, 1 \leq i, j \leq k).(x₁, \ldots, xₖ) := (\sum_{j=1}^{k} a_{i,j}x_{j}, i = 1, \ldots, k)\]

In particular, the diagonal action of G in gₜ commutes with the action of GLₜ(𝕜).

- For a reductive Lie algebra, its rank is denoted by rkₜ and the dimension of its Borel subalgebras is denoted by bₜ. In particular, dima = 2bₜ − rkₜ.
- If E is a subset of a vector space V, denote by span(E) the vector subspace of V generated by E. The grassmanian of all d-dimensional subspaces of V is denoted by Grₜ(V). By definition, a cone of V is a subset of V invariant under the natural action of 𝕜ⁿ := 𝕜 \ {0} and a multicone of Vₜ is a subset of Vₜ invariant under the natural action of (𝕜ⁿ)ₜ on Vₜ.

**Lemma 1.6.** Let X be an open cone of V and let S be a closed multicone of X × Vₜ⁻¹. Denoting by S’ the image of S by the first projection, S’ × {0} = S ∩ (X × {0}). In particular, S’ is closed in X.

**Proof.** For x in X, x is in S’ if and only if for some (v₂, ..., vₖ) in Vₜ⁻¹, (x, tv₂, ..., tvₖ) is in S for all t in 𝕜 since S is a closed multicone of X × Vₜ⁻¹, whence the lemma. □

- The dual g* of g identifies with g by a given non degenerate, invariant, symmetric bilinear form ⟨., .⟩ on g × g extending the Killing form of [g, g].
- Let 𝕜 be the root system of h in g and 𝕜₋ the positive root system of 𝕜 defined by b. The Weyl group of 𝕜 is denoted by W(𝕜) and the basis of 𝕜₋ is denoted by Π. The neutral elements of G and W(𝕜) are denoted by 1g and 1b respectively. For α in 𝕜, the corresponding root subspace is denoted by gα and a generator xα of gα is chosen so that ⟨xα, xα⟩ = 1 for all α in 𝕜.
- The normalizers of b and h in G are denoted by B and N_G(b) respectively. For x in b, h is the element of h such that x − h is in the nilpotent radical u of b.
- For X an algebraic B-variety, denote by G ×_b X the quotient of G × X under the right action of B given by (g, x).b := (gb, b⁻¹.x). More generally, for k positive integer and for X an algebraic Bᵏ-variety, denote by Gᵏ ×_b X the quotient of Gᵏ × X under the right action of Bᵏ given by (g, x).b := (gb, b⁻¹.x) with g and b in Gᵏ and Bᵏ respectively.

**Lemma 1.7.** Let P and Q be parabolic subgroups of G such that P is contained in Q. Let X be a Q-variety and let Y be a closed subset of X, invariant under P. Then Q.Y is a closed subset of X. Moreover, the canonical map from Q ×ᵢ P Y to Q.Y is a projective morphism.

**Proof.** Since P and Q are parabolic subgroups of G and since P is contained in Q, Q/P is a projective variety. Denote by Q ×ᵢ P X and Q ×ᵢ P Y the quotients of Q × X and Q × Y under the right action of P given by (g, x).p := (gp, p⁻¹.x). Let g ↦→ g be the quotient map from Q to Q/P. Since X is a Q-variety, the map

\[Q × X \longrightarrow Q/P × X, \quad (g, x) ↦→ (g, g.x)\]

defines through the quotient an isomorphism from Q ×ᵢ P X to Q/P × X. Since Y is a P-invariant closed subset of X, Q ×ᵢ P Y is a closed subset of Q ×ᵢ P X and its image by the above isomorphism is closed. Hence Q.Y is a closed subset of X since Q/P is a projective variety. From the commutative
we deduce that the map $Q \times_p Y \to Q \cdot Y$ is a projective morphism. □

- For $k \geq 1$ and for the diagonal action of $B$ in $b^k$, $b^k$ is a $B$-variety. The image of $(g, x_1, \ldots, x_k)$ in $G \times b^k$ in $G \times b^k$ is denoted by $(g, x_1, \ldots, x_k)$. The sets $B(k)$ and $N(k)$ are the images of $G \times b^k$ and $G \times u^k$ respectively by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$ so that $B(k)$ and $N(k)$ are closed subsets of $g^k$ by Lemma 1.7. Let $B_n(k)$ be the normalization of $B(k)$ and let $\eta_n$ be the normalization morphism. The map

$$G \times b^k \to B(k), \quad (g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$$

defines through the quotient a morphism $\gamma : G \times_B b^k \to B(k)$ and we have the commutative diagram:

$$
\begin{array}{ccc}
G \times_B b^k & \xrightarrow{\gamma_n} & B_n(k) \\
\downarrow{\gamma} & & \downarrow{\eta_n} \\
B(k) & & \\
\end{array}
$$

where $\gamma_n$ is uniquely defined by this diagram. Let $N_n(k)$ be the normalization of $N(k)$ and let $\kappa$ be the normalization morphism. We have the commutative diagram:

$$
\begin{array}{ccc}
G \times_B u^k & \xrightarrow{\nu_n} & N_n(k) \\
\downarrow{\nu} & & \downarrow{\kappa} \\
N(k) & & \\
\end{array}
$$

with $\nu$ the restriction of $\gamma$ to $G \times_B u^k$ and $\nu_n$ is uniquely defined by this diagram.

- Let $i$ be the injection $(x_1, \ldots, x_k) \mapsto (1, x_1, \ldots, x_k)$ from $b^k$ to $G \times_B b^k$. Then $i := \gamma \circ i$ is the identity of $b^k$ and $i_n := \gamma_n \circ i$ is a closed embedding of $b^k$ into $B_n(k)$. In particular, $B(k) = G.t(B)$ and $B_n(k) = G.t_n(b^k)$.

- Let $e$ be the sum of the $x\beta \,' s$, $\beta$ in $\Pi$, and let $h$ be the element of $\mathfrak{h} \cap [g, g]$ such that $\beta(h) = 2$ for all $\beta$ in $\Pi$. Then there exists a unique $f$ in $[g, g]$ such that $(e, h, f)$ is a principal $\mathfrak{s}_2$-triple. The one-parameter subgroup of $G$ generated by $ad\ h$ is denoted by $t \mapsto h(t)$. The Borel subalgebra containing $f$ is denoted by $b_-$ and its nilpotent radical is denoted by $u_-$. Let $B_-$ be the normalizer of $b_-$ in $G$ and let $U$ and $U_-$ be the unipotent radicals of $B$ and $B_-$ respectively.

**Lemma 1.8.** Let $k \geq 2$ be an integer. Let $X$ be an affine variety and set $Y := b^k \times X$. Let $Z$ be a closed $B$-invariant subset of $Y$ under the group action given by $g.(v_1, \ldots, v_k, x) = (g(v_1), \ldots, g(v_k), x)$ with $(g, v_1, \ldots, v_k)$ in $B \times b^k$ and $x$ in $X$. Then $Z \cap b^k \times X$ is the image of $Z$ by the projection $(v_1, \ldots, v_k, x) \mapsto (\overline{v_1}, \ldots, \overline{v_k}, x)$.

**Proof.** For all $v$ in $b$,

$$\overline{v} = \lim_{t \to 0} h(t)(v)$$

whence the lemma since $Z$ is closed and $B$-invariant. □
• For $x \in \mathfrak{g}$, let $x_s$ and $x_n$ be the semisimple and nilpotent components of $x$ in $\mathfrak{g}$. Denote by $\mathfrak{g}^x$ and $G^x$ the centralizers of $x$ in $\mathfrak{g}$ and $G$ respectively. For $a$ a subalgebra of $\mathfrak{g}$ and for $A$ a subgroup of $G$, set:

$$a^x := a \cap \mathfrak{g}^x \quad A^x := A \cap G^x$$

The set of regular elements of $\mathfrak{g}$ is

$$\mathfrak{g}_{\text{reg}} := \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = \ell\}$$

and denote by $\mathfrak{g}_{\text{reg}, \text{ss}}$ the set of regular semisimple elements of $\mathfrak{g}$. Both $\mathfrak{g}_{\text{reg}}$ and $\mathfrak{g}_{\text{reg}, \text{ss}}$ are $G$-invariant dense open subsets of $\mathfrak{g}$. Setting $b_{\text{reg}} := b \cap \mathfrak{g}_{\text{reg}}$, $b_{\text{reg}} := b \cap \mathfrak{g}_{\text{reg}}$, $u_{\text{reg}} := u \cap \mathfrak{g}_{\text{reg}}$, $\mathfrak{g}_{\text{reg}, \text{ss}} = G(b_{\text{reg}})$, $\mathfrak{g}_{\text{reg}} = G(b_{\text{reg}})$ and $G(u_{\text{reg}})$ is the set of regular elements of the nilpotent cone $\mathfrak{N}_\mathfrak{g}$ of $\mathfrak{g}$.

**Lemma 1.9.** Let $k \geq 2$ be an integer and let $x$ be in $\mathfrak{g}^k$. For $O$ open subset of $\mathfrak{g}_{\text{reg}}$, $E_x \cap O$ is not empty if and only if for some $g$ in $\text{GL}_k(\mathbb{k})$, the first component of $g.x$ is in $O$.

**Proof.** Since the components of $g.x$ are in $E_x$ for all $g$ in $\text{GL}_k(\mathbb{k})$, the condition is sufficient. Suppose that $E_x \cap O$ is not empty and denote by $x_1, \ldots, x_k$ the components of $x$. For some $(a_1, \ldots, a_k)$ in $\mathbb{k}^k \setminus \{0\}$,

$$a_1x_1 + \cdots + a_kx_k \in O$$

Let $i$ be such that $a_i \neq 0$ and let $\tau$ be the transposition such that $\tau(1) = i$. Denoting by $g$ the element of $\text{GL}_k(\mathbb{k})$ such that $g_{1,j} = a_{\tau(j)}$ for $j = 1, \ldots, k$, $g_{j,j} = 1$ for $j = 2, \ldots, k$ and $g_{j,l} = 0$ for $j \geq 2$ and $j \neq \ell$, the first component of $g.\tau.x$ is in $O$. \hfill \Box

• Denote by $S(\mathfrak{g})^g$ the algebra of $\mathfrak{g}$-invariant elements of $S(\mathfrak{g})$. Let $p_1, \ldots, p_\ell$ be homogeneous generators of $S(\mathfrak{g})^g$ of degree $d_1, \ldots, d_\ell$ respectively. Choose the polynomials $p_1, \ldots, p_\ell$ so that $d_1 \leq \cdots \leq d_\ell$. For $i = 1, \ldots, \ell$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$, consider a shift of $p_i$ in the direction $y$: $p_i(x + ty)$ with $t \in \mathbb{k}$. Expanding $p_i(x + ty)$ as a polynomial in $t$, we obtain

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x, y)t^m; \quad \forall (t, x, y) \in \mathbb{k} \times \mathfrak{g} \times \mathfrak{g}$$

where $y \mapsto (m)!p_i^{(m)}(x, y)$ is the derivative at $x$ of $p_i$ at the order $m$ in the direction $y$. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $S(\mathfrak{g}) \otimes_\mathbb{k} S(\mathfrak{g})$ under the diagonal action of $G$ in $\mathfrak{g} \times \mathfrak{g}$. Remark that $p_i^{(0)}(x, y) = p_i(x)$ while $p_i^{(d_i)}(x, y) = p_i(y)$ for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

**Remark 1.10.** The family $\mathcal{P}_x := \{p_i^{(m)}(x, \cdot) ; 1 \leq i \leq \ell, 1 \leq m \leq d_i\}$ for $x \in \mathfrak{g}$, is a Poisson-commutative family of $S(\mathfrak{g})$ by Mishchenko-Fomenko [MF78]. We say that the family $\mathcal{P}_x$ is constructed by the argument shift method.

• Let $i \in \{1, \ldots, \ell\}$. For $x$ in $\mathfrak{g}$, denote by $\mathcal{E}_i(x)$ the element of $\mathfrak{g}$ given by

$$\langle \mathcal{E}_i(x), y \rangle = \frac{d}{dt} p_i(x + ty) |_{t=0}$$

for all $y$ in $\mathfrak{g}$. Thereby, $\mathcal{E}_i$ is an invariant element of $S(\mathfrak{g}) \otimes_\mathbb{k} \mathfrak{g}$ under the canonical action of $G$. According to [Ko63, Theorem 9], for $x$ in $\mathfrak{g}$, $x$ is in $\mathfrak{g}_{\text{reg}}$ if and only if $\mathcal{E}_1(x), \ldots, \mathcal{E}_\ell(x)$ are linearly independent. In this case, $\mathcal{E}_1(x), \ldots, \mathcal{E}_\ell(x)$ is a basis of $\mathfrak{g}^x$.

Denote by $\mathfrak{z}_\mathfrak{g}$ the center of $\mathfrak{g}$ and for $x$ in $\mathfrak{g}$ by $\mathfrak{z}_x$ the center of $\mathfrak{g}^x$. As $\mathcal{E}_1, \ldots, \mathcal{E}_\ell$ are invariant, for all $x$ in $\mathfrak{g}$, $\mathcal{E}_1(x), \ldots, \mathcal{E}_\ell(x)$ are in $\mathfrak{z}_x$. 6
Denote by $\varepsilon_i^{(m)}$, for $0 \leq m \leq d_i - 1$, the elements of $S(\mathfrak{g} \times \mathfrak{g}) \otimes_{k} \mathfrak{g}$ defined by the equality:

$$(2) \quad \varepsilon_i(x + ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x, y)t^m, \quad \forall (t, x, y) \in k \times \mathfrak{g} \times \mathfrak{g}$$

and set:

$$V_{x,y} := \text{span} \{ \varepsilon_i^{(0)}(x, y), \ldots, \varepsilon_i^{(d_i-1)}(x, y), \ i = 1, \ldots, \ell \}$$

for $(x, y) \in \mathfrak{g} \times \mathfrak{g}$. According to [Bol91, Corollary 2], $V_{x,y}$ has dimension $b_{\mathfrak{g}}$ if and only if $E_{x,y}$ has dimension 2 and $E_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\text{reg}}$.

2. On the varieties $\mathcal{B}^{(k)}$

Let $k \geq 2$ be an integer. According to the above notations, we have the commutative diagrams:

Since the Borel subalgebras of $\mathfrak{g}$ are conjugate under $G$, $\mathcal{B}^{(k)}$ is the subset of elements of $\mathfrak{g}^k$ whose components are in a same Borel subalgebra and $\mathcal{N}^{(k)}$ are the elements of $\mathcal{B}^{(k)}$ whose all the components are nilpotent.

**Lemma 2.1.** (i) The morphism $\gamma$ from $G \times_B b^k$ to $\mathcal{B}^{(k)}$ is projective and birational. In particular, $G \times_B b^k$ is a desingularization of $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(k)}$ has dimension $kb_{\mathfrak{g}} + n$.

(ii) The morphism $\nu$ from $G \times_B u^k$ to $\mathcal{N}^{(k)}$ is projective and birational. In particular, $G \times_B u^k$ is a desingularization of $\mathcal{N}^{(k)}$ and $\mathcal{N}^{(k)}$ has dimension $(k + 1)n$.

**Proof.** (i) Denote by $\Omega_{\mathfrak{g}}^{(2)}$ the subset of elements $(x, y)$ of $\mathfrak{g}^2$ such that $E_{x,y}$ has dimension 2 and such that $E_{x,y} \setminus \{0\}$ is contained in $\mathfrak{g}_{\text{reg}}$. According to Lemma 1.7, $\gamma$ is a projective morphism. For $1 \leq i < j \leq k$, let $\Omega_{i,j}^{(k)}$ be the inverse image of $\Omega_{\mathfrak{g}}^{(2)}$ by the projection

$$(x_1, \ldots, x_k) \mapsto (x_i, x_j)$$

Then $\Omega_{i,j}^{(k)}$ is an open subset of $\mathfrak{g}^k$ whose intersection with $\mathcal{B}^{(k)}$ is not empty. Let $\Omega_{\mathfrak{g}}^{(k)}$ be the union of the $\Omega_{i,j}^{(k)}$. According to [Bol91, Corollary 2] and [Ko63, Theorem 9], for $(x, y)$ in $\Omega_{\mathfrak{g}}^{(2)} \cap \mathcal{B}^{(2)}$, $V_{x,y}$ is the unique Borel subalgebra of $\mathfrak{g}$ containing $x$ and $y$ so that the restriction of $\gamma$ to $\gamma^{-1}(\Omega_{\mathfrak{g}}^{(k)})$ is a bijection onto $\Omega_{\mathfrak{g}}^{(k)}$. Hence $\gamma$ is birational. Moreover, $G \times_B b^k$ is a smooth variety as a vector bundle over the smooth variety $G/B$, whence the assertion since $G \times_B b^k$ has dimension $kb_{\mathfrak{g}} + n$.

(ii) According to Lemma 1.7, $\nu$ is a projective morphism. Let $\mathcal{N}_{\text{reg}}^{(k)}$ be the subset of elements of $\mathcal{N}^{(k)}$ whose at least one component is a regular element of $\mathfrak{g}$. Then $\mathcal{N}_{\text{reg}}^{(k)}$ is an open subset of $\mathcal{N}^{(k)}$. Since a regular nilpotent element is contained in one and only one Borel subalgebra of $\mathfrak{g}$, the restriction of $\nu$ to $\nu^{-1}(\mathcal{N}_{\text{reg}}^{(k)})$ is a bijection onto $\mathcal{N}_{\text{reg}}^{(k)}$. Hence $\nu$ is birational. Moreover, $G \times_B u^k$ is a smooth variety as a vector bundle over the smooth variety $G/B$, whence the assertion since $G \times_B u^k$ has dimension $(k + 1)n$.  \qed
2.1. Let $\kappa$ be the map 
\[ U_\kappa \times u_{\text{reg}} \to \mathfrak{g}_{\text{reg}} \quad (g, x) \mapsto g(x) \]

**Lemma 2.2.** Let $V$ be the set of elements of $\mathcal{N}^{(k)}$ whose first component is in $U_{\kappa}(u_{\text{reg}})$ and let $V_k$ be the set of elements $x$ of $\mathcal{N}^{(k)}$ such that $E_x \cap \mathfrak{g}_{\text{reg}}$ is not empty.

(i) The image of $\kappa$ is a smooth open subset of $\mathfrak{g}_{\text{reg}}$ and $\kappa$ is an isomorphism onto $U_{\kappa}(u_{\text{reg}})$.
(ii) The subset $V$ of $\mathcal{N}^{(k)}$ is open.
(iii) The open subset $V$ of $\mathcal{N}^{(k)}$ is smooth.
(iv) The set $V_k$ is a smooth open subset of $\mathcal{N}^{(k)}$.

**Proof.** (i) Since $\mathfrak{g}_{\text{reg}}$ is the nullvariety of $p_1, \ldots, p_{\ell}$ in $\mathfrak{g}$, $\mathfrak{g}_{\text{reg}} \cap \mathfrak{g}_{\text{reg}}$ is a smooth open subset of $\mathfrak{g}_{\text{reg}}$ by [Ko63, Theorem 9]. For $(g, x)$ in $U_\kappa \times u_{\text{reg}}$ such that $g(x)$ is in $u$, $b^{-1} g$ is in $G^\kappa$ for some $b$ in $B$ since $B(x) = u_{\text{reg}}$. Hence $g = 1_\mathfrak{g}$ since $G^\kappa$ is contained in $B$ and since $U_\kappa \cap B = \{1_\mathfrak{g}\}$. As a result, $\kappa$ is an injective morphism from the smooth variety $U_\kappa \times u_{\text{reg}}$ to the smooth variety $\mathfrak{g}_{\text{reg}} \cap \mathfrak{g}_{\text{reg}}$. Hence $\kappa$ is an open immersion by Zariski’s Main Theorem [Mu88, §9].

(ii) By definition, $V$ is the intersection of $\mathcal{N}^{(k)}$ and $U_{\kappa}(u_{\text{reg}}) \times \mathfrak{g}_{\text{reg}}^{k-1}$. So, by (i), it is an open subset of $\mathcal{N}^{(k)}$.

(iii) Let $(x_1, \ldots, x_{k})$ be in $u^k$ and let $g$ be in $G$ such that $(g(x_1), \ldots, g(x_{k}))$ is in $V$. Then $x_i$ is in $u_{\text{reg}}$ and for some $(g', b)$ in $U_\kappa \times B$, $g' b(x_i) = g(x_i)$. Hence $g^{-1} g' b$ is in $G^{x_i}$ and $g$ is in $U_{\kappa}$ since $G^{x_i}$ is contained in $B$. As a result, the map 
\[ U_\kappa \times u_{\text{reg}} \times u^k \to V \quad (g, x_1, \ldots, x_{k}) \mapsto (g(x_1), \ldots, g(x_{k})) \]

is an isomorphism whose inverse is given by 
\[ V \to U_\kappa \times u_{\text{reg}} \times u^k \quad (x_1, \ldots, x_{k}) \mapsto (\kappa^{-1}(x_1)_1, \kappa^{-1}(x_1)_2, \ldots, \kappa^{-1}(x_{k})) \]

with $\kappa^{-1}$ the inverse of $\kappa$ and $\kappa^{-1}(x_1)_1$ the component of $\kappa^{-1}(x_1)$ on $U_\kappa$, whence the assertion since $U_\kappa \times u_{\text{reg}} \times u^k$ is smooth.

(iv) According to Lemma 1.9, $V_k = \text{GL}_{k}(k). V$, whence the assertion by (iii). \[
\square
\]

**Corollary 2.3.** (i) The subvariety $\mathcal{N}^{(k)} \setminus V_k$ of $\mathcal{N}^{(k)}$ has codimension $k + 1$.

(ii) The restriction of $\nu$ to $\nu^{-1}(V_k)$ is an isomorphism onto $V_k$.

(iii) The subset $\nu^{-1}(V_k)$ is a big open subset of $G \times_B u^k$.

**Proof.** (i) By definition, $\mathcal{N}^{(k)} \setminus V_k$ is the subset of elements $x$ of $\mathcal{N}^{(k)}$ such that $E_x$ is contained in $\mathfrak{g} \setminus \mathfrak{g}_{\text{reg}}$. Hence $\mathcal{N}^{(k)} \setminus V_k$ is contained in the image of $G \times_B (u \setminus u_{\text{reg}})^k$ by $\nu$. Let $(x_1, \ldots, x_{k})$ be in $u^k \cap (\mathcal{N}^{(k)} \setminus V_k)$. Then, for all $(a_1, \ldots, a_{k})$ in $k^k$, 
\[ \langle x_\beta, a_1 x_1 + \cdots + a_{k} x_{k} \rangle = 0 \]

for some $\beta$ in $\Pi$. Since $\Pi$ is finite, $E_x$ is orthogonal to $x_\beta$ for some $\beta$ in $\Pi$. As a result, the subvariety of Borel subalgebras of $\mathfrak{g}$ containing $x_1, \ldots, x_{k}$ has positive dimension. Hence 
\[ \dim(\mathcal{N}^{(k)} \setminus V_k) < \dim G \times_B (u \setminus u_{\text{reg}})^k = n + k(n - 1) \]

Moreover, for $\beta$ in $\Pi$, denoting by $u_\beta$ the orthogonal complement of $\mathfrak{g}^{-\beta}$ in $u$, $\nu(G \times_B (u_\beta)^k)$ is contained in $\mathcal{N}^{(k)} \setminus V_k$ and its dimension equal $(k + 1)(n - 1)$ since the variety of Borel subalgebras containing $u_\beta$ has dimension 1, whence the assertion.

(ii) For $x$ in $\mathcal{N}^{(k)}$, $E_x$ is contained in all Borel subalgebra of $\mathfrak{g}$, containing the components of $x$. Then the restriction of $\nu$ to $\nu^{-1}(V_k)$ is injective since all regular nilpotent element of $\mathfrak{g}$ is contained in a single Borel subalgebra of $\mathfrak{g}$, whence the assertion by Zariski’s Main Theorem [Mu88, §9] since $V_k$ is a smooth open subset of $\mathcal{N}^{(k)}$ by Lemma 2.2,(iv).
(iii) Identify \( U_- \) with the open subset \( U_- B / B \) of \( G / B \) and denote by \( \pi_0 \) the bundle projection from \( G \times_B u^k \to G / B \). Since \( \nu^{-1}(V_k) \) is \( G \)-invariant, it suffices to prove that \( \nu^{-1}(V_k) \cap \pi_0^{-1}(U_-) \) is a big open subset of \( \pi_0^{-1}(U_-) \). Let \( V_0 \) be the subset of elements \( x \) of \( u^k \) such that \( E_x \cap g_{\text{reg}} \) is not empty. Then \( u^k \setminus V_0 \) is contained in \( (u \setminus u_{\text{reg}})^k \) and has codimension at least 2 in \( u^k \) since \( k \geq 2 \). As a result, \( U_- \times V_0 \) is a big open subset of \( U_- \times u^k \). The open subset \( \pi_0^{-1}(U_-) \) of \( G \times_B u^k \) identifies with \( U_- \times u^k \) and \( \nu^{-1}(V_k) \cap \pi_0^{-1}(U_-) \) identifies with \( U_- \times V_0 \), whence the assertion. \( \square \)

2.2. Denote by \( \pi_0 : g \to g / G \) and \( \pi_0 : h \to h / W(\mathbb{R}) \) the quotient maps, i.e. the morphisms defined by the invariants. Recall \( g / G = h / W(\mathbb{R}) \), and let \( \overline{\mathcal{X}} \) be the following fiber product:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\overline{\nu}} & g \\
\pi_0 & \downarrow & \\
\mathfrak{b} & \xrightarrow{\pi_0} & h / W(\mathbb{R})
\end{array}
\]

where \( \overline{\mathcal{X}} \) and \( \overline{\mathfrak{b}} \) are the restriction maps. The actions of \( G \) and \( W(\mathbb{R}) \) on \( g \) and \( h \) respectively induce an action of \( G \times W(\mathbb{R}) \) on \( \mathcal{X} \): \( (g, w). (x, y) := (g(x), w(y)) \).

