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To cite this version:
Jean-Yves Charbonnel, Mouchira Zaiter. On the Commuting variety of a reductive Lie algebra and other related varieties. 62 pages. 2012. <hal-00684336>

HAL Id: hal-00684336
https://hal.archives-ouvertes.fr/hal-00684336
Submitted on 1 Apr 2012

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ON THE COMMUTING VARIETY OF A REDUCTIVE LIE ALGEBRA AND OTHER RELATED VARIETIES.

JEAN-YVES CHARBONNEL AND MOUCHIRA ZAITER

Abstract. In this note, one discusses about some varieties which are constructed analogously to the isospectral commuting varieties. These varieties are subvarieties of varieties having very simple desingularizations. For instance, this is the case of the nullcone of any cartesian power of a reductive Lie algebra and one proves that it has rational singularities. Moreover, as a byproduct of these investigations and the Ginzburg’s results, one gets that the normalizations of the isospectral commuting variety and the commuting variety have rational singularities.

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1. Introduction.

In this note, the base field $\mathbb{k}$ is algebraically closed of characteristic 0, $\mathfrak{g}$ is a reductive Lie algebra of finite dimension, $\ell$ is its rank, $\dim \mathfrak{g} = \ell + 2n$ and $G$ is its adjoint group. The neutral element of $G$ is denoted by $1_\mathfrak{g}$.

Date: April 1, 2012.
1991 Mathematics Subject Classification. 14A10, 14L17, 22E20, 22E46.
Key words and phrases. polynomial algebra, complex, commuting variety, desingularization, Gorenstein, Cohen-Macaulay, rational singularities, cohomology.
1.1. Notations. • For $V$ a vector space, its dual is denoted by $V^*$ and the augmentation ideal of its symmetric algebra $S(V)$ is denoted by $S_+(V)$.

• All topological terms refer to the Zariski topology. If $Y$ is a subset of a topological space $X$, let denote by $\overline{Y}$ the closure of $Y$ in $X$. For $Y$ an open subset of the algebraic variety $X$, $Y$ is called a big open subset if the codimension of $X \setminus Y$ in $X$ is bigger than 2. For $Y$ a closed subset of an algebraic variety $X$, its dimension is the biggest dimension of its irreducible components and its codimension in $X$ is the smallest codimension in $X$ of its irreducible components. For $X$ an algebraic variety, $\mathcal{O}_X$ is its structural sheaf, $\mathcal{K}[X]$ is the algebra of regular functions on $X$ and $\mathcal{K}(X)$ is the field of rational functions on $X$ when $X$ is irreducible. When $X$ is smooth, the sheaf of regular differential forms of top degree on $X$ is denoted by $\Omega^*_X$.

• For $X$ an algebraic variety and for $\mathcal{M}$ a sheaf on $X$, $\Gamma(V, \mathcal{M})$ is the space of local sections of $\mathcal{M}$ over the open subset $V$ of $X$. For $i$ a nonnegative integer, $H^i(X, \mathcal{M})$ is the $i$-th group of cohomology of $\mathcal{M}$. For example, $H^0(X, \mathcal{M}) = \Gamma(X, \mathcal{M})$.

Lemma 1.1. Let $X$ be an irreducible affine algebraic variety and let $Y$ be a desingularization of $X$. Then $H^0(Y, \mathcal{O}_Y)$ is the integral closure of $\mathcal{K}[X]$ in its fraction field.

Proof. Let $X_\alpha$ be the normalization of $X$. According to [H77, Ch. II, Exercise 3.8], the desingularization morphism factorizes through $X_\alpha$ so that $Y$ is a desingularization of $X_\alpha$. So one can suppose $X = X_\alpha$. Then $\mathcal{K}[X]$ is a subalgebra of $H^0(Y, \mathcal{O}_Y)$. Moreover, $H^0(Y, \mathcal{O}_Y)$ is a subalgebra of $\mathcal{K}(X)$ since $Y$ is a desingularization of $X$. According to [H77, Ch. II, Proposition 4.1], a morphism of affine varieties is separated. Then, according to [EGAII, Corollaire 5.4.3], $H^0(Y, \mathcal{O}_Y)$ is a finite extension of $\mathcal{K}[X]$ since it is finitely generated and since the desingularization morphism is projective by definition, whence the lemma. □

• For $K$ a group and for $E$ a set with a group action of $K$, $E^K$ is the set of invariant elements of $E$ under $K$.

Lemma 1.2. Let $A$ be an algebra generated by the subalgebras $A_1$ and $A_2$. Let $K$ be a group with a group action of $K$ on $A_2$. Let suppose that the following conditions are verified:

1. $A_1 \cap A_2$ is contained in $A_2^K$;
2. $A$ is a free $A_2$-module having a basis contained in $A_1$;
3. $A_1$ is a free $A_1 \cap A_2$-module having the same basis.

Then there exists a unique group action of $K$ on the algebra $A$ extending the action of $K$ on $A_2$ and fixing all the elements of $A_1$. Moreover, if $A_1 \cap A_2 = A_2^K$ then $A^K = A_1$.

Proof. Let $m_i, l \in L$ be a basis of the $A_2$-module $A$, contained in $A_1$, and let $M$ be the subspace of $A$ generated by the $m_i$’s so that the canonical morphisms

$$M \otimes_k A_2 \longrightarrow A \quad M \otimes_k (A_1 \cap A_2) \longrightarrow A_1$$

are isomorphisms by Conditions (2) and (3). Hence there exists a unique group action of $K$ on the space $A$ fixing all the elements of $M$ and extending the action of $K$ on $A_2$. For $(i, j)$ in $L^2$, let denote by $a_{i,j,k}$ the coordinate of $m_im_j$ at $m_k$ in the basis $m_i, l \in L$. According to Conditions (1) and (3), the $a_{i,j,k}$’s are
invariant under $K$. Let $a, a'$ be in $A$. Denoting by $a_i$ and $a'_i$ the coordinates of $a$ and $a'$ at $m_i$ in the basis $m_l, l \in L$ respectively, for all $g$ in $K$, one has
\[
g.aa' = g.(\sum_{i,j} a_i a'_j) = g.(\sum_{i,j} m_i m_j a_i a'_j) = \sum_{i,j} m_i m_j g.(a_i a'_j) = \sum_{i,j} m_i m_j (g.a_i) (g.a'_j) = (g.a)(g.a')
\]
so that the action of $K$ is an action on the algebra $A$, fixing all element of $A_1$. Furthermore, $a$ is in $A^K$ if and only if the $a_i$’s are in $A_2^K$ since the $m_i$’s are invariant under $K$. Hence $A^K = A_1$ if $A_1 \cap A_2 = A_2^K$. □

- For $E$ a set and $k$ a positive integer, $E^k$ denotes its $k$-th cartesian power. If $E$ is finite, its cardinality is denoted by $|E|$. If $E$ is a vector space, for $x = (x_1, \ldots, x_k)$ in $E^k$, $P_x$ is the subspace of $E$ generated by $x_1, \ldots, x_k$. Moreover, there is a canonical action of $GL_k(\mathbb{k})$ in $E^k$ given by:
\[
(a_{i,j}, 1 \leq i, j \leq k). (x_1, \ldots, x_k) := \left( \sum_{j=1}^k a_{i,j} x_j, i = 1, \ldots, k \right)
\]

In particular, the diagonal action of $G$ in $g^k$ commutes with the action of $GL_k(\mathbb{k})$.

- For a reductive Lie algebra, its rank is denoted by $\ell_a$ and the dimension of its Borel subalgebras is denoted by $\ell_a$. In particular, $\dim a = 2\ell_a - \ell_a$.

- If $E$ is a subset of a vector space $V$, let denote by $\text{span}(E)$ the vector subspace of $V$ generated by $E$. The grassmanian of all $d$-dimensional subspaces of $V$ is denoted by $\text{Gr}_d(V)$. By definition, a cone of $V$ is a subset of $V$ invariant under the natural action of $\mathbb{k}^* := \mathbb{k} \setminus \{0\}$ and a multicone of $V^k$ is a subset of $V^k$ invariant under the natural action of $(\mathbb{k}^*)^k$ on $V^k$.

**Lemma 1.3.** Let $X$ be an open cone of $V$ and let $S$ be a closed multicone of $X \times V^{k-1}$. Denoting by $S_1$ the image of $S$ by the first projection, $S_1 \times \{0\} = S \cap (X \times \{0\})$. In particular, $S_1$ is closed in $X$.

**Proof.** For $x$ in $X$, $x$ is in $S_1$ if and only if for some $(v_2, \ldots, v_k)$ in $V^{k-1}$, $(x, tv_2, \ldots, tv_k)$ is in $S$ for all $t$ in $\mathbb{k}$ since $S$ is a closed multicone of $X \times V^{k-1}$, whence the lemma. □

- The dual of $\mathfrak{g}$ is denoted by $\mathfrak{g}^*$ and it identifies with $\mathfrak{g}$ by a given non degenerate, invariant, symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g} \times \mathfrak{g}$ extending the Killing form of $[\mathfrak{g}, \mathfrak{g}]$.

- Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$. Denote by $\mathcal{R}$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and let denote by $\mathcal{R}_+$ the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$ is denoted by $W(\mathcal{R})$ and the basis of $\mathcal{R}_+$ is denoted by $\Pi$. The neutral element of $W(\mathcal{R})$ is denoted by $\mathbf{1}_\mathfrak{h}$. For $\alpha$ in $\mathcal{R}$, the corresponding root subspace is denoted by $\mathfrak{g}^\alpha$ and a generator $x_\alpha$ of $\mathfrak{g}^\alpha$ is chosen so that $(x_\alpha, x_{-\alpha}) = 1$ for all $\alpha$ in $\mathcal{R}$.

- The normalizers of $\mathfrak{b}$ and $\mathfrak{h}$ in $G$ are denoted by $B$ and $N_G(\mathfrak{h})$ respectively. For $x$ in $\mathfrak{b}$, $\mathfrak{U}$ is the element of $\mathfrak{b}$ such that $x - \mathfrak{U}$ is in the nilpotent radical $\mathfrak{u}$ of $\mathfrak{b}$.

- For $X$ an algebraic $B$-variety, let denote by $G \times_B X$ the quotient of $G \times X$ under the right action of $B$ given by $(g, x).b := (gb, b^{-1}.x)$. More generally, for $k$ positive integer and for $X$ an algebraic $B^k$-variety,
let denote by $G^k \times_P X$ the quotient of $G^k \times X$ under the right action of $B^k$ given by $(g, x) \cdot b := (gb, b^{-1}x)$ with $g$ and $b$ in $G^k$ and $B^k$ respectively.

**Lemma 1.4.** Let $P$ and $Q$ be parabolic subgroups of $G$ such that $P$ is contained in $Q$. Let $X$ be a $Q$-variety and let $Y$ be a closed subset of $X$, invariant under $P$. Then $Q. Y$ is a closed subset of $X$. Moreover, the canonical map from $Q \times_P Y$ to $Q.Y$ is a projective morphism.

**Proof.** Since $P$ and $Q$ are parabolic subgroups of $G$ and since $P$ is contained in $Q$, $Q/P$ is a projective variety. Let denote by $Q \times_P Y$ and $Q \times Y$ the quotients of $Q \times X$ and $Q \times Y$ under the right action of $P$ given by $(g, x) \cdot p := (gp, p^{-1}x)$. Let $g \mapsto \overline{g}$ be the quotient map from $Q$ to $Q/P$. Since $X$ is a $Q$-variety, the map

$$Q \times X \rightarrow Q/P \times X \quad (g, x) \mapsto (\overline{g}, g \cdot x)$$

defines through the quotient an isomorphism from $Q \times_P Y$ to $Q \times_P X$. Since $Y$ is a $P$-invariant closed subset of $X$, $Q \times_P Y$ is a closed subset of $Q \times_P X$ and its image by the above isomorphism equals $Q \times_P Q. Y$. Hence $Q \times_P Q. Y$ is a closed subset of $X$ since $Q \times_P Q. Y$ is a projective variety. From the commutative diagram

$$Q \times_P Y \quad Q/P \times Q. Y \quad Q. Y$$

one deduces that the map $Q \times_P Y \rightarrow Q. Y$ is a projective morphism. \hfill $\Box$

- For $k \geq 1$ and for the diagonal action of $B$ in $b^k$, $b^k$ is a $B$-variety. The canonical map from $G \times b^k$ to $G \times_B b^k$ is denoted by $(g, x_1, \ldots, x_k) \mapsto (g, x_1, \ldots, x_k)$. Let $B^{(k)}$ and $N^{(k)}$ be the images of $G \times b^k$ and $G \times u^k$ respectively by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$ so that $B^{(k)}$ and $N^{(k)}$ are closed subsets of $g^k$ by Lemma 1.4. Let $B^{(k)}_n$ be the normalization of $B^{(k)}$ and let $\eta$ be the normalization morphism. One has a commutative diagram:

$$G \times_B b^k \quad \gamma \downarrow \quad B^{(k)}_n \quad \eta$$

Let $N^{(k)}_n$ be the normalization of $N^{(k)}$ and let $\kappa$ be the normalization morphism. One has a commutative diagram:

$$G \times_B u^k \quad \iota \downarrow \quad N^{(k)}_n \quad \kappa$$

with $\iota$ the restriction of $\gamma$ to $G \times_B u^k$.

- Let $i$ be the injection $(x_1, \ldots, x_k) \mapsto (1, x_1, \ldots, x_k)$ from $b^k$ to $G \times_B b^k$. Then $i := \gamma \circ i$ and $\iota_n := \gamma \circ i$ are closed embeddings of $b^k$ into $B^{(k)}_n$ and $B^{(k)}_n$ respectively. In particular, $B^{(k)} = G \cdot i(b^k)$ and $B^{(k)}_n = G \cdot i_n(b^k)$.

- Let $e$ be the sum of the $x_0$'s, $\beta$ in $\Pi$, and let $h$ be the element of $b \cap [g, g]$ such that $\beta(h) = 2$ for all $\beta$ in $\Pi$. Then there exists a unique $f$ in $[g, g]$ such that $(e, h, f)$ is a principal $sl_2$-triple. The one parameter
subgroup of $G$ generated by $\text{ad}h$ is denoted by $t \mapsto h(t)$. The Borel subalgebra containing $f$ is denoted by $b_-$ and its nilpotent radical is denoted by $u_-$. Let $B_-$ be the normalizer of $b_-$ in $G$ and let $U$ and $U_-$ be the unipotent radicals of $B$ and $B_-$ respectively.