Lemma 2.4. (i) There exists a well defined \( G \)-equivariant morphism \( \chi_n \) from \( G \times_B \mathfrak{b} \) to \( \overline{\mathcal{X}} \) such that \( \gamma \) is the composition of \( \chi_n \) and \( \overline{\mathcal{X}} \).

(ii) The morphism \( \chi_n \) is projective and birational. Moreover, \( \mathcal{X} \) is irreducible.

(iii) The subscheme \( \mathcal{X} \) is normal. Moreover, every element of \( g_{\text{reg}} \times h \cap \mathcal{X} \) is a smooth point of \( \mathcal{X} \).

(iv) The algebra \( k[\mathcal{X}] \) is the space of global sections of \( \mathcal{O}_{G \times_B \mathfrak{b}} \) and \( k[\mathcal{X}]^G = S(\mathfrak{b}) \).

Proof. (i) Since the map \( (g, x) \mapsto (g(x), \overline{x}) \) is constant on the \( B \)-orbits, there exists a uniquely defined morphism \( \chi_n \) from \( G \times_B \mathfrak{b} \) to \( g \times \overline{h} \) such that \( (g(x), \overline{x}) \) is the image of \( \chi_n \) of the image of \( (g, x) \) in \( G \times_B \mathfrak{b} \). The image of \( \chi_n \) is contained in \( \mathcal{X} \) since for all \( p \) in \( S(g)^G \), \( p(\overline{x}) = p(x) = p(g(x)) \). Furthermore, \( \chi_n \) verifies the condition of the assertion.

(ii) According to Lemma 1.7, \( \chi_n \) is a projective morphism. Let \( (x, y) \) be in \( g \times h \) such that \( p(x) = p(y) \) for all \( p \) in \( S(g)^G \). For some \( g \) in \( G \), \( g(x) \) is in \( h \) and its semisimple component is \( y \) so that \( (x, y) \) is in the image of \( \chi_n \). As a result, \( \mathcal{X} \) is irreducible as the image of the irreducible variety \( G \times_B \mathfrak{b} \). Since for all \( (x, y) \) in \( \mathcal{X} \cap h_{\text{reg}} \times h_{\text{reg}} \), there exists a unique \( w \) in \( W(\mathbb{R}) \) such that \( y = w(x) \), the fiber of \( \chi_n \) at any element \( \mathcal{X} \cap G . (h_{\text{reg}} \times h_{\text{reg}}) \) has one element. Hence \( \chi_n \) is birational, whence the assertion.

(iii) The morphism \( \pi_0 \) is finite, and so is \( \overline{\mathcal{X}} \). Moreover \( \pi_0 \) is smooth over \( h_{\text{reg}} \), \( \overline{\mathcal{X}} \) is smooth over \( g_{\text{reg}} \). Finally, \( \pi_0 \) is flat and all fibers are normal and Cohen-Macaulay. Thus the same holds for the morphism \( \overline{\mathfrak{b}} \). Since \( h \) is smooth this implies that \( \mathcal{X} \) is normal and Cohen-Macaulay by [MA86, Ch. 8, §23].

(iv) According to (ii), (iii) and Lemma 1.4, \( k[\mathcal{X}] = H^0(G \times_B \mathfrak{b}, \mathcal{O}_{G \times_B \mathfrak{b}}) \). Under the action of \( G \) in \( g \times h \), \( k[\mathfrak{g} \times h]^G = S(g)^G \otimes_k S(\mathfrak{b}) \) and its image in \( k[\mathcal{X}] \) by the quotient morphism is equal to \( S(\mathfrak{b}) \). Moreover, since \( G \) is reductive, \( k[\mathcal{X}]^G \) is the image of \( k[\mathfrak{g} \times h]^G \) by the quotient morphism, whence the assertion. \( \square \)

Proposition 2.5. [He76, Theorem B and Corollary] The variety \( \mathcal{X} \) has rational singularities.

Corollary 2.6. (i) Let \( x \) and \( x' \) be in \( h_{\text{reg}} \) such that \( x' \) is in \( G . x \) and \( \overline{x} = \overline{x} \). Then \( x' \) is in \( B(x) \).

(ii) For all \( w \) in \( W(\mathbb{R}) \), the map

\[
U_- \times h_{\text{reg}} \to \mathcal{X}, \quad (g, x) \mapsto (g(x), w(\overline{x}))
\]

is an isomorphism onto a smooth open subset of $\mathcal{X}$.

**Proof.** (i) The semisimple components of $x$ and $x'$ are conjugate under $B$ since they are conjugate to $\mathcal{X}$ under $B$. Let $b$ and $b'$ be in $B$ such that $\mathcal{X}$ is the semisimple component of $b(x)$ and $b'(x')$. Then the nilpotent components of $b(x)$ and $b'(x')$ are regular nilpotent elements of $\mathfrak{g}^\mathcal{X}$, belonging to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^\mathcal{X}$ of $\mathfrak{g}^\mathcal{X}$. Hence $x'$ is in $B(x)$ since regular nilpotent elements of a Borel subalgebra of a reductive Lie algebra are conjugate under the corresponding Borel subgroup.

(ii) Since the action of $G$ and $W(\mathcal{R})$ on $\mathcal{X}$ commute, it suffices to prove the assertion for $w = 1_b$. Denote by $\theta$ the map

$$U_- \times b_{\text{reg}} \longrightarrow \mathcal{X}, \quad (g, x) \longmapsto (g(x), \mathcal{X}).$$

Let $(g, x)$ and $(g', x')$ be in $U_- \times b_{\text{reg}}$ such that $\theta(g, x) = \theta(g', x')$. By (i), $x' = b(x)$ for some $b$ in $B$. Hence $g^{-1}g'$ is in $G^x$. Since $x$ is in $b_{\text{reg}}$, $G^x$ is contained in $B$ and $g^{-1}g'$ is in $U_- \cap B$, whence $(g, x) = (g', x')$ since $U_- \cap B = \{1_b\}$. As a result, $\theta$ is a dominant injective map from $U_- \times b_{\text{reg}}$ to the normal variety $\mathcal{X}$. Hence $\theta$ is an isomorphism onto a smooth open subset of $\mathcal{X}$, by Zariski’s Main Theorem [Mu88, §9]. \qed

2.3. According to Lemma 2.1,(i), $G \times_B b^k$ is a desingularization of $\mathcal{B}^{(k)}$ and we have the commutative diagram:

$$\begin{array}{ccc}
G \times_B b^k & \longrightarrow & \mathcal{B}^{(k)} \\
\gamma & \downarrow & \downarrow \eta_n \\
\mathcal{B}^{(k)} & \longrightarrow & \mathcal{B}^{(k)}
\end{array}$$

**Lemma 2.7.** Let $\sigma$ be the canonical projection from $\mathcal{X}^k$ to $\mathfrak{g}^k$. Denote by $\iota_k$ the map

$$b^k \longrightarrow \mathfrak{g}^k, \quad (x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k, \overline{x_1}, \ldots, \overline{x_k}).$$

(i) The map $\iota_k$ is a closed embedding of $b^k$ into $\mathfrak{g}^k$.

(ii) The subvariety $\iota_k(b^k)$ of $\mathfrak{g}^k$ is an irreducible component of $\sigma^{-1}(b^k)$.

(iii) The subvariety $\sigma^{-1}(b^k)$ of $\mathfrak{g}^k$ is invariant under the canonical action of $W(\mathcal{R})^k$ in $\mathfrak{g}^k$ and this action induces a simply transitive action of $W(\mathcal{R})^k$ on the set of irreducible components of $\sigma^{-1}(b^k)$.

**Proof.** (i) The map

$$b^k \longrightarrow G^k \times b^k, \quad (x_1, \ldots, x_k) \longmapsto (1_g, \ldots, 1_g, x_1, \ldots, x_k)$$

defines through the quotient a closed embedding of $b^k$ in $G^k \times_B b^k$. Denote it by $\iota'$. Let $\chi_n^{(k)}$ be the map

$$G^k \times_B b^k \longrightarrow \mathfrak{g}^k, \quad (x_1, \ldots, x_k) \longmapsto (\chi_n(x_1), \ldots, \chi_n(x_k)).$$

Then $\iota_k = \chi_n^{(k)} \circ \iota'$. Since $\chi_n$ is a projective morphism, $\iota_k$ is a closed morphism. Moreover, it is an embedding since $\sigma \circ \iota_k$ is the identity of $b^k$.

(ii) Since $S(\mathfrak{b})$ is a finite extension of $S(\mathfrak{b})^{W(\mathcal{R})}$, $\sigma$ is a finite morphism. So $\sigma^{-1}(b^k)$ and $b^k$ have the same dimension. According to (i), $\iota_k(b^k)$ is an irreducible subvariety of $\sigma^{-1}(b^k)$ of the same dimension, whence the assertion.

(iii) Since all the fibers of $\sigma$ are invariant under the action of $W(\mathcal{R})^k$ on $\mathfrak{g}^k$, $\sigma^{-1}(b^k)$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\sigma^{-1}(b^k)$. For $w$ in $W(\mathcal{R})^k$, set
$Z_w := w.\iota_k(b^k)$. If $Z_w = \iota_k(b^k)$, then for all $(x_1, \ldots, x_k)$ in $b^k_{\text{reg}}$, $(x_1, \ldots, x_k, w.(x_1, \ldots, x_k))$ is in $\iota_k(b^k)$ so that $(x_1, \ldots, x_k)$ is invariant under $w$ and $w$ is the identity.

Let $Z$ be an irreducible component of $\sigma^{-1}(b^k)$ and let $Z_0$ be its image by the map

$$(x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto (\overline{x}_1, \ldots, \overline{x}_k, y_1, \ldots, y_k).$$

Since $\sigma$ is $G^k$-equivariant and $b^k$ is invariant under $B^k$, $\sigma^{-1}(b^k)$ and $Z$ are invariant under $B^k$. Hence by Lemma 1.8, $Z_0$ is closed. Moreover, since the image of the map

$Z_0 \times u^k \to \mathcal{X}^k, \quad ((x_1, \ldots, x_k, y_1, \ldots, y_k), (u_1, \ldots, u_k)) \mapsto (x_1 + u_1, \ldots, x_k + u_k, y_1, \ldots, y_k)$

is an irreducible subset of $\sigma^{-1}(b^k)$ containing $Z$, $Z$ is the image of this map. Since $Z_0$ is contained in $\mathcal{X}^k$, $Z_0$ is contained in the image of the map

$b^k \times W(\mathcal{R})^k \to b^k \times b^k, \quad (x_1, \ldots, x_k, w_1, \ldots, w_k) \mapsto (x_1, \ldots, x_k, w_1(x_1), \ldots, w_k(x_k))$.

As $W(\mathcal{R})$ is finite and $Z_0$ is irreducible, for some $w$ in $W(\mathcal{R})^k$, $Z_0$ is the image of $b^k$ by the map

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, w.(x_1, \ldots, x_k))$$

and $Z = Z_w$, whence the assertion. □

Set $\mathcal{Y} := G^k \times_{G^k} b^k$. The map

$$G \times b^k \to G^k \times b^k, \quad (g, v_1, \ldots, v_k) \mapsto (g, g, v_1, \ldots, v_k)$$

defines through the quotient a closed immersion from $G \times_B b^k$ to $\mathcal{Y}$. Denote it by $\nu$. Consider the diagonal action of $G$ on $g^k$ and $\mathcal{X}^k$: $g.(x_1, \ldots, x_k, y_1, \ldots, y_k) := (g(x_1), \ldots, g(x_k), y_1, \ldots, y_k)$, and identify $G \times_B b^k$ with $\nu(G \times_B b^k)$ by the closed immersion $\nu$.

**Corollary 2.8.** Set $\mathcal{B}^{(k)}_k := G.\iota_k(b^k)$.

(i) The subset $\mathcal{B}^{(k)}_k$ is the image of $G \times_B b^k$ by $\chi^{(k)}_n$. Moreover, the restriction of $\chi^{(k)}_n$ to $G \times_B b^k$ is a projective birational morphism from $G \times_B b^k$ onto $\mathcal{B}^{(k)}_k$.

(ii) The subset $\mathcal{B}^{(k)}_k$ of $\mathcal{X}^k$ is an irreducible component of $\sigma^{-1}(\mathcal{B}^{(k)})$.

(iii) The subvariety $\sigma^{-1}(\mathcal{B}^{(k)})$ of $\mathcal{X}^k$ is invariant under $W(\mathcal{R})^k$ and this action induces a simply transitive action of $W(\mathcal{R})^k$ on the set of irreducible components of $\sigma^{-1}(\mathcal{B}^{(k)})$.

(iv) The subalgebra $k[\mathcal{B}^{(k)}_k]$ of $k[\sigma^{-1}(\mathcal{B}^{(k)})]$ equals $k[\sigma^{-1}(\mathcal{B}^{(k)})]^{W(\mathcal{R})^k}$ with respect to the action of $W(\mathcal{R})^k$ on $\sigma^{-1}(\mathcal{B}^{(k)})$.

**Proof.** (i) Let $\gamma_x$ be the restriction of $\chi^{(k)}_n$ to $G \times_B b^k$. Since $\iota_k = \chi^{(k)}_n \circ u'$, $G \times_B b^k = G.\iota_k(b^k)$ and $\chi^{(k)}_n$ is $G$-equivariant, $\mathcal{B}^{(k)}_k = \gamma_x(G \times_B b^k)$. Hence $\mathcal{B}^{(k)}_k$ is closed in $\mathcal{X}^k$ and $\gamma_x$ is a projective morphism from $G \times_B b^k$ to $\mathcal{B}^{(k)}_k$ since $\chi^{(k)}_n$ is a projective morphism. According to Lemma 2.1,(i), $\sigma \circ \gamma_x$ is a birational morphism onto $\mathcal{B}^{(k)}_k$. Then $\gamma_x$ is birational since $\sigma(\mathcal{B}^{(k)}_k) = \mathcal{B}^{(k)}_k$, whence the assertion.

(ii) Since $\sigma$ is a finite morphism, $\sigma^{-1}(\mathcal{B}^{(k)}_k)$, $\mathcal{B}^{(k)}_k$ and $\mathcal{B}^{(k)}_k$ have the same dimension, whence the assertion since $\mathcal{B}^{(k)}_k$ is irreducible as an image of an irreducible variety.

(iii) Since the fibers of $\sigma$ are invariant under $W(\mathcal{R})^k$, $\sigma^{-1}(\mathcal{B}^{(k)}_k)$ is invariant under this action and $W(\mathcal{R})^k$ permutes the irreducible components of $\sigma^{-1}(\mathcal{B}^{(k)}_k)$. Let $Z$ be an irreducible component of $\sigma^{-1}(\mathcal{B}^{(k)}_k)$. As $\sigma$ is $G^k$-equivariant, $\sigma^{-1}(\mathcal{B}^{(k)}_k)$ and $Z$ are invariant under the diagonal action of $G$. Moreover, $Z = G.(Z \cap \sigma^{-1}(b^k))$ since $\mathcal{B}^{(k)}_k = G.b^k$. Hence for some irreducible component $Z_0$ of $Z \cap \sigma^{-1}(b^k)$, $Z = G.Z_0$. According to Lemma 2.7,(iii), $Z_0$ is contained in $w.\iota_k(b^k)$ for some $w$ in $W(\mathcal{R})^k$. Hence $Z = w.\mathcal{B}^{(k)}_k$ since the actions of $G^k$ and $W(\mathcal{R})^k$ on $\mathcal{X}^k$ commute and $Z$ is an irreducible component of $\sigma^{-1}(\mathcal{B}^{(k)}_k)$. 11
Let \( w=(w_1,\ldots,w_k) \) be in \( W(\mathcal{R})^k \) such that \( w_B^\ell(w) = B^\ell \). Let \( x \) be in \( b_{\text{reg}} \) and let \( i = 1,\ldots,k \). Set:

\[
z := (x_1,\ldots,x_k,\overline{x_1},\ldots,\overline{x_k}) \text{ with } x_j := \begin{cases} x & \text{if } j = i \\ x_j & \text{otherwise} \end{cases}.
\]

Then there exists \((y_1,\ldots,y_k)\) in \( b^k \) and \( g \) in \( G \) such that

\[
w.z = (g(y_1),\ldots,g(y_k),\overline{y_1},\ldots,\overline{y_k}).
\]

For some \( b \) in \( B \), \( b(y_i) = \overline{y_i} \) since \( y_i \) is a regular semisimple element, belonging to \( b \). As a result, \( g b^{-1}(\overline{y_i}) = x \) and \( w_i(x) = \overline{y_i} \). Hence \( g b^{-1} \) is an element of \( N_G(b) \) representing \( w_{\overline{x}}^{-1} \). Furthermore, since \( g b^{-1}(b(y_j)) = e \) for \( j \neq i \), \( b(y_j) \) is a regular nilpotent element belonging to \( b \). Then, since there is one and only one Borel subalgebra containing a regular nilpotent element, \( g b^{-1}(b) = b \) and \( w_i = 1_b \). As a result, \( w \) is the identity of \( W(\mathcal{R})^k \), whence the assertion.

(iv) Since the fibers of \( \overline{\sigma} \) are invariant under \( W(\mathcal{R})^k \), \( k[\overline{B}(b)] \) is contained in \( k[\overline{\sigma}^{-1}(\overline{B}(b))]|W(\mathcal{R})^k \).

Let \( p \) be in \( k[\overline{\sigma}^{-1}(\overline{B}(b))]|W(\mathcal{R})^k \). Since \( W(\mathcal{R}) \) is a finite group, \( p \) is the restriction to \( \overline{\sigma}^{-1}(\overline{B}(b)) \) of an element \( q \) of \( k[\overline{X}]^F \), invariant under \( W(\mathcal{R})^k \). Since \( k[\overline{X}]^F \) is in \( k[\overline{G}] \), \( q \) is in \( k[\overline{G}]^g \) by Lemma 2.1,(iv), and \( p \) is in \( k[\overline{B}(b)] \), whence the assertion.

2.4. For \( \alpha \) a positive root, denote by \( b_{\alpha} \) the kernel of \( \alpha \) and by \( S_{\alpha} \) the closure in \( b \) of the image of the map

\[
U \times b_{\alpha} \longrightarrow b, \quad (g, x) \longmapsto g(x).
\]

For \( \beta \) in \( \Pi \), set:

\[
u_\beta := \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} g^\alpha, \quad b_\beta := b_\beta \oplus u_\beta.
\]

Lemma 2.9. For \( \alpha \) in \( \mathcal{R}_+ \), let \( b_{\alpha}' \) be the set of subregular elements belonging to \( b_{\alpha} \).

(i) For \( \alpha \) in \( \mathcal{R}_+ \), \( S_{\alpha} \) is a subvariety of codimension 2 of \( b \). Moreover, it is contained in \( b \setminus b_{\text{reg}} \).

(ii) For \( \beta \) in \( \Pi \), \( S_{\beta} = b_{\beta} \).

(iii) The \( S_{\alpha} \)'s, \( \alpha \in \mathcal{R}_+ \), are the irreducible components of \( b \setminus b_{\text{reg}} \).

Proof. (i) For \( x \) in \( b_{\alpha}' \), \( b^v = b \oplus kx_\alpha \). Hence \( U(b_{\alpha}' \cup b_{\beta}) \) has dimension \( n - 1 + \ell - 1 \), whence the assertion since \( U(b_{\alpha}') \) is dense in \( S_{\alpha} \) and \( b_{\alpha}' \) is contained in \( b \setminus b_{\text{reg}} \).

(ii) For \( \beta \) in \( \Pi \), \( U(b_{\beta} \cup b_{\beta}) \) is contained in \( b_{\beta} \) since \( b_{\beta} \) is an ideal of \( b \), whence the assertion by (i).

(iii) According to (i), it suffices to prove that \( b \setminus b_{\text{reg}} \) is the union of the \( S_{\alpha} \)'s. Let \( x \) be in \( b \setminus b_{\text{reg}} \). According to [Y72], for some \( g \) in \( G \) and for some \( \beta \) in \( \Pi \), \( x \) is in \( g(b_{\beta}) \). Since \( b_{\beta} \) is an ideal of \( b \), by Bruhat’s decomposition of \( G \), for some \( b \) in \( B \) and for some \( w \) in \( W(\mathcal{R}) \), \( b^{-1}(x) \) is in \( w(b_{\beta}) \cap b \).

By definition,

\[
w(b_{\beta}) = w(b_{\beta}) \oplus u_{w(\beta)} = b_{w(\beta)} \oplus \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} g^u(\alpha).
\]

So,

\[
w(b_{\beta}) \cap b = b_{w(\beta)} \oplus u_0 \text{ with } u_0 := \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \{\beta\}} g^u(\alpha).
\]

The subspace \( u_0 \) of \( u \) is a subalgebra, not containing \( g^u(\beta) \). Then, denoting by \( U_0 \) the closed subgroup of \( U \) whose Lie algebra is \( \text{ad}u_0 \),

\[
\overline{U_0(b_{w(\beta)})} = w(b_{\beta}) \cap b
\]
since the left hand side is contained in the right hand side and has the same dimension. As a result, $x$ is in $S_{w(b)}$ since $S_{w(b)}$ is $B$-invariant, whence the assertion.

Recall that $\theta$ is the map

$$U_\times b_{\text{reg}} \longrightarrow X \, , \quad (g, x) \mapsto (g(x), \overline{x})$$

and denote by $W'_k$ the inverse image of $\theta(U_\times b_{\text{reg}})$ by the projection

$$B_x^{(k)} \longrightarrow X \, , \quad (x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto (x_1, y_1).$$

**Lemma 2.10.** Let $W_k$ be the subset of elements $(x, y)$ of $B_x^{(k)}$ $(x \in g^k, y \in b^k)$ such that $E_x \cap g_{\text{reg}}$ is not empty.

(i) The subset $W'_k$ of $B_x^{(k)}$ is a smooth open subset. Moreover, the map

$$U_\times b_{\text{reg}} \times b^{k-1} \longrightarrow W'_k \, , \quad (g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k), \overline{x}_1, \ldots, \overline{x}_k).$$

is an isomorphism of varieties.

(ii) The subset $B_x^{(k)}$ of $g^k \times b^k$ is invariant under the canonical action of $GL_k(k)$.

(iii) The subset $W_k$ of $B_x^{(k)}$ is a smooth open subset. Moreover, $W_k$ is the $G \times GL_k(k)$-invariant set generated by $W'_k$.

(iv) The subvariety $B_x^{(k)} \setminus W_k$ of $B_x^{(k)}$ has codimension at least $2k$.

**Proof.** (i) According to Corollary 2.6,(ii), $\theta$ is an isomorphism onto a smooth open subset of $X$. As a result, $W'_k$ is an open subset of $B_x^{(k)}$ and the map

$$U_\times b_{\text{reg}} \times b^{k-1} \longrightarrow W'_k \, , \quad (g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k), \overline{x}_1, \ldots, \overline{x}_k)$$

is an isomorphism whose inverse is given by

$$W'_k \longrightarrow U_\times b_{\text{reg}} \times b^{k-1} \, , \quad (x_1, \ldots, x_k) \mapsto (\theta^{-1}(x_1, \overline{x}_1), \theta^{-1}(x_1, \overline{x}_1)(x_1), \ldots, \theta^{-1}(x_1, \overline{x}_1)(x_k))$$

with $\theta^{-1}$ the inverse of $\theta$ and $\theta^{-1}(x_1, \overline{x}_1)$ the component of $\theta^{-1}(x_1, \overline{x}_1)$ on $U_\times$, whence the assertion since $U_\times b_{\text{reg}} \times b^{k-1}$ is smooth.

(ii) For $(x_1, \ldots, x_k)$ in $b^k$ and for $(a_{i,j}, 1 \leq i, j \leq k)$ in $GL_k(k)$,

$$\sum_{j=1}^k a_{i,j}x_j = \sum_{j=1}^k a_{i,j}\overline{x}_j$$

so that $\psi_k(b^k)$ is invariant under the action of $GL_k(k)$ in $g^k \times b^k$ defined by

$$(a_{i,j}, 1 \leq i, j \leq k), (x_1, \ldots, x_k, y_1, \ldots, y_k) := (\sum_{j=1}^k a_{i,j}x_j, j = 1, \ldots, k, \sum_{j=1}^k a_{i,j}y_j, j = 1, \ldots, k),$$

whence the assertion since $B_x^{(k)} = G.\psi_k(b^k)$ and the actions of $G$ and $GL_k(k)$ in $g^k \times b^k$ commute.

(iii) According to (i), $G.W'_k$ is a smooth open subset of $B_x^{(k)}$. Moreover, $G.W'_k$ is the subset of elements $(x, y)$ such that the first component of $x$ is regular. So, by (ii) and Lemma 1.9, $W_k = GL_k(k)(G.W'_k)$, whence the assertion.
(iv) According to Corollary 2.8(i), \( \mathcal{B}_x^{(k)} \) is the image of \( G \times_B b^k \) by the restriction \( \gamma_x \) of \( \chi_n^{(k)} \) to \( G \times_B b^k \). Then \( \mathcal{B}_x^{(k)} \setminus W_k \) is contained in the image of \( G \times_B (b \setminus b_{reg})^k \) by \( \gamma_x \). As a result, by Lemma 2.9, 
\[
\dim \mathcal{B}_x^{(k)} \setminus W_k \leq n + k(b_0 - 2),
\]
whence the assertion. \( \square \)

**Corollary 2.11.** The restriction of \( \gamma_x \) to \( \gamma_x^{-1}(W_k) \) is an isomorphism onto \( W_k \). Moreover, \( \gamma_x^{-1}(W_k) \) is a big open subset of \( G \times_B b^k \).

**Proof.** Since the subset of Borel subalgebras containing a regular element is finite, the fibers of \( \gamma_x \) over the elements of \( W_k \) are finite. In particular, the restriction of \( \gamma_x \) to \( \gamma_x^{-1}(W_k) \) is a quasi finite surjective morphism onto \( W_k \). So, by Zariski’s Main Theorem [Mu88, §9], it is an isomorphism since \( W_k \) is smooth by Lemma 2.10(iii).

Recall that \( G \times_B b^k \) identifies with a closed subset of \( G/B \times g^k \). For \( u \in G/B \) and \( x \) in \( g^k \) such that \( (u, x) \) is in \( G \times_B b^k \), \( (u, x) \) is in \( \gamma_x^{-1}(W_k) \) if and only if \( E_x \cap g_{reg} \) is not empty. Denote by \( \pi \) the bundle projection of the vector bundle \( G \times_B b^k \) over \( G/B \). Let \( \Sigma \) be an irreducible component of \( G \times_B b^k \setminus \gamma_x^{-1}(W_k) \). For \( u \) in \( \pi(\Sigma) \), set:
\[
\Sigma_u := \{ x \in g^k \mid (u, x) \in \Sigma \}.
\]
Since \( W_k \) is a cone, for all \( u \) in \( \pi(\Sigma) \), \( \Sigma_u \) is a closed cone of \( u \), whence \( \pi(\Sigma) \times \{0\} = \Sigma \cap G/B \times \{0\} \) so that \( \pi(\Sigma) \) is a closed subset of \( G/B \). Suppose that \( \Sigma \) has codimension 1 in \( G \times_B b^k \). A contradiction is expected. Then \( \pi(\Sigma) \) has codimension at most 1 in \( G/B \). If \( \pi(\Sigma) \) has codimension 1 in \( G/B \), then for all \( u \) in \( \pi(\Sigma) \), \( \Sigma_u = u \). It is impossible since \( u \cap g_{reg} \) is not empty. As a result, for all \( u \) in a dense open subset of \( G/B \), \( \Sigma_u \) is closed of codimension 1 in \( u \). According to Lemma 2.9, \( u \setminus g_{reg} \) has codimension 2 in \( u \) and \( \Sigma_u \) is contained in \( (u \setminus g_{reg})^k \), whence the contradiction. \( \square \)

2.5. For \( E \) a \( B \)-module, denote by \( \mathcal{L}_0(E) \) the sheaf of local sections of the vector bundle \( G \times_B E \) over \( G/B \). Let \( \Delta \) be the diagonal of \( (G/B)^k \) and let \( \mathcal{J}_\Delta \) be its ideal of definition in \( \mathcal{O}_{(G/B)^k} \). The variety \( G/B \) identifies with \( \Delta \) so that \( \mathcal{O}_{(G/B)^k}/\mathcal{J}_\Delta \) is isomorphic to \( \mathcal{O}_{G/B} \). For \( E \) a \( B^k \)-module, denote by \( \mathcal{L}(E) \) the sheaf of local sections of the vector bundle \( G^k \times_{B^k} E \) over \( (G/B)^k \).

**Lemma 2.12.** Let \( E \) be a \( B^k \)-module. Denote by \( \overline{E} \) the \( B \)-module defined by the diagonal action of \( B \) on \( E \). The short sequence of \( \mathcal{O}_{(G/B)^k} \)-modules
\[
0 \rightarrow \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^k} \mathcal{L}(E) \rightarrow \mathcal{L}(E) \rightarrow \mathcal{L}_0(\overline{E}) \rightarrow 0
\]
is exact.

**Proof.** Since \( \mathcal{L}(E) \) is a locally free \( \mathcal{O}_{(G/B)^k} \)-module, the short sequence of \( \mathcal{O}_{(G/B)^k} \)-modules
\[
0 \rightarrow \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^k} \mathcal{L}(E) \rightarrow \mathcal{L}(E) \rightarrow \mathcal{O}_\Delta \otimes \mathcal{O}_{(G/B)^k} \mathcal{L}(E) \rightarrow 0
\]
is exact, whence the assertion since \( \mathcal{O}_\Delta \otimes \mathcal{O}_{(G/B)^k} \mathcal{L}(E) \) is isomorphic to \( \mathcal{L}_0(\overline{E}) \). \( \square \)

From Lemma 2.12 results a canonical morphism
\[
H^0((G/B)^k, \mathcal{L}(E)) \longrightarrow H^0(G/B, \mathcal{L}_0(\overline{E}))
\]
for all \( B^k \)-module \( E \). According to the identification of \( g \) and \( g^* \) by \( \langle \cdot, \cdot \rangle \), the duals of \( b \) and \( u \) identify with \( b_- \) and \( u_- \) respectively so that \( b_- \) and \( u_- \) are \( B \)-modules.
Lemma 2.13. (i) The algebra $\mathbb{k}[\mathcal{B}_{n}^{(k)}]$ is equal to $H^{0}(G/B, \mathcal{L}_{0}(S(b^{k})))$.

(ii) The algebra $\mathbb{k}[\mathcal{N}_{n}^{(k)}]$ is equal to $H^{0}(G/B, \mathcal{L}_{0}(S(u^{k})))$.

(iii) The algebra $\mathbb{k}[\mathcal{B}_{b}^{(k)}]$ is the image of the morphism

$$H^{0}((G/B)^{k}, \mathcal{L}(S(b^{k}))) \longrightarrow H^{0}(G/B, \mathcal{L}_{0}(S(b^{k}))) \, .$$

(iv) The algebra $\mathbb{k}[\mathcal{N}_{b}^{(k)}]$ is the image of the morphism

$$H^{0}((G/B)^{k}, \mathcal{L}(S(u^{k}))) \longrightarrow H^{0}(G/B, \mathcal{L}_{0}(S(u^{k}))) \, .$$

Proof. (i) Since $G \times_{B} b^{k}$ is a desingularization of the normal variety $\mathcal{B}_{n}^{(k)}$, $\mathbb{k}[\mathcal{B}_{n}^{(k)}]$ is the space of global sections of $\mathcal{O}_{G \times_{B} b^{k}}$ by Lemma 1.4. Let $\pi$ be the bundle projection of the fiber bundle $G \times_{B} b^{k}$. Since $S(b^{k})$ is the space of polynomial functions on $b^{k}$,

$$\pi_{*}(\mathcal{O}_{G \times_{B} b^{k}}) = \mathcal{L}_{0}(S(b^{k})),$$

whence the assertion.

(ii) By Lemma 2.1, (ii), $G \times_{B} u^{k}$ is a desingularization of $\mathcal{N}_{n}^{(k)}$ so that $\mathbb{k}[\mathcal{N}_{n}^{(k)}]$ is the space of global sections of $\mathcal{O}_{G \times_{B} u^{k}}$ by Lemma 1.4. Denoting by $\pi_{0}$ the bundle projection of $G \times_{B} u^{k}$,

$$\pi_{0*}(\mathcal{O}_{G \times_{B} u^{k}}) = \mathcal{L}_{0}(S(u^{k})),$$

whence the assertion.

(iii) Since $G^{k} \times_{b^{k}} b^{k}$ is isomorphic to $(G \times_{B} b)^{k}$,

$$H^{0}((G/B)^{k}, \mathcal{O}_{G^{k} \times_{b^{k}} b^{k}}) = H^{0}(G/B, \mathcal{O}_{G \times_{B} b})^{\otimes k}.$$  

By (i),

$$H^{0}(G/B, \mathcal{O}_{G \times_{B} b}) = H^{0}(G/B, \mathcal{L}(S(b\ldots))) = \mathbb{k}[X]$$

since $G \times_{B} b$ is a desingularization of $X$ by Lemma 2.1, (i) and (ii), whence

$$H^{0}((G/B)^{k}, \mathcal{L}(S(b^{k}))) = \mathbb{k}[X^{k}].$$

By definition, $\mathcal{B}_{x}^{(k)}$ is a closed subvariety of $X^{k}$. According to Corollary 2.8, $\mathbb{k}[\mathcal{B}_{x}^{(k)}]$ is a subalgebra of $\mathbb{k}[\mathcal{B}_{x}^{(k)}]$ having the same fraction field and $\mathbb{k}[\mathcal{B}_{x}^{(k)}]$ is a finite extension of $\mathbb{k}[\mathcal{B}^{(k)}]$. Hence $\mathbb{k}[\mathcal{B}_{x}^{(k)}]$ is a subalgebra of $\mathbb{k}[\mathcal{B}_{x}^{(k)}]$. For $\varphi \in \mathbb{k}[\mathcal{B}_{x}^{(k)}]$, $\varphi$ is the restriction to $\mathcal{B}_{x}^{(k)}$ of an element $\psi$ of $\mathbb{k}[X^{k}]$. As mentioned above, $\psi$ is a global section of $\mathcal{L}(b^{k})$. Denoting by $\overline{\psi}$ its restriction to the diagonal of $(G/B)^{k}$, $\overline{\psi}$ is a global section of $\mathcal{L}_{0}(S(b^{k}))$ so that $\overline{\psi}$ is in $\mathbb{k}[\mathcal{B}_{x}^{(k)}]$. Moreover, for all smooth point $x$ of $\mathcal{B}_{x}^{(k)}$, $\overline{\psi}(x) = \varphi(x)$. Hence $\varphi$ is in the image of the morphism

$$H^{0}((G/B)^{k}, \mathcal{L}(S(b^{k}))) \longrightarrow H^{0}(G/B, \mathcal{L}_{0}(S(b^{k}))) \, .$$

Conversely, for $\varphi$ image of $\psi$ in $H^{0}((G/B)^{k}, \mathcal{L}(S(b^{k})))$ by this morphism, $\psi$ is in $\mathbb{k}[X^{k}]$ and $\varphi(x) = \psi(x)$ for all smooth point $x$ of $\mathcal{B}_{x}^{(k)}$ so that $\varphi$ is the restriction of $\psi$ to $\mathcal{B}_{x}^{(k)}$.

(iv) Let $\varphi$ be in $\mathbb{k}[\mathcal{N}_{r}^{(k)}]$. Since $\mathcal{N}_{r}^{(k)}$ is a closed subvariety of $\mathcal{B}_{r}^{(k)}$, $\varphi$ is the restriction to $\mathcal{N}_{r}^{(k)}$ of an element $\psi$ of $\mathbb{k}[\mathcal{N}_{r}^{(k)}]$. As mentioned above, $\psi$ is a global section of $\mathcal{L}(S(u^{k}))$. Denoting by $\overline{\psi}$ the restriction of $\psi$ to the diagonal of $(G/B)^{k}$, $\overline{\psi}$ is a global section of $\mathcal{L}_{0}(S(u^{k}))$ so that $\overline{\psi}$ is in $\mathbb{k}[\mathcal{N}_{r}^{(k)}]$. Moreover, for all smooth point $x$ of $\mathcal{N}_{r}^{(k)}$, $\overline{\psi}(x) = \varphi(x)$. Hence $\varphi$ is in the image of the morphism

$$H^{0}((G/B)^{k}, \mathcal{L}(S(u^{k}))) \longrightarrow H^{0}(G/B, \mathcal{L}_{0}(S(u^{k}))) \, .$$
Conversely, for $\varphi$ image of $\psi$ in $H^0((G/B)^k, \mathcal{L}(S(u^k)))$ by this morphism, $\psi$ is in $\mathbb{K}[\mathfrak{M}^k]$ and $\varphi(x) = \psi(x)$ for all smooth point $x$ of $N^k$ so that $\varphi$ is the restriction of $\psi$ to $N^k$. □

Let $x_1, \ldots, x_{k\ell}$ be a basis of $\mathfrak{h}^k$ verifying the following conditions for $j = 1, \ldots, k$:

1. $x_{j\ell} = 0$ for $l < (j - 1)\ell$ and $l > j\ell$
2. $x_{j\ell} = h$,
3. $x_{j(j-1)\ell+1}, \ldots, x_{j(j-1)\ell-1}$ is a basis of the orthogonal complement to $h$ in $\mathfrak{h}$,

with $x_{j\ell}$ the component of $x_i$ on the $j$-th factor $\mathfrak{h}$. Set:

$$E_0 := \{0\}, \quad F_0 := \mathfrak{b}^k, \quad E_i := \text{span}(x_1, \ldots, x_i), \quad F_i := \mathfrak{b}^k/E_i$$

for $i = 1, \ldots, k\ell$. In the $B$-module $b_-$, $\mathfrak{h}$ is the subspace of invariant elements and $u_-$ is the quotient of $b_-$ by $\mathfrak{h}$. So for $i = 0, \ldots, k\ell$, $F_i$ is a $B^k$-module. As a matter of fact, because of the choice of the basis $x_1, \ldots, x_{k\ell}$, $F_i = F_{i,1} \times \cdots \times F_{i,k}$ where $F_{i,1}, \ldots, F_{i,k}$ are $B$-modules quotient of $b_-$ and the action of $B^k$ on $F_i$ is the product action. For $i = 0, \ldots, k\ell$, set:

$$A_i := H^0(G/B, \mathcal{L}_0(S(F_{i,1}))) \quad \text{and} \quad C_i := H^0((G/B)^k, \mathcal{L}_0(S(F_{i,1}))).$$

Denote by $B_i$ the image of $C_i$ by the restriction map to the diagonal of $(G/B)^k$. Then $A_i, B_i, C_i$ are integral graded algebras. Moreover, by Lemma 1.4, $A_i$ and $C_i$ are normal as spaces of global sections of structural sheaves of vector bundles over $G/B$ and $(G/B)^k$.

For $i < k\ell$, the $B^k$-module $F_{i+1}$ is a quotient of the $B^k$-module $F_i$ so that $S(F_{i+1})$ is a quotient of $S(F_i)$ as a $B^k$-algebra and $L(S(F_{i+1}))$ and $L_0(S(F_{i+1}))$ are quotients of $L(S(F_i))$ and $L_0(S(F_i))$ respectively, whence morphisms of algebras

$$C_i \xrightarrow{\nu_i} C_{i+1} \quad \text{and} \quad A_i \xrightarrow{\nu_i} A_{i+1}.$$

For $i = 0, \ldots, k\ell - 1$ and $m$ positive integer, denote again by $x_{i+1}$ the image of $x_{i+1}$ in $F_i$ by the quotient morphism and by $J_{m,i}$ the ideal of $S(F_i)$ generated by $x_{i+1}^m$. As $x_{i+1}$ is a fixed point of the $B^k$-module $F_i$, $J_{m,i}$ is a $B^k$-submodule of $S(F_i)$.

**Lemma 2.14.** Let $i = 0, \ldots, k\ell - 1$ and $m$ a positive integer.

(i) The algebra $\mathbb{K}[x_{i+1}]$ is canonically embedded in $A_i$ and $C_i$.

(ii) The space $H^0(G/B, \mathcal{L}_0(J_{m,i}))$ is the ideal of $A_i$ generated by $x_{i+1}^m$ and the image of the canonical morphism

$$H^0(G/B, \mathcal{L}_0(J_{m,i})) \longrightarrow H^0(G/B, \mathcal{L}_0(S(F_{i+1}) \otimes S \mathbb{K} x_{i+1}^m))$$

is equal to $\nu_{i,0}(A_i) \otimes S \mathbb{K} x_{i+1}^m$.

(iii) The space $H^0((G/B)^k, \mathcal{L}(J_{m,i}))$ is the ideal of $C_i$ generated by $x_{i+1}^m$ and the image of the canonical morphism

$$H((G/B)^k, \mathcal{L}(J_{m,i})) \longrightarrow H((G/B)^k, \mathcal{L}(S(F_{i+1}) \otimes S \mathbb{K} x_{i+1}^m))$$

is equal to $C_{i+1} \otimes S \mathbb{K} x_{i+1}^m$.

(iv) Let $v_1, \ldots, v_l$ be in $A_i$ such that $\nu_{i,0}(v_1), \ldots, \nu_{i,0}(v_l)$ are linearly free over $\mathbb{K}$. Then $v_1, \ldots, v_l$ are linearly free over $\mathbb{K}[x_{i+1}]$.

(v) Let $w_1, \ldots, w_l$ be in $C_i$ such that $\nu_1(w_1), \ldots, \nu_1(w_l)$ are linearly free over $\mathbb{K}$. Then $w_1, \ldots, w_l$ are linearly free over $\mathbb{K}[x_{i+1}]$. 16
Proof. (i) Since \( x_{i+1} \) is a fixed point of the \( B^k \)-module \( S(F_i) \), \( \mathcal{L}(k[x_{i+1}]) \) and \( \mathcal{L}_0(k[x_{i+1}]) \) are sub-modules of \( \mathcal{L}(S(F_i)) \) and \( \mathcal{L}_0(S(F_i)) \) respectively. Moreover they are isomorphic to \( \mathcal{O}_{(G/B)^k} \otimes_k k[x_{i+1}] \mathcal{O}_{G/B} \otimes_k k[x_{i+1}] \) respectively, whence the assertion.

(ii) Since \( F_{i+1} \) is the quotient of \( F_i \) by \( k[x_{i+1}] \), we have the exact sequence of \( B^k \)-modules,
\[
0 \to J_{m+1,i} \to J_{m,i} \to S(F_{i+1}) \otimes_k k[x_{i+1}] \to 0,
\]
whence the exact sequence of \( \mathcal{O}_{G/B} \)-modules,
\[
0 \to \mathcal{L}_0(J_{m+1,i}) \to \mathcal{L}_0(J_{m,i}) \to \mathcal{L}_0(S(F_{i+1}) \otimes_k k[x_{i+1}]) \to 0,
\]
and whence the canonical morphism
\[
H^0(G/B, \mathcal{L}_0(J_{m,i})) \to H^0(G/B, \mathcal{L}_0(S(F_{i+1}) \otimes_k k[x_{i+1}])).
\]
In particular, \( \nu_i,0(A_j) \otimes k[x_{i+1}^m] \) is contained in its image since the image of \( ax_{i+1}^m \) is equal to \( \nu_i,0(a) \otimes x_{i+1}^m \) for all \( a \) in \( A_j \).

Let \( a \) be in \( H^0(G/B, \mathcal{L}_0(J_{m,i})) \). Let \( O_1, \ldots, O_l \) be a cover of \( G/B \) by affine trivialization open subsets of the vector bundle \( G \times_B S(F_i) \). For \( j = 1, \ldots, l \), denoting by \( \Phi_j \) a trivialization over \( O_j \), we have a commutative diagram:
\[
\begin{array}{ccc}
\pi^{-1}_i(O_j) & \xrightarrow{\Phi_j} & O_j \times S(F_i) \\
& \searrow_{\pi_i} & \downarrow_{\pi_i} \\
& & O_j
\end{array}
\]
with \( \pi_i \) the bundle projection. Since \( x_{i+1} \) is invariant under \( B \), for \( \varphi \) local section of \( \mathcal{L}_0(S(F_i)) \) above \( O_j \), \( \Phi_j(x_{i+1}^m \varphi) = x_{i+1}^m \Phi_j(\varphi) \), whence
\[
\Phi_j,*(\mathcal{O}_{G/B}(O_j, \mathcal{L}_0(J_{m,i}))) = k[O_j] \otimes_k S(F_i)x_{i+1}^m.
\]
As a result, for some local section \( a_{j} \) above \( O_j \) of \( \mathcal{L}_0(S(F_i)) \), \( a = x_{i+1}^m \varphi \). Moreover, \( a_{j} \) is uniquely defined by this equality. Then for all \( j, j', a_j \) and \( a_{j'} \) have the same restriction to \( O_j \cap O_j' \). Denoting by \( a' \) the global section of \( \mathcal{L}_0(S(F_i)) \) extending \( a_{1}, \ldots, a_{l}, a = a'x_{i+1}^m \), whence the assertion.

(iii) According to [He76, Theorem B and Corollary], for a \( B \)-module quotient \( V \) of \( b_- \) having \( u_- \) as quotient, \( H^1(G/B, \mathcal{L}_0(V)) = 0 \). Hence \( C_{i+1} \) is the image of \( \nu_i \). From the exact sequence of \( (G/B)^k \)-modules
\[
0 \to J_{m+1,i} \to J_{m,i} \to S(F_{i+1}) \otimes_k k[x_{i+1}] \to 0,
\]
we deduce the exact sequence of \( \mathcal{O}_{G/B} \)-modules,
\[
0 \to \mathcal{L}(J_{m+1,i}) \to \mathcal{L}(J_{m,i}) \to \mathcal{L}(S(F_{i+1}) \otimes_k k[x_{i+1}]) \to 0,
\]
and whence the canonical morphism
\[
H^0((G/B)^k, \mathcal{L}(J_{m,i})) \to H^0((G/B)^k, \mathcal{L}(S(F_{i+1}) \otimes_k k[x_{i+1}])).
\]
In particular, \( C_{i+1} \otimes_k k[x_{i+1}^m] \) is its image since the image of \( ax_{i+1}^m \) is equal to \( \nu_i(a) \otimes x_{i+1}^m \) for all \( a \) in \( C_{i} \) and \( C_{i+1} \) is the image of \( \nu_i \).

Let \( a \) be in \( H^0(G/B, \mathcal{L}(J_{m,i})) \). Prove by induction on \( l \) that for some \( a_l \) in \( C_l \), \( a = a_lx_{i+1}^m \) is in \( H^0(G/B, \mathcal{L}(J_{m+l,i})) \). It is true for \( l = 0 \). Suppose that it is true for \( l \). By the above result for \( m + l \), for some \( a'_{l} \) in \( C_{i} \), \( a = a_{l}x_{i+1}^m - a'_{l}x_{i+1}^{m+l} \) is in \( H^0(G/B, \mathcal{L}(J_{m+l+1,i})) \), whence the statement. As \( C_i \) is a
graded algebra and $H^0(G/B, \mathcal{L}(J_{m+1,i}))$ is a graded subspace whose elements have degree at least $m + l$, for $l$ sufficiently big, $a = ax_{i+1}^m$, whence the assertion.

(iv) Suppose that $v_1, \ldots, v_l$ is not linearly free over $k[x_{i+1}]$. A contradiction is expected. Let $a_1, \ldots, a_l$ be in $k[x_{i+1}]$ such that $a_i \neq 0$ for some $i$ and

$$a_1v_1 + \cdots + a_lv_l = 0.$$ 

Suppose that $m$ is the biggest integer such that $x_{i+1}^m$ divides $a_1, \ldots, a_l$ in $k[x_{i+1}]$. For $j = 1, \ldots, l$, denote by $c_j$ the element of $k$ such that $x_{i+1}^m$ divides $a_j - c_jx_{i+1}^m$. Then by (ii),

$$c_1v_{i,0}(v_1) + \cdots + c_lv_{i,0}(v_l) = 0,$$

whence a contradiction by the maximality of $m$.

(v) Suppose that $w_1, \ldots, w_l$ is not linearly free over $k[x_{i+1}]$. A contradiction is expected. Let $a_1, \ldots, a_l$ be in $k[x_{i+1}]$ such that $a_i \neq 0$ for some $i$ and

$$a_1w_1 + \cdots + a_lw_l = 0.$$ 

Suppose that $m$ is the biggest integer such that $x_{i+1}^m$ divides $a_1, \ldots, a_l$ in $k[x_{i+1}]$. For $j = 1, \ldots, l$, denote by $c_j$ the element of $k$ such that $x_{i+1}^m$ divides $a_j - c_jx_{i+1}^m$. Then by (iii),

$$c_1v_i(w_1) + \cdots + c_lv_i(w_l) = 0,$$

whence a contradiction by the maximality of $m$.

Corollary 2.15. Let $i = 0, \ldots, k\ell - 1$.

(i) The algebra $A_i$ is a free extension of $k[x_{i+1}]$ and $v_{i,0}(A_i)$ is the quotient of $A_i$ by the ideal generated by $x_{i+1}$.

(ii) The algebra $C_i$ is a free extension of $k[x_{i+1}]$ and $C_{i+1}$ is the quotient of $C_i$ by the ideal generated by $x_{i+1}$.

(iii) The algebra $B_i$ is a free extension of $k[x_{i+1}]$ and $B_{i+1}$ is the quotient of $B_i$ by the ideal generated by $x_{i+1}$.