**Lemma 1.5.** Let $k \geq 2$ be an integer. Let $X$ be an affine variety and let set $Y := b^k \times X$. Let $Z$ be a closed $B$-invariant subset of $Y$ for the group action given by $g.(v_1, \ldots, v_k, x) = (g(v_1), \ldots, g(v_k), x)$ with $(g, v_1, \ldots, v_k)$ in $B \times b^k$ and $x$ in $X$. Then $Z \cap b^k \times X$ is the image of $Z$ by the projection $(v_1, \ldots, v_k, x) \mapsto (\overline{v_1}, \ldots, \overline{v_k}, x)$.

**Proof.** For all $v$ in $b$,

$$\overline{v} = \lim_{t \to 0} h(t)(v)$$

whence the lemma since $Z$ is closed and $B$-invariant. \hfill $\square$

- For $x \in g$, let $x_s$ and $x_n$ be the semisimple and nilpotent components of $x$ in $g$. Let denote by $g^x$ and $G^x$ the centralizers of $x$ in $g$ and $G$ respectively. For $a$ a subalgebra of $g$ and for $A$ a subgroup of $G$, let set:

$$a^x := a \cap g^x \quad A^x := A \cap G^x$$

The set of regular elements of $g$ is

$$g_{reg} := \{x \in g \mid \text{dim } g^x = \ell\}$$

and let denote by $g_{reg,ss}$ the set of regular semisimple elements of $g$. Both $g_{reg}$ and $g_{reg,ss}$ are $G$-invariant dense open subsets of $g$. Setting $b_{reg} := b \cap g_{reg}$, $b_{reg} := b \cap g_{reg}$, $u_{reg} := u \cap g_{reg}$, $g_{reg,ss} = G(b_{reg})$, $g_{reg} = G(b_{reg})$ and $G(u_{reg})$ is the set of regular elements in the nilpotent cone $\mathfrak{g}$ of $g$.

**Lemma 1.6.** Let $k \geq 2$ be an integer and let $x$ be in $g^k$. For $O$ open subset of $g_{reg}$, $P_x \cap O$ is not empty if and only if for some $g$ in $\text{GL}_k(\mathbb{K})$, the first component of $g.x$ is in $O$.

**Proof.** Since the components of $g.x$ are in $P_x$ for all $g$ in $\text{GL}_k(\mathbb{K})$, the condition is sufficient. Let suppose that $P_x \cap O$ is not empty and let denote by $x_1, \ldots, x_k$ the components of $x$. For some $(a_1, \ldots, a_k)$ in $\mathbb{K}^k \setminus \{0\}$,

$$a_1 x_1 + \cdots + a_k x_k \in O$$

Let $i$ be such that $a_i \neq 0$ and let $\tau$ be the transposition of $\mathbb{K}_k$ such that $\tau(1) = i$. Denoting by $g$ the element of $\text{GL}_k(\mathbb{K})$ such that $g_{1,j} = a_{e(j)}$ for $j = 1, \ldots, k$, $g_{j,j} = 1$ for $j = 2, \ldots, k$ and $g_{j,l} = 0$ for $j \geq 2$ and $j \neq l$, the first component of $g.x$ is in $O$. \hfill $\square$

- Let denote by $S(g)^0$ the algebra of $g$-invariant elements of $S(g)$. Let $p_1, \ldots, p_l$ be homogeneous generators of $S(g)^0$ of degree $d_1, \ldots, d_l$ respectively. Let choose the polynomials $p_1, \ldots, p_l$ so that $d_1 \leq \cdots \leq d_l$. For $i = 1, \ldots, d_l$ and $(x,y) \in g \times g$, let consider a shift of $p_i$ in direction $y$: $p_i(x + ty)$ with $t \in \mathbb{K}$. Expanding $p_i(x + ty)$ as a polynomial in $t$, one obtains

$$p_i(x + ty) = \sum_{m=0}^{d_i} p_i^{(m)}(x,y)t^m; \quad \forall (t, x, y) \in \mathbb{K} \times g \times g$$

where $y \mapsto (m!) p_i^{(m)}(x,y)$ is the derivate at $x$ of $p_i$ at the order $m$ in the direction $y$. The elements $p_i^{(m)}$ defined by (1) are invariant elements of $S(g) \otimes_{\mathbb{K}} S(g)$ under the diagonal action of $G$ in $g \times g$. Let remark that $p_i^{(0)}(x,y) = p_i(x)$ while $p_i^{(d_i)}(x,y) = p_i(y)$ for all $(x,y) \in g \times g$. 
Remark 1.7. The family \( \mathcal{P}_x := \{ p_i^{(m)}(x, \cdot) ; 1 \leq i \leq \ell, 1 \leq m \leq d_i \} \) for \( x \in g \), is a Poisson-commutative family of \( \mathfrak{S}(g) \) by Mishchenko-Fomenko [MF78]. One says that the family \( \mathcal{P}_x \) is constructed by the argument shift method.

- Let \( i \in \{1, \ldots, \ell \} \) be. For \( x \in g \), let denote by \( \varepsilon_i(x) \) the element of \( g \) given by

\[
\langle \varepsilon_i(x), y \rangle = \frac{d}{dt} p_i(x + ty) \big|_{t=0}
\]

for all \( y \) in \( g \). Thereby, \( \varepsilon_i \) is an invariant element of \( S(\mathfrak{g}) \otimes \mathbb{K} g \) under the canonical action of \( G \). According to [Ko63, Theorem 9], for \( x \in g \), \( x \) is in \( g_{\text{reg}} \) if and only if \( \varepsilon_1(x), \ldots, \varepsilon_\ell(x) \) are linearly independent. In this case, \( \varepsilon_1(x), \ldots, \varepsilon_\ell(x) \) is a basis of \( g^x \).

Let denote by \( \varepsilon_i^{(m)} \), for \( 0 \leq m \leq d_i - 1 \), the elements of \( S(\mathfrak{g} \times \mathfrak{g}) \otimes \mathbb{K} g \) defined by the equality:

\[
\varepsilon_i(x + ty) = \sum_{m=0}^{d_i-1} \varepsilon_i^{(m)}(x, y)^m, \quad \forall (t, x, y) \in \mathbb{K} \times g \times g
\]

and let set:

\[
V_{x,y} := \text{span}(\varepsilon_i^{(0)}(x, y), \ldots, \varepsilon_i^{(d_i-1)}(x, y), \ i = 1, \ldots, \ell)
\]

for \( (x, y) \) in \( g \times g \). According to [Bol91, Corollary 2], \( V_{x,y} \) has dimension \( b_\mathfrak{g} \) if and only if \( P_{x,y} \setminus \{0\} \) is contained in \( g_{\text{reg}} \).

1.2. Main result. By definition, \( \mathfrak{B}^{(k)} \) is the subset of elements \( (x_1, \ldots, x_k) \) of \( g^k \) such that \( x_1, \ldots, x_k \) are in a same Borel subalgebra of \( g \). This subset of \( g^k \) is closed and contains two interesting subsets: the generalized commuting variety of \( g \), denoted by \( \mathcal{C}^{(k)} \) and the nullcone of \( g^k \) denoted by \( \mathcal{N}^{(k)} \). According to [Mu88, Ch.2, §1, Theorem], for \( (x_1, \ldots, x_k) \) in \( \mathfrak{B}^{(k)} \), \( (x_1, \ldots, x_k) \) is in \( \mathcal{N}^{(k)} \) if and only if \( x_1, \ldots, x_k \) are nilpotent. By definition, \( \mathcal{C}^{(k)} \) is the closure in \( g^k \) of the set of elements whose all components are in a same Cartan subalgebra. According to a Richardson Theorem [Ri79], \( \mathcal{C}^{(2)} \) is the commuting variety of \( g \).

There is a natural projective morphism \( G \times_B b^k \to \mathfrak{B}^{(k)} \). For \( k = 1 \), this morphism is not birational but for \( k \geq 2 \), it is birational. Furthermore, denoting by \( \mathfrak{X} \) the subvariety of elements \( (x, y) \) of \( g \times \mathfrak{h} \) such that \( y \) is in the closure of the orbit of \( x \) under \( G \), the canonical morphism \( G \times_B b \to \mathfrak{X} \) is projective and birational and \( \mathfrak{g} \) is the categorical quotient of \( \mathfrak{X} \) under the action of \( W(\mathfrak{R}) \) on the factor \( \mathfrak{h} \). For \( k \geq 2 \), the inverse image of \( \mathfrak{B}^{(k)} \) by the canonical projection from \( \mathfrak{X}^k \) to \( g^k \) is not irreducible but the canonical action of \( W(\mathfrak{R})^k \) on \( \mathfrak{X}^k \) induces a simply transitive action on the set of its irreducible components. Denoting by \( \mathfrak{B}^{(k)}_{\mathfrak{X}} \) one of these components, one has a commutative diagram

\[
\begin{array}{ccc}
G \times_B b^k & \xrightarrow{\gamma} & \mathfrak{B}^{(k)}_{\mathfrak{X}} \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
\mathfrak{B}^{(k)} & & \\
\end{array}
\]

with \( \sigma \) the restriction to \( \mathfrak{B}^{(k)}_{\mathfrak{X}} \) of the canonical projection from \( \mathfrak{X}^k \) to \( g^k \). The first main theorem of this note is the following theorem:
Theorem 1.8. (i) The variety $\mathcal{N}^{(k)}$ has rational singularities.

(ii) The variety $B^{(k)}_n$ has rational singularities. Moreover, for $k \geq 2$, $B^{(k)}_X$ is the normalization of $B^{(k)}_n$ and $\pi$ is the normalization morphism.

(iii) The restriction of $\eta$ to $\eta^{-1}(\mathcal{N}^{(k)})$ is an isomorphism onto $\mathcal{N}^{(k)}$ and the ideal of definition of $\eta^{-1}(\mathcal{N}^{(k)})$ in $k[B^{(k)}_n]$ is generated by the homogeneous elements of positive degree of $k[B^{(k)}_n]^G$.

From Theorem 1.8, one deduces that for $k \geq 2$, the ideal of definition of $\mathcal{N}^{(k)}$ in $k[B^{(k)}_n]$ is not generated by the homogeneous elements of positive degree of $k[B^{(k)}_n]^G$. Moreover, according to a Joseph’s result [J07], $k[B^{(k)}_n]^G$ is isomorphic to $\text{S}(\mathfrak{h}^k)^{W(\mathfrak{g})}$ for the diagonal action of $W(\mathfrak{g})$ in $\mathfrak{h}^k$.

In the study of the generalized commuting variety, the closure in $\text{Gr}_r(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under the action of $G$ plays an important role. Denoting by $X$ the closure in $\text{Gr}_r(\mathfrak{b})$ of the orbit of $\mathfrak{h}$ under $B$, $G.X$ is the closure of the orbit of $G.\mathfrak{h}$ and one has the following result:

Theorem 1.9. Let $X'$ be the set of centralizers of regular elements of $\mathfrak{g}$ whose semisimple components is regular or subregular. Let $X_n$ and $(G.X)_n$ be the normalizations of $X$ and $G.X$ respectively. Let denote by $\theta_0$ and $\theta$ the normalization morphisms $X_n \to X$ and $(G.X)_n \to G.X$ respectively.

(i) All element of $X$ is a commutative algebraic subalgebra of $\mathfrak{g}$.

(ii) For $x$ in $\mathfrak{g}$ and for $v'$ a regular linear form on $\mathfrak{g}^*$, the stabilizer of $v'$, with respect to the coadjoint action of $\mathfrak{g}^*$, is in $G.X$.

(iii) For $x$ in $\mathfrak{g}$, the set of elements of $G.X$ containing $x$ has dimension at most $\dim \mathfrak{g}^* - \ell$.

(iv) The set $X'$ is an open subset of $X$ and $X \setminus X'$ has codimension at least 2 in $X$.

(v) All irreducible component of $X \setminus B.\mathfrak{h}$ has a nonempty intersection with $X'$.

(vi) The set $G.X'$ is an open subset of $G.X$ and $G.X \setminus G.X'$ has codimension at least 2 in $G.X$.

(vii) All irreducible component of $G.X \setminus G.\mathfrak{h}$ has a nonempty intersection with $G.X'$.

(viii) The restriction of $\theta$ to $\theta_0^{-1}(G.X')$ is a homeomorphism onto $G.X'$ and $\theta_0^{-1}(G.X')$ is a smooth open subset of $G.X_n$.

(ix) The restriction of $\theta_0$ to $\theta_0^{-1}(X')$ is a homeomorphism onto $X'$ and $\theta_0^{-1}(X')$ is a smooth open subset of $X_n$.

Let $X_{0,k}$ be the closure in $\mathfrak{b}^k$ of $B.\mathfrak{h}^k$ and let $\Gamma$ be a desingularization of $X$ in the category of $B$-varieties. Let $E^{(k)}$ be the inverse image of the canonical vector bundle over $X$. Then $E^{(k)}$ is a desingularization of $X_{0,k}$. Let set: $\mathcal{C}^{(k)} := \eta^{-1}(\mathcal{C}^{(k)})$. The following theorem is the second main result of this note:

Theorem 1.10. (i) The variety $X_{0,k}$ has rational singularities.

(ii) The variety $\mathcal{C}^{(k)}$ is irreducible and $G \times_B E^{(k)}$ is a desingularization of $\mathcal{C}^{(k)}$.

(iii) The normalization morphisms of $\mathcal{C}^{(k)}$ and $\mathcal{C}^{(k)}$ are homeomorphisms.

(iv) For $k = 2$, the normalizations of $\mathcal{C}^{(2)}_n$ and $\mathcal{C}^{(2)}$ have rational singularities.

The proof of Assertion (iv) is an easy consequence of the proof of Assertion (i), and the deep result of Ginzburg [Gi11] which asserts that the normalization of $\mathcal{C}^{(2)}_n$ is Gorenstein.
2. Cohomological results

Let $k \geq 2$ be an integer. According to the above notations, one has the commutative diagrams:

\[
\begin{array}{ccc}
G \times B^k & \xrightarrow{\gamma} & B^{(k)}_n \\
\downarrow \gamma & & \downarrow \eta \\
\mathcal{B}^{(k)}_n & & B^{(k)}_n
\end{array}
\quad \begin{array}{ccc}
G \times B^k & \xrightarrow{\nu} & N^{(k)}_n \\
\downarrow \nu & & \downarrow \pi \\
\mathcal{N}^{(k)}_n & & B^{(k)}_n
\end{array}
\]

2.1. Since the Borel subalgebras of $\mathfrak{g}$ are conjugate under $G$, $\mathcal{B}^{(k)}$ is the subset of elements of $\mathfrak{g}^k$ whose components are in a same Borel subalgebra and $\mathcal{N}^{(k)}$ are the elements of $\mathcal{B}^{(k)}$ whose all the components are nilpotent.