Proof. (i) Let $K_0$ be a $k$-subspace of $A_i$ such that the restriction of $v_{i,0}$ to $K_0$ is an isomorphism onto the $k$-space $v_{i,0}(A_i)$. Prove by induction on $m$ the equality

$$A_i = K_0k[x_{i+1}] + H^0(G/B, \mathcal{L}_0(J_{m,i})).$$

The equality is true for $m = 0$. Suppose that it is true for $m$. Let $a$ be in $H^0(G/B, \mathcal{L}_0(J_{m,i}))$. By Lemma 2.14,(ii), for some $b$ in $A_i$, $a - bx_{i+1}^m$ is in $H^0(G/B, \mathcal{L}_0(J_{m+1,i}))$, whence the equality. Since $A_i$ is graded with $x_{i+1}$ having degree 1, $A_i = K_0k[x_{i+1}]$. So $A_i$ is a free $k[x_{i+1}]$-module by Lemma 2.14,(iv). Again by Lemma 2.14,(ii), $v_{i,0}(A_i)$ is the quotient of $A_i$ by the ideal generated by $x_{i+1}$.

(ii) Let $K$ be a $k$-subspace of $C_i$ such that the restriction of $v_i$ to $K$ is an isomorphism onto the $k$-space $C_{i+1}$. Prove by induction on $m$ the equality

$$C_i = Kk[x_{i+1}] + H^0((G/B)^k, \mathcal{L}(J_{m,i})).$$

The equality is true for $m = 0$. Suppose that it is true for $m$. Let $a$ be in $H^0((G/B)^k, \mathcal{L}(J_{m,i}))$. By Lemma 2.14,(iii), for some $b$ in $C_i$, $a - bx_{i+1}^m$ is in $H^0((G/B)^k, \mathcal{L}(J_{m+1,i}))$, whence the equality. Since $C_i$ is graded with $x_{i+1}$ having degree 1, $C_i = Kk[x_{i+1}]$. So $C_i$ is a free $k[x_{i+1}]$-module by Lemma 2.14,(v). Again by Lemma 2.14,(iii), $C_{i+1}$ is the quotient of $C_i$ by the ideal generated by $x_{i+1}$.
(iii) We have the commutative diagram

\[
\begin{array}{ccc}
C_i & \xrightarrow{\nu_i} & C_{i+1} \\
\downarrow & & \downarrow \\
B_i & \xrightarrow{\nu'_i} & B_{i+1}
\end{array}
\]

where the vertical arrows are the restriction morphisms to the diagonal of \((G/B)^k\) and \(\nu'_i\) is the restriction of \(\nu_i\) to \(B_i\). In particular \(\nu'_i\) is surjective since so is \(\nu_i\). Let \(K'_0\) be a \(\mathbb{K}\)-subspace of \(B_i\) such that the restriction of \(\nu'_i\) to \(K'_0\) is an isomorphism onto the \(\mathbb{K}\)-space \(B_{i+1}\). For \(m\) positive integer, denote by \(\Sigma_{m,i}\) the image of the restriction morphism to the diagonal of \((G/B)^k\),

\[
H^0((G/B)^k, \mathcal{L}(J_{m,i})) \rightarrow B_i.
\]

Prove by induction on \(m\) the equality

\[
B_i = K'_0 \mathbb{K}[x_{i+1}] + \Sigma_{m,i}.
\]

The equality is true for \(m = 0\). Suppose that it is true for \(m\). Let \(a\) be in \(\Sigma_{m,i}\). Then \(a\) is the image of an element \(a'\) of \(H^0((G/B)^k, \mathcal{L}(J_{m,i}))\). By Lemma 2.14,(iii), for some \(b'\) in \(C_i\), \(a' - b' x_{i+1}^m\) is in \(H^0((G/B)^k, \mathcal{L}(J_{m+1,i}))\). Denoting by \(b\) the image of \(b'\) in \(B_i\), \(a - b x_{i+1}^m\) is in \(\Sigma_{m+1,i}\), whence the equality. Since \(B_i\) is graded with \(x_{i+1}\) having degree 1, \(B_i = K'_0 \mathbb{K}[x_{i+1}]\). So \(B_i\) is a free \(\mathbb{K}[x_{i+1}]\)-module by Lemma 2.14,(iv).

Since \(B_{i+1}\) is contained in \(\nu_i(\mathbb{K})\), \(K'_0\) can be chosen contained in \(K_0\). Let \(v_i, l \in L\) be a basis of \(K_0\) such that \(v_i, l \in L'\) is a basis of \(K'_0\) for some subset \(L'\) of \(L\). Let \(a\) be in the kernel of \(\nu'_i\). Then \(a\) is in the kernel of \(\nu_i\) so that \(a = b x_{i+1}\) for some \(b\) in \(A_i\) by Lemma 2.14,(ii). By (i) and the freeness of the extension \(B_i\) of \(\mathbb{K}[x_{i+1}]\),

\[
b = \sum_{l \in L} v_l p_l \quad \text{and} \quad a = \sum_{l \in L'} v_l q_l
\]

with \(p_l, l \in L\) and \(q_l, l \in L'\) in \(\mathbb{K}[x_{i+1}]\) with finite supports. Then \(q_l = p_l x_{i+1}\) for all \(l \in L'\) and \(p_l = 0\) for \(l \in L \setminus L'\) so that \(a\) is in \(B_i x_{i+1}\), whence the assertion. \(\square\)

For \(j = 0, \ldots, k\ell\), set \(F_j^* := \text{Specm}(S(F_j))\). By definition, \(F_j^*\) is the subspace of elements \((y_1, \ldots, y_k)\) of \(\mathbb{B}^k\) such that for \(m = 1, \ldots, k\) and \(l = 1, \ldots, j\), \(\langle y_m, x_{m,i} \rangle = 0\).

**Lemma 2.16.** Let \(i = 0, \ldots, k\ell - 1\). Denote by \(T\) the annihilator of \(x_{i+1}\) in the \(A_i\)-module \(H^1(G/B, \mathcal{L}_0(S(F_i)))\).

(i) The algebra \(A_i\) is the integral closure of \(B_i\) in its fraction field.

(ii) There is a well defined morphism \(\mu : T \rightarrow A_{i+1}\) of \(A_i\)-modules.

(iii) The \(A_i\)-module \(A_{i+1}\) is the direct sum of \(\mu(T)\) and \(\nu_{i+1}(A_i)\).

**Proof.** Since \(A_i\) is the space of global sections of \(\mathcal{L}_0(S(F_i))\), for all integer \(m\), \(H^m(G/B, \mathcal{L}_0(S(F_i)))\) is an \(A_i\)-module.

(i) According to the proof of Lemma 2.1, \(\Omega_0^k\) is an open subset of \(\mathbb{g}^k\) such that for all \(x\) in \(\Omega_0^k \cap B^k\), there exists only one Borel subalgebra of \(\mathbb{g}\) containing \(E_x\). By definition, \(F_{i}^*\) is the subspace of elements \((y_1, \ldots, y_k)\) of \(\mathbb{B}^k\) such that for \(j = 1, \ldots, k\) and \(l = 1, \ldots, i\), \(\langle y_j, x_{i,l} \rangle = 0\). By Lemma 1.7, \(G.F_i^*\) is a closed subset of \(\mathbb{g}^k\) and the morphism \(G \times_B F_i^* \rightarrow G.F_i^*\) is projective. By Conditions (1), (2), (3), for some \(y_2, \ldots, y_{k-1}\) in \(\mathbb{B}_i\), \((e, y_2, \ldots, y_{k-1}, h)\) is in \(F_i^*\). Hence \(\Omega_0^k \cap G.F_i^*\)
is a dense open subset of $G.F_i^*$ and the above morphism is birational. So, by Lemma 1.4, $A_i$ is the integral closure of $k[G.F_i^*]$ in its fraction field.

By Lemma 1.7, $G_i^*.F_i^*$ is a closed subset of $G_i^*$ containing $G.F_i^*$. Then for all $\varphi$ in $k[G.F_i^*]$, $\varphi$ is the restriction to $G.F_i^*$ of an element $\psi$ of $k[G_i^*.F_i^*]$ so that $\psi$ is a global section of $\mathcal{L}(S(F_i^*))$. Denoting by $\overline{\psi}$ the restriction of $\psi$ to the diagonal of $(G/B)^k$, $\varphi = \overline{\psi}$. Hence $k[G.F_i^*]$ is contained in $B_i$ so that $A_i$ is the integral closure of $B_i$ in its fraction field.

(ii) Let $O := O_1, \ldots, O_m$ be a cover of $G/B$ by open subsets isomorphic to the affine space of dimension $n$ so that $O_1$ is a trivialization open subset of the vector bundles $G \times_B S(F_i^*)$ and $G \times_B S(F_{i+1})$. Denote by $\mathcal{Z}_1$ the space of cocycles of degree 1 of the complex $C^*$ of Čech cohomology of $O$ with values in $\mathcal{L}_0(S(F_i^*))$.

Let $\overline{a}$ be in $T$ and $a$ a representative of $\overline{a}$ in $\mathcal{Z}_1$. Since $\overline{a}$ is in $T$, $x_{i+1}a$ is the boundary of an element $b$ of $C^0$. For $l = 1, \ldots, m$, denote by $b_l$ the component of $b$ in $\Gamma(O_l, \mathcal{L}_0(S(F_i^*)))$ and by $\overline{b_l}$ its image in $\Gamma(O_l, \mathcal{L}_0(S(F_i^*)))$ by the quotient morphism. Then for $1 \leq l, l' \leq m$, $b_l$ and $\overline{b_l}$ have the same restriction to $O_l \cap O_{l'}$ so that $\overline{b_l}$ is the restriction to $O_l$ of an element $\overline{b}_l$ of $A_{i+1}$. If $a'$ is a representative of $\overline{a}$ in $\mathcal{Z}_1$, $a' - a$ is the boundary of an element $b'$ in $C^0$ and $x_{i+1}a$ is the boundary of $b + x_{i+1}b'$. Hence $b$ only depends on $\overline{a}$, whence a well defined map $T \rightarrow A_{i+1}$. It is clearly a morphism of $A_i$-modules.

(iii) From the exact sequence of $\mathcal{O}_{G/B}$-modules

$$0 \rightarrow \mathcal{L}_0(x_{i+1}S(F_i^*)) \rightarrow \mathcal{L}_0(S(F_i^*)) \rightarrow \mathcal{L}_0(S(F_{i+1})) \rightarrow 0$$

we deduce the long exact sequence of cohomology

$$\cdots \rightarrow A_i \rightarrow A_{i+1} \rightarrow x_{i+1}H^1(G/B, \mathcal{L}_0(S(F_i^*))) \rightarrow H^1(G/B, \mathcal{L}_0(S(F_i^*))) \rightarrow \cdots$$

since $x_{i+1}$ is a global section of $\mathcal{L}_0(S(F_i^*))$ by Lemma 2.14,(i). Since the $\mathcal{O}_{G/B}$-modules $\mathcal{L}_0(x_{i+1}S(F_i^*))$ and $\mathcal{L}_0(S(F_i^*))$ are isomorphic, we have an isomorphism

$$H^1(G/B, \mathcal{L}_0(x_{i+1}S(F_i^*))) \rightarrow H^1(G/B, \mathcal{L}_0(S(F_i^*)))$$

and the image of the kernel of the arrow

$$x_{i+1}H^1(G/B, \mathcal{L}_0(S(F_i^*))) \rightarrow H^1(G/B, \mathcal{L}_0(S(F_i^*)))$$

by this isomorphism is equal to $T$, whence an exact sequence

$$A_i \rightarrow A_{i+1} \rightarrow T \rightarrow 0.$$ 

By (ii), from the definition of the arrow

$$H^0(G/B, \mathcal{L}_0(S(F_{i+1}))) \rightarrow x_{i+1}H^1(G/B, \mathcal{L}_0(S(F_i^*)))$$

we deduce that for $a$ in $T$, the image of $\mu(a)$ is equal to $a$. Hence $A_{i+1}$ is the direct sum of $\mu(T)$ and $\nu_{i,0}(A_i)$.

\[ \square \]

**Corollary 2.17.** For $i = 0, \ldots, k\ell - 1$, $\nu_{i,0}(A_i)$ is equal to $A_{i+1}$.

**Proof.** According to Lemma 2.16, $A_{i+1}$ is the direct sum of $\mu(T)$ and $\nu_{i,0}(A_i)$. Let $a$ be in $\mu(T)$. By Lemma 2.16,(i) $A_{i+1}$ and $\nu_{i,0}(A_i)$ have the same fraction field since $B_{i+1}$ is contained in $\nu_{i,0}(A_i)$ by the proof of Corollary 2.15,(iii). So for some $b$ in $\nu_{i,0}(A_i)$, $ba$ is in $\nu_{i,0}(A_i)$, whence $ba = 0$ by Lemma 2.16,(ii) and (iii). As a result, $\mu(T) = \{0\}$ and $\nu_{i,0}(A_i) = A_{i+1}$. \[ \square \]
Proposition 2.18. (i) The algebra \( \kappa[B_{n}^{(k)}] \) is a free extension of \( S(b^{k}) \) and \( \kappa[N_{n}^{(k)}] \) is the quotient of \( \kappa[B_{n}^{(k)}] \) by the ideal generated by \( S_{r}(b^{k}) \).

(ii) The algebra \( \kappa[B_{x}^{(k)}] \) is a free extension of \( S(b^{k}) \) and \( \kappa[N^{(k)}] \) is the quotient of \( \kappa[B_{x}^{(k)}] \) by the ideal generated by \( S_{r}(b^{k}) \).

Proof. (i) According to Lemma 2.13,(i) and (ii), \( A_0 = \kappa[B_{n}^{(k)}] \) and \( A_{k_{r}} = \kappa[N_{n}^{(k)}] \). Moreover, \( S(b^{k}) = \kappa[x_1, \ldots, x_{k_{r}}] \) by definition. So the assertion will result from the following claim:

Claim 2.19. For \( i = 1, \ldots, k \ell \), \( A_0 \) is a free extension of \( \kappa[x_1, \ldots, x_i] \) and \( A_i \) is the quotient of \( A_0 \) by the ideal generated by \( x_1, \ldots, x_i \).

Prove the claim by induction on \( i \). According to Corollary 2.15,(i) and Corollary 2.17, the claim is true for \( i = 1 \). Suppose that it is true for \( i \). By Corollary 2.15,(i) and Corollary 2.17, \( A_{i+1} \) is the quotient of \( A_i \) by the ideal generated by \( x_{i+1} \). So by induction hypothesis, \( A_{i+1} \) is the quotient of \( A_0 \) by the ideal generated by \( x_1, \ldots, x_{i+1} \). For \( j = 1, \ldots, i+1 \), denote by \( \mu_j \) the quotient morphism \( A_0 \rightarrow A_j \). Let \( K_{i+1} \) be a \( \kappa \)-subspace of \( A_0 \) such that the restriction of \( \mu_{i+1} \) to \( K_{i+1} \) is a \( \kappa \)-linear isomorphism onto \( A_{i+1} \). Then \( A_0 = K_{i+1} + A_0 x_1 + \cdots + A_0 x_{i+1} \). So by induction on \( m \),

\[
A_0 = K_{i+1}[x_1, \ldots, x_{i+1}] + A_0 \mathcal{S}_m
\]

with \( \mathcal{S}_m \) the ideal of \( \kappa[x_1, \ldots, x_{i+1}] \) generated by its monomials of degree \( m \). As a result, \( A_0 = K_{i+1}[x_1, \ldots, x_{i+1}] \) since \( A_0 \) is a graded algebra.

Let \( v_1, \ldots, v_l \) be linearly free over \( \kappa \) in \( K_{i+1} \) and let \( a_1, \ldots, a_l \) be in \( \kappa[x_1, \ldots, x_{i+1}] \) such that

\[
a_1 v_1 + \cdots + a_l v_l = 0.
\]

For \( j = 1, \ldots, l \), \( a_j \) has an expansion

\[
a_j = \sum_{m \in \mathbb{N}} a_{jm} x_{i+1}^m
\]

with \( a_{jm}, m \in \mathbb{N} \) in \( \kappa[x_1, \ldots, x_i] \) with finite support. According to Corollary 2.15,(i), the sequence \( x_{i+1}^m \mu_j(v_j), (j, m) \in \{1, \ldots, l\} \times \mathbb{N} \) is linearly free over \( \kappa \). So, by induction hypothesis, \( a_{jm} = 0 \) for all \( (j, m) \). As a result, \( A_0 \) is a free extension of \( \kappa[x_1, \ldots, x_{i+1}] \), whence the claim.

(ii) According to Lemma 2.13,(iii) and (iv), \( B_0 = \kappa[B_{x}^{(k)}] \) and \( B_{k_{r}} = \kappa[N^{(k)}] \). So the assertion will result from the following claim:

Claim 2.20. For \( i = 1, \ldots, k \ell \), \( B_0 \) is a free extension of \( \kappa[x_1, \ldots, x_i] \) and \( B_i \) is the quotient of \( B_0 \) by the ideal generated by \( x_1, \ldots, x_i \).

Prove the claim by induction on \( i \). According to Corollary 2.15,(iii), the claim is true for \( i = 1 \). Suppose that it is true for \( i \). By Corollary 2.15,(iii), \( B_{i+1} \) is the quotient of \( B_i \) by the ideal generated by \( x_{i+1} \). So by induction hypothesis, \( B_{i+1} \) is the quotient of \( B_0 \) by the ideal generated by \( x_1, \ldots, x_{i+1} \). For \( j = 0, \ldots, k \ell \), \( B_j \) is contained in \( A_j \) and the quotient morphism \( B_0 \rightarrow B_j \) is the restriction of \( \mu_j \) to \( B_0 \). Let \( K'_{i+1} \) be a \( \kappa \)-subspace of \( B_0 \) such that the restriction of \( \mu_{i+1} \) to \( K'_{i+1} \) is a \( \kappa \)-linear isomorphism onto \( B_{i+1} \). Then \( B_0 = K'_{i+1} + B_0 x_1 + \cdots + B_0 x_{i+1} \). So by induction on \( m \),

\[
B_0 = K'_{i+1}[x_1, \ldots, x_{i+1}] + B_0 \mathcal{S}_m.
\]

As a result, \( B_0 = K'_{i+1}[x_1, \ldots, x_{i+1}] \) since \( B_0 \) is a graded algebra. Moreover by (i), a basis of \( K'_{i+1} \) is linearly free over \( \kappa[x_1, \ldots, x_{i+1}] \) so that \( B_0 \) is a free extension of \( \kappa[x_1, \ldots, x_{i+1}] \), whence the claim. □
Remark 2.21. According to Proposition 2.18, \( \mathcal{S}(b^k) \) is embedded in \( \mathcal{K}[B_x^k] \) and by Lemma 2.13,(iii), the embedding is given by the map

\[
S(b^k) \longrightarrow \mathcal{K}[B_x^k], \quad p \mapsto ((x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto p(y_1, \ldots, y_k).
\]

Denote by \( \Phi \) this map.

**Corollary 2.22.** (i) The image of \( \Phi \) is equal to \( \mathcal{K}[B_x^k]^G \). Moreover, \( \mathcal{K}[B_x^k] \) is generated by \( \mathcal{K}[B^k] \) and \( \mathcal{K}[B_x^k]^G \).

(ii) The image of \( \Phi \) is equal to \( \mathcal{K}[B_n^k]^G \).

(iii) The subalgebras \( \mathcal{K}[B^k]^G \) and \( \Phi((S(h)^{\otimes k})W(\mathcal{R})) \) of \( \mathcal{K}[B_x^k]^G \) are equal.

**Proof.** (i) Since \( B_x^k \) is a closed subvariety of \( X^k \) and \( \mathcal{K}[X] \) is generated by \( \mathcal{S}(a) \) and \( \mathcal{S}(b) \), \( \mathcal{K}[B_x^k] \) is generated by \( \mathcal{S}(b^k) \) and the image of \( \mathcal{S}(g^k) \) in \( \mathcal{K}[B_x^k] \) which is equal to \( \mathcal{K}[B^k] \). For \( p \in \mathcal{K}[B_x^k] \), denote by \( \overline{p} \) the element of \( \mathcal{S}(b)^{\otimes k} \) such that

\[
\overline{p}(x_1, \ldots, x_k) := p(x_1, \ldots, x_k, x_1, \ldots, x_k).
\]

Then the restriction of \( p - \Phi(\overline{p}) \) to \( t_k(b^k) \) is equal to 0. Moreover, if \( p \in \mathcal{K}[B_x^k]^G \), \( p - \Phi(\overline{p}) \) is \( G \)-invariant so that \( p = \Phi(\overline{p}) \) since \( G.t_k(b^k) \) is dense in \( B_x^k \), whence the assertion.

(ii) Since \( \mathcal{K}[B_x^k] \) is contained in \( \mathcal{K}[B_n^k] \), so is \( \mathcal{K}[B_x^k]^G \) by (i). Since \( G \) is reductive, there exists a projection \( \mathcal{K}[B_n^k] \longrightarrow \mathcal{K}[B_n^k]^G \) which is \( \mathcal{K}[B_n^k]^G \)-linear. As a result, \( \mathcal{K}[B_n^k]^G \) is the integral extension of \( \mathcal{K}[B_x^k]^G \) in \( \mathcal{K}[B_n^k] \) since \( \mathcal{K}[B_n^k] \) is an integral extension of \( \mathcal{K}[B_x^k] \). Let \( J \) be the ideal of augmentation of \( \mathcal{K}[B_x^k]^G \) and set \( J' := \mathcal{K}[B_n^k]J \). By (i) and Proposition 2.18,(i), \( J' \) is a prime ideal. Suppose that \( \mathcal{K}[B_x^k]^G \) is strictly contained in \( \mathcal{K}[B_n^k]^G \). A contradiction is expected. The algebras \( \mathcal{K}[B_n^k]^G \) and \( \mathcal{K}[B_x^k]^G \) are graded subalgebras of \( \mathcal{K}[B_n^k] \). Let \( a \) be a homogeneous element in \( \mathcal{K}[B_n^k]^G \setminus \mathcal{K}[B_x^k]^G \) of minimal degree. Then \( a \) has positive degree. As a result, it is in \( J' \) since \( J' \) is radical and \( a \) satisfies a dependence integral equation over \( \mathcal{K}[B_x^k]^G \). Since \( \mathcal{K}[B_n^k]^G J \) is the image of \( J' \) by the projection \( \mathcal{K}[B_n^k] \longrightarrow \mathcal{K}[B_n^k]^G \), \( a \) is in \( \mathcal{K}[B_n^k]^G J \). By the minimality of the degree of \( a \), \( a \) is in \( \mathcal{K}[B_x^k]^G \), whence the contradiction.

(iii) For \( (x_1, \ldots, x_k) \in h^k \), for \( w \in W(\mathcal{R}) \) and for \( g_w \) a representative of \( w \) in \( N_G(h) \), we have

\[
(w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k)) = g_w(x_1, \ldots, x_k, x_1, \ldots, x_k)
\]

so that the subalgebra \( \mathcal{K}[B^k]^G \) of \( \mathcal{K}[B_x^k]^G \) is contained in \( \Phi((S(h^k))^W(\mathcal{R})) \) by (i). Moreover, since \( G \) is reductive, \( \mathcal{K}[B^k]^G \) is the image of \( (S(h)^{\otimes k})W(\mathcal{R})) \) by the restriction morphism. According to [J07, Theorem 2.9 and some remark], the restriction morphism \( (S(h)^{\otimes k})W(\mathcal{R}) \rightarrow (S(h)^{\otimes k})W(\mathcal{R}) \) is surjective, whence the equality \( \mathcal{K}[B^k]^G = \Phi((S(h)^{\otimes k})W(\mathcal{R})) \). □

According to Proposition 2.18,(ii) and Corollary 2.22,(i), \( \mathcal{K}[B_x^k] \) is a free extension of \( \mathcal{K}[B_x^k]^G = S(h^k) \).

**Corollary 2.23.** Let \( M \) be a graded complement to \( \mathcal{K}[B_x^k]^G \mathcal{K}[B^k] \) in \( \mathcal{K}[B^k] \).

(i) The space \( M \) contains a basis of \( \mathcal{K}[B_x^k] \) over \( S(h)^{\otimes k} \).

(ii) The intersection of \( M \) and \( S_+(h^k) \mathcal{K}[B_x^k] \) is different from \( \{0\} \).

**Proof.** (i) Since \( M \) is a graded complement to \( \mathcal{K}[B_x^k]^G \mathcal{K}[B^k] \) in \( \mathcal{K}[B^k] \), by induction on \( l \),

\[
\mathcal{K}[B^k] = M \mathcal{K}[B^k]^G + (\mathcal{K}[B^k]^G)^l \mathcal{K}[B^k].
\]
Hence $k[B^{(k)}] = MS(h)^G$ since $k[B^{(k)}]$ is graded. Then, by Corollary 2.22, (i) and (iii),
\[
 k[B^{(k)}_x] = MS(h)^G_k \quad \text{and} \quad k[B^{(k)}]_x = M + S_x(h^k)k[B^{(k)}_x].
\]
Then $M$ contains a graded complement $M'$ to $S_x(h^k)k[B^{(k)}_x]$ in $k[B^{(k)}_x]$, whence the assertion.

(ii) Suppose that $M' = M$. We expect a contradiction. According to (i), the canonical maps
\[
 M \otimes_k S(h)^G \to k[B^{(k)}_x], \quad M \otimes_k k[B^{(k)}]_x \to k[B^{(k)}]
\]
are isomorphisms. Then, according to Lemma 1.5, there exists a group action of $W(R)$ on $k[B^{(k)}]$ extending the diagonal action of $W(R)$ on $S(h)^G_k$ and such that $k[B^{(k)}]_{W(R)} = k[B^{(k)}]$ since $k[B^{(k)}] \cap S(h)^G_k = (S(h)^G_k)^{W(R)}$ by Corollary 2.22, (iii). Moreover, since $W(R)$ is finite, the subfield of invariant elements of the fraction field of $k[B^{(k)}]$ is the fraction field of $k[B^{(k)}]^{W(R)}$. Hence the action of $W(R)$ in $k[B^{(k)}]$ is trivial since $k[B^{(k)}]$ and $k[B^{(k)}]$ have the same fraction field, whence the contradiction since $(S(h)^G_k)^{W(R)}$ is strictly contained in $S(h)^G_k$. $\Box$

3. On the nullcone

Let $k \geq 2$ be an integer. Let $I$ be the ideal of $k[B^{(k)}_n]$ generated by $1 \in S_x(h^k)$.

Lemma 3.1. Let $N$ be the subscheme of $B^{(k)}_n$ defined by $I$.

(i) The ideal $I$ is prime and $N$ is isomorphic to $N^{(k)}_n$.

(ii) The variety $N$ is the inverse image of $N^{(k)}$ by $\eta_n$.

Proof. (i) By Proposition 2.18, (i), $k[N] = k[N^{(k)}_n]$, whence the assertion.

(ii) By (i), $N$ is reduced hence a variety. According to Remark 2.21, for $(g, x_1, \ldots, x_\ell)$ in $G \times h^\ell$, $\gamma_n(g, x_1, \ldots, x_\ell)$ is a zero of $I$ if and only if $x_1, \ldots, x_\ell$ are nilpotent, whence the assertion. $\Box$

Theorem 3.2. (i) The variety $N^{(k)}$ is normal if and only if so is $B^{(k)}_x$. If so, $\gamma_n = \gamma_x$ and the restriction of $\sigma$ to $B^{(k)}_x$ is the normalization morphism of $B^{(k)}$.

(ii) The variety $N^{(k)}$ is Cohen-Macaulay if and only if so is $B^{(k)}_x$.

(iii) The variety $N^{(k)}$ has rational singularities if and only if it is Cohen-Macaulay.

(iv) The variety $B^{(k)}_x$ has rational singularities if and only if it is Cohen-Macaulay.

(v) Let $I_0$ be the ideal of $k[B^{(k)}]$ generated by $k[B^{(k)}]^{G}_x$. Then $I_0$ is strictly contained in the ideal of definition of $N^{(k)}$ in $k[B^{(k)}]$.

Proof. (i) According to Proposition 2.18, (ii), $k[B^{(k)}_x]$ is a free extension of $S(h^k)$ and $k[N^{(k)}_n]$ is the quotient of $k[B^{(k)}_x]$ by the ideal generated by $S_x(h^k)$. So by [MA86, Ch. 8, Theorem 23.9 and Corollary], $0$ is a normal point of $B^{(k)}_x$ if $N^{(k)}$ is normal. As a result $B^{(k)}_x$ is normal if so is $N^{(k)}$ since $B^{(k)}_x$ is a cone and its set of normal points is open. Conversely, suppose that $B^{(k)}_x$ is normal so that $B^{(k)}_n = B^{(k)}_x$ and $\gamma_n = \gamma_x$. Moreover, by Corollary 2.8, (i), the restriction of $\sigma$ to $B^{(k)}_x$ is the normalization morphism of $B^{(k)}$. According to Proposition 2.18, (i), $k[N^{(k)}_n]$ is the image of $k[B^{(k)}_n]$ by a morphism and by Proposition 2.18, (ii), $k[N^{(k)}]$ is the image of $k[B^{(k)}]$ by this morphism, whence $k[N^{(k)}] = k[N^{(k)}_n]$.

(ii) Suppose that $N^{(k)}$ is Cohen-Macaulay. Then the localization of $k[B^{(k)}]$ at $0$ is Cohen-Macaulay by [MA86, Ch. 8, Theorem 23.9] and Proposition 2.18, (ii). By [MA86, Ch. 8, Theorem 24.5], the set of points of $B^{(k)}_x$ at which the localization is Cohen-Macaulay is open. Hence $B^{(k)}_x$ is Cohen-Macaulay since its is a cone.

Conversely suppose that $B^{(k)}_x$ is Cohen-Macaulay. According to Proposition 2.18, (ii), any basis in $S(h^k)$ is a regular sequence in $k[B^{(k)}_x]$ and $k[N^{(k)}_n]$ is the quotient of $k[B^{(k)}_x]$ by the ideal generated
by this sequence. Then the localization at 0 of $\mathbb{k}[N^{(k)}]$ is Cohen-Macaulay by [MA86, Ch. 6, Theorem 17.4 and Ch. 5, Theorem 14.1]. So, again by [MA86, Ch. 8, Theorem 24.5], $N^{(k)}$ is Cohen-Macaulay since it is a cone.

(iii) According to [KK73, p. 50], $N^{(k)}$ is Cohen-Macaulay if it has rational singularities. Suppose that $N^{(k)}$ is Cohen-Macaulay. By Lemma 2.2, (iv) and Corollary 2.3, (i), $N^{(k)}$ is smooth in codimension 1. Then, by Serre’s normality criterion [Bou98, §1, no 10, Théorème 4], $N^{(k)}$ is normal. So, by [KK73, p.50], it remains to prove that for $U$ open subset of $N^{(k)}$ and $\omega$ a regular differential form of top degree on the smooth locus of $U$, $\nu^*(\omega)$ has a regular extension to $\nu^{-1}(U)$.

Let $U'$ be the smooth locus of $U$. According to Lemma 2.2, (iv), $U \cap V_k$ is contained in $U'$. So by Corollary 2.3, (iii), $\nu^{-1}(U')$ is a big open subset of $\nu^{-1}(U)$. Let $\Omega_{\nu^{-1}(U)}$ be the sheaf of regular differential forms of top degree on $\nu^{-1}(U)$. For some open cover $O_1, \ldots, O_m$ of $\nu^{-1}(U)$, for $i = 1, \ldots, m$, the restriction of $\Omega_{\nu^{-1}(U)}$ to $O_i$ is a free $O_{O_i}$-module of rank 1. Denoting by $\omega_i$ a generator, for some regular function $a_i$ on $O_i \cap \nu^{-1}(U')$,

$$\omega \big|_{O_i \cap \nu^{-1}(U')} = a_i(\omega_i \big|_{O_i \cap \nu^{-1}(U')}).$$

Since $O_i$ is normal and $O_i \cap \nu^{-1}(U')$ is a big open subset of $O_i$, $a_i$ has a regular extension to $O_i$. Denoting again by $a_i$ this extension, $a_i \omega_i$ is a regular differential form of top degree on $O_i$ having the same restriction as $\nu^*(\omega)$ to $O_i \cap \nu^{-1}(U')$. As a result, since $\Omega_{\nu^{-1}(U)}$ is torsion free and $\nu^{-1}(U)$ is irreducible, for $1 \leq i, j \leq m$, $a_i \omega_i$ and $a_j \omega_j$ have the same restriction to $O_i \cap O_j$. Denoting by $\omega'$ the global section of $\Omega_{\nu^{-1}(U)}$ whose restriction to $O_i$ is $a_i \omega_i$ for $i = 1, \ldots, m$, $\nu^*(\omega)$ is the restriction of $\omega'$ to $\nu^{-1}(U')$, whence the assertion.

(iv) According to [KK73, p. 50], $B_x^{(k)}$ is Cohen-Macaulay if it has rational singularities. Suppose that $B_x^{(k)}$ is Cohen-Macaulay. By Lemma 2.10, (iv), $B_x^{(k)}$ is smooth in codimension 1. Then, by Serre’s normality criterion [Bou98, §1, no 10, Théorème 4], $B_x^{(k)}$ is normal. So, by [KK73, p.50], it remains to prove that for $U$ open subset of $B_x^{(k)}$ and $\omega$ a regular differential form of top degree on the smooth locus of $U$, $\gamma_x^{(k)}(\omega)$ has a regular extension to $\gamma_x^{(k)}(U)$.

Let $U'$ be the smooth locus of $U$. According to Lemma 2.10, (iv), $U \cap W_k$ is contained in $U'$. So by Corollary 2.11, $\gamma_x^{(k)}(U')$ is a big open subset of $\gamma_x^{(k)}(U)$. Let $\Omega_{\gamma_x^{(k)}(U)}$ be the sheaf of regular differential forms of top degree on $\gamma_x^{(k)}(U)$. For some open cover $O_1, \ldots, O_m$ of $\gamma_x^{(k)}(U)$, for $i = 1, \ldots, m$, the restriction of $\Omega_{\gamma_x^{(k)}(U)}$ to $O_i$ is a free $O_{O_i}$-module of rank 1. Denoting by $\omega_i$ a generator, for some regular function $a_i$ on $O_i \cap \gamma_x^{(k)}(U')$,

$$\omega \big|_{O_i \cap \gamma_x^{(k)}(U')} = a_i(\omega_i \big|_{O_i \cap \gamma_x^{(k)}(U')}).$$

Since $O_i$ is normal and $O_i \cap \gamma_x^{(k)}(U')$ is a big open subset of $O_i$, $a_i$ has a regular extension to $O_i$. Denoting again by $a_i$ this extension, $a_i \omega_i$ is a regular differential form of top degree on $O_i$ having the same restriction as $\gamma_x^{(k)}(\omega)$ to $O_i \cap \gamma_x^{(k)}(U')$. As a result, since $\Omega_{\gamma_x^{(k)}(U)}$ is torsion free and $\gamma_x^{(k)}(U)$ is irreducible, for $1 \leq i, j \leq m$, $a_i \omega_i$ and $a_j \omega_j$ have the same restriction to $O_i \cap O_j$. Denoting by $\omega'$ the global section of $\Omega_{\gamma_x^{(k)}(U)}$ whose restriction to $O_i$ is $a_i \omega_i$ for $i = 1, \ldots, m$, $\gamma_x^{(k)}(\omega)$ is the restriction of $\omega'$ to $\gamma_x^{(k)}(U')$, whence the assertion.

(v) Since $\mathbb{k}[B^{(k)}] \subset S_+(b^k)$, $I_0$ is contained in $I \cap \mathbb{k}[B^{(k)}]$. According to Lemma 3.1, (ii) and (i), $I \cap \mathbb{k}[B^{(k)}]$ is the ideal of definition of $N^{(k)}$ in $\mathbb{k}[B^{(k)}]$. Let $M$ be a graded complement of $\mathbb{k}[B^{(k)}] \subset \mathbb{k}[B^{(k)}]$ in $\mathbb{k}[B^{(k)}]$. According to Corollary 2.23, (ii), $I \cap M$ is different from $\{0\}$. Hence $I_0$ is strictly contained in $I \cap \mathbb{k}[B^{(k)}]$, whence the assertion.

\[ \square \]

Remark 3.3. According to [VX15, 6.2], for $g$ simple of type $B_2$, $N^{(2)}$ is not normal and according to [VX15, Theorem 6.1], for $g = sl_3$, $N^{(k)}$ has rational singularities for all $k$. 

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Summarizing the results of the preceding subsections, Theorem 1.1.(i), (ii), (iii), (iv), (vii) are
given by Theorem 3.2, Theorem 1.1.(v) is given by Proposition 2.18,(ii) and Corollary 2.22,(i) and
Theorem 1.1.(vi) is given by Corollary 2.22,(iii). To complete Theorem 1.1, recall that \( \kappa \) is
the normalization morphism of \( N^{(k)} \) and denote by \( \eta \) the normalization morphism of \( B^{(k)}_x \).

**Proposition 3.4.** (i) The morphism \( \eta \) is a homeomorphism.

(ii) The morphism \( \kappa \) is a homeomorphism.

*Proof.* Recall that the morphisms

\[
G \times_B b \longrightarrow G/B \times g \quad \text{and} \quad G \times_B b^k \longrightarrow G/B \times g^k
\]

are closed embeddings. For \( x = (x_1, \ldots, x_k, y_1, \ldots, y_k) \) in \( B^{(k)}_x \), denote by \( B_x \) the
subset of Borel subalgebras \( b' \) of \( g \) such that \( \chi_n(b', x_i) = (x_i, y_i) \) for \( i = 1, \ldots, k \). Then \( \gamma_x^{-1}(x) = B_x \times \{(x_1, \ldots, x)\} \). From the two commutative diagrams

\[
\begin{array}{ccc}
G \times_B b^k & \xrightarrow{\gamma_n} & B^{(k)}_n, \\
\downarrow{\gamma_x} & \downarrow{\eta} & \\
B^{(k)}_x & \xrightarrow{\nu} & N^{(k)}_n \\
\end{array}
\]

we deduce that it suffices to prove that \( B_x \) is connected for all \( x \) in \( B^{(k)}_x \) since \( \nu \) is the restriction of \( \gamma_x \) to \( G \times_B u^k \).

For \( \beta \) a simple root, denote by \( s_\beta \) the reflection of \( h \) with respect to \( \beta \). For \( w \) in \( W(\mathbb{R}) \) denote by \( l(w) \) its length with respect to the set of simple roots, \( n_w \) a representative of \( w \) in \( N_G(h) \) and set \( w(b) := n_w(b) \). Let \( x = (x_1, \ldots, x_k; y_1, \ldots, y_k) \) be in \( B^{(k)}_x \) and \( b' \) and \( b'' \) in \( B_x \). By Bruhat decomposition of \( G \), for some \( (g, b, w) \) in \( G \times B \times W(\mathbb{R}) \), \( b' = g(b) \), \( b'' = gw(b) \). Set:

\[
l(w) := q, \quad u_i := g^{-1}(x_i), \quad v_i := b^{-1}(u_i)
\]

for \( i = 1, \ldots, k \). In particular, \( v := (v_1, \ldots, v_k, y_1, \ldots, y_k) \) is in \( B^{(k)}_x \) and \( b \) and \( w(b) \) are in \( B_v \) and it
suffices to prove \( b \) and \( w(b) \) are in the same connected component of \( B_v \) since \( x = gbv \). It will be a consequence of the following claim.

**Claim 3.5.** There exist a sequence \( L_1, \ldots, L_q \) of projective lines contained in \( B_v \) and a sequence \( b_0, \ldots, b_q \) in \( B_v \) such that

\[
b = b_0, \quad w(b) = b_q, \quad b_{i-1} \in L_i, \quad b_i \in L_i
\]

for \( i = 1, \ldots, q \).

Prove the claim by induction on \( q \). For \( q = 0 \), \( b = w(b) \). Suppose that \( q = 1 \) and \( w = s_\beta \) for some
simple root \( \beta \). Then \( v_1, \ldots, v_k \) are in \( b \cap s_\beta(b) \) and for \( i = 1, \ldots, k \),

\[
\chi_n(b, v_i) = \chi_n(s_\beta(b), v_i) = (v_i, y_i).
\]

Let \( p_\beta \) be the parabolic subalgebra \( g^{-\beta} + b \) and \( l_\beta \) the reductive factor containing \( b \). The set \( L_\beta \) of
Borel subalgebras of \( g \), contained in \( p_\beta \), is a projective line. In the case \( g = l_\beta \), \( N^{(k)}_b \) is normal and \( \eta \) is
an isomorphism by Theorem 3.2,(i). Then, by Zariski’s Main Theorem [Mu88, §9], the fibers of \( \gamma_x \) are connected. So, \( L_\beta \) is contained in \( B_v \) since \( b \) and \( s_\beta(b) \) are two different points of \( B_v \cap L_\beta \).
Suppose the claim true for the integers smaller than \( q \). Let \( w = s_1 \cdots s_q \) be a reduced decomposition of \( w \) and set \( w' := s_1 \cdots s_{q-1} \). For \( i = 1, \ldots, q \), let \( \beta_i \) be the simple root such that \( s_i = s_{\beta_i} \). For \( i = 1, \ldots, k \),

\[
 v_i \in \mathfrak{h} \oplus \bigoplus_{y \in \mathbb{R}_+} g^s(y) \quad \text{and} \quad n_w^{-1}(v_i) \in \mathfrak{h} \oplus \bigoplus_{y \in \mathbb{R}_+} g^s(y).
\]

Since \( s_1 \cdots s_q \) is the reduced decomposition of \( w \), \( w(\beta_q) \) is a negative root, whence

\[
 \{ \gamma \in \mathbb{R}_+ \mid w(\gamma) \in \mathbb{R}_+ \} \subset \mathbb{R}_+ \setminus \{ \beta_q \}.
\]

As a result \( n_w^{-1}(v_1), \ldots, n_w^{-1}(v_k) \) are in \( \mathfrak{b} \). So, by induction hypothesis, there exist a sequence \( L_0, \ldots, L_{q-1} \) of projective lines contained in \( \mathfrak{B}_v \) and a sequence \( b_0, \ldots, b_{q-1} \) in \( \mathfrak{B}_v \) such that

\[
 b = b_0, \quad w'(b) = b_{q-1}, \quad b_{i-1} \in L_i, \quad b_i \in L_i
\]

for \( i = 1, \ldots, q - 1 \). By the case \( q = 1 \), for some projective line \( L'_q \), contained in \( \mathfrak{B}_{n_{w'}^{-1}, v} \), \( b \) and \( s_q(b) \) are in \( L'_q \). Then, setting \( b_q = w(b) \) and \( L_q := n_{w'}^{-1}L'_q \), the sequences \( L_1, \ldots, L_q \) and \( b_0, \ldots, b_q \) verify the conditions of the claim. \( \square \)

4. **Main varieties**

Denote by \( X \) the closure in \( \text{Gr}_s(\mathfrak{g}) \) of the orbit of \( \mathfrak{h} \) under \( B \). According to Lemma 1.7, \( G.X \) is the closure in \( \text{Gr}_s(\mathfrak{g}) \) of the orbit of \( \mathfrak{h} \) under \( G \). Let \( \mathcal{E}_0 \) and \( \mathcal{E} \) be the restrictions to \( X \) and \( G.X \) respectively of the tautological vector bundle over \( \text{Gr}_s(\mathfrak{g}) \). By definition, \( \mathcal{E} \) is the subvariety of elements \((V, x)\) of \( G.X \times \mathfrak{g} \) such that \( x \) is in \( V \) and \( \mathcal{E}_0 \) is the intersection of \( \mathcal{E} \) and \( X \times \mathfrak{b} \). In this section, we give geometric properties of \( X \) and \( G.X \). These varieties play an important role in the study of the generalized commuting varieties and isospectrale commuting varieties as is suggested by Theorem 1.3 and it will be shown in two future notes.

4.1. For \( \alpha \) in \( \mathcal{R} \), set \( V_{\alpha} := \mathfrak{h} \oplus g^\alpha \) and denote by \( X_\alpha \) the closure in \( \text{Gr}_s(\mathfrak{g}) \) of the orbit of \( V_{\alpha} \) under \( B \).

**Lemma 4.1.** Let \( \alpha \) be in \( \mathcal{R}_+ \). Let \( \mathfrak{p} \) be a parabolic subalgebra containing \( \mathfrak{b} \) and let \( P \) be its normalizer in \( G \).

(i) The subset \( P.X \) of \( \text{Gr}_s(\mathfrak{g}) \) is the closure in \( \text{Gr}_s(\mathfrak{g}) \) of the orbit of \( \mathfrak{h} \) under \( P \).

(ii) The closed set \( X_\alpha \) of \( \text{Gr}_s(\mathfrak{g}) \) is an irreducible component of \( X \setminus B.\mathfrak{h} \).

(iii) The set \( P.X_\alpha \) is an irreducible component of \( P.X \setminus P.\mathfrak{h} \).

(iv) The varieties \( X \setminus B.\mathfrak{h} \) and \( P.X \setminus P.\mathfrak{h} \) are equidimensional of codimension 1 in \( X \) and \( P.X \) respectively.

**Proof.** (i) Since \( X \) is a \( B \)-invariant closed subset of \( \text{Gr}_s(\mathfrak{g}) \), \( P.X \) is a closed subset of \( \text{Gr}_s(\mathfrak{g}) \) by Lemma 1.7. Hence \( P.\mathfrak{h} \) is contained in \( P.X \) since \( \mathfrak{h} \) is in \( X \), whence the assertion since \( P.\mathfrak{h} \) is a \( P \)-invariant subset containing \( X \).

(ii) Denoting by \( H_\alpha \) the coroot of \( \alpha \),

\[
 \lim_{t \to \infty} \exp(t \text{ad} x_\alpha)(-1/2t H_\alpha) = x_\alpha.
\]

So \( V_{\alpha} \) is in the closure of the orbit of \( \mathfrak{h} \) under the one parameter subgroup of \( G \) generated by \( \text{ad} x_\alpha \). As a result, \( X_\alpha \) is a closed subset of \( X \setminus B.\mathfrak{h} \) since \( V_{\alpha} \) is not a Cartan subalgebra. Moreover, \( X_\alpha \) has dimension \( n - 1 \) since the normalizer of \( V_{\alpha} \) in \( \mathfrak{g} \) is \( \mathfrak{h} + g^\alpha \). Hence \( X_\alpha \) is an irreducible component of \( X \setminus B.\mathfrak{h} \) since \( X \) has dimension \( n \).
(iii) Since $X_\alpha$ is a $B$-invariant closed subset of $\text{Gr}_k(\mathfrak{g})$, $P.X_\alpha$ is a closed subset of $\text{Gr}_k(\mathfrak{g})$ by Lemma 1.7. According to (ii), $P.X_\alpha$ is contained in $P.X \setminus P.\mathfrak{h}$ and it has dimension $\dim \mathfrak{p} - \ell - 1$, whence the assertion since $P.X$ has dimension $\dim \mathfrak{p} - \ell$.

(iv) Let $P_0$ be the unipotent radical of $P$ and let $L$ be the reductive factor of $P$ whose Lie algebra contains $\text{ad}\mathfrak{b}$. Denote by $N_\ell(\mathfrak{b})$ the normalizer of $\mathfrak{b}$ in $L$. Since $B.\mathfrak{h}$ and $P.\mathfrak{h}$ are isomorphic to $U$ and $L/N_\ell(\mathfrak{b}) \times P_0$ respectively, they are affine open subsets of $X$ and $P.X$ respectively, whence the assertion by [EGAIV, Corollaire 21.12.7].

For $x$ in $\mathfrak{g}$, set:

$$V_x := \text{span}(\varepsilon_1(x), \ldots, \varepsilon_\ell(x)).$$

**Lemma 4.2.** (i) For $(V, x)$ in $X \times \mathfrak{b}$, $(V, x)$ is in the closure of $B.(\{\mathfrak{b}\} \times \text{reg})$ in $\text{Gr}_k(\mathfrak{b}) \times \mathfrak{b}$ if and only if $x$ is in $V$.

(ii) The set $\mathcal{E}$ is the closure in $\text{Gr}_k(\mathfrak{g}) \times \mathfrak{g}$ of $G.(\{\mathfrak{b}\} \times \text{reg})$.

(iii) For $(V, x)$ in $\mathcal{E}$, $V_x$ is contained in $V$.

**Proof.** (i) Let $\mathcal{E}_0'$ be the closure of $B.(\{\mathfrak{b}\} \times \text{reg})$ in $\text{Gr}_k(\mathfrak{b}) \times \mathfrak{b}$. Then $\mathcal{E}_0'$ is a closed subset of $\mathcal{E}_0$. Let $(V, x)$ be in $\mathcal{E}_0$. Let $E$ be a complement to $V$ in $\mathfrak{b}$ and let $\Omega_E$ be the set of complements to $E$ in $\mathfrak{g}$. Then $\Omega_E$ is an open neighborhood of $V$ in $\text{Gr}_k(\mathfrak{b})$. Moreover, the map

$$\text{Hom}_k(V, E) \xrightarrow{k} \Omega_E, \quad \varphi \mapsto \kappa(\varphi) := \text{span}(v + \varphi(v) \mid v \in V).$$

is an isomorphism of varieties. Let $\Omega_E'$ be the inverse image of the set of Cartan subalgebras. Then $0$ is in the closure of $\Omega_E'$ in $\text{Hom}_k(V, E)$ since $V$ is in $X$. For all $\varphi$ in $\Omega_E', (\kappa(\varphi), x + \varphi(x))$ is in $\mathcal{E}_0'$. Hence $(V, x)$ is in $\mathcal{E}_0'$.

(ii) Let $(V, x)$ be in $\mathcal{E}$. For some $g$ in $G$, $g(V)$ is in $X$. So by (i), $(g(V), g(x))$ is in $\mathcal{E}_0$ and $(V, x)$ is in the closure of $G.(\{\mathfrak{b}\} \times \text{reg})$ in $\text{Gr}_k(\mathfrak{g}) \times \mathfrak{g}$, whence the assertion.

(iii) For $i = 1, \ldots, \ell$, let $\mathcal{E}_i$ be the set of elements $(V, x)$ of $\mathcal{E}$ such that $\varepsilon_i(x)$ is in $V$. Then $\mathcal{E}_i$ is a closed subset of $G.X \times \mathfrak{g}$, invariant under the action of $G$ on $\text{Gr}_k(\mathfrak{g}) \times \mathfrak{g}$ since $\varepsilon_i$ is a $G$-equivariant map. For all $(g, x)$ in $G \times \text{reg}$, $(g(h), (g)(x))$ is in $\mathcal{E}_i$ since $\varepsilon_i(g(x))$ centralizes $(g, x)$. Hence $\mathcal{E}_i = \mathcal{E}$ since $G.(\{\mathfrak{b}\} \times \{0\})$ is dense in $\mathcal{E}$ by (ii). As a result, for all $V$ in $G.X$ and for all $x$ in $V$, $\varepsilon_1(x), \ldots, \varepsilon_\ell(x)$ are in $V$.