**Lemma 2.1.** (i) The morphism $\gamma$ from $G \times B^k$ to $\mathcal{B}^{(k)}$ is projective and birational. In particular, $G \times B^k$ is a desingularization of $\mathcal{B}^{(k)}$ and $\mathcal{B}^{(k)}$ has dimension $k_b + n$.

(ii) The morphism $\nu$ from $G \times B^k$ to $\mathcal{N}^{(k)}$ is projective and birational. In particular, $G \times B^k$ is a desingularization of $\mathcal{N}^{(k)}$ and $\mathcal{N}^{(k)}$ has dimension $(k + 1)n$.

**Proof.** (i) According to Lemma 1.4, $\gamma$ is a projective morphism. For $1 \leq i < j \leq k$, let $\Omega^{(k)}_{ij}$ be the inverse image of $\Omega_0$ by the projection

\[
(x_1, \ldots, x_k) \mapsto (x_i, x_j)
\]

Then $\Omega^{(k)}_{ij}$ is an open subset of $\mathfrak{g}^k$ whose intersection with $\mathcal{B}^{(k)}_n$ is not empty. Let $\Omega^{(k)}_0$ be the union of the $\Omega^{(k)}_{ij}$. According to [Bol91, Corollary 2] and [Ko63, Theorem 9], for $(x, y) \in \Omega_0 \cap \mathcal{B}^{(2)}$, $V_{x,y}$ is the unique Borel subalgebra of $\mathfrak{g}$ containing $x$ and $y$ so that the restriction of $\gamma$ to $\gamma^{-1}(\Omega^{(k)}_0)$ is a bijection onto $\Omega^{(k)}_0$. Hence $\gamma$ is birational. Moreover, $G \times B^k$ is a smooth variety as a vector bundle over the smooth variety $G/B$, whence the assertion since $G \times B^k$ has dimension $k_b + n$.

(ii) According to Lemma 1.4, $\nu$ is a projective morphism. Let $\mathcal{N}^{(k)}_{\text{reg}}$ be the subset of elements of $\mathcal{N}^{(k)}$ whose at least one component is a regular element of $\mathfrak{g}$. Then $\mathcal{N}^{(k)}_{\text{reg}}$ is an open subset of $\mathcal{N}^{(k)}$. Since a regular nilpotent element is contained in one and only one Borel subalgebra of $\mathfrak{g}$, the restriction of $\nu$ to $\nu^{-1}(\mathcal{N}^{(k)}_{\text{reg}})$ is a bijection onto $\mathcal{N}^{(k)}_{\text{reg}}$. Hence $\nu$ is birational. Moreover, $G \times B^k$ is a smooth variety as a vector bundle over the smooth variety $G/B$, whence the assertion since $G \times B^k$ has dimension $(k + 1)n$. \qed

Let $\kappa$ be the map

\[
U_- \times u_{\text{reg}} \longrightarrow \mathfrak{R}_n \\
(g, x) \longmapsto g(x)
\]

**Lemma 2.2.** Let $V$ be the set of elements of $\mathcal{N}^{(k)}$ whose first component is in $U_-(u_{\text{reg}})$ and let $V_\kappa$ be the set of elements $x$ of $\mathcal{N}^{(k)}$ such that $P_x \cap u_{\text{reg}}$ is not empty.

(i) The image of $\kappa$ is a smooth open subset of $\mathfrak{R}_n$ and $\kappa$ is an isomorphism onto $U_-(u_{\text{reg}})$.

(ii) The subset $V$ of $\mathcal{N}^{(k)}$ is open.

(iii) The open subset $V$ of $\mathcal{N}^{(k)}$ is smooth.

(iv) The set $V_\kappa$ is a smooth open subset of $\mathcal{N}^{(k)}$.

**Proof.** (i) Since $\mathfrak{R}_n$ is the nullvariety of $p_1, \ldots, p_T$ in $\mathfrak{g}$, $\mathfrak{R}_n \cap u_{\text{reg}}$ is a smooth open subset of $\mathfrak{R}_n$ by [Ko63, Theorem 9]. For $(g, x)$ in $U_- \times u_{\text{reg}}$ such that $g(x)$ is in $u$, $b^{-1}g$ is in $G^+$ for some $b$ in $B$ since $B(x) = u_{\text{reg}}$. 


Hence \( g = 1_{\mathfrak{g}} \) since \( G^k \) is contained in \( B \) and since \( U_{-} \cap B = \{1_{\mathfrak{g}}\} \). As a result, \( \kappa \) is an injective morphism from the smooth variety \( U_{-} \times u_{\text{reg}} \) to the smooth variety \( \mathfrak{g}_{\text{reg}} \cap \mathfrak{g}_{\text{reg}} \). Hence \( \kappa \) is an open immersion by Zariski Main Theorem [Mu88, §9].

(ii) By definition, \( V \) is the intersection of \( N^{(k)} \) and \( U_{-}(u_{\text{reg}}) \times \mathfrak{g}^{k-1}_{\text{reg}} \). So, by (i), it is an open subset of \( N^{(k)} \).

(iii) Let \( (x_1, \ldots, x_k) \) be in \( u^k \) and let \( g \) be in \( G \) such that \( (g(x_1), \ldots, g(x_k)) \) is in \( V \). Then \( x_1 \) is in \( u_{\text{reg}} \) and for some \( (g, b) \) in \( U_{-} \times B \), \( g^b(x_1) = g(x_1) \). Hence \( g^{-1}g^b \) is in \( G^{k_1} \) and \( g \) is in \( U_{-}B \) since \( G^{k_1} \) is contained in \( B \). As a result, the map

\[
U_{-} \times u_{\text{reg}} \times u^{k-1} \rightarrow V \quad (g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))
\]

is an isomorphism whose inverse is given by

\[
V \rightarrow U_{-} \times u_{\text{reg}} \times u^{k-1} \quad (x_1, \ldots, x_k) \mapsto (\kappa^{-1}(x_1)_1, \kappa^{-1}(x_1)_1(x_1), \ldots, \kappa^{-1}(x_1)_1(x_k))
\]

with \( \kappa^{-1} \) the inverse of \( \kappa \) and \( \kappa^{-1}(x_1)_1 \) the component of \( \kappa^{-1}(x_1) \) on \( U_{-} \), whence the assertion since \( U_{-} \times u_{\text{reg}} \times u^{k-1} \) is smooth.

(iv) According to Lemma 1.6, \( V_k = \text{GL}_k(\mathbb{k}).V \), whence the assertion by (iii). \( \square \)

**Corollary 2.3.**

(i) The subvariety \( N^{(k)} \setminus V_k \) has codimension \( k + 1 \).

(ii) The restriction of \( \nu \) to \( \nu^{-1}(V_k) \) is an isomorphism onto \( V_k \).

(iii) The subset \( \nu^{-1}(V_k) \) is a big open subset of \( G \times_B u^k \).

**Proof.**

(i) By definition, \( N^{(k)} \setminus V_k \) is the subset of elements \( x \) of \( N^{(k)} \) such that \( P_x \) is contained in \( \mathfrak{g} \setminus \mathfrak{g}_{\text{reg}} \). Hence \( N^{(k)} \setminus V_k \) is contained in the image of \( G \times_B (u \setminus u_{\text{reg}})^k \) by \( \nu \). Let \( (x_1, \ldots, x_k) \) be in \( u^k \cap (N^{(k)} \setminus V_k) \). Then, for all \( (a_1, \ldots, a_k) \) in \( k^k \),

\[
\langle x_\beta, a_1x_1 + \cdots + a_kx_k \rangle = 0
\]

for some \( \beta \in \Pi \). Since \( \Pi \) is finite, \( P_x \) is orthogonal to \( x_\beta \) for some \( \beta \) in \( \Pi \). As a result, the subvariety of Borel subalgebras of \( \mathfrak{g} \) containing \( x_1, \ldots, x_k \) has positive dimension. Hence

\[
\dim(N^{(k)} \setminus V_k) < \dim G \times_B (u \setminus u_{\text{reg}})^k = n + k(n - 1)
\]

Moreover, for \( \beta \) in \( \Pi \), denoting by \( u_{\beta} \) the orthogonal complement of \( \mathfrak{g}^{-\beta} \) in \( u \), \( u(G \times_B (u_{\beta})^k) \) is contained in \( N^{(k)} \setminus V_k \) and its dimension equal \( (k + 1)(n - 1) \) since the variety of Borel subalgebras containing \( u_{\beta} \) has dimension 1, whence the assertion.

(ii) For \( x \) in \( N^{(k)} \), \( P_x \) is contained in all Borel subalgebra of \( \mathfrak{g} \), containing the components of \( x \). Then the restriction of \( \nu \) to \( \nu^{-1}(V_k) \) is injective since all regular nilpotent element of \( \mathfrak{g} \) is contained in a single Borel subalgebra of \( \mathfrak{g} \), whence the assertion by Zariski Main Theorem [Mu88, §9] since \( V_k \) is a smooth open subset of \( N^{(k)} \) by Lemma 2.2.(iv).

(iii) Let identify \( U_{-} \) with the open subset \( U_{-}B/B \) of \( G/B \) and let denote by \( \psi \) the canonical projection from \( G \times_B u^k \) to \( G/B \). Since \( \nu^{-1}(V_k) \) is \( G \)-invariant, it suffices to prove that \( \nu^{-1}(V_k) \cap \psi^{-1}(U_{-}) \) is a big open subset of \( \psi^{-1}(U_{-}) \).

The open subset \( \psi^{-1}(U_{-}) \) of \( G \times_B u^k \) identifies with \( U_{-} \times u^k \) and \( \nu^{-1}(V_k) \cap \psi^{-1}(U_{-}) \) identifies with the set of \( (g, x) \) such that \( P_x \cap \mathfrak{g}_{\text{reg}} \) is not empty. Let denote by \( V_0 \) the subset of elements \( x \) of \( u^k \) such that \( P_x \cap \mathfrak{g}_{\text{reg}} \) is not empty. Then \( u^k \setminus V_0 \) is contained in \( (u \setminus u_{\text{reg}})^k \) and has codimension at least 2 in \( u^k \) since \( k \geq 2 \). As a result, \( U_{-} \times V_0 \) is a big open subset of \( U_{-} \times u^k \), whence the assertion. \( \square \)
Theorem 2.4. Let $k \geq 2$ be an integer and let $\mathcal{N}_n^{(k)}$ be the normalization of $\mathcal{N}^{(k)}$. Then $\mathcal{N}_n^{(k)}$ has rational singularities.

Proof. Since $G \times_B u^k$ is a desingularization of $\mathcal{N}^{(k)}$ by Lemma 2.1(ii), one has a commutative diagram

\[
\begin{array}{ccc}
G \times_B u^k & \overset{\nu}{\longrightarrow} & \mathcal{N}_n^{(k)} \\
\downarrow{\nu} & & \downarrow{\pi} \\
\mathcal{N}^{(k)} & & \\
\end{array}
\]

with $\pi$ the normalization morphism. Moreover, $\nu_n$ is a projective birational morphism. According to Corollary 2.3, $\nu^{-1}(V_k)$ is a smooth big open subset of $\mathcal{N}_n^{(k)}$, $\nu^{-1}(V_k)$ is a big open subset of $G \times_B u^k$ and the restriction of $\nu_n$ to $\nu^{-1}(V_k)$ is an isomorphism onto $\nu^{-1}(V_k)$. Hence, by Proposition C.2, with $Y = G \times_B u^k$, $\mathcal{N}_n^{(k)}$ has rational singularities. \hfill $\Box$

2.2. For $E$ a finite dimensional $B$-module, let denote by $\mathcal{L}_0(E)$ the sheaf of local sections of the vector bundle $G \times_B E$ over $G/B$. For $(k, l)$ in $\mathbb{N}^2$, let set:

\[
E_k := (b^*)^\otimes k \quad E_{k,l} := (b^*)^\otimes k \otimes_B u^{2l}
\]

so that $E_k$ and $E_{k,l}$ are $B$-modules. According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by $\langle \ldots \rangle$, the dual of $u$ identifies with $u^*$ so that $u^*$ is a $B$-module.

Proposition 2.5. Let $k, l$ be nonnegative integers.

(i) For all positive integer $i$, $H^i(G/B, \mathcal{L}_0(u^k)) = 0$.

(ii) For all positive integer $i$, $H^i(G/B, \mathcal{L}_0(E_k)) = 0$.

(iii) For all positive integer $i$, $H^{i+l}(G/B, \mathcal{L}_0(E_{k,l})) = 0$.

Proof. (i) First of all, since $H^1(G/B, \mathcal{O}_{G/B}) = 0$ for all positive integer by Borel-Weil-Bott’s Theorem [Dem68], one can suppose $k > 0$. According to the identification of $u^*$ and $u_-$, $S(u_+)$ is the algebra of polynomial functions on $u^k$. Then, since $G \times_B u^k$ is a vector bundle over $G/B$, for all nonnegative integer $i$,

\[
H^i(G \times_B u^k, \mathcal{O}_{G \times_B u^k}) = H^i(G/B, \mathcal{L}_0(S(u_+^k))) = \bigoplus_{q \geq 0} H^i(G/B, \mathcal{L}_0(S^q(u_+)))
\]

According to Theorem 2.4, for $i > 0$, the left hand side equals 0 since $G \times_B u^k$ is a desingularization of $\mathcal{N}^{(k)}$ by Lemma 2.1(ii). As a result, for $i > 0$,

\[
H^i(G/B, \mathcal{L}_0(S^k(u_+))) = 0
\]

The decomposition of $u_+^k$ as a direct sum of $k$ copies of $u_-$ induces a multigradation of $S(u_+)$ such that each subspace of multidegree $(j_1, \ldots, j_k)$ is a $B$-submodule. Denoting this subspace by $S_{j_1, \ldots, j_k}$, one has

\[
S^k(u_+^k) = \bigoplus_{(j_1, \ldots, j_k) \neq 0} S_{j_1, \ldots, j_k} \text{ and } S_{1, \ldots, 1} = u_+^k
\]

Hence for $i > 0$,

\[
0 = H^i(G/B, \mathcal{L}_0(S^k(u_+))) = \bigoplus_{(j_1, \ldots, j_k) \neq 0} H^i(G/B, \mathcal{L}_0(S_{j_1, \ldots, j_k}))
\]
whence the assertion.