**Corollary 4.3.** Let $(V, x)$ be in $\mathcal{E}$.

(i) The space $\mathfrak{z}_x$ is contained in $V_x$ and $V$.

(ii) The space $V$ is an algebraic, commutative subalgebra of $\mathfrak{g}$.

**Proof.** (i) If $x$ is regular semisimple, $V$ is a Cartan subalgebra of $\mathfrak{g}$ whence the assertion in this case by Lemma 4.2.(iii) and [Ko63, Theorem 9]. Suppose that $x$ is not regular semisimple. Let $\mathfrak{g}^{ts}$ be the nilpotent cone of $\mathfrak{g}^{\text{reg}}$ and let $\Omega_{\text{reg}}$ be the regular nilpotent orbit of $\mathfrak{g}^{\text{reg}}$. For all $y$ in $\Omega_{\text{reg}}$, $x + y$ is in $\mathfrak{z}_{\text{reg}}$ and $\varepsilon_1(x + y), \ldots, \varepsilon_\ell(x + y)$ is a basis of $\mathfrak{g}^{x + y}$ by [Ko63, Theorem 9]. Then for all $z$ in $\mathfrak{z}_x$, there exist regular functions on $\Omega_{\text{reg}}$, $a_{1,z}, \ldots, a_{\ell,z}$, such that

$$z = a_{1,z}(y)\varepsilon_1(x + y) + \cdots + a_{\ell,z}(y)\varepsilon_\ell(x + y)$$

for all $y$ in $\Omega_{\text{reg}}$. Furthermore, these functions are uniquely defined by this equality. Since $\mathfrak{g}^{\text{reg}}$ is a normal variety and $\mathfrak{g}^{\text{reg}} \setminus \Omega_{\text{reg}}$ has codimension 2 in $\mathfrak{g}^{\text{reg}}$, the functions $a_{1,z}, \ldots, a_{\ell,z}$ have regular extensions to $\mathfrak{g}^{\text{reg}}$. Denoting again by $a_{i,z}$ the regular extension of $a_{i,z}$ for $i = 1, \ldots, \ell$,

$$z = a_{1,z}(y)\varepsilon_1(x + y) + \cdots + a_{\ell,z}(y)\varepsilon_\ell(x + y)$$

for all $y$ in $\mathfrak{g}^{\text{reg}}$. As a result, $\mathfrak{z}_x$ is contained in $V_x$. Hence $\mathfrak{z}_x$ is contained in $V$ by Lemma 4.2.(iii).
(ii) Since the set of commutative subalgebras of dimension $\ell$ is closed in $\text{Gr}_r(\mathfrak{g})$, $V$ is a commutative subalgebra of $\mathfrak{g}$. According to (i), the semisimple and nilpotent components of the elements of $V$ are contained in $V$. For $x \in V \setminus \mathfrak{n}_\mathfrak{g}$, all the replica of $x$ are contained in the center of $\mathfrak{g}$. Hence $V$ is an algebraic subalgebra of $\mathfrak{g}$ by (i).

\[ \square \]

4.2. For $s$ in $\mathfrak{h}$, denote by $X^s$ the subset of elements of $X$, contained in $\mathfrak{g}^s$.

**Lemma 4.4.** Let $s$ be in $\mathfrak{h}$.

(i) The set $X^s$ is the closure in $\text{Gr}_r(\mathfrak{g}^s)$ of the orbit of $\mathfrak{h}$ under $B^s$.

(ii) The set of elements of $G.X$ containing $3_\mathfrak{s}$ is the closure in $\text{Gr}_r(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under $G^s$.

**Proof.** (i) Set $p := \mathfrak{g}^s + \mathfrak{b}$, let $P$ be the normalizer of $p$ in $G$ and let $p_u$ be the nilpotent radical of $P$. For $g$ in $P$, denote by $\overline{g}$ its image by the canonical projection from $P$ to $G^s$. Let $Z$ be the closure in $\text{Gr}_r(\mathfrak{g}) \times \text{Gr}_r(\mathfrak{g})$ of the image of the map

$$B \longrightarrow \text{Gr}_r(\mathfrak{g}) \times \text{Gr}_r(\mathfrak{g}), \quad g \mapsto (g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$$

and let $Z'$ be the subset of elements $(V, V')$ of $\text{Gr}_r(\mathfrak{g}) \times \text{Gr}_r(\mathfrak{g})$ such that

$$V' \subset \mathfrak{g}^s \cap \mathfrak{b} \quad \text{and} \quad V \subset V' \oplus p_u.$$

Then $Z'$ is a closed subset of $\text{Gr}_r(\mathfrak{g}) \times \text{Gr}_r(\mathfrak{g})$ and $Z$ is contained in $Z'$ since $(g(\mathfrak{h}), \overline{g}(\mathfrak{h}))$ is in $Z'$ for all $g$ in $B$. Since $\text{Gr}_r(\mathfrak{b})$ is a projective variety, the images of $Z$ by the projections $(V, V') \mapsto V$ and $(V, V') \mapsto V'$ are closed in $\text{Gr}_r(\mathfrak{b})$ and they are equal to $X$ and $\mathfrak{B}.\mathfrak{h}$ respectively. Furthermore, $\mathfrak{B}.\mathfrak{h}$ is contained in $X^s$.

Let $V$ be in $X^s$. For some $V'$ in $\text{Gr}_r(\mathfrak{b})$, $(V, V')$ is in $Z$. Since

$$V \subset \mathfrak{g}^s, \quad V' \subset \mathfrak{g}^s, \quad V \subset V' \oplus p_u,$$

$V = V'$ so that $V$ is in $\mathfrak{B}.\mathfrak{h}$, whence the assertion.

(ii) Since $3_\mathfrak{s}$ is contained in $\mathfrak{h}$, all element of $\overline{\mathfrak{g}}^s.\mathfrak{h}$ is an element of $G.X$ containing $3_\mathfrak{s}$. Let $V$ be in $G.X$, containing $3_\mathfrak{s}$. Since $V$ is a commutative subalgebra of $\mathfrak{g}^s$ and since $\mathfrak{g}^s \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}^s$, for some $g$ in $\mathfrak{g}^s$, $g(V)$ is contained in $\mathfrak{b} \cap \mathfrak{g}^s$. So, one can suppose that $V$ is contained in $\mathfrak{b}$. According to the Bruhat decomposition of $G$, since $X$ is $B$-invariant, for some $b$ in $U$ and for some $w$ in $W(\mathbb{R})$, $V$ is in $b.w.X$. Set:

$$\mathcal{D}_+ := \{ \alpha \in \mathcal{D}_+ \mid w(\alpha) \in \mathcal{D}_+ \}, \quad \mathcal{D}_+ := \{ \alpha \in \mathcal{D}_+ \mid w(\alpha) \notin \mathcal{D}_+ \},$$

$$u_1 := \bigoplus_{\alpha \in \mathcal{D}_+} \mathfrak{g}^w(\alpha), \quad u_2 := \bigoplus_{\alpha \in \mathcal{D}_-} \mathfrak{g}^w(\alpha), \quad u_3 := \bigoplus_{\alpha \notin \mathcal{D}_-} \mathfrak{g}^w(\alpha),$$

$$b^w := wBw^{-1}, \quad b^w := \mathfrak{h} \oplus u_1 \oplus u_3,$$

so that $\text{ad}b^w$ is the Lie algebra of $B^w$ and $w.X$ is the closure in $\text{Gr}_r(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under $b^w$. Moreover, $u$ is the direct sum of $u_1$ and $u_2$. For $i = 1, 2$, denote by $U_i$ the closed subgroup of $U$ whose Lie algebra is $\text{ad}u_i$. Then $U = U_2U_1$ and $b = b_2b_1$ with $b_1$ in $U_i$ for $i = 1, 2$. Since $w^{-1}(u_1)$ is contained in $u$ and $X$ is invariant under $B, b_2b_1w.X = b_2w.X$. Then $b_2^{-1}(V)$ is in $w.X$ and

$$b_2^{-1}(V) \subset b \cap b^w = \mathfrak{h} \oplus u_1$$

since $V$ is contained in $\mathfrak{b}$. Set:

$$u_{2,1} := u_2 \cap \mathfrak{g}^s, \quad u_{2,2} := u_2 \cap p_u$$

and for $i = 1, 2$, denote by $U_{2,i}$ the closed subgroup of $U_2$ whose Lie algebra is $\text{ad}u_{2,i}$. Then $u_2$ is the direct sum of $u_{2,1}$ and $u_{2,2}$ and $U_2 = U_{2,1}U_{2,2}$ so that $b_2 = b_{2,1}b_{2,2}$ with $b_{2,i}$ in $U_{2,i}$ for $i = 1, 2$. 

\[ \square \]
As a result, $\mathfrak{z}_s$ is contained in $b_{2,1}^{-1}(V)$ and $b_{2,2}^{-1}(\mathfrak{z}_s)$ is contained $\mathfrak{h} \oplus \mathfrak{u}_1$. Hence $b_{2,2}^{-1}(\mathfrak{z}_s) = \mathfrak{z}_s$ since $u_1 \cap u_{2,2} = \{0\}$.

Suppose $b_{2,2} \neq 1_\mathfrak{g}$. We expect a contradiction. For some $x \in u_{2,2}$, $b_{2,2} = \exp(\text{ad}x)$. The space $u_{2,2}$ is a direct sum of root spaces since so are $u_2$ and $\mathfrak{p}_u$. Let $\alpha_1, \ldots, \alpha_m$ be the positive roots such that the corresponding root spaces are contained in $u_{2,2}$. They are ordered so that for $i \leq j$, $\alpha_j - \alpha_i$ is a positive root if it is a root. For $i = 1, \ldots, m$, let $c_i$ be the coordinate of $x$ at $x_{\alpha_i}$, and let $i_0$ be the smallest integer such that $c_{i_0} \neq 0$. For all $z$ in $\mathfrak{z}_s$,

$$b_{2,2}^{-1}(z) - z - c_{i_0} \alpha_{i_0}(z) x_{\alpha_{i_0}} \in \bigoplus_{j \neq i_0} \mathfrak{g}^j,$$

whence the contradiction since for some $z$ in $\mathfrak{z}_s$, $\alpha_{i_0}(z) \neq 0$. As a result, $b_{2,1}^{-1}(V)$ is an element of $w.\mathcal{X} = \overline{\mathcal{B}^w \cdot \mathfrak{h}}$, contained in $\mathfrak{g}^i$. So, by (i), $b_{2,1}^{-1}(V)$ and $V$ are in $\overline{G^w \cdot \mathfrak{h}}$, whence the assertion.

Define a torus of $\mathfrak{g}$ as a commutative algebraic subalgebra of $\mathfrak{g}$ whose all elements are semisimple. For $\Lambda$ subset of $\mathfrak{R}$, denote by $\mathfrak{h}_\Lambda$ the intersection of the kernels of the elements of $\Lambda$.

**Corollary 4.5.** Let $V$ be in $X$. Then for some subset $\Lambda$ of $\mathfrak{R}$ and for some $g$ in $B$, $g(V)$ is the direct sum of $\mathfrak{h}_\Lambda$ and $g(V) \cap \mathfrak{u}$.

**Proof.** By Corollary 4.3,(ii), $V$ is the direct sum of a subtorus of $\mathfrak{b}$ and its intersection with $\mathfrak{u}$. So for some $g$ in $B$,

$$g(V) = g(V) \cap \mathfrak{h} \oplus g(V) \cap \mathfrak{u}.$$

Let $\Lambda$ be the set of roots such that $g(V) \cap \mathfrak{h}$ is contained in $\mathfrak{h}_\Lambda$. If $\Lambda = \mathfrak{R}$, $g(V)$ is contained in $\mathfrak{u}$. Suppose $\Lambda$ strictly contained in $\mathfrak{R}$. For some $s$ in $g(V) \cap \mathfrak{h}$, $\alpha(s) \neq 0$ for all $\alpha$ in $\mathfrak{R} \setminus \Lambda$. Since $g(V)$ is a commutative algebra, $g(V)$ is contained in $\mathfrak{g}^s$. So, by Lemma 4.4,(i), $g(V)$ is in $\overline{B^s \cdot \mathfrak{h}}$. In particular, by Corollary 4.3,(ii), $\mathfrak{h}_\Lambda$ is contained in $g(V)$ since $\mathfrak{h}_\Lambda$ is the center of $\mathfrak{g}^s$, whence $\mathfrak{h}_\Lambda = g(V) \cap \mathfrak{h}$ and $g(V)$ is the direct sum of $\mathfrak{h}_\Lambda$ and $g(V) \cap \mathfrak{u}$.

4.3. For $x$ in $\mathfrak{g}$, denote by $Z_x$ the subset of elements of $G.X$ containing $x$ and by $(G^s)_0$ the identity component of $G^s$.

**Lemma 4.6.** Let $x$ be in $\mathfrak{N}_x$ and let $Z$ be an irreducible component of $Z_x$. Suppose that some element of $Z$ is not contained in $\mathfrak{R}_x$.

(i) For some torus $s$ of $\mathfrak{g}^s$, all element of a dense open subset of $Z$ contains a conjugate of $s$ under $(G^s)_0$.

(ii) For some $s$ in $\mathfrak{z}$ and for some irreducible component $Z_1$ of $Z_{s+x}$, $Z$ is the closure in $\text{Gr}_x(\mathfrak{g})$ of $(G^s)_0 Z_1$.

(iii) If $Z_1$ has dimension smaller than $\dim \mathfrak{g}^{s+x} - \ell$, then $Z$ has dimension smaller than $\dim \mathfrak{g}^x - \ell$.

**Proof.** (i) After some conjugation by an element of $G$, we can suppose that $\mathfrak{g}^s \cap \mathfrak{b}$ and $\mathfrak{g}^s \cap \mathfrak{h}$ are a Borel subalgebra and a maximal torus of $\mathfrak{g}^s$ respectively. Let $Z_0$ be the subset of elements of $Z$ contained in $\mathfrak{b}$ and let $(B^s)_0$ be the identity component of $B^s$. Since $Z$ is an irreducible component of $Z_x$, $Z$ is invariant under $(G^s)_0$ and $Z = (G^s)_0 Z_0$. Since $(G^s)_0/(B^s)_0$ is a projective variety, according to the proof of Lemma 1.7, $(G^s)_0 Z_0$ is a closed subset of $Z$ for all closed subset $Z_0$ of $Z$. Hence for some irreducible component $Z_1$ of $Z_0$, $Z = (G^s)_0 Z_1$.

For $\Lambda$ subset of $\mathfrak{R}$, denote by $Z_{x, \Lambda}$ the subset of elements $V$ of $Z_x$ such that

$$g(V) = \mathfrak{h}_\Lambda \oplus g(V) \cap \mathfrak{u}$$
for some $g$ in $(B^0)_0$. According to Corollary 4.5, $Z_*$ is the union of $Z_{s,A}, \Lambda \subset \mathcal{R}$. Since all element of $Z_{s,A}$ is contained in $\mathfrak{h}_A + u$,

$$\overline{Z_{s,A}} \subset \bigcup_{R \supset \Lambda} Z_{s,A}.$$

So, by induction on $|\mathcal{R} \setminus \Lambda|$, $Z_{s,A}$ is a constructible subset of $Z_*$. Then, since $\mathcal{R}$ is finite, for some subset $A$ of $\mathcal{R}$, $Z_{s,A}$ is dense in $Z_*$. As a result, $(G^s)_0 Z_{s,A}$ contains a dense open subset of $Z$ and for all $V$ in $(G^s)_0 Z_{s,A}$, the biggest torus contained in $V$ is conjugate to $\mathfrak{h}_A$ under $(G^s)_0$.

(ii) For some $s$ in $s$, $\mathfrak{g}^s$ is the centralizer of $s$ in $g$. Let $Z'$ be the subset of elements of $Z$ containing $s$. Then $Z'$ is contained in $Z_{s,+}$ and according to Corollary 4.3, $Z'$ is the subset of elements of $Z$, containing $s$. By (i), for some irreducible component $Z'_1$ of $Z'$, $(G^s)_0 Z'_1$ is dense in $Z$. Let $Z_1$ be an irreducible component of $Z_{s,+}$, containing $Z'_1$. According to Corollary 4.3(ii), $Z_1$ is contained in $Z_*$ since $x$ is the nilpotent component of $s + x$. So $Z_1 = Z'_1$ and $(G^s)_0 Z_1$ is dense in $Z$.

(iii) Since $Z_1$ is an irreducible component of $Z_{s,+}$, $Z_1$ is invariant under the identity component of $G^{s,+}$. Moreover, $G^{s,+}$ is contained in $G^s$ since $x$ is the nilpotent component of $s + x$. As a result, by (ii),

$$\dim Z \leq \dim \mathfrak{g}^x - \dim \mathfrak{g}^{s,+} + \dim Z_1,$$

whence the assertion. \qed

Denote by $C_h$ the $G$-invariant closed cone generated by $h$ with $h$ in $\mathfrak{h}$ such that $\beta(h) = 2$ for all $\beta$ in $\Pi$.

**Lemma 4.7.** Suppose $\mathfrak{g}$ semisimple. Let $\Gamma$ be the closure in $\text{Gr}_t(\mathfrak{g}) \times \mathfrak{g}$ of the image of the map

$$\mathfrak{k}^* \times G \longrightarrow \text{Gr}_t(\mathfrak{g}) \times \mathfrak{g}, \quad (t, g) \mapsto (g(\mathfrak{h}), tg(h))$$

and $\Gamma_0$ the intersection of $\Gamma$ and $\text{Gr}_t(\mathfrak{g}) \times \mathfrak{h}$.

(i) The subvariety $\Gamma$ of $\text{Gr}_t(\mathfrak{g}) \times \mathfrak{g}$ has dimension $2n + 1$. Moreover, $\Gamma$ is contained in $\mathcal{E}$.

(ii) The varieties $G.X$ and $C_h$ are the images of $\Gamma$ by the first and second projections respectively.

(iii) The subvariety $\Gamma_0$ of $\Gamma$ is equidimensional of codimension 1.

(iv) For $x$ nilpotent in $\mathfrak{g}$, the subvariety of elements $V$ of $G.X$, containing $x$ and contained in $G(x)$, has dimension at most $\dim \mathfrak{g}^x - \ell$.

**Proof.** (i) Since the stabilizer of $(\mathfrak{h}, h)$ in $\mathfrak{k}^* \times G$ equals $\{1\} \times H$, $\Gamma$ has dimension $2n + 1$. Since $tg(h)$ is in $g(\mathfrak{h})$ for all $(t, g)$ in $\mathfrak{k}^* \times G$ and $\mathcal{E}$ is a closed subset of $\text{Gr}_t(\mathfrak{g}) \times \mathfrak{g}$, $\Gamma$ is contained in $\mathcal{E}$.

(ii) Since $\text{Gr}_t(\mathfrak{g})$ is a projective variety, the image of $\Gamma$ by the second projection is closed in $\mathfrak{g}$. So, it equals $C_h$ since it is contained in $C_h$ and it contains the cone generated by $G.h$. Let $Y$ be the image of $\Gamma$ by the first projection. Since $\Gamma$ is a closed subset of $\text{Gr}_t(\mathfrak{g}) \times \mathfrak{g}$, invariant by the automorphisms $(V, x) \mapsto (V, tx)$ with $t$ in $\mathfrak{k}^*$, $X \times \{0\}$ is the intersection of $\Gamma$ and $\text{Gr}_t(\mathfrak{g}) \times \{0\}$. Then $Y$ is a closed subset of $\text{Gr}_t(\mathfrak{g})$ containing $G.\mathfrak{h}$. Moreover $\Gamma$ is contained in the closed subset $G.X \times \mathfrak{g}$ of $\text{Gr}_t(\mathfrak{g}) \times \mathfrak{g}$. Hence $Y = G.X$.

(iii) The subvariety $C_h$ of $\mathfrak{g}$ has dimension $2n + 1$ and the nullvariety of $p_1$ in $C_h$ is contained in $\mathfrak{h}$ since it is the nullvariety in $\mathfrak{g}$ of the polynomials $p_1, \ldots, p_\ell$. Hence $\mathfrak{h}_0$ is the nullvariety of $p_1$ in $C_h$ and $\Gamma_0$ is the nullvariety in $\Gamma$ of the function $(V, x) \mapsto p_1(x)$. So $\Gamma_0$ is equidimensional of codimension 1 in $\Gamma$.

(iv) Let $T$ be the subset of elements $V$ of $G.X$, containing $x$ and contained in $G(x)$. Denote by $\Gamma_T$ the inverse image of $G.T$ by the projection $\Gamma \longrightarrow G.X$. Then $\Gamma_T$ is contained in $\Gamma_0$. Since all
element of $T$ contains $x$ and is contained in $G(x)$ and since $\Gamma_T$ is invariant under $G$, the image of $\Gamma_T$ by the second projection is equal to $G(x)$. Moreover, $T \times \{x\} \subset G.X \times \{x\} \cap \Gamma_T$. Hence

$$\dim \Gamma_T \geq \dim T + \dim g - \dim g^x.$$ 

By (i) and (iii),

$$\dim \Gamma_T \leq \dim g - \ell$$

since $\Gamma_T$ is contained in $\Gamma_0$. Hence $T$ has dimension at most $\dim g^x - \ell$. \hfill\qed

When $g$ is semisimple, denote by $(G.X)_u$ the subset of elements of $G.X$ contained in $\mathfrak{R}_g$.

**Corollary 4.8.** Suppose $g$ semisimple. Let $x$ be in $\mathfrak{R}_g$.

(i) The variety $(G.X)_u$ has dimension at most $2n - \ell$.

(ii) The variety $Z_x \cap (G.X)_u$ has dimension at most $\dim g^x - \ell$.

**Proof.** (i) Let $T$ be an irreducible component of $(G.X)_u$ and let $E_T$ be the restriction to $T$ of the vector bundle $E$ over $G.X$. Then $E_T$ is irreducible and has dimension $\dim T + \ell$. Denoting by $Y$ the image of the projection $E_T \longrightarrow g$, $Y$ is an irreducible closed subvariety of $g$ contained in $\mathfrak{R}_g$. The subvariety $(G.X)_u$ of $G.X$ is invariant under $G$ since so is $\mathfrak{R}_g$. Hence $E_T$ and $Y$ are $G$-invariant and for some $y$ in $\mathfrak{R}_g$, $Y = \overline{G(y)}$ since $\mathfrak{R}_g$ is a finite union of orbits. Denoting by $F_y$ the fiber at $y$ of the projection $E_T \longrightarrow Y$, $V$ is contained in $\overline{G(y)}$ and contains $y$ for all $V$ in $F_y$. So, by Lemma 4.7,(iv),

$$\dim F_y \leq \dim g^y - \ell.$$ 

Since the projection is $G$-equivariant, this inequality holds for the fibers at the elements of $G(y)$. Hence,

$$\dim E_T \leq \dim g - \ell \text{ and } \dim T \leq 2n - \ell.$$ 

(ii) Let $Z$ be an irreducible component of $Z_x \cap (G.X)_u$ and let $T$ be an irreducible component of $(G.X)_u$, containing $Z$. Let $E_T$ and $Y$ be as in (i). Then $G(x)$ is contained in $Y$ and the inverse image of $G(x)$ in $E_T$ has dimension at least $\dim G(x) + \dim Z$. So, by (i),

$$\dim G(x) + \dim Z \leq \dim g - \ell,$$

whence the assertion. \hfill\qed

**Theorem 4.9.** For $x$ in $g$, the variety of elements of $G.X$, containing $x$, has dimension at most $\dim g^x - \ell$.

**Proof.** Prove the theorem by induction on $\dim g$. If $g$ is commutative, $G.X = \{g\}$. If the derived Lie algebra of $g$ is simple of dimension 3, $G.X$ has dimension 2 and for $x$ not in the center of $g$, $Z_x = \{g^x\}$. Suppose the theorem true for all reductive Lie algebra of dimension strictly smaller than $\dim g$. Let $x$ be in $g$. Since $G.X$ has dimension $\dim g - \ell$, we can suppose that $x$ is not in the center of $g$. Suppose that $x$ is not nilpotent. Then $g^x$ has dimension strictly smaller than $\dim g$ and all element of $G.X$ containing $x$ is contained in $g^x$ and contains the center of $g^x$ by Corollary 4.3.(i). So, by Lemma 4.4,(ii), $Z_x$ is contained in $G^{x,\overline{\mathfrak{h}}}$, whence the theorem in this case by induction hypothesis. As a result, by Lemma 4.6, for all $x$ in $g$, all irreducible component of $Z_x$, containing an element not contained in $\mathfrak{R}_g$, has dimension at most $\dim g^x - \ell$.

Let $x$ be a nilpotent element of $g$. Denoting by $Z'_x$ the subset of elements of $G.(\mathfrak{h} \cap [g, g])$ containing $x$, $Z_x$ is the image of $Z'_x$ by the map $V \mapsto V + \delta_x$, whence the theorem by Corollary 4.8. \hfill\qed
4.4. Let \( s \) be in \( \mathfrak{h} \setminus \{0\} \). Set \( \mathfrak{p} := \mathfrak{g}^s + \mathfrak{b} \) and denote by \( \mathfrak{p}_u \) the nilpotent radical of \( \mathfrak{p} \). Let \( P \) be the normalizer of \( \mathfrak{p} \) in \( G \) and let \( P_u \) be its unipotent radical. For a nilpotent orbit \( \Omega \) of \( G^s \) in \( \mathfrak{g}^s \), denote by \( \Omega^\# \) the induced orbit by \( \Omega \) from \( \mathfrak{g}^s \) to \( \mathfrak{g} \).