(ii) Let \( i \) be a positive integer. Let prove by induction on \( j \) that for \( k \geq j \),

\[
H^i(G/B, \mathcal{L}_0(E_j \otimes_k u^{-1}_0)) = 0
\]

By (i), it is true for \( j = 0 \). Let suppose \( j > 0 \) and (3) true for \( j - 1 \) and for all \( k \geq j - 1 \). From the exact sequence of \( B \)-modules

\[
0 \rightarrow b \rightarrow b^* \rightarrow u_0 \rightarrow 0
\]

one deduces the exact sequence of \( B \)-modules

\[
0 \rightarrow E_{j-1} \otimes_k b \otimes_k u^{-1}_0 \rightarrow E_j \otimes_k u^{-1}_0 \rightarrow E_{j-1} \otimes_k u^{-1}_0 \rightarrow 0
\]

whence the exact sequence of \( \mathcal{O}_{G/B} \)-modules

\[
0 \rightarrow \mathcal{L}_0(E_j \otimes_k b \otimes_k u^{-1}_0) \rightarrow \mathcal{L}_0(E_j \otimes_k u^{-1}_0) \rightarrow \mathcal{L}_0(E_{j-1} \otimes_k u^{-1}_0) \rightarrow 0
\]

since \( \mathcal{L}_0 \) is an exact functor. From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

\[
H^i(G/B, \mathcal{L}_0(E_j \otimes_k u^{-1}_0)) \rightarrow H^i(G/B, \mathcal{L}_0(E_j \otimes_k u^{-1}_0)) \rightarrow H^i(G/B, \mathcal{L}_0(E_{j-1} \otimes_k u^{-1}_0))
\]

By induction hypothesis, the last term equals 0. Since \( b \) is a trivial \( B \)-module,

\[
\mathcal{L}_0(E_j \otimes_k b \otimes_k u^{-1}_0) = b \otimes_k \mathcal{L}_0(E_j \otimes_k u^{-1}_0)
\]

\[
H^i(G/B, \mathcal{L}_0(E_j \otimes_k b \otimes_k u^{-1}_0)) = b \otimes_k H^i(G/B, \mathcal{L}_0(E_j \otimes_k u^{-1}_0))
\]

Then, by induction hypothesis again, the first term of the last exact sequence equals 0, whence Equality (3) and whence the assertion since it is true for \( k = 0 \) by Borel-Weil-Bott’s Theorem.

(iii) Let \( k \) be a nonnegative integer. Let prove by induction on \( j \) that for \( i > 0 \) and for \( l \geq j \),

\[
H^{i+j}(G/B, \mathcal{L}_0(E_{k+l-j})) = 0
\]

By (ii) it is true for \( j = 0 \). Let suppose \( j > 0 \) and (4) true for \( j - 1 \) and for all \( l \geq j - 1 \). From the short exact sequence of \( B \)-modules

\[
0 \rightarrow u \rightarrow g \rightarrow b^* \rightarrow 0
\]

one deduces the short exact sequence of \( B \)-modules

\[
0 \rightarrow E_{k+l-j} \rightarrow g \otimes_k E_{k+l-j} \rightarrow E_{k+l-j+1} \rightarrow 0
\]

whence the exact sequence of \( \mathcal{O}_{G/B} \)-modules

\[
0 \rightarrow \mathcal{L}_0(E_{k+l-j}) \rightarrow \mathcal{L}_0(g \otimes_k E_{k+l-j}) \rightarrow \mathcal{L}_0(E_{k+l-j+1}) \rightarrow 0
\]

since \( \mathcal{L}_0 \) is an exact functor. From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

\[
H^{i+j}(G/B, \mathcal{L}_0(E_{k+l-j})) \rightarrow H^{i+j}(G/B, \mathcal{L}_0(E_{k+l-j})) \rightarrow H^{i+j}(G/B, \mathcal{L}_0(g \otimes_k E_{k+l-j}))
\]
for all positive integer $i$. By induction hypothesis, the first term equals 0 for all $i > 0$. Since $g$ is a $G$-module,

$$L_0(g \otimes_k E_{k+i-j-1}) = g \otimes_k L_0(E_{k+i-j-1})$$

$$H^{i+j}(G/B, L_0(g \otimes_k E_{k+i-j-1})) = g \otimes_k H^{i+j}(G/B, L_0(E_{k+i-j-1}))$$

Then by induction hypothesis again, the last term of the last exact sequence equals 0, whence Equality (4) and whence the assertion for $j = l$. \hfill \Box

**Corollary 2.6.** Let $V$ be a subspace of $b$ containing $u$ and let $i$ be a positive integer.

(i) For all nonnegative integers $k, l$, $H^{i+l}(G/B, L_0((b^*)^\otimes k \otimes_k V^{(l)}) = 0$.

(ii) For all nonnegative integer $m$ and for all positive integer $k$,

$$H^{i+m}(G/B, L_0(\wedge^m(V^k))) = 0$$

**Proof.** (i) Let prove by induction on $j$ that for $l \geq j$,

$$H^{i+l}(G/B, L_0((b^*)^\otimes k \otimes_k V^{(j)}) = 0$$

According to Proposition 2.5,(iii), it is true for $j = 0$. Let suppose that it is true for $j - 1$. From the exact sequence of $B$-modules

$$0 \rightarrow u \rightarrow V \rightarrow V/u \rightarrow 0$$

one deduces the exact sequence of $B$-modules

$$0 \rightarrow (b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l+j-1)} \rightarrow (b^*)^\otimes k \otimes_k V^{(j)} \otimes_k u^{(l-j)}$$

$$\hspace{1cm} \rightarrow V/u \otimes_k (b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l-j)} \rightarrow 0$$

whence the exact sequence of $O_{G/B}$-modules

$$0 \rightarrow L_0((b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l+j-1)}) \rightarrow L_0((b^*)^\otimes k \otimes_k V^{(j)} \otimes_k u^{(l-j)})$$

$$\hspace{1cm} \rightarrow L_0(V/u \otimes_k (b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l-j)}) \rightarrow 0$$

From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

$$H^{i+l}(G/B, L_0((b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l+j-1)})) \rightarrow H^{i+j}(G/B, L_0((b^*)^\otimes k \otimes_k V^{(j)} \otimes_k u^{(l-j)}))$$

$$\hspace{1cm} \rightarrow H^{i+l}(G/B, L_0(V/u \otimes_k (b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l-j)}))$$

By induction hypothesis, the first term equals 0. Since $V/u$ is a trivial $B$-module,

$$L_0(V/u \otimes_k (b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l-j)}) = V/u \otimes_k L_0((b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l-j)})$$

$$\hspace{1cm} = H^{i+l}(G/B, L_0((b^*)^\otimes k \otimes_k V^{(j-1)} \otimes_k u^{(l-j)}))$$

Then, by induction hypothesis again, the last term of the last exact sequence equals 0, whence Equality (5) and whence the assertion for $j = l$. \hfill \Box
(ii) Since
\[ \bigwedge^m(V^k) = \bigoplus_{(j_1, \ldots, j_k) \in \mathbb{N}^k} \bigwedge^{j_1}(V) \otimes \cdots \otimes \bigwedge^{j_k}(V) \]
(ii) results from (i) and Proposition B.2. \( \square \)

3. On the varieties \( \mathcal{B}(k) \).

Let \( \mathcal{X} \) be the closed subvariety of \( g \times h \) such that \( k[X] = S(g) \otimes_{S(h)^{W(\mathfrak{r})}} S(\mathfrak{h}) \). Let \( k \geq 2 \) be an integer and let \( \mathcal{B}_n(k) \) be the normalization of \( \mathcal{B}(k) \). The goal of the section is to prove that \( \mathcal{B}_n(k) \) is a closed subvariety of \( \mathcal{X}_k \) and to give some consequences of this fact.

3.1. According to the notations of Subsection 1.1, \( \gamma \) is the morphism from \( G \times_B B \) to \( g \) defined by the map \((g, x) \mapsto g(x)\) through the quotient map.

**Lemma 3.1.** (i) The subvariety \( \mathcal{X} \) of \( g \times h \) is invariant under the \( G \)-action on the first factor and the \( W(\mathfrak{r}) \)-action on the second factor. Furthermore, these actions commute.

(ii) There exists a well defined \( G \)-equivariant morphism \( \gamma_n \) from \( G \times_B B \) to \( \mathcal{X} \) such that \( \gamma \) is the compound of \( \gamma_n \) and the canonical projection from \( \mathcal{X} \) to \( g \).

(iii) The variety \( \mathcal{X} \) is irreducible and the morphism \( \gamma_n \) is projective and birational.

(iv) The variety \( \mathcal{X} \) is normal. Moreover, all element of \( \mathfrak{g}_{\text{reg}} \times h \cap \mathcal{X} \) is a smooth point of \( \mathcal{X} \).

(v) The algebra \( k[X] \) is the space of global sections of \( O_{G \times_B B} \) and \( k[X]^{G} = S(\mathfrak{h}) \).

**Proof.** (i) By definition, for \((x, y) \in g \times h \), \((x, y) \in \mathcal{X} \) if and only if \( p(x) = p(y) \) for all \( p \in S(g)^G \). Hence \( \mathcal{X} \) is invariant under the \( G \)-action on the first factor and the \( W(\mathfrak{r}) \)-action on the second factor. Moreover, these two actions commute.

(ii) Since the map \((g, x) \mapsto (g(x), \mathfrak{X})\) is constant on the \( B \)-orbits, there exists a uniquely defined morphism \( \gamma_n \) from \( G \times_B B \) to \( g \times h \) such that \((g(x), \mathfrak{X})\) is the image by \( \gamma_n \) of the image of \((g, x)\) in \( G \times_B B \). The image of \( \gamma_n \) is contained in \( \mathcal{X} \) since for all \( p \in S(g)^G \), \( p(\mathfrak{X}) = p(x) = p(g(x)) \). Furthermore, \( \gamma_n \) verifies the condition of the assertion.

(iii) According to Lemma 1.4, \( \gamma_n \) is a projective morphism. Let \((x, y) \in g \times h \) such that \( p(x) = p(y) \) for all \( p \in S(g)^G \). For some \( g \in G \), \( g(x) \) is in \( B \) and its semi-simple component is \( y \) so that \((x, y) \) is in the image of \( \gamma_n \). As a result, \( \mathcal{X} \) is irreducible as the image of the irreducible variety \( G \times_B B \). Since for all \((x, y) \in \mathcal{X} \cap \mathfrak{g}_{\text{reg}} \times h_{\text{reg}} \), there exists a unique \( w \) in \( W(\mathfrak{r}) \) such that \( w = w(x) \), the fiber of \( \gamma_n \) at any element \( \mathcal{X} \cap G.(\mathfrak{b}_{\text{reg}} \times h_{\text{reg}}) \) has one element. Hence \( \gamma_n \) is birational, whence the assertion.

(iv) Let \( I \) be the ideal of \( k[g \times h] \) generated by the functions \((x, y) \mapsto p_i(x) - p_i(y), i = 1, \ldots, \ell \) and let \( \mathcal{X}_I \) be the subscheme of \( g \times h \) defined by \( I \). Then \( \mathcal{X} \) is the nullvariety of \( I \) in \( g \times h \). Since \( \mathcal{X} \) has codimension \( \ell \) in \( g \times h \), \( \mathcal{X}_I \) is a complete intersection. Let \((x, y) \) be in \( \mathcal{X} \) such that \( x \) is a regular element of \( g \) and let \( T_{x,y} \) be the tangent space at \((x, y) \) of \( \mathcal{X}_I \). For \( i = 1, \ldots, \ell \), the differential at \((x, y) \) of the function \((x, y) \mapsto p_i(x) - p_i(y) \) is
\[ (v, w) \mapsto \langle e_i(x), v \rangle - \langle e_i(y), w \rangle \]
For \( w \) in \( h \), if \((v, w) \) and \((v', w) \) are in \( T_{x,y} \) then \( v - v' \) is orthogonal to \( e_1(x), \ldots, e_\ell(x) \). Since \( x \) is regular, \( e_1(x), \ldots, e_\ell(x) \) is a basis of \( g^* \) by [Ko63, Theorem 9]. Hence
\[ \dim T_{x,y} \leq \dim g - \ell + \dim h \]
As a result, $X \cap g_{\reg} \times \mathfrak{h}$ is contained in the subset of smooth points of $X_I$. According to [V72], $g \setminus g_{\reg}$ has codimension 3 in $g$. Hence $X_I$ is regular in codimension 1 and according to Serre’s normality criterion [Bou98, §1, no 10, Théorème 4], $X_I$ is normal. In particular, $I$ is prime and $X = X_I$, whence the assertion.

(v) According to (iii), (iv) and Lemma 1.1, $k[X] = H^0(G \times_B b, \mathcal{O}_{G \times_B b})$. Under the action of $G$ in $g \times \mathfrak{h}$, $k[\mathfrak{g} \times \mathfrak{h}]^G = S(\mathfrak{g})^G \otimes_k S(\mathfrak{h})$ and its image in $k[X]$ by the quotient morphism equals $S(\mathfrak{h})$. Moreover, since $G$ is reductive, $k[X]^G$ is the image of $k[\mathfrak{g} \times \mathfrak{h}]^G$ by the quotient morphism, whence the assertion. □

The following proposition is given by [He76, Theorem B and Corollary].

**Proposition 3.2.** (i) For $i > 0$, $H^i(G/B, \mathcal{L}_0(S(\mathfrak{b}^r)))$ equals 0.

(ii) The variety $X$ has rational singularities.

**Corollary 3.3.** (i) Let $x$ and $x'$ be in $b_{\reg}$ such that $(x', \overline{x})$ is in $G \cdot (x, \overline{x})$. Then $x'$ is in $B(x)$.

(ii) For all $w$ in $W(\mathfrak{R})$, the map

$$U_- \times b_{\reg} \to X \quad (g, x) \mapsto (g(x), w(\overline{x}))$$

is an isomorphism onto a smooth open subset of $X$.

**Proof.** (i) The semisimple components of $x$ and $x'$ are conjugate under $B$ since they are conjugate to $\overline{x}$ under $B$. Let $b$ and $b'$ be in $B$ such that $\overline{x}$ is the semisimple component of $b(x)$ and $b'(x')$. Then the nilpotent components of $b(x)$ and $b'(x')$ are regular nilpotent elements of $\mathfrak{g}^\mathbb{T}$, belonging to the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^\mathbb{T}$ of $\mathfrak{g}^\mathbb{T}$. Hence $x'$ is in $B(x)$.

(ii) Since the action of $G$ and $W(\mathfrak{R})$ on $X$ commute, it suffices to prove the corollary for $w = 1_b$. Let denote by $\theta$ the map

$$U_- \times b_{\reg} \to X \quad (g, x) \mapsto (g(x), \overline{x})$$

Let $(g, x)$ and $(g', x')$ be in $U_- \times b_{\reg}$ such that $\theta(g, x) = \theta(g', x')$. By (i), $x' = b(x)$ for some $b$ in $B$. Hence $g^{-1}g'b$ is in $G^x$. Since $x$ is in $b_{\reg}$, $G^x$ is contained in $B$ and $g^{-1}g'$ is in $U_- \cap B$, whence $(g, x) = (g', x')$ since $U_- \cap B = \{1_g\}$. As a result, $\theta$ is a dominant injective map from $U_- \times b_{\reg}$ to the normal variety $X$. Hence $\theta$ is an isomorphism onto a smooth open subset of $X$, by Zariski Main Theorem [Mu88, §9]. □

### 3.2. Let $\Delta$ be the diagonal of $(G/B)^k$ and let $\mathcal{J}_\Lambda$ be its ideal of definition in $\mathcal{O}_{(G/B)^k}$. The variety $G/B$ identifies with $\Delta$ so that $\mathcal{O}_{(G/B)^k}/\mathcal{J}_\Lambda$ is isomorphic to $\mathcal{O}_{G/B}$. For $E$ a $B^k$-module, let denote by $\mathcal{L}(E)$ the sheaf of local sections of the vector bundle $G^k \times_{\mathfrak{g}^k} E$ over $(G/B)^k$.

**Lemma 3.4.** Let $E$ be a finite dimensional $B^k$-module and let $E_\bullet$ be an acyclic complex of finite dimensional $B^k$-modules. Let denote by $\mathcal{L}$ the $B$-module defined by the diagonal action of $B$ on $E$.

(i) The short sequence of $\mathcal{O}_{(G/B)^k}$-modules

$$0 \to \mathcal{J}_\Lambda \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E) \to \mathcal{L}(E) \to \mathcal{L}_0(\mathcal{E}) \to 0$$

is exact.

(ii) The complex $\mathcal{J}_\Lambda \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(E_\bullet)$ is acyclic.
Proof. (i) Since $\mathcal{L}(E)$ is a locally free $\mathcal{O}_{(G/B)^*}$-module, the short sequence of $\mathcal{O}_{(G/B)^*}$-modules

$$0 \rightarrow \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E) \rightarrow \mathcal{L}(E) \rightarrow \mathcal{O}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E) \rightarrow 0$$

is exact, whence the assertion since $\mathcal{O}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E)$ is isomorphic to $\mathcal{L}_0(E)$.

(ii) Since $\Delta$ is a smooth subvariety of the smooth variety $(G/B)^*$, it is a locally complete intersection. Hence locally, $\mathcal{J}_\Delta$ has a free resolution by a Koszul complex

$$K \rightarrow \mathcal{J}_\Delta \rightarrow 0$$

Locally, one has a double complex $C := K \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E)$. Since $\mathcal{L}(E)$ is an acyclic complex of locally free modules, the complex $C_i$ is acyclic for all $i$ and the complex $C_i \otimes \mathcal{O}_{(G/B)^*}$ is acyclic for all $i > 0$, whence the assertion since the exactness of the complex of the assertion is a local property.

Corollary 3.5. Let $V$ be a subspace of $b$ containing $u$ and let $i$ be a positive integer.

(i) For all nonnegative integer $m$,

$$H^{i+m+1}((G/B)^k, \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(\wedge^m(V^k))) = 0$$

(ii) For all nonnegative integer $m$,

$$H^{i+1}((G/B)^k, \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(S^m((b^*)^k))) = 0$$

Proof. The spaces $(b^*)^k$ and $V^k$ are naturally $B^k$-modules. Then it is so for $S^m((b^*)^k)$ and $\wedge^m(V^k)$. Let denote by $E$ one of these two modules and let denote by $\mathcal{T}$ the $B$-module defined by the diagonal action of $B$ on $E$. According to Lemma 3.4,(i), the short sequence of $\mathcal{O}_{(G/B)^*}$-modules

$$0 \rightarrow \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E) \rightarrow \mathcal{L}(E) \rightarrow \mathcal{L}_0(\mathcal{T}) \rightarrow 0$$

is exact whence the cohomology long exact sequence

$$\cdots \rightarrow H^i((G/B)^k, \mathcal{L}(E)) \rightarrow H^i(G/B, \mathcal{L}_0(\mathcal{T})) \rightarrow H^{i+1}((G/B)^k, \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E)) \rightarrow H^{i+1}((G/B)^k, \mathcal{L}(E)) \rightarrow \cdots$$

Since

$$\wedge^m(V^k) = \bigoplus_{j_1, \ldots, j_k \geq 0} \bigotimes_{j_1 + \cdots + j_k = m} \wedge^{h_1}(V) \otimes \cdots \otimes \wedge^{h_k}(V)$$

$$S^m((b^*)^k) = \bigoplus_{j_1, \ldots, j_k \geq 0} S^{h_1}(b^*) \otimes \cdots \otimes S^{h_k}(b^*)$$

$H^j((G/B)^k, \mathcal{L}(E)) = 0$ for $j > m$ and for $E = \wedge^m(V^k)$ by Corollary 2.6,(ii) and for $j > 0$ and for $E = S^m((b^*)^k)$ by Proposition 2.5,(ii) and Proposition B.2. As a result, the sequence

$$0 \rightarrow H^j(G/B, \mathcal{L}_0(\mathcal{T})) \rightarrow H^{j+1}((G/B)^k, \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^*} \mathcal{L}(E)) \rightarrow 0$$

is exact with $j = i + m$ for $E = \wedge^m(V^k)$ and with $j = i$ for $E = S^m((b^*)^k)$, whence Assertion (i) by Corollary 2.6,(ii) and Assertion (ii) by Proposition 2.5,(ii) and Proposition B.2.

Let set $\mathcal{Y} := G^k \times_{B^k} b^k$. The map

$$G \times b^k \rightarrow G^k \times b^k \quad (g, v_1, \ldots, v_k) \mapsto (g, \ldots, g, v_1, \ldots, v_k)$$

defines through the quotient a closed immersion from $G \times_{B} b^k$ to $\mathcal{Y}$. Let denote it by $\nu$. 
Corollary 3.6. Let \( \mathfrak{J} \) be the ideal of definition in \( \mathcal{O}_G \) of \( v \times_B b^k \). Then \( H^i(\mathfrak{J}, \mathfrak{J}) = 0 \) for all positive integer \( i \).

Proof. Let denote by \( \kappa \) the canonical projection from \( \mathfrak{J} \) to \( (G/B)^k \). Then
\[
\kappa_*(\mathfrak{J}) = \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(S((b^*)^k))
\]
so that
\[
H^i(\mathfrak{J}, \mathfrak{J}) = H^i((G/B)^k, \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(S((b^*)^k)))
\]
for all \( i \). According to Corollary 3.5,(ii), the both sides equal 0 for \( i \geq 2 \).

Since \( \langle \ldots \rangle \) identifies \( g^* \) and its dual, one has a short exact sequence of \( B^k \)-modules:
\[
0 \rightarrow u^k \rightarrow (g^*)^k \rightarrow (b^*)^k \rightarrow 0
\]
From this exact sequence, on deduces the exact Koszul complex
\[
\cdots \rightarrow K_2 \rightarrow K_1 \rightarrow K_0 \rightarrow S((b^*)^k) \rightarrow 0
\]
with
\[
K_m := S((g^*)^k) \otimes \mathbb{E} \wedge^m((u)^k)
\]
\[
d a_0 \wedge \cdots \wedge a_m := \sum_{i=0}^{m} (-1)^i a_i a_0 \wedge \cdots \wedge a_i \wedge \cdots \wedge a_m
\]
This complex \( K_* \) is canonically graded by
\[
K_* := \sum_{q} K^q_* \text{ with } K^q_m := S^{q-m}((g^*)^k) \otimes \mathbb{E} \wedge^m((u)^k)
\]
so that the sequence
\[
\cdots \rightarrow K^q_2 \rightarrow K^q_1 \rightarrow K^q_0 \rightarrow S^q((b^*)^k) \rightarrow 0
\]
is exact. According to Lemma 3.4,(ii), the sequence of \( \mathcal{O}_{(G/B)^k} \)-modules:
\[
\cdots \rightarrow \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K^q_2) \rightarrow \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K^q_1) \rightarrow \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K^q_0) \rightarrow \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(S^q((b^*)^k)) \rightarrow 0
\]
is exact. Since \( H_* \) is an exact \( \delta \)-functor, for \( i \) nonnegative integer,
\[
H^i((G/B)^k, \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(S((b^*)^k))) = 0
\]
if
\[
H^{i+m}((G/B)^k, \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(K^q_m)) = 0
\]
for all nonnegative integers \( q \) and \( m \). Since \( (g^*)^k \) is a \( G \)-module, for all nonnegative integers \( q \) and \( m \), \( \mathcal{L}(K^q_m) \) is isomorphic to
\[
S^{q-m}((g^*)^k) \otimes \mathbb{E} \wedge((u)^k)
\]
As a result,
\[
H^i((G/B)^k, \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(S((b^*)^k))) = 0
\]
if
\[
H^{i+m}((G/B)^k, \mathfrak{J}_\Delta \otimes_{\mathcal{O}_{(G/B)^k}} \mathcal{L}(\wedge^m((u)^k))) = 0
\]
for all nonnegative integer \( m \). According to Corollary 3.5,(i),
\[ H^{1+m}((G/B)^k, \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^k} \mathcal{L}(\wedge^m(u^k))) = 0 \]
for all positive integer \( m \). From the cohomology long exact sequence deduced from the short exact sequence of Lemma 3.4,(i), with \( E \) the trivial module of dimension 1, the sequence
\[
0 \rightarrow H^0((G/B)^k, \mathcal{J}_\Delta) \rightarrow H^0((G/B)^k, \mathcal{O}_{(G/B)^k}) \rightarrow H^0(G/B, \mathcal{O}_{G/B})
\]
\[
\rightarrow H^1((G/B)^k, \mathcal{J}_\Delta) \rightarrow H^1((G/B)^k, \mathcal{O}_{(G/B)^k})
\]
is exact. According to Borel-Weil-Bott’s Theorem [Dem68],
\[ H^0((G/B)^k, \mathcal{J}_\Delta) = k \quad H^0(G/B, \mathcal{O}_{G/B}) = k \quad H^1((G/B)^k, \mathcal{O}_{(G/B)^k}) = 0 \]
whence
\[ H^1((G/B)^k, \mathcal{J}_\Delta) = 0 \]
As a result,
\[ H^1((G/B)^k, \mathcal{J}_\Delta \otimes \mathcal{O}_{(G/B)^k} \mathcal{L}(S((b^k)^k))) = 0 \]
whence the corollary.

3.3. According to Lemma 2.1,(i), \( G \times_B b^k \) is a desingularization of \( B^{(k)} \) and one has a commutative diagram

\[
\begin{array}{ccc}
G \times_B b^k & \xrightarrow{\gamma_n} & B^{(k)} \\
\gamma & \downarrow & \downarrow \eta \\
B^{(k)} & & B^{(k)}
\end{array}
\]

**Lemma 3.7.** Let \( \sigma \) be the canonical projection from \( \mathcal{X}^k \) to \( g^k \). Let denote by \( \iota_1 \) the map
\[ b^k \rightarrow \mathcal{X}^k \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \bar{x}_1, \ldots, \bar{x}_k) \]

(i) The map \( \iota_1 \) is a closed embedding of \( b^k \) into \( \mathcal{X}^k \).

(ii) The subvariety \( \iota_1(b^k) \) of \( \mathcal{X}^k \) is an irreducible component of \( \sigma^{-1}(b^k) \).

(iii) The subvariety \( \sigma^{-1}(b^k) \) of \( \mathcal{X}^k \) is invariant under the canonical action of \( W(\mathcal{R})^k \) in \( \mathcal{X}^k \) and this action induces a simply transitive action of \( W(\mathcal{R})^k \) on the set of irreducible components of \( \sigma^{-1}(b^k) \).

**Proof.** (i) The map
\[ b^k \rightarrow G^k \times b^k \quad (x_1, \ldots, x_k) \mapsto (1_{\mathcal{R}}, \ldots, 1_{\mathcal{R}}, x_1, \ldots, x_k) \]
defines through the quotient a closed embedding of \( b^k \) in \( G^k \times_B b^k \). Let denote it by \( \iota' \). Let \( \gamma_n^{(k)} \) be the map
\[ G^k \times_B b^k \rightarrow \mathcal{X}^k \quad (x_1, \ldots, x_k) \mapsto (\gamma_n(x_1), \ldots, \gamma_n(x_k)) \]
Then \( \iota_1 = \gamma_n^{(k)} \circ \iota' \). Since \( \gamma_n \) is a projective morphism, \( \iota_1 \) is a closed morphism. Moreover, it is injective since \( \sigma \circ \iota_1 \) is the identity of \( b^k \).

(ii) According to Lemma 3.1,(ii) and Lemma 1.4, \( \sigma \) is a finite morphism. So \( \sigma^{-1}(b^k) \) and \( b^k \) have the same dimension. According to (i), \( \iota_1(b^k) \) is an irreducible subvariety of \( \omega^{-1}(b^k) \) of the same dimension, whence the assertion.
(iii) Since all the fibers of \( \sigma \) are invariant under the action of \( W(\mathcal{R})^k \), \( \sigma^{-1}(b^k) \) is invariant under this action and \( W(\mathcal{R})^k \) permutes the irreducible components of \( \sigma^{-1}(b^k) \). For \( w \) in \( W(\mathcal{R})^k \), let set \( Z_w := \eta_1(b^k) \). Then \( Z_w \) is an irreducible component of \( \sigma^{-1}(b^k) \) for all \( w \) in \( W(\mathcal{R})^k \) by (ii). For \( w \) in \( W(\mathcal{R})^k \) such that \( Z_w = \eta_1(b^k) \), for all \( (x_1, \ldots, x_k) \) in \( b^k \), \( (x_1, \ldots, x_k, w.(x_1, \ldots, x_k)) \) is in \( \eta_1(b^k) \) so that \( (x_1, \ldots, x_k) \) is invariant under \( w \) and \( w \) is the identity.