**Lemma 4.10.** Let \( Y \) be a \( G \)-invariant irreducible closed subset of \( \mathfrak{g} \) and let \( Y' \) be the union of \( G \)-orbits of maximal dimension in \( Y \). Suppose that \( s \) is the semisimple component of an element \( x \) of \( Y' \).

(i) By [Ko63, §3.2, Lemma 5], \( G^s \) is connected and \( P = P_uG^s \). For all \( y \) in \( \mathfrak{p} \) and for all \( g \) in \( P_u \), \( g(y) \) is in \( y + \mathfrak{p}_u \). Hence \( Y_1 \) is invariant under \( P \) since it is invariant under \( G^s \). Moreover, it is a closed subset of \( \mathfrak{p} \) since \( \overline{\mathfrak{g}}^s + \overline{\Omega} \) is a closed subset of \( \mathfrak{g}^s \).

(ii) According to (i) and Lemma 1.7, \( G(Y_1) \) is a closed subset of \( \mathfrak{g} \). According to [CMa93, Theorem 7.1.1], \( G^s \cap (\Omega + \mathfrak{p}_u) \) is a \( P \)-orbit and the centralizers in \( \mathfrak{g} \) of its elements are contained in \( \mathfrak{p} \). For \( y \) in \( \Omega^\# \cap (\Omega + \mathfrak{p}_u) \) and for \( g \) in \( G \), if \( g(y) \) is in \( Y_1 \) then it is in \( \Omega + \mathfrak{p}_u \) since it is nilpotent. So, for \( y \) in \( \Omega^\# \cap (\Omega + \mathfrak{p}_u) \), the subset of elements \( g \) of \( G \) such that \( g(y) \) is in \( Y_1 \) has dimension \( \dim \mathfrak{p} \). As a result,

\[
\dim G(Y_1) = \dim G \times_P Y_1 = \dim \mathfrak{p}_u + \dim Y_1.
\]

Since \( \dim \mathfrak{g}^s = \dim \mathfrak{g}^s - \dim \Omega \),

\[
\begin{align*}
\dim Y_1 &= \dim \mathfrak{g}^s + \dim \mathfrak{p}_u + \dim \mathfrak{g}^s - \dim \mathfrak{g}^s \\
\dim G(Y_1) &= \dim \mathfrak{g}^s + 2\dim \mathfrak{p}_u + \dim \mathfrak{g}^s - \dim \mathfrak{g}^s \\
&= \dim \mathfrak{g}^s + \dim G(x).
\end{align*}
\]

(iii) Let \( \tau \) be the canonical morphism from \( \mathfrak{g} \) to its categorical quotient \( \mathfrak{g}/G \) under \( G \) and let \( Z \) be the closure in \( \mathfrak{g}/G \) of \( \tau(Y) \). Since \( Y \) is irreducible, \( Z \) is irreducible and there exists an irreducible component \( \overline{Z} \) of the preimage of \( Z \) in \( \mathfrak{h} \) whose image in \( \mathfrak{g}/G \) equals \( Z \). Since the set of conjugacy classes under \( G \) of the centralizers of the elements of \( \mathfrak{h} \) in \( \mathfrak{g} \) is finite, for some nonempty open subset \( Z^\# \) of \( \overline{Z} \), the centralizers of its elements are conjugate under \( G \). The image of \( Z^\# \) in \( \mathfrak{g}/G \) contains a dense open subset \( Z' \) of \( Z \). Let \( Y'' \) be the inverse image of \( Z' \) by the restriction of \( \tau \) to \( Y' \). Then \( Y'' \) is a dense open subset of \( Y \) and the centralizers in \( \mathfrak{g} \) of the semisimple components of its elements are conjugate under \( G \).

(iv) Suppose that \( x \) is in \( Y'' \). Let \( Z_Y \) be the set of elements \( y \) of \( Y'' \) such that \( \mathfrak{g}^y = \mathfrak{g}^x \). Then \( G.Z_Y = Y'' \). For all nilpotent orbit \( \Omega \) of \( G^s \) in \( \mathfrak{g}^s \), set:

\[
Y_\Omega = \mathfrak{g}^s + \overline{\Omega} + \mathfrak{p}_u
\]

Then \( Z_Y \) is contained in the union of the \( Y_\Omega \)'s. Hence \( Y'' \) is contained in the union of the \( G(Y_\Omega) \)'s. According to (ii), \( G(Y_\Omega) \) is a closed subset of \( \mathfrak{g} \). Hence \( Y \) is contained in the union of the \( G(Y_\Omega) \)'s since \( Y'' \) is dense in \( Y \). Then \( Y \) is contained in \( G(Y_\Omega) \) for some \( \Omega \) since \( Y \) is irreducible and there are finitely many nilpotent orbits in \( \mathfrak{g}^s \), whence the assertion. \( \Box \)

**Theorem 4.11.** (i) The variety \( G.X \) is the union of \( G.\mathfrak{h} \) and the \( G.X_\beta \)'s, \( \beta \in \Pi \).

(ii) The variety \( X \) is the union of \( U.\mathfrak{h} \) and the \( X_\alpha \)'s, \( \alpha \in \mathcal{R}_+ \).
Proof. Let $\mu$ be the map

$$Gr_\ell([g, g]) \rightarrow Gr_\ell(g), \quad V \mapsto \nu \ell + V$$

with $\ell'$ the rank of $[g, g]$ and set:

$$X_d := B.(\mathfrak{b} \cap [g, g]), \quad \nu X_{\alpha, d} := B.(V_\alpha \cap [g, g])$$

for $\alpha$ in $R_+$. Then $X, G.X, X_\alpha, G.X_\alpha$ are the images of $X_d, G.X_d, X_{\alpha, d}, G.X_{\alpha, d}$ by $\mu$ respectively. So we can suppose $g$ semisimple.

(i) For $\ell = 1$, $g$ is simple of dimension 3. In this case, $G.X$ is the union of $G.\mathfrak{b}$ and $G.\mathfrak{g}^\ell$. So, we can suppose $\ell \geq 2$. According to Lemma 4.1, for $\alpha$ in $R_+, G.X_\alpha$ is an irreducible component of $G.X \setminus G.\mathfrak{b}$. Moreover, for all $\beta$ in $\Pi \cap W(\mathfrak{X})(\alpha)$, $G.X_\alpha = G.X_\beta$ since $V_\alpha$ and $V_\beta$ are conjugate under $N_G(\mathfrak{b})$.

Let $T$ be an irreducible component of $G.X \setminus G.\mathfrak{b}$. Set:

$$E_T := E \cap T \times g$$

and denote by $Y$ the image of $E_T$ by the second projection. Then $Y$ is closed in $g$ since $Gr_\ell(g)$ is a projective variety. Since $E_T$ is a vector bundle over $T$ and since $T$ is irreducible, $E_T$ is irreducible and so is $Y$. Since $T$ is an irreducible component of $G.X \setminus G.\mathfrak{b}$, $E_T$ and $Y$ are $G$-invariant. According to Lemma 4.1, $T$ has codimension 1 in $G.X$. Hence, by Corollary 4.8, $Y$ is not contained in the nilpotent cone since $\ell \geq 2$. Let $Y'$ be the set of elements $x$ of $Y$ such that $g^x$ has minimal dimension. According to Lemma 4.10, for some $x$ in $Y'$,

$$\dim Y \leq \dim G(x) + \dim \nu \ell$$

and according to Theorem 4.9,

$$\dim E_T \leq \dim G(x) + \nu \ell + \dim g^x - \ell = \dim g + \dim \nu \ell - \ell$$

Hence $E_T$ has dimension at most $2n + \dim \nu \ell$ and $\dim \nu \ell = \ell - 1$ since $T$ has codimension 1 in $G.X$. As a result, $x_\ell$ is subregular and for some $g$ in $G$, $g(\nu \ell)$ is the kernel of a positive root $\alpha$. Denoting by $s_\alpha$ the subalgebra of $g$ generated by $g^\alpha$ and $g^{-\alpha}$, $g(\nu \ell, \alpha)$ is the direct sum of $h_\alpha$ and $s_\alpha$. Since the maximal commutative subalgebras of $s_\alpha$ have dimension 1, a commutative subalgebra of dimension $\ell$ of $g(\nu \ell, \alpha)$ is either a Cartan subalgebra of $g$ or conjugate to $V_\alpha$ under the adjoint group of $g(\nu \ell, \alpha)$. As a result, $V_\alpha$ is in $T$ and $T = G.V_\alpha = G.X_\alpha$ since $T$ is $G$-invariant, whence the assertion.

(ii) According to Lemma 4.1, for $\alpha$ in $R_+$, $X_\alpha$ is an irreducible component of $X \setminus B.\mathfrak{b}$. Let $g_1, \ldots, g_m$ be the simple factors of $g$. For $j = 1, \ldots, m$, denote by $X_j$ the closure in $Gr_{\mathfrak{h}_j}(\mathfrak{g}_j)$ of the orbit of $\mathfrak{b} \cap \mathfrak{g}_j$. Then $X = X_1 \times \cdots \times X_m$ and the complement to $B.\mathfrak{b}$ in $X$ is the union of the

$$X_1 \times \cdots \times X_{j-1} \times (X_j \setminus B.(\mathfrak{b} \cap \mathfrak{g}_j)) \times X_{j+1} \times \cdots \times X_m$$

So, we can suppose $g$ simple. Consider

$$b = p_0 \subset \cdots \subset p_\ell = g$$

an increasing sequence of parabolic subalgebras verifying the following condition: for $i = 0, \ldots, \ell - 1$, there is no parabolic subalgebra $q$ of $g$ such that

$$p_i \subset q \subset p_{i+1}.$$
Prove by induction on $\ell - i$ that for all sequence of parabolic subalgebras verifying the above condition, the $X_{i,\alpha}$’s, $\alpha \in R_+$, are the irreducible components of $X_i \setminus P_i.b$.

For $i = \ell$, it results from (i). Suppose that it is true for $i + 1$. According to Lemma 4.1.(iii), the $X_{i,\alpha}$’s are irreducible components of $X_i \setminus P_i.b$.

Claim 4.12. Let $T$ be an irreducible component of $X_i \setminus P_i.b$ such that $P_i$ is its stabilizer in $P_{i+1}$. Then $T = X_{i,\alpha}$ for some $\alpha \in R_+$.

Proof. According to the induction hypothesis, $T$ is contained in $X_{i+1,\alpha}$ for some $\alpha \in R_+$. According to Lemma 4.1.(iv), $T$ has codimension 1 in $X_i$ so that $P_{i+1}.T$ and $X_{i+1,\alpha}$ have the same dimension. Then they are equal and $T$ contains $g^+$ for some $x$ in $b_{\text{reg}}$ such that $x$ is a subregular element belonging to $b$. Denoting by $\alpha'$ the positive root such that $\alpha'(x) = 0$, $g^+ = V_{\alpha'}$ since $V_{\alpha'}$ is the commutative subalgebra contained in $b$ and containing $h_{\alpha'}$, which is not Cartan, so that $T = X_{i,\alpha'}$.

Suppose that $X_i \setminus P_i.b$ is not the union of the $X_{i,\alpha}$’s, $\alpha \in R_+$. We expect a contradiction. Let $T$ be an irreducible component of $X_i \setminus P_i.b$, different from $X_{i,\alpha}$ for all $\alpha$. According to Claim 4.12 and according to the condition verified by the sequence, $T$ is invariant under $P_{i+1}$. Moreover, according to Claim 4.12, it is so for all sequence $p'_0, \ldots, p'_\ell$ of parabolic subalgebras verifying the above condition and such that $p'_j = p_j$ for $j = 0, \ldots, i$. As a result, for all simple root $\beta$ such that $g^{-\beta}$ is not in $p_i$, $T$ is invariant under the one parameter subgroup of $G$ generated by $\text{ad}g^{-\beta}$. Hence $T$ is invariant under $G$. It is impossible since for $x$ in $g \setminus \{0\}$, the orbit $G(x)$ is not contained in $p_i$ since $g$ is simple, whence the assertion. \hfill $\square$

4.5. Let $X'$ be the subset of $g^+$ with $x$ in $b_{\text{reg}}$ such that $x$ is regular or subregular. For $\alpha \in R_+$, denote by $\theta_{\alpha}$ the map

$$
\mathfrak{k} \longrightarrow \mathcal{X}, \quad t \mapsto \exp(\text{rad} x_{\alpha}.h).
$$

According to [Sh94, Ch. VI, Theorem 1], $\theta_{\alpha}$ has a regular extension to $\mathbb{P}^1(\mathfrak{k})$, also denoted by $\theta_{\alpha}$. Set $Z_{\alpha} := \theta_{\alpha}(\mathbb{P}^1(\mathfrak{k}))$ and $X'_\alpha := B.Z_{\alpha}$ so that $X'_\alpha = U.h \cup B.V_{\alpha}$.

Lemma 4.13. Let $\alpha$ be in $R_+$ and let $V$ be in $X$. Then $V$ is in $B.Z_{\alpha}$ if and only if $g(V)$ contains $h_{\alpha}$ for some $g$ in $B$.

Proof. The condition is necessary by definition. Suppose that $V$ contains $h_{\alpha}$. Since $V$ is commutative by Corollary 4.3.(ii), $V$ is contained in $h + g^\alpha$. If $V$ is a Cartan subalgebra, then $V = \theta_{\alpha}(t)$ for some $t$ in $\mathfrak{k}$. Otherwise, $V = \theta_{\alpha}(\infty)$, whence the lemma. \hfill $\square$

Corollary 4.14. Let $\alpha$ be a positive root.

(i) The sets $X'_\alpha$ and $G.X'_\alpha$ are open subsets of $X$ and $G.X$ respectively.

(ii) The sets $X'$ and $G.X'$ are big open subsets of $X$ and $G.X$ respectively.

Proof. (i) Since $X'_\alpha$ is a $B$-invariant subset containing the open subset $U.h$, it suffices to prove that $X'_\alpha$ is a neighborhood of $V_{\alpha}$ in $X$. Denote by $H_{\alpha}$ the coroot of $\alpha$ and set:

$$
E' := \bigoplus_{\gamma \in R_+ \setminus \{\alpha\}} g^\gamma, \quad E := \mathfrak{k}H_{\alpha} \oplus E'.
$$

Let $\Omega_E$ be the set of subspaces $V$ of $b$ such that $E$ is a complement to $V$ in $b$ and let $\Omega'_E$ be the complement in $X \cap \Omega_E$ to the union of $X_\gamma$, $\gamma \in R_+ \setminus \{\alpha\}$. Then $\Omega'_E$ is an open neighborhood of $V_{\alpha}$ in $X$. Since $X'_\alpha$ contains $U.h$, $X'_\alpha$ contains all the Cartan subalgebras contained in $\Omega'_E$. Let $V$ be in $\Omega'_E$ such that $V$ is not a Cartan subalgebra. According to Corollary 4.5, for some nonempty subset
\( \Lambda \) of \( \mathcal{R} \), \( V \) is contained \( b_\Lambda + u \) and contains a conjugate of \( b_\Lambda \) under \( B \). Then \( b = kH_a + b_\Lambda \) since \( V \) is in \( \Omega_E \). As a result, \( b_\Lambda = b_\gamma \) for some positive root \( \gamma \) and \( V \) is conjugate to \( V_\gamma \) under \( B \) by Lemma 4.13. Since \( V \) is not in \( X_\delta \) for all \( \delta \) in \( \mathcal{R}_+ \setminus \{a\} \), \( \gamma = a \) and \( V \) is in \( X'_a \). Then \( X'_a \) contains \( \Omega'_E \). As a result, \( X'_a \) is an open subset of \( X \) and \( G(X \setminus X'_a) \) is a closed subset of \( G.X \) by Lemma 1.7, whence the assertion.

(ii) By definition, \( X' \) is the union of \( X'_a, a \in \mathcal{R}_+ \). Hence \( X' \) is an open subset of \( X \) by (i). Moreover, by Theorem 4.11,(ii), \( X \setminus X' \) is the union of \( X_a \setminus X', a \in \mathcal{R}_+ \). Then \( X' \) is a big open subset of \( X \) since, for all \( a, X_a \setminus X' \) is strictly contained in the irreducible subvariety \( X_a \) of \( X \).

Since \( G.X' \) is the union of \( G.X'_a, a \in \mathcal{R}_+ \), \( G.X' \) is an open subset of \( G.X \) by (i). Moreover, by Theorem 4.11,(i), \( G.X \setminus G.X' \) is the union of \( G.X_a \setminus G.X', a \in \mathcal{R}_+ \). Hence \( G.X' \) is a big open subset of \( G.X \) since, for all \( a, G.X_a \setminus G.X' \) is strictly contained in the irreducible subvariety \( G.X'_a \) of \( G.X \).

**Proposition 4.15.** The sets \( X' \) and \( G.X' \) are smooth big open subsets of \( X \) and \( G.X \) respectively.

*Proof.* According to Corollary 4.14,(ii), it remains to prove that \( X' \) and \( G.X' \) are smooth open subsets of \( X \) and \( G.X \) respectively. Denote by \( \pi \) the bundle projection of the vector bundle \( E \) over \( G.X \). Recall \( \mathcal{E}_0 := \pi^{-1}(X) \). Let \( \mu \) be the map

\[
g_{\text{reg}} \longrightarrow \text{Gr}_r(g) \quad x \mapsto g^x
\]

and let \( \mu_0 \) be its restriction to \( b_{\text{reg}} \). Then \( \mu \) is a regular map. Let \( \Gamma_\mu \) and \( \Gamma_{\mu_0} \) be the images of the graphs of \( \mu \) and \( \mu_0 \) respectively by the isomorphism

\[
g \times \text{Gr}_r(g) \longrightarrow \text{Gr}_r(g) \times g \quad (x, V) \mapsto (V, x).
\]

Then \( \Gamma_\mu \) and \( \Gamma_{\mu_0} \) are smooth varieties contained in \( \mathcal{E} \) and \( \mathcal{E}_0 \) respectively since for \( x \) in \( g_{\text{reg}, ss} \), \( g^x \) is a Cartan subalgebra, contained in \( b \) when \( x \) is in \( b \). Set:

\[
\Gamma'_\mu := \Gamma_\mu \cap \pi^{-1}(G.X') = \mathcal{E} \cap G.X' \times g_{\text{reg}} \quad \text{and} \quad \Gamma'_{\mu_0} := \Gamma_{\mu_0} \cap \pi^{-1}(X') = \mathcal{E} \cap X' \times b_{\text{reg}}.
\]

Then \( \Gamma'_{\mu} \) is a smooth variety as an open subset of \( \Gamma_\mu \) and \( \Gamma'_{\mu} \) is an open subset of \( \pi^{-1}(G.X') \) such that \( \pi(\Gamma'_{\mu}) = G.X' \) since all element of \( G.X' \) contains regular elements. In the same way, \( \Gamma'_{\mu_0} \) is a smooth open subset of \( \pi^{-1}(X') \) such that \( \pi(\Gamma'_{\mu_0}) = X' \). As a result, \( \Gamma'_{\mu} \) and \( \Gamma'_{\mu_0} \) are smooth open subsets of vector bundles over \( G.X' \) and \( X' \) respectively since \( \mathcal{E} \) and \( \mathcal{E}_0 \) are vector bundles over \( G.X \) and \( X \) respectively. Hence \( G.X' \) and \( X' \) are smooth varieties by [MA86, Ch. 8, Theorem 23.7].

Summarizing the results of the section, Theorem 1.2,(i) is given by Corollary 4.3,(ii), Theorem 1.2,(ii) is given by Theorem 4.9, Theorem 1.2,(iii) is given by Lemma 4.1,(iv) since \( X \) and \( G.X \) have dimension \( n \) and \( 2n \) respectively and Theorem 1.2,(iv) is given by Proposition 4.15.

5. **On the generalized isospectral commuting variety**

Let \( k \geq 2 \) be an integer. According to Section 2, we have the commutative diagram

\[
\begin{array}{ccc}
G \times_B b^k & \xrightarrow{\gamma} & \mathcal{B}^{(k)}_x \\
\downarrow{\gamma} & & \downarrow{\eta} \\
\mathcal{B}^{(k)} & \xrightarrow{\eta} & \mathcal{B}^{(k)}_x
\end{array}
\]

By Lemma 2.7,(i), \( t_k \) is a closed embedding of \( b^k \) into \( \mathcal{B}^{(k)}_x \), by Corollary 2.8,(i) \( \mathcal{B}^{(k)}_x = G.t_k(b^k) \) is closed in \( \mathcal{X}^{(k)}_x \) and \( \eta \) is the restriction to \( \mathcal{B}^{(k)}_x \) of the canonical projection from \( \mathcal{X}^{(k)}_x \) to \( g^k \). Denote by \( \mathcal{C}_x^{(k)} \) the closure of \( G.b^k \) in \( g^k \) with respect to the diagonal action of \( G \) in \( g^k \) and set \( \mathcal{C}_x^{(k)} := \eta^{-1}(\mathcal{C}_x^{(k)}) \).
The varieties \( \mathcal{C}^{(k)} \) and \( \mathcal{C}^{(k)}_{x} \) are called generalized commuting variety and generalized isospectral commuting variety respectively. For \( k = 2 \), \( \mathcal{C}^{(k)}_{x} \) is the isospectral commuting variety considered by M. Haiman in [Ha99, §8] and [Ha02, §7.2].

5.1. Set:
\[
\mathcal{E}^{(k)}_{0} := \{(u, x_1, \ldots, x_k) \in X \times b^k \mid u \ni x_1, \ldots, u \ni x_k\}.
\]

**Lemma 5.1.** Denote by \( \mathcal{E}^{(k,s)}_{0} \) the intersection of \( \mathcal{E}^{(k)}_{0} \) and \( U.\mathfrak{h} \times (\mathfrak{g}_{\text{reg,ss}} \cap b)^k \) and for \( w \) in \( W(\mathcal{R}) \), denote by \( \theta_w \) the map
\[
\mathcal{E}^{(k)}_{0} \longrightarrow b^k \times b^k, \quad (u, x_1, \ldots, x_k) \longmapsto (x_1, \ldots, x_k, w(x_1), \ldots, w(x_k))
\]

(i) Denoting by \( \mathfrak{x}_{0,k} \) the image of \( \mathcal{E}^{(k)}_{0} \) by the projection \( (u, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k) \), \( \mathfrak{x}_{0,k} \) is the closure of \( B.\mathfrak{h}^k \) in \( b^k \) and \( \mathcal{E}^{(k)} \) is the image of \( G \times \mathfrak{x}_{0,k} \) by the map \( (g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k)) \).

(ii) For all \( w \) in \( W(\mathcal{R}) \), \( \theta_w(\mathcal{E}^{(k,s)}_{0}) \) is dense in \( \theta_w(\mathcal{E}^{(k)}_{0}) \).

**Proof.** (i) Since \( X \) is a projective variety, \( \mathfrak{x}_{0,k} \) is a closed subset of \( b^k \). The variety \( \mathcal{E}^{(k)}_{0} \) is irreducible of dimension \( n + k^2 \) as a vector bundle of rank \( k^2 \) over the irreducible variety \( X \). So, \( B.((\mathfrak{h} \times b^k) \) is dense in \( \mathcal{E}^{(k)}_{0} \) and \( \mathfrak{x}_{0,k} \) is the closure of \( B.\mathfrak{h}^k \) in \( b^k \), whence the assertion by Lemma 1.7.

(ii) Since \( U.\mathfrak{h} \times (\mathfrak{g}_{\text{reg,ss}} \cap b)^k \) is an open subset of \( X \times b^k \), \( \mathcal{E}^{(k,s)}_{0} \) is an open subset of \( \mathcal{E}^{(k)}_{0} \). Moreover, it is a dense open subset since \( \mathcal{E}^{(k)}_{0} \) is irreducible, whence the assertion since \( \theta_w \) is a morphism of algebraic varieties.

\[\square\]

5.2. Let \( x \) be in \( \mathfrak{h} \). According to [Ko63, §3.2, Lemma 5], \( G^{s} \) is connected. Denote by \( \mathcal{R}_{x} \) the set of roots whose kernel contains \( x \) and denote by \( W(\mathcal{R}_{x}) \) the Weyl group of \( \mathcal{R}_{x} \).

**Lemma 5.2.** Let \( x = (x_1, \ldots, x_k) \) be in \( \mathcal{C}^{(k)} \) verifying the following conditions:

1. \( s \) is the semisimple component of \( x_{1} \).
2. for \( z \) in \( E_{x} \), the centralizer in \( \mathfrak{g} \) of the semisimple component of \( z \) has dimension at least \( \dim \mathfrak{g}^{s} \).

Then for \( i = 1, \ldots, k \), the semisimple component of \( x_{i} \) is in \( \mathfrak{g}^{s} \).