Let \( Z \) be an irreducible component of \( \sigma^{-1}(b^k) \) and let \( Z_0 \) be its image by the map

\[
(x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto (\overline{x_1}, \ldots, \overline{x_k}, y_1, \ldots, y_k)
\]

Since \( \sigma \) is \( G^k \)-equivariant and since \( b^k \) is invariant under \( B^k \), \( \sigma^{-1}(b^k) \) and \( Z \) are invariant under \( B^k \). Hence by Lemma 1.5, \( Z_0 \) is closed. Moreover, since the image of the map

\[
Z_0 \times u^k \rightarrow \mathcal{X}^k \quad (x_1, \ldots, x_k, y_1, \ldots, y_k) \mapsto (x_1 + u_1, \ldots, x_k + u_k, y_1, \ldots, y_k)
\]

is an irreducible subset of \( \sigma^{-1}(b^k) \) containing \( Z \), \( Z \) is the image of this map. Since \( Z_0 \) is contained in \( \mathcal{X}^k \), \( Z_0 \) is contained in the image of the map

\[
b^k \times W(\mathcal{R})^k \rightarrow b^k \times b^k \quad (x_1, \ldots, x_k, w_1, \ldots, w_k) \mapsto (x_1, \ldots, x_k, w_1(x_1), \ldots, w_k(x_k))
\]

Then, since \( W(\mathcal{R})^k \) is finite and since \( Z_0 \) is irreducible, for some \( w \) in \( W(\mathcal{R})^k \), \( Z_0 \) is the image of \( b^k \) by the map

\[
(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, w.(x_1, \ldots, x_k))
\]

Then \( Z = Z_w \), whence the assertion. \( \square \)

Let consider the diagonal action of \( G \) on \( \mathcal{X}^k \) and let identify \( G \times_B b^k \) with \( \nu(G \times_B b^k) \) by the closed immersion \( \nu \).

**Corollary 3.8.** Let set \( \mathcal{B}^k_X := G.\eta_1(b^k) \).

(i) The subset \( \mathcal{B}^k_X \) is the image of \( G \times_B b^k \) by \( \gamma^k_\mathcal{X} \). Moreover, the restriction of \( \gamma^k_\mathcal{X} \) to \( G \times_B b^k \) is a projective birational morphism from \( G \times_B b^k \) onto \( \mathcal{B}^k_X \).

(ii) The subset \( \mathcal{B}^k_X \) of \( \mathcal{X}^k \) is an irreducible component of \( \sigma^{-1}(\mathcal{B}^k) \).

(iii) The subvariety \( \sigma^{-1}(\mathcal{B}^k) \) of \( \mathcal{X}^k \) is invariant under \( W(\mathcal{R})^k \) and this action induces a simply transitive action of \( W(\mathcal{R})^k \) on the set of irreducible components of \( \sigma^{-1}(\mathcal{B}^k) \).

(iv) The subalgebra \( k[\mathcal{B}^k_X] \) of \( k[\sigma^{-1}(\mathcal{B}^k)] \) equals \( k[\sigma^{-1}(\mathcal{B}^k)]^{W(\mathcal{R})^k} \) with respect to the action of \( W(\mathcal{R})^k \) on \( \sigma^{-1}(\mathcal{B}^k) \).

**Proof.** (i) Let \( \gamma^k_\mathcal{X} \) be the restriction of \( \gamma^k_\mathcal{X} \) to \( G \times_B b^k \). Since \( \eta_1 = \gamma^k_\mathcal{X} \cdot u', \) since \( G \times_B b^k = G.\eta^k(a^k) \) and since \( \gamma^k_\mathcal{X} \) is \( G^k \)-equivariant, \( \mathcal{B}^k_X \) = \( \gamma^k_\mathcal{X}(G \times_B b^k) \). Hence \( \mathcal{B}^k_X \) is closed in \( \mathcal{X}^k \) and \( \gamma^k_\mathcal{X} \) is a projective morphism from \( G \times_B b^k \) to \( \mathcal{B}^k_X \) since \( \gamma^k_\mathcal{X} \) is a projective morphism. According to Lemma 2.1(i), \( \sigma.\gamma^k_\mathcal{X} \) is a birational morphism onto \( \mathcal{B}^k_X \). Then \( \gamma^k_\mathcal{X} \) is birational since \( \sigma(\mathcal{B}^k_X) = \mathcal{B}^k_X \), whence the assertion.

(ii) Since \( \sigma \) is a finite morphism, \( \sigma^{-1}(\mathcal{B}^k) \), \( \mathcal{B}^k_X \) and \( \mathcal{B}^k_X \) have the same dimension, whence the assertion since \( \mathcal{B}^k_X \) is irreducible as an image of an irreducible variety.

(iii) Since the fibers of \( \sigma \) are invariant under \( W(\mathcal{R})^k \), \( \sigma^{-1}(\mathcal{B}^k) \) is invariant under this action and \( W(\mathcal{R})^k \) permutes the irreducible components of \( \sigma^{-1}(\mathcal{B}^k) \). Let \( Z \) be an irreducible component of \( \sigma^{-1}(\mathcal{B}^k) \). Since \( \sigma \) is \( G^k \)-equivariant, \( \mathcal{B}^k_X \) and \( Z \) are invariant under the diagonal action of \( G \). Moreover,
Then let denote by $\mathcal{B}_X^{(k)}$ an irreducible component of $\mathcal{B}_X$. Let $W$ be in $W(\mathcal{R})^k$ such that $w.\mathcal{B}_X^{(k)} = \mathcal{B}_X^{(k)}$. Let $x$ be in $b_{reg}$ and let $i$ be equal to $1, \ldots, k$. Let set

$$w.x := (y_1, \ldots, y_k) \in b^k$$

Then there exists $(y_1, \ldots, y_k)$ in $b^k$ and $g$ in $G$ such that

$$w.x = (g(y_1), \ldots, g(y_k), y_1, \ldots, y_k)$$

(iv) Since the fibers of $\mathcal{R}$ are invariant under $W(\mathcal{R})^k$, $k[\mathcal{B}_X^{(k)}]$ is contained in $k[\mathcal{R}^{-1}(\mathcal{B}_X^{(k)})]^W(\mathcal{R})^k$. Let $p$ be in $k[\mathcal{R}^{-1}(\mathcal{B}_X^{(k)})]^W(\mathcal{R})^k$. Since $W(\mathcal{R})$ is a finite group, $p$ is the restriction to $\mathcal{R}^{-1}(\mathcal{B}_X^{(k)})$ of an element $q$ of $k[X]^W$, invariant under $W(\mathcal{R})^k$. Since $k[X]^W = S(\mathcal{R})$, $q$ is in $S(\mathcal{R})^W$ and $p$ is in $k[\mathcal{B}_X^{(k)}]$, whence the assertion.

Let recall that $\theta$ is the map

$$U_+ \times b_{reg} \longrightarrow X \quad (g, x) \longmapsto (g(x), \overline{x})$$

and let denote by $W_k'$ the inverse image of $\theta(U_+ \times b_{reg})$ by the projection

$$\mathcal{B}_X^{(k)} \longrightarrow X \quad (x_1, \ldots, x_k, y_1, \ldots, y_k) \longmapsto (x_1, y_1)$$

**Lemma 3.9.** Let $W_k$ be the subset of elements $(x, y)$ of $\mathcal{B}_X^{(k)}$ $(x \in \mathfrak{g}_k, y \in \mathfrak{b}_k)$ such that $P_k \cap b_{reg}$ is not empty.

(i) The subset $W_k'$ of $\mathcal{B}_X^{(k)}$ is a smooth open subset. Moreover, the map

$$U_+ \times b_{reg} \times b^{k-1} \longrightarrow W_k' \quad (g, x_1, \ldots, x_k) \longmapsto (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k})$$

is an isomorphism of varieties.

(ii) The subset $\mathcal{B}_X^{(k)}$ of $\mathfrak{g}_k \times \mathfrak{b}_k$ is invariant under the canonical action of $GL_4(k)$.

(iii) The subset $W_k$ of $\mathcal{B}_X^{(k)}$ is a smooth open subset. Moreover, $W_k$ is the $G \times GL_4(k)$-invariant set generated by $W_k'$.

(iv) The subvariety $\mathcal{B}_X^{(k)} \setminus W_k$ has codimension at least $2k$.

**Proof.** (i) According to Corollary 3.3.(ii), the image of $\theta$ is an open subset of $X$. Hence $W_k'$ is an open subset of $\mathcal{B}_X^{(k)}$. Let $(x_1, \ldots, x_k)$ be in $b^k$ and let $g$ be in $G$ such that $(g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k})$ is in $W_k'$. Then $x_1$ is in $b_{reg}$ and for some $g'$ in $U_-$ and for some $x_i'$ in $b_{reg}$, $g.(x_1, \overline{x_1}) = g'.(x_i', \overline{x_i'})$. Hence, according
Then \( b \) to Corollary 3.3,(i), for some \( B \), \( x'_1 = b(x_1) \). So, \( g^{-1}g' b \) is in \( G^{x_1} \) and \( g \) is in \( U_- B \) since \( G^{x_1} \) is contained in \( B \). As a result, the map

\[
U_- \times \text{b}_{\text{reg}} \times b^{k-1} \rightarrow W'_k \quad (g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k), \overline{x}_1, \ldots, \overline{x}_k)
\]

is an isomorphism whose inverse is given by

\[
W'_k \rightarrow U_- \times \text{b}_{\text{reg}} \times b^{k-1} \quad (x_1, \ldots, x_k) \mapsto (\theta^{-1}(x_1, \overline{x}_1), \theta^{-1}(x_1, \overline{x}_1)1(x_1), \ldots, \theta^{-1}(x_1, \overline{x}_1)(x_k))
\]

with \( \theta^{-1} \) the inverse of \( \theta \) and \( \theta^{-1}(x_1, \overline{x}_1) \) the component of \( \theta^{-1}(x_1, \overline{x}_1) \) on \( U_- \), whence the assertion since \( U_- \times \text{b}_{\text{reg}} \times b^{k-1} \) is smooth.

(ii) For \( (x_1, \ldots, x_k) \) in \( b^k \) and for \( (a_{i,j}, 1 \leq i, j \leq k) \) in \( \text{GL}_k(\mathbb{k}) \),

\[
\sum_{j=1}^{k} a_{i,j} x_j = \sum_{j=1}^{k} a_{i,j} \overline{x}_j
\]

So, \( t_1(b^k) \) is invariant under the action of \( \text{GL}_k(\mathbb{k}) \) in \( g^k \times b^k \) defined by

\[
(a_{i,j}, 1 \leq i, j \leq k). (x_1, \ldots, x_k, y_1, \ldots, y_k) := (\sum_{j=1}^{k} a_{i,j} x_j, j = 1, \ldots, k, \sum_{j=1}^{k} a_{i,j} y_j, j = 1, \ldots, k)
\]

whence the assertion since \( \mathcal{B}_X^{(k)} = G.t_1(b^k) \) and since the actions of \( G \) and \( \text{GL}_k(\mathbb{k}) \) in \( g^k \times b^k \) commute.

(iii) According to (i), \( G.W'_k \) is a smooth open subset of \( \mathcal{B}_X^{(k)} \). Moreover, \( G.W'_k \) is the subset of elements \((x, y)\) such that the first component of \( x \) is regular. So, by (ii) and Lemma 1.6, \( W_k = \text{GL}_k(\mathbb{k}).(G.W'_k) \), whence the assertion.

(iv) According to Corollary 3.8,(i), \( \mathcal{B}_X^{(k)} \) is the image of \( G \times_B b^k \) by the restriction \( \gamma_X \) of \( \gamma_n^{(k)} \) to \( G \times_B b^k \). Then \( \mathcal{B}_X^{(k)} \setminus W_k \) is contained in the image of \( G \times_B (b \setminus \text{b}_{\text{reg}})^k \) by \( \gamma_X \). As a result (see Lemma 8.1),

\[
\dim \mathcal{B}_X^{(k)} \setminus W_k \leq n + k(b_n - 2)
\]

whence the assertion. \( \square \)

**Proposition 3.10.** (i) The varieties \( \mathcal{B}_n^{(k)} \) and \( \mathcal{B}_X^{(k)} \) are equal. Moreover, \( \gamma_n = \gamma_X \).

(ii) The restriction to \( S(b)^{\text{tr}} \) of the quotient morphism \( \mathbb{k}[\mathcal{X}]^{\text{tr}} \rightarrow \mathbb{k}[\mathcal{B}_X^{(k)}] \) is an embedding.

(iii) The algebra \( \mathbb{k}[\mathcal{B}_n^{(k)}] \) is generated by \( \mathbb{k}[\mathcal{B}_X^{(k)}] \) and \( S(b)^{\text{tr}} \). Moreover, \( \eta \) is the restriction of \( \varpi \) to \( \mathcal{B}_X^{(k)} \).

(iv) The restriction of \( \gamma_X \) to \( \gamma_X^{-1}(W_k) \) is an isomorphism onto \( W_k \).

**Proof.** (i) According to Corollary 3.6, from the short exact sequence

\[
0 \rightarrow \mathfrak{j} \rightarrow \mathcal{O}_{G \times_B b^k} \rightarrow \mathcal{O}_{G \times_B b^k} \rightarrow 0
\]

one deduces the short exact sequence

\[
0 \rightarrow H^0(G \times_B b^k, \mathfrak{j}) \rightarrow H^0(G \times_B b^k, \mathcal{O}_{G \times_B b^k}) \rightarrow H^0(G \times_B b^k, \mathcal{O}_{G \times_B b^k}) \rightarrow 0
\]

In particular, the restriction map

\[
H^0(G \times_B b^k, \mathcal{O}_{G \times_B b^k}) \rightarrow H^0(G \times_B b^k, \mathcal{O}_{G \times_B b^k})
\]
is surjective. Since $k[X]$ equals $H^0(G \times_B b, \mathcal{O}_{G \times_B b})$ by Lemma 3.1.(v), the image of this map equals $k[\mathcal{B}_n^{(k)}]$ by Corollary 3.8.(i). Moreover, according to Lemma 1.1, $k[\mathcal{B}_n^{(k)}] = H^0(G \times_B b, \mathcal{O}_{G \times_B b})$ since $G \times_B b^{(k)}$ is a desingularization of the normal variety $\mathcal{B}_n^{(k)}$ by Lemma 2.1.(i). Hence $k[\mathcal{B}_n^{(k)}] = k[\mathcal{B}_n^{(k)}]$ and $\gamma_n = \gamma_X$.