**Proof.** Since \( x \) is in \( \mathcal{C}^{(k)} \), \( [x_{j}, x_{j}] = 0 \) for all \( (i, j) \). Suppose that for some \( i \), the semisimple component \( x_{i,s} \) of \( x_{i} \) is not in \( \mathfrak{g}^{s} \). A contradiction is expected. Since \( [x_{1}, x_{i}] = 0 \), for all \( t \) in \( \mathfrak{z} \), \( s + tx_{i,s} \) is the semisimple component of \( x_{1} + tx_{i} \). Moreover, after conjugation by an element of \( G^{s} \), we can suppose that \( x_{i,s} \) is in \( \mathfrak{h} \). Since \( \mathcal{R} \) is finite, there exists \( t \) in \( \mathfrak{k}^{s} \) such that the subset of roots whose kernel contains \( s + tx_{i,s} \) is contained in \( \mathcal{R}_{x} \). Since \( x_{i,s} \) is not in \( \mathfrak{g}^{s} \), for some \( \alpha \) in \( \mathcal{R}_{x} \), \( \alpha(s + tx_{i,s}) \neq 0 \) that is \( \mathfrak{g}^{s+tx_{i,s}} \) is strictly contained in \( \mathfrak{g}^{s} \), whence the contradiction. \[\square\]

For \( w \) in \( W(\mathcal{R}) \), set:
\[
C_{w} := G^{s}wB/B, \quad B^{w} := wBw^{-1}.
\]

**Lemma 5.3.** [Hu95, §6.17, Lemma] Let \( \mathcal{B} \) be the set of Borel subalgebras of \( \mathfrak{g} \) and let \( \mathcal{B}_{x} \) be the set of Borel subalgebras of \( \mathfrak{g} \) containing \( s \).

(i) For all \( w \) in \( W(\mathcal{R}) \), \( C_{w} \) is a connected component of \( \mathcal{B}_{x} \).

(ii) For \( (u, w') \) in \( W(\mathcal{R}) \times W(\mathcal{R}) \), \( C_{u} = C_{w'} \) if and only if \( w'w^{-1} \) is in \( W(\mathcal{R}_{x}) \).

(iii) The variety \( C_{w} \) is isomorphic to \( G^{s}/(G^{s} \cap B^{w}) \).

For \( x \) in \( \mathcal{B}^{(k)} \), denote by \( \mathcal{B}_{x} \) the subset of Borel subalgebras containing \( E_{x} \).
Corollary 5.4. Let \( x = (x_1, \ldots, x_k) \) be in \( \mathcal{C}^{(k)} \). Suppose that \( x \) verifies Conditions (1) and (2) of Lemma 5.2. Then \( \{C_w \cap B_x \mid w \in W(R)\} \) is the set of connected components of \( B_x \).

Proof. Since a Borel subalgebra contains the semisimple component of its elements and since \( s \) is the semisimple component of \( x_1, B_x \) is contained in \( \mathfrak{B}_x \). As a result, according to Lemma 5.3,(i), every connected component of \( B_x \) is contained in \( C_w \) for some \( w \) in \( W(R) \). Set \( x_n := (x_{1,n}, \ldots, x_{k,n}) \).

Since \( [x_i, x_j] = 0 \) for all \( (i, j) \), \( E_i \) is contained in \( \mathfrak{g}^s \). Let \( \mathfrak{B}^s \) be the set of Borel subalgebras of \( \mathfrak{g}^s \) and for \( y \) in \( (\mathfrak{g}^s)^k \), let \( \mathfrak{B}^s_y \) be the set of Borel subalgebras of \( \mathfrak{g}^s \) containing \( E_y \). According to [Hu95, Theorem 6.5], \( \mathfrak{B}^s_x \) is connected. Moreover, according to Lemma 5.2, the semisimple components of \( x_1, \ldots, x_k \) are in \( \mathfrak{g}^s \) so that \( \mathfrak{B}^s_x = \mathfrak{B}^s_y \). Let \( w \) be in \( W(R) \). According to Lemma 5.3,(iii), there is an isomorphism from \( \mathfrak{B}^s \) to \( C_w \). Moreover, the image of \( \mathfrak{B}^s_x \) by this isomorphism is equal to \( C_w \cap B_x \), whence the corollary.

Corollary 5.5. Let \( x = (x_1, \ldots, x_k) \) be in \( \mathcal{C}^{(k)} \) verifying Conditions (1) and (2) of Lemma 5.2. Then \( \mathfrak{g}^{-1}(x) \) is contained in \( \{x_1, \ldots, x_k, w(x_{1,s}), \ldots, w(x_{k,s}) \mid w \in W(R)\} \).

Proof. Since \( \gamma = \mathfrak{g}^{-1}(x) \) is the image of \( \gamma^{-1}(x) \) by \( \mathfrak{g} \). Furthermore, \( \mathfrak{g} \) is constant on the connected components of \( \gamma^{-1}(x) \) since \( \mathfrak{g}^{-1}(x) \) is finite. Let \( C \) be a connected component of \( \gamma^{-1}(x) \). Identifying \( G \times B \) with the subvariety of elements \( (u, x) \) of \( B \times \mathfrak{g}^k \) such that \( E_x \) is contained in \( u \), \( C \) identifies with \( (C_w \cap B_x) \times \{x\} \) for some \( w \) in \( W(R) \) by Corollary 5.4. Then for some \( g \) in \( G' \) and for some representative \( g_w \) of \( w \) in \( N_G(b) \), \( g_w(b) \) contains \( E_x \) so that

\[
\mathfrak{g}^{-1}(C) = \{(x_1, \ldots, x_k, (gg_w)^{-1}(x_1), \ldots, (gg_w)^{-1}(x_k))\}.
\]

By Lemma 5.2, \( x_1,s, \ldots, x_k,s \) are in \( \mathfrak{g}^s \) so that \( w^{-1}(x_{i,s}) \) is the semisimple component of \( (gg_w)^{-1}(x_i) \) for \( i = 1, \ldots, k \). Hence

\[
\mathfrak{g}^{-1}(C) = \{(x_1, \ldots, x_k, w^{-1}(x_{1,s}), \ldots, w^{-1}(x_{k,s}))\},
\]

whence the corollary.

Proposition 5.6. The variety \( \mathcal{C}_x^{(k)} \) is irreducible and equal to the closure of \( G_{\eta}(\mathfrak{b}^k) \) in \( B_x^{(k)} \).

Proof. Denote by \( \overline{G_{\eta}(\mathfrak{b}^k)} \) the closure of \( G_{\eta}(\mathfrak{b}^k) \) in \( B_x^{(k)} \). Then \( \overline{G_{\eta}(\mathfrak{b}^k)} \) is irreducible as the closure of an irreducible set. Since \( \eta \) is \( G \)-equivariant, \( \eta(G_{\eta}(\mathfrak{b}^k)) = G_{\eta}(\mathfrak{b}^k) \). Hence \( \eta(\overline{G_{\eta}(\mathfrak{b}^k)}) = \overline{\mathcal{C}^{(k)}} \) since \( \eta \) is a finite morphism and \( \mathcal{C}^{(k)} \) is the closure of \( G_{\eta}(\mathfrak{b}^k) \) in \( \mathfrak{g}^k \) by definition. So, it remains to prove that for all \( x \) in \( \mathcal{C}^{(k)} \), \( \eta^{-1}(x) \) is contained in \( G_{\eta}(\mathfrak{b}^k) \). There is a canonical action of \( GL_k(\mathfrak{b}) \) on \( \mathfrak{g}^k \) and \( \mathfrak{X}^k \). Since this action commutes with the action of \( G \) in \( \mathfrak{X}^k \), \( B_x^{(k)} \) is invariant under \( GL_k(\mathfrak{b}) \) and \( \eta \) is \( GL_k(\mathfrak{b}) \)-equivariant. As a result, since \( \mathcal{C}^{(k)} \) and \( G_{\eta}(\mathfrak{b}^k) \) are invariant under \( GL_k(\mathfrak{b}) \), for \( x \) in \( \mathcal{C}^{(k)} \), \( \eta^{-1}(x') \) is contained in \( G_{\eta}(\mathfrak{b}^k) \) for all \( x' \) in \( E_x^{(k)} \) such that \( E_{x'} = E_x \) if \( \eta^{-1}(x) \) is contained in \( G_{\eta}(\mathfrak{b}^k) \).

Then, according to Lemma 5.2, since \( \eta \) is \( G \)-equivariant, it suffices to prove that \( \eta^{-1}(x) \) is contained in \( G_{\eta}(\mathfrak{b}^k) \) for \( x \) in \( \mathcal{C}^{(k)} \cap \mathfrak{b}^k \) verifying Conditions (1) and (2) of Lemma 5.2 for some \( s \) in \( \mathfrak{b} \).

According to Corollary 5.5,

\[
\eta^{-1}(x) \subset \{(x_1, \ldots, x_k, w(x_{1,s}), \ldots, w(x_{k,s})) \mid w \in W(R)\} \text{ with } x = (x_1, \ldots, x_k).
\]

For \( s \) regular, \( E_x \) is contained in \( \mathfrak{b} \) and \( x_i = x_{i,s} \) for \( i = 1, \ldots, k \). By definition,

\[
(w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k)) \in \iota_k(\mathfrak{b}^k)
\]

and for \( g_w \) a representative of \( w \) in \( N_G(\mathfrak{b}) \),

\[
g_w^{-1}(w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k)) = (x_1, \ldots, x_k, w(x_1), \ldots, w(x_k)).
\]
Hence $\eta^{-1}(x)$ is contained in $G.\ell_k(\mathfrak{b}^k)$. As a result, according to the notations of Lemma 5.1, for all $w$ in $W(\mathfrak{r})$, $\theta_w(\mathfrak{c}^{(k)}_0)$ is contained in $G.\ell_k(\mathfrak{b}^k)$. Hence, by Lemma 5.1(ii), $\theta_w(\mathfrak{c}^{(k)}_0)$ is contained in $G.\ell_k(\mathfrak{b}^k)$, whence the proposition. \hfill $\square$

5.3. Let $\sigma$ be the canonical projection from $X^k$ to $\mathfrak{g}^k$. By Corollary 2.6(ii), $B^{(k)}_x$ is an irreducible component of $\sigma^{-1}(B^{(k)})$ and the action of $W(\mathfrak{R})^k$ on $X^k$ induces a simply transitive action on the set of irreducible components of $\sigma^{-1}(B^{(k)})$. According to Remark 2.21, there is an embedding $\Phi$ of $S(\mathfrak{b})^\otimes_k$ into $K[\mathfrak{b}^{(k)}]$ given by

\[ p \mapsto ((x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto p(y_1, \ldots, y_k)). \]

By Corollary 2.22(i), this embedding identifies $S(\mathfrak{b})^\otimes_k$ with $K[\mathfrak{b}^{(k)}]^G$.

**Lemma 5.7.** Let $\Psi$ be the restriction to $S(\mathfrak{b})^\otimes_k$ of the canonical map from $K[\mathfrak{b}^{(k)}]$ to $K[\mathfrak{c}^{(k)}]$. 

(i) The subvariety $C^{(k)}_x$ of $X^k$ is invariant under the diagonal action of $W(\mathfrak{R})$ in $X^k$.

(ii) The map $\Psi$ is an embedding of $S(\mathfrak{b})^\otimes_k$ into $K[\mathfrak{c}^{(k)}]$. Moreover, $\Psi(S(\mathfrak{b})^\otimes_k)$ is equal to $K[\mathfrak{c}^{(k)}]^G$.

(iii) The image of $(S(\mathfrak{b})^\otimes_k)^W(\mathfrak{R})$ by $\Psi$ equals $K[\mathfrak{c}^{(k)}]^G$.

**Proof.** (i) For $x$ in $B^{(k)}_x$ and $w$ in $W(\mathfrak{R})$, $\eta(x) = \eta(w.x)$, whence the assertion by Proposition 5.6.

(ii) For $P$ in $S(\mathfrak{b})^\otimes_k$, $P = 0$ if $P(x) = 0$ for all $x$ in $\ell_k(\mathfrak{b}^k)$. Hence $\Psi$ is injective. Since $G$ is reductive, $K[\mathfrak{c}^{(k)}]^G$ is the image of $K[\mathfrak{b}^{(k)}]^G$ by the quotient morphism, whence the assertion.

(iii) Since $G$ is reductive, $K[\mathfrak{c}^{(k)}]^G$ is the image of $K[\mathfrak{b}^{(k)}]^G$ by the quotient morphism, whence the assertion since $(S(\mathfrak{b})^\otimes_k)^W(\mathfrak{R})$ is equal to $K[\mathfrak{b}^{(k)}]^G$ by Corollary 2.22(iii). \hfill $\square$

Identify $S(\mathfrak{b})^\otimes_k$ with $K[\mathfrak{c}^{(k)}]^G$ by $\Psi$.

**Proposition 5.8.** Let $\overline{C}^{(k)}_x$ and $\overline{C}^{(k)}$ be the normalizations of $C^{(k)}_x$ and $C^{(k)}$.

(i) The variety $\overline{C}^{(k)}_x$ is the categorical quotient of $C^{(k)}_x$ under the action of $W(\mathfrak{R})$.

(ii) The variety $\overline{C}^{(k)}$ is the categorical quotient of $C^{(k)}$ under the action of $W(\mathfrak{R})$.

**Proof.** (i) According to Corollary 2.22(i), $K[\mathfrak{b}^{(k)}]$ is generated by $K[\mathfrak{b}^{(k)}]$ and $S(\mathfrak{b})^\otimes_k$. Since $C^{(k)}_x = \eta^{-1}(C^{(k)})$ by Proposition 5.6, the image of $K[\mathfrak{b}^{(k)}]$ in $K[\mathfrak{c}^{(k)}]$ by the quotient morphism is equal to $K[\mathfrak{c}^{(k)}]$. Hence $K[\mathfrak{c}^{(k)}]$ is generated by $K[\mathfrak{c}^{(k)}]$ and $S(\mathfrak{b})^\otimes_k$. Then, by Lemma 5.7(iii), $K[\mathfrak{c}^{(k)}]^{W(\mathfrak{R})} = K[\mathfrak{c}^{(k)}]$.

(ii) Let $K$ be the fraction field of $K[\mathfrak{c}^{(k)}]$. Since $\overline{C}^{(k)}_x$ is a $W(\mathfrak{R})$-variety, there is an action of $W(\mathfrak{R})$ in $K$ and $K^{W(\mathfrak{R})}$ is the fraction field of $K[\mathfrak{c}^{(k)}]^{W(\mathfrak{R})}$ since $W(\mathfrak{R})$ is finite. As a result, the integral closure $K[\overline{C}^{(k)}_x]$ of $K[\overline{C}^{(k)}]$ in $K$ is invariant under $W(\mathfrak{R})$ and $K[\overline{C}^{(k)}]$ is contained in $K[\mathfrak{c}^{(k)}]^{W(\mathfrak{R})}$ by (i).

Let $a$ be in $K[\overline{C}^{(k)}]^{W(\mathfrak{R})}$. Then $a$ verifies a dependence integral equation over $K[\mathfrak{c}^{(k)}]$, 

\[ a^m + a_{m-1}a^{m-1} + \cdots + a_0 = 0 \]

whence

\[ a^m + \frac{1}{|W(\mathfrak{R})|} \sum_{w \in W(\mathfrak{R})} w.a_{m-1}a^{m-1} + \cdots + \frac{1}{|W(\mathfrak{R})|} \sum_{w \in W(\mathfrak{R})} w.a_0 = 0 \]

since $a$ is invariant under $W(\mathfrak{R})$ so that $a$ is in $K[\overline{C}^{(k)}]$ by (i), whence the assertion. \hfill $\square$
6. Desingularization

Let \( k \geq 2 \) be an integer. Let \( X, X' \) be as in Subsection 4.5. Denote by \( X_n \) the normalization of \( X \) and by \( \theta_0 \) the normalization morphism. According to Proposition 4.15, \( X' \) identifies with a smooth big open subset of \( X_n \) and according to [Hir64], there exists a desingularization \((\Gamma, \pi_n)\) of \( X_n \) in the category of \( B \)-varieties such that the restriction of \( \pi_n \) to \( \pi_n^{-1}(X') \) is an isomorphism onto \( X' \). Set \( \pi = \theta_0 \circ \pi_n \) so that \((\Gamma, \pi)\) is a desingularization of \( X \) in the category of \( B \)-varieties. Recall that \( \mathcal{E}_0 \) is the restriction to \( X \) of the tautological vector bundle over \( \text{Gr}_\ell(g) \) and \( \mathcal{X}_{0,k} \) is the closure in \( b^k \) of \( B \cdot b^k \). Set \( \mathcal{X}_k := G \times_B \mathcal{X}_{0,k} \). Then \( \mathcal{X}_k \) is a closed subvariety of \( G \times_B b^k \).

**Lemma 6.1.** Let \( \tau' \) be the canonical morphism from \( \mathcal{E}_0 \) to \( b \).

(i) The morphism \( \tau' \) is projective and birational.

(ii) Let \( \nu \) be the canonical map from \( \pi^*(\mathcal{E}_0) \) to \( \mathcal{E}_0 \). Then \( \nu \) and \( \tau := \tau' \circ \nu \) are \( B \)-equivariant birational projective morphisms from \( \pi^*(\mathcal{E}_0) \) to \( \mathcal{E}_0 \) and \( b \) respectively. In particular, \( \pi^*(\mathcal{E}_0) \) is a desingularization of \( \mathcal{E}_0 \) and \( b \).

**Proof.** (i) Since \( X \) is a projective variety, \( \tau' \) is a projective morphism and \( \tau'(\mathcal{E}_0) \) is closed in \( b \). Moreover, \( \tau'(\mathcal{E}_0) \) is \( B \)-invariant since \( \tau' \) is a \( B \)-equivariant morphism and it contains \( b \) since \( b \) is in \( X \). For \( x \) in \( b_{\text{reg}} \), \((\tau')^{-1}(x) = \{(b, x)\}\). Hence \( \tau' \) is a birational morphism and \( \tau'(\mathcal{E}_0) = b \) since \( B(b) \) is an open subset of \( b \).

(ii) Since \( \mathcal{E}_0 \) is a vector bundle over \( X \) and since \( \pi \) is a projective birational morphism, \( \nu \) is a projective birational morphism. Then \( \tau \) is a projective birational morphism from \( \pi^*(\mathcal{E}_0) \) to \( b \) by (i). It is \( B \)-equivariant since so are \( \nu \) and \( \tau' \). Moreover, \( \pi^*(\mathcal{E}_0) \) is a desingularization of \( \mathcal{E}_0 \) and \( b \) since \( \pi^*(\mathcal{E}_0) \) is smooth as a vector bundle over a smooth variety. \( \square \)

Denote by \( \psi \) the canonical projection from \( \pi^*(\mathcal{E}_0) \) to \( \Gamma \). Then, according to the above notations, we have the commutative diagram:

\[
\begin{array}{ccc}
\pi^*(\mathcal{E}_0) & \xrightarrow{\psi} & \Gamma \\
\tau' \downarrow & & \downarrow \pi \\
\mathcal{E}_0 & \xrightarrow{\nu} & X
\end{array}
\]

Recall that \( \mathcal{E}_0^{(k)} \) is the subvariety of \( X \times b^k \):

\[
\mathcal{E}_0^{(k)} := \{(u, x_1, \ldots, x_k) \in X \times b^k \mid u \ni x_1, \ldots, u \ni x_k\}.
\]

As \( \mathcal{E}_0 \) is a vector bundle over \( X \), so is \( \mathcal{E}_0^{(k)} \).

**Lemma 6.2.** Set \( \mathcal{E}_s^{(k)} := \pi^*(\mathcal{E}_0^{(k)}) \). Let \( \tau_k \) be the canonical morphism from \( \mathcal{E}_s^{(k)} \) to \( b^k \).

(i) The vector bundle \( \mathcal{E}_s^{(k)} \) over \( \Gamma \) is a vector subbundle of the trivial bundle \( \Gamma \times b^k \). Moreover, \( \mathcal{E}_s^{(k)} \) has dimension \( k\ell + n \).

(ii) The morphism \( \tau_k \) is a projective birational morphism from \( \mathcal{E}_s^{(k)} \) onto \( \mathcal{X}_{0,k} \). Moreover, \( \mathcal{E}_s^{(k)} \) is a desingularization of \( \mathcal{X}_{0,k} \) in the category of \( B \)-varieties.

**Proof.** (i) By definition, \( \mathcal{E}_s^{(k)} \) is the subvariety of \( \Gamma \times b^k \). Since \( X \) and \( \Gamma \) have dimension \( n \), \( \mathcal{E}_s^{(k)} \) has dimension \( k\ell + n \) as a vector bundle of rank \( k\ell \) over \( \Gamma \).

(ii) Since \( \Gamma \) is a projective variety, \( \tau_k \) is a projective morphism and \( \tau_k(\mathcal{E}_s^{(k)}) = \mathcal{X}_{0,k} \) by Lemma 5.1,(i). For \((x_1, \ldots, x_k) \) in \( b^k_{\text{reg,ss}} \cap \mathcal{X}_{0,k} \), \( \tau_k^{-1}(x_1, \ldots, x_k) = \{(\pi^{-1}(g^k), (x_1, \ldots, x_k))\} \) since
\( g^{\ast} \) is a Cartan subalgebra. Hence \( \tau_k \) is a birational morphism, whence the assertion since \( E_s^{(k)} \) is a smooth \( B \)-variety as a vector bundle over the smooth \( B \)-variety \( \Gamma \). \( \Box \)

Set \( \mathcal{Y} := G \times_B (\Gamma \times b^k) \). The canonical projections from \( G \times \Gamma \times b^k \) to \( G \times \Gamma \) and \( G \times b^k \) define through the quotients morphisms from \( \mathcal{Y} \) to \( G \times_B \Gamma \) and \( G \times_B b^k \). Denote by \( \zeta \) and \( \iota \) these morphisms. Then we have the following diagram:

\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\zeta} & G \times_B b^k \\
\downarrow{\iota} & & \downarrow{\gamma_x} \\
G \times_B \Gamma & & \mathcal{B}_x^{(k)}
\end{array}
\]

The map \((g, x) \mapsto (g, \tau_k(x))\) from \( G \times E_s^{(k)} \) to \( G \times b^k \) defines through the quotient a morphism \( \overline{\tau_k} \) from \( G \times_B E_s^{(k)} \) to \( \mathcal{X}_k \).

**Proposition 6.3.** Set \( \xi := \gamma_x \circ \overline{\tau_k} \).

(i) The variety \( G \times_B E_s^{(k)} \) is a closed subvariety of \( \mathcal{Y} \).

(ii) The variety \( G \times_B E_s^{(k)} \) is a vector bundle of rank \( k \ell \) over \( G \times_B \Gamma \). Moreover, \( G \times_B \Gamma \) and \( G \times_B E_s^{(k)} \) are smooth varieties.

(iii) The morphism \( \xi \) is a projective birational morphism from \( G \times_B E_s^{(k)} \) onto \( C_s^{(k)} \).

**Proof.** (i) According to Lemma 6.2,(i), \( E_s^{(k)} \) is a closed subvariety of \( \Gamma \times b^k \), invariant under the diagonal action of \( B \). Hence \( G \times E_s^{(k)} \) is a closed subvariety of \( G \times \Gamma \times b^k \), invariant under the action of \( B \), whence the assertion.

(ii) Since \( E_s^{(k)} \) is a \( B \)-equivariant vector bundle over \( \Gamma \), \( G \times_B E_s^{(k)} \) is a \( G \)-equivariant vector bundle over \( G \times_B \Gamma \). Since \( G \times_B \Gamma \) is a fiber bundle over the smooth variety \( G/B \) with smooth fibers, \( G \times_B \Gamma \) is a smooth variety. As a result, \( G \times_B E_s^{(k)} \) is a smooth variety.

(iii) According to Lemma 6.2,(ii) and Lemma 1.7, \( \overline{\tau_k} \) is a projective birational morphism from \( G \times_B E_s^{(k)} \) to \( \mathcal{X}_k \). Since \( \mathcal{X}_{0,k} \) is a \( B \)-invariant closed subvariety of \( b^k \), \( \mathcal{X}_k \) is closed in \( G \times_B b^k \). According to Lemma 5.1,(i), \( \gamma(\mathcal{X}_k) = C_s^{(k)} \). Moreover, \( \gamma_x(\mathcal{X}_k) \) is an irreducible closed subvariety of \( B_x^{(k)} \) since \( \gamma_x \) is a projective morphism by Lemma 1.7. Hence \( \gamma_x(\mathcal{X}_k) = C_s^{(k)} \) by Proposition 5.6. For all \( z \in G.\xi_k(b^{k \text{reg}}_\text{reg}) \), \( |\gamma_x^{-1}(z)| = 1 \). Hence the restriction of \( \gamma_x \) to \( \mathcal{X}_k \) is a birational morphism onto \( C_s^{(k)} \) since \( G.\xi_k(b^{k \text{reg}}_\text{reg}) \) is dense in \( C_s^{(k)} \). Moreover, this morphism is projective since \( \gamma_x \) is projective. As a result, \( \xi \) is a projective birational morphism from \( G \times_B E_s^{(k)} \) onto \( C_s^{(k)} \).

Theorem 1.3 results from Proposition 5.6 and Proposition 6.3,(ii) and (iii) and the following corollary results from Lemma 6.2,(ii), Proposition 6.3,(ii) and (iii), and Lemma 1.4.

**Corollary 6.4.** Let \( \mathcal{X}_{0,k} \) and \( C_s^{(k)} \) be the normalizations of \( \mathcal{X}_{0,k} \) and \( C_s^{(k)} \) respectively. Then \( \mathcal{X}_{0,k} \) and \( C_s^{(k)} \) are the spaces of global sections of \( \mathcal{O}_{E_s^{(k)}} \) and \( \mathcal{O}_{G \times_B E_s^{(k)}} \) respectively.

**References**


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