(ii) According to (i), $\iota_n(b^k)$ is a closed subvariety of $\mathcal{B}_n^{(k)}$ and for $p$ in $S(b)^{\otimes k}$, the restriction to $\iota_n(b^k)$ of its image in $k[\mathcal{B}_n^{(k)}]$ is the function

$$(x_1, \ldots, x_k, \overline{x_1}, \ldots, \overline{x_k}) \mapsto p(x_1, \ldots, x_k)$$

Hence the restriction to $S(b)^{\otimes k}$ of the quotient map $k[X]^{\otimes k} \to k[\mathcal{B}_n^{(k)}]$ is an embedding.

(iii) The comorphism of the restriction of $\sigma$ to $\mathcal{B}_n^{(k)}$ is the embedding of to $k[\mathcal{B}_n^{(k)}]$ into $k[\mathcal{B}_n^{(k)}]$ so that $\eta$ is the restriction of $\sigma$ to $\mathcal{B}_n^{(k)}$ by (i). Since $k[X]$ is generated by $S(g)^{\otimes k}$ and $S(b)^{\otimes k}$ and since the image of $S(g)^{\otimes k}$ by the quotient morphism equals $k[\mathcal{B}_n^{(k)}]$, $k[\mathcal{B}_n^{(k)}]$ is generated by $k[\mathcal{B}_n^{(k)}]$ and $S(b)^{\otimes k}$.

(iv) Since the subset of Borel subalgebras containing a regular element is finite, the fibers of $\gamma_X$ over the elements of $W_k$ are finite. Indeed, according to Zariski Main Theorem [Mu88, §9], they have only one element since $\mathcal{B}_n^{(k)}$ is normal by (i) and since $\gamma_X$ is projective and birational. So, the restriction of $\gamma_X$ to $\gamma_X^{-1}(W_k)$ is a bijection onto the open subset $W_k$, whence the assertion by Zariski Main Theorem [Mu88, §9].

\[ \square \]

Remark 3.11. By Proposition 3.10,(i) and (iii), $\mathcal{B}_n^{(k)}$ identifies with $\mathcal{B}_n^{(k)}$ and $\eta$ identifies with the restriction of $\sigma$ to $\mathcal{B}_n^{(k)}$ so that $\gamma_n = \gamma_X$ and $\iota_n = \iota_1$.

Let consider on $b^k$ the diagonal action of $W(R)$.

Corollary 3.12. (i) The subalgebra $S(b)^{\otimes k}$ of $k[\mathcal{B}_n^{(k)}]$ equals $k[\mathcal{B}_n^{(k)}]^G$.

(ii) The subalgebras $k[\mathcal{B}_n^{(k)}]^G$ and $(S(b)^{\otimes k})^{W(R)}$ of $k[\mathcal{B}_n^{(k)}]^G$ are equal.

Proof. (i) Let $p$ be in $k[\mathcal{B}_n^{(k)}]^G$ such that its restriction to $\iota_n(b^k)$ equals 0. Since

$$\lim_{t \to 0} h(t)(x_1, \ldots, x_k) = (\overline{x_1}, \ldots, \overline{x_k})$$

for all $(x_1, \ldots, x_k)$ in $b^k$, the restriction of $p$ to $\iota_n(b^k)$ equals 0 and $p = 0$ since $\mathcal{B}_n^{(k)} = G \cdot \iota_n(b^k)$.

According to Lemma 3.1.(v), $S(b)^{\otimes k} = (k[X])^{\otimes k}$, whence $S(b)^{\otimes k}$ is a subalgebra of $k[\mathcal{B}_n^{(k)}]^G$ since $k[\mathcal{B}_n^{(k)}]$ is a $G$-equivariant quotient of $k[X]^{\otimes k}$. For $p$ in $k[\mathcal{B}_n^{(k)}]$, let denote by $\overline{p}$ the element of $S(b)^{\otimes k}$ such that

$$\overline{p}(x_1, \ldots, x_k) := p(x_1, \ldots, x_k, \overline{x_1}, \ldots, \overline{x_k})$$

Then the restriction of $p - \overline{p}$ to $\iota_n(b^k)$ equals 0. Moreover, if $p$ is in $k[\mathcal{B}_n^{(k)}]^G$, $p - \overline{p}$ is $G$-invariant. Hence $k[\mathcal{B}_n^{(k)}]^G = S(b)^{\otimes k}$.

(ii) According to (i), the restriction from $\mathcal{B}_n^{(k)}$ to $b^k$ induces an embedding of $k[\mathcal{B}_n^{(k)}]^G$ into $(S(b)^{\otimes k})^{W(R)}$. Moreover, since $G$ is reductive, $k[\mathcal{B}_n^{(k)}]^G$ is the image of $(S(g)^{\otimes k})^G$ by the restriction morphism. According to [J07, Theorem 2.9 and some remark], the restriction morphism $(S(g)^{\otimes k})^G \to (S(b)^{\otimes k})^{W(R)}$ is surjective. Hence the restriction morphism $k[\mathcal{B}_n^{(k)}]^G \to (S(b)^{\otimes k})^{W(R)}$ is surjective. Then the injection $k[\mathcal{B}_n^{(k)}]^G \to (S(b)^{\otimes k})^{W(R)}$ is bijective since $k[\mathcal{B}_n^{(k)}]$ and $S(b)^{\otimes k}$ are graded quotients of $S(g)^{\otimes k}$.

\[ \square \]
3.4. The natural action of $k^*$ in $\mathfrak{g}^k$ induces an action of $k^*$ on $\mathfrak{h}^k$, $\mathcal{B}(k)$, $\mathcal{B}_n(k)$ and $G \times \mathfrak{h}^k$ $b^k$. In particular, $k[\mathcal{B}(k)]$ is a graded subalgebra of the graded algebra $k[\mathcal{B}_n(k)]$.

**Proposition 3.13.** The variety $\mathcal{B}_n(k)$ has rational singularities.

**Proof.** From the short exact sequence of $O_{\mathcal{G} \times \mathfrak{h}^k b^k}$-modules

$$0 \to \mathfrak{z} \to O_{\mathcal{G} \times \mathfrak{h}^k b^k} \to O_{\mathcal{G} \times \mathfrak{h}^k b^k} \to 0$$

one deduces the cohomology long exact sequence

$$\cdots \to H^i(G \times \mathfrak{h}^k b^k, O_{\mathcal{G} \times \mathfrak{h}^k b^k}) \to H^i(\mathcal{G} \times \mathfrak{h}^k b^k, O_{\mathcal{G} \times \mathfrak{h}^k b^k}) \to H^{i+1}(G \times \mathfrak{h}^k b^k, \mathfrak{z}) \to \cdots$$

By Borel-Weil-Bott’s Theorem [Dem68], for $i > 0$, the first term equals 0 and by Corollary 3.6, the third term equals 0. Hence $H^i(G \times \mathfrak{h}^k b^k, O_{\mathcal{G} \times \mathfrak{h}^k b^k}) = 0$ for all $i > 0$, whence the proposition since $(G \times \mathfrak{h}^k b^k, \gamma_n)$ is a desingularization of $\mathcal{B}_n(k)$ by Lemma 2.1,(i). □

**Corollary 3.14.** Let $M$ be a graded complement of $k[\mathcal{B}(k)]/G \cdot k[\mathcal{B}(k)]$ in $k[\mathcal{B}(k)]$.

(i) The algebra $k[\mathcal{B}(k)]$ is a free extension of $S(b)^{\mathfrak{h}^k}$. Moreover, $M$ contains a basis of $k[\mathcal{B}_n(k)]$ over $S(b)^{\mathfrak{h}^k}$.

(ii) The intersection of $M$ and $S_+(b^k)k[\mathcal{B}_n(k)]$ is different from 0.

**Proof.** (i) Let recall that $N^{(k)}$ is the subset of elements $(x_1, \ldots, x_k)$ of $\mathcal{B}(k)$ such that $x_1, \ldots, x_k$ are nilpotent and let recall that $\eta$ is the canonical morphism from $\mathcal{B}_n(k)$ to $\mathcal{B}(k)$. Let denote by $\tau$ the morphism from $\mathcal{B}_n(k)$ to $\mathfrak{h}^{\mathfrak{h}^k}$ whose isomorphism is the injection of $S(b)^{\mathfrak{h}^k}$ in $k[\mathcal{B}_n(k)]$. First of all, $\mathcal{B}_n(k)$, $\mathfrak{h}^{\mathfrak{h}^k}$ and $N^{(k)}$ have dimension $kb_n + n$, $k\ell$, $(k + 1)n$ respectively. Moreover, $\eta^{-1}(N^{(k)})$ is the nullvariety of $S_+(b^k)$ in $\mathcal{B}_n(k)$. In particular, the fiber at $(0, \ldots, 0)$ of $\tau$ has minimal dimension. Since $\tau$ is an equivariant morphism with respect to the actions of $k^*$ and since $(0, \ldots, 0)$ is in the closure of all orbit of $k^*$ in $\mathfrak{h}^k$, $\tau$ is an equidimensional morphism of dimension $\dim \mathcal{B}_n(k) - \dim \mathfrak{h}^{\mathfrak{h}^k}$. According to Proposition 3.13 and [El78], $\mathcal{B}_n(k)$ is Cohen-Macaulay. Then, by [MA86, Theorem 23.1], $k[\mathcal{B}(k)]$ is a flat extension of $S(b)^{\mathfrak{h}^k}$.

The action of $k^*$ on $\mathcal{B}(k)$ induces a $\mathbb{N}$-gradation of the algebra $k[\mathcal{B}_n(k)]$ compatible with the gradations of $k[\mathcal{B}(k)]$ and $S(b)^{\mathfrak{h}^k}$ since $\tau$ is equivariant. Since $M$ is a graded complement of $k[\mathcal{B}(k)]/G \cdot k[\mathcal{B}(k)]$ in $k[\mathcal{B}(k)]$, by induction on $l$,

$$k[\mathcal{B}_n(k)] = M k[\mathcal{B}(k)]^G + (k[\mathcal{B}(k)]^G) k[\mathcal{B}(k)]$$

Hence $k[\mathcal{B}(k)] = M k[\mathcal{B}(k)]^G$ since $k[\mathcal{B}(k)]$ is graded. Then, by Proposition 3.10,(iii) and Corollary 3.12,(ii), $k[\mathcal{B}_n(k)] = M S_+(b^k)^{\mathfrak{h}^k}$. In particular,

$$k[\mathcal{B}_n(k)] = M + S_+(b^k)k[\mathcal{B}_n(k)]$$

Then $M$ contains a graded complement $M'$ of $S_+(b^k)k[\mathcal{B}_n(k)]$ in $k[\mathcal{B}_n(k)]$. Arguing as before, $k[\mathcal{B}_n(k)] = M' S(b)^{\mathfrak{h}^k}$ since $k[\mathcal{B}_n(k)]$ is graded. By flatness, from the short exact sequence

$$0 \to S_+(b^k) \to S(b)^{\mathfrak{h}^k} \to k \to 0$$

one deduces the short exact sequence

$$0 \to k[\mathcal{B}_n(k)] \otimes S(b)^{\mathfrak{h}^k} S_+(b^k) \to k[\mathcal{B}_n(k)] \to k[\mathcal{B}_n(k)] \otimes S(b)^{\mathfrak{h}^k} k \to 0$$

As a result, the canonical map $M' \otimes_k S(b)^{\mathfrak{h}^k} \to k[\mathcal{B}_n(k)]$ is injective. Hence all basis of $M'$ is a basis of the $S(b)^{\mathfrak{h}^k}$-module $k[\mathcal{B}_n(k)]$, whence the assertion.
(ii) Let suppose that $M' = M$. One expects a contradiction. According to (i), the canonical maps

$$M \otimes_k S(b)^{\otimes k} \longrightarrow k[B_n^{(k)}] \quad M \otimes_k k[B_n^{(k)}]^G \longrightarrow k[B_n^{(k)}]$$

are isomorphisms. Then, according to Lemma 1.2, there exists a group action of $W(R)$ on $k[B_n^{(k)}]$ extending the diagonal action of $W(R)$ in $S(b)^{\otimes k}$ and such that $k[B_n^{(k)}]^{W(R)} = k[B_n^{(k)}]$ since $k[B_n^{(k)}] \cap S(b)^{\otimes k} = (S(b)^{\otimes k})^{W(R)}$ by Corollary 3.12(ii). Moreover, since $W(R)$ is finite, the subfield of invariant elements of the fraction field of $k[B_n^{(k)}]$ is the fraction field of $k[B_n^{(k)}]^{W(R)}$. Hence the action of $W(R)$ in $k[B_n^{(k)}]$ is trivial since $k[B_n^{(k)}]$ and $k[B_n^{(k)}]$ have the same fraction field, whence the contradiction since $(S(b)^{\otimes k})^{W(R)}$ is strictly contained in $S(b)^{\otimes k}$.}

4. On the nullcone.

Let $k \geq 2$ be an integer. Let $I$ be the ideal of $k[B_n^{(k)}]$ generated by $S_+(b^k)$ and let $N$ be the subscheme of $B_n^{(k)}$ defined by $I$.

**Lemma 4.1.** Let set $\mathcal{N}_X^{(k)} := \eta^{-1}(N^{(k)})$.

(i) The variety $\mathcal{N}_X^{(k)}$ equals $\gamma_\eta(G \times_B u^k)$.

(ii) The null variety of $I$ in $\mathcal{B}_n^{(k)}$ equals $\mathcal{N}_X^{(k)}$.

(iii) The scheme $N$ is smooth in codimension 1.

**Proof.** (i) By definition, $\gamma^{-1}(\mathcal{N}_X^{(k)}) = G \times_B u^k$. Then, since $\gamma = \gamma_\eta \gamma_\eta = \gamma_\eta(G \times_B u^k)$.

(ii) Let $V_I$ be the nullity of $I$ in $\mathcal{B}_n^{(k)}$. According to Proposition 3.10(ii), for $(g, x_1, \ldots, x_k)$ in $G \times b^k$, $\gamma_\eta((g, x_1, \ldots, x_k))$ is a zero of $I$ if and only if $x_1, \ldots, x_k$ are nilpotent, whence the assertion.

(iii) According to Lemma 3.9(i), one has an isomorphism of varieties

$$U_\times b_{\text{reg}} \times b^{(k-1)} \longrightarrow W'_k \quad (g, x_1, \ldots, x_k) \longmapsto (g(x_1), \ldots, g(x_k), x_1, \ldots, x_k)$$

Let $J$ be the ideal of $k[b_{\text{reg}} \times b^{(k-1)}]$ generated by the functions $(x_1, \ldots, x_k) \mapsto (v, x_i), i = 1, \ldots, k, v \in b$ and let $N_0$ be the subscheme of $b_{\text{reg}} \times b^{(k-1)}$ defined by the ideal $J$. Then the above map induces an isomorphism of $U_\times N_0$ onto the open subset $W_k' \cap N$ of $N$. For all $x$ in $u_{\text{reg}} \times u^{(k-1)}$, the tangent space of $N_0$ at $x$ equals $u^k$. Hence $N_0$ is smooth and $W_k' \cap N$ is smooth. Then, since $N$ is a subscheme of $\mathcal{B}_n^{(k)}$ invariant under the actions of $G$ and $GL_k(\mathbb{K})$, the open subset $W_k' \cap N$ of $N$ is smooth by Lemma 3.9(ii). By definition, $W_k \cap N = \eta^{-1}(V_k)$, whence the assertion by Corollary 2.3(i) since $\eta$ is finite. 

**Proposition 4.2.** The variety $\mathcal{N}_X^{(k)}$ is a normal variety and $I$ is its ideal of definition in $k[B_n^{(k)}]$. In particular, $I$ is prime.

**Proof.** According to Corollary 3.14(i), $k[B_n^{(k)}]$ is a flat extension of $S(b)^{\otimes k}$. Since $B_n^{(k)}$ is Cohen Macaulay, $N$ is Cohen Macaulay by [MA86, Corollary of Theorem 23.2]. According to Lemma 4.1(iii), $N$ is smooth in codimension 1. Hence $N$ is a normal scheme by Serre’s normality criterion [Bou98, §1, no 10, Théorème 4]. According to Lemma 4.1(ii), $\mathcal{N}_X^{(k)}$ is the nullity of $I$ in $\mathcal{B}_n^{(k)}$. Moreover, $\mathcal{N}_X^{(k)}$ is irreducible as image of the irreducible variety $G \times_B u^k$ by Lemma 4.1(i). Hence $I$ is prime and $\mathcal{N}_X^{(k)}$ is a normal variety.
Theorem 4.3. Let $I_0$ be the ideal of $k[B(k)]$ generated by $k[B(k)]^G$.

(i) The ideal $I_0$ is strictly contained in the ideal of definition of $N(k)$ in $k[B(k)]$.

(ii) The nullcone $N(k)$ has rational singularities.

Proof. (i) Since $k[B(k)]^G$ is contained in $S_+(h^\ell)$, $I_0$ is contained in $I \cap k[B(k)]$. According to Lemma 4.1,(ii) and Proposition 4.2, $I \cap k[B(k)]$ is the ideal of definition of $N(k)$ in $k[B(k)]$. Let $M$ be a graded complement of $k[B(k)]^G \subset k[B(k)]$ in $k[B(k)]$. According to Corollary 3.14,(ii), $I \cap M$ is different from 0. Hence $I_0$ is strictly contained in $I \cap k[B(k)]$, whence the assertion.

(ii) According to Proposition 3.10,(iii) and Proposition 4.2, the restriction to $k[B(k)]$ of the quotient map from $k[B(k)]^G$ to $k[N(k)]$ is surjective. Furthermore, the image of $k[B(k)]$ by this morphism equals $k[N(k)]$ since $N(k) = \eta^{-1}(N(k))$, whence $k[N(k)] = k[N(k)]$. As a result, $N(k)$ has rational singularities since $N(k)$ is normal and since the normalization of $N(k)$ has rational singularities by Theorem 2.4.

5. Main varieties.

Let denote by $X$ the closure in $Gr_r(g)$ of the orbit of $b$ under $B$. According to Lemma 1.4, $G.X$ is the closure in $Gr_r(g)$ of the orbit of $b$ under $G$.

5.1. For $\alpha$ in $\mathcal{R}$, let denote by $b_\alpha$ the kernel of $\alpha$. Let set $V_\alpha := h_\alpha \oplus g^\alpha$ and let denote by $X_\alpha$ the closure in $Gr_r(g)$ of the orbit of $b_\alpha$ under $B$.

Lemma 5.1. Let $\alpha$ be in $\mathcal{R}_+$. Let $\mathfrak{p}$ be a parabolic subalgebra containing $b$ and let $P$ be its normalizer in $G$.

(i) The subset $P.X$ of $Gr_r(g)$ is the closure in $Gr_r(g)$ of the orbit of $b$ under $P$.

(ii) The closed set $X_\alpha$ of $Gr_r(g)$ is an irreducible component of $X \setminus B.b$.

(iii) The set $P.X_\alpha$ is an irreducible component of $P.X \setminus P.b$.

(iv) The varieties $X \setminus B.b$ and $P.X \setminus P.b$ are equidimensional of codimension 1 in $X$ and $P.X$ respectively.

Proof. (i) Since $X$ is a $B$-invariant closed subset of $Gr_r(g)$, $P.X$ is a closed subset of $Gr_r(g)$ by Lemma 1.4. Hence $P.b$ is contained in $P.X$ since $b$ is in $X$, whence the assertion since $P.b$ is a $P$-invariant subset containing $X$.

(ii) Denoting by $H_\alpha$ the coroot of $\alpha$,

$$\lim_{t \to \infty} \exp(tad x_\alpha) \left( \frac{1}{2t} H_\alpha \right) = x_\alpha$$

So $V_\alpha$ is in the closure of the orbit of $b$ under the one parameter subgroup of $G$ generated by $ad x_\alpha$. As a result, $X_\alpha$ is a closed subset of $X \setminus B.b$ since $V_\alpha$ is not a Cartan subalgebra. Moreover, $X_\alpha$ has dimension $n - 1$ since the normalizer of $V_\alpha$ in $g$ is $b + g^\alpha$. Hence $X_\alpha$ is an irreducible component of $X \setminus B.b$ since $X$ has dimension $n$.

(iii) Since $X_\alpha$ is a $B$-invariant closed subset of $Gr_r(g)$, $P.X_\alpha$ is a closed subset of $Gr_r(g)$ by Lemma 1.4. According to (ii), $P.X_\alpha$ is contained in $P.X \setminus P.b$ and it has dimension $\dim p - \ell - 1$, whence the assertion since $P.X$ has dimension $\dim p - \ell$.

(iv) Let $P_u$ be the unipotent radical of $P$ and let $L$ be the reductive factor of $P$ whose Lie algebra contains $ad b$. Let denote by $N_L(b)$ the normalizer of $b$ in $L$. Since $B.b$ and $P.b$ are isomorphic to $U$ and
For $x$ in $V$, let set:

$$V_x := \text{span}([e_1(x), \ldots, e_{\ell}(x)])$$

**Lemma 5.2.** Let $\Delta$ be the set of elements $(x, V)$ of $g \times G.X$ such that $x$ is in $V$.

(i) For $(x, V)$ in $b \times X$, $(x, V)$ is in the closure of $B.(b_{\text{reg}} \times \{b\})$ in $b \times \text{Gr}_t(b)$ if and only if $x$ is in $V$.

(ii) The set $\Delta$ is the closure in $g \times \text{Gr}_t(g)$ of $G.(b_{\text{reg}} \times \{b\})$.

(iii) For $(x, V)$ in $\Delta$, $V_x$ is contained in $V$.

**Proof.** (i) Let $\Delta'$ be the subset of elements $(x, V)$ of $b \times X$ such that $x$ is in $V$ and let $\Delta'_0$ be the closure of $B.(b_{\text{reg}} \times \{b\})$ in $b \times \text{Gr}_t(b)$. Then $\Delta'$ is a closed subset of $b \times \text{Gr}_t(b)$ containing $\Delta'_0$. Let $(x, V)$ be in $\Delta'$. Let $E$ be a complement of $V$ in $b$ and let $\Omega_E$ be the set of complements of $E$ in $g$. Then $\Omega_E$ is an open neighborhood of $V$ in $\text{Gr}_t(b)$. Moreover, the map

$$\text{Hom}_b(V, E) \xrightarrow{\kappa} \Omega_E \quad \varphi \mapsto \kappa(\varphi) := \text{span}([v + \varphi(v) \mid v \in V])$$

is an isomorphism of varieties. Let $\Omega'_0$ be the inverse image of the set of Cartan subalgebras. Then 0 is in the closure of $\Omega'_0$ in $\text{Hom}_b(V, E)$ since $V$ is in $X$. For all $\varphi$ in $\Omega'_0$, $(x + \varphi(x), \kappa(\varphi))$ is in $\Delta'_0$. Hence $(x, V)$ is in $\Delta'_0$.

(ii) Let $(x, V)$ be in $\Delta$. For some $g$ in $G$, $g(V)$ is in $X$. So by (i), $(g(x), g(V))$ is in $\Delta'_0$ and $(x, V)$ is in the closure of $G.(b_{\text{reg}} \times \{b\})$ in $g \times \text{Gr}_t(g)$, whence the assertion.

(iii) For $i = 1, \ldots, \ell$, let $\Delta_i$ be the set of elements $(x, V)$ of $\Delta$ such that $e_i(x)$ is in $V$. Then $\Delta_i$ is a closed subset of $g \times G.X$, invariant under the action of $G$ in $g \times \text{Gr}_t(g)$ since $e_i$ is a $G$-equivariant map. For all $(g, x)$ in $G \times b_{\text{reg}}$, $(g(x), g(b))$ is in $\Delta$ since $e_i(g(x))$ centralizes $g(x)$. Hence $\Delta_i = \Delta$ since $G.(b_{\text{reg}} \times \{b\})$ is dense in $\Delta$ by (ii). As a result, for all $V$ in $G.X$ and for all $x$ in $V$, $e_1(x), \ldots, e_{\ell}(x)$ are in $V$. □

**Corollary 5.3.** Let $(x, V)$ be in $\Delta$ and let $\mathfrak{z}$ be the centre of $\mathfrak{g}^{x+}$.

(i) The subspace $\mathfrak{z}$ is contained in $V_x$ and $V$.

(ii) The space $V$ is an algebraic, commutative subalgebra of $g$.

**Proof.** (i) If $x$ is regular semisimple, $V$ is a Cartan subalgebra of $g$. Let suppose that $x$ is not regular semisimple. Let denote by $\mathfrak{z}$ the centre of $\mathfrak{g}^{x+}$. Let $\mathfrak{z}_{\text{reg}}$ be the nilpotent cone of $\mathfrak{g}^{x+}$ and let $\Omega_{\text{reg}}$ be the regular nilpotent orbit of $\mathfrak{g}^{x+}$. For all $y$ in $\Omega_{\text{reg}}$, $x_s + y$ is in $\mathfrak{z}_{\text{reg}}$ and $e_1(x_s + y), \ldots, e_{\ell}(x_s + y)$ is a basis of $\mathfrak{g}^{x+s}$ by [Ko63, Theorem 9]. Then for all $z$ in $\mathfrak{z}$, there exist regular functions on $\Omega_{\text{reg}}$, $a_1, \ldots, a_{\ell}$, such that

$$z = a_1(z)(x_s + y) + \cdots + a_{\ell}(z)(x_s + y)$$

for all $y$ in $\Omega_{\text{reg}}$. Furthermore, these functions are uniquely defined by this equality. Since $\mathfrak{z}_{\text{reg}}$ is a normal variety and since $\mathfrak{z}_{\text{reg}} \setminus \Omega_{\text{reg}}$ has codimension 2 in $\mathfrak{z}_{\text{reg}}$, the functions $a_1, \ldots, a_{\ell}$ have regular extensions to $\mathfrak{z}_{\text{reg}}$. Denoting again by $a_i$ the regular extension of $a_i$ for $i = 1, \ldots, \ell$,

$$z = a_1(z)(x_s + y) + \cdots + a_{\ell}(z)(x_s + y)$$

for all $y$ in $\mathfrak{z}_{\text{reg}}$. As a result, $\mathfrak{z}$ is contained in $V_x$. Hence $\mathfrak{z}$ is contained in $V$ by Lemma 5.2,(iii).
(ii) Since the set of commutative subalgebras of dimension $\ell$ is closed in $\text{Gr}_r(\mathfrak{g})$, $V$ is a commutative subalgebra of $\mathfrak{g}$. According to (i), the semisimple and nilpotent components of the elements of $V$ are contained in $V$. For $x$ in $V \setminus \mathfrak{n}_g$, all the replica of $x$ are contained in $\mathfrak{z}$. Hence $V$ is an algebraic subalgebra of $\mathfrak{g}$ by (i).

$\blacksquare$

5.2. For $s$ in $\mathfrak{h}$, let denote by $X^s$ the subset of elements of $X$, contained in $\mathfrak{g}^s$.

**Lemma 5.4.** Let $s$ be in $\mathfrak{h}$ and let $\mathfrak{z}$ be the centre of $\mathfrak{g}^s$.

(i) The set $X^s$ is the closure in $\text{Gr}_r(\mathfrak{g}')$ of the orbit of $\mathfrak{h}$ under $B'$.

(ii) The set of elements of $G.X$ containing $\mathfrak{z}$ is the closure in $\text{Gr}_r(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under $G^s$.

**Proof.**

(i) Let set $\mathfrak{v} := g^s + \mathfrak{h}$, let $P$ be the normalizer of $\mathfrak{v}$ in $G$ and let $\mathfrak{p}_u$ be the nilpotent radical of $\mathfrak{v}$. For $g$ in $P$, let denote by $\mathfrak{g}(\mathfrak{h})$ its image by the canonical projection from $P$ to $G^s$. Let $Z$ be the closure in $\text{Gr}_r(\mathfrak{g}) \times \text{Gr}_r(\mathfrak{g})$ of the image of the map

$$B \rightarrow \text{Gr}_r(\mathfrak{b}) \times \text{Gr}_r(\mathfrak{b}) \quad g \mapsto (g(\mathfrak{h}), \mathfrak{g}(\mathfrak{h}))$$

and let $Z'$ be the subset of elements $(V, V')$ of $\text{Gr}_r(\mathfrak{b}) \times \text{Gr}_r(\mathfrak{b})$ such that

$$V' \subseteq g^s \cap \mathfrak{b} \text{ and } V \subseteq V' \oplus \mathfrak{p}_u$$

Then $Z'$ is a closed subset of $\text{Gr}_r(\mathfrak{b}) \times \text{Gr}_r(\mathfrak{b})$ and $Z$ is contained in $Z'$ since $(g(\mathfrak{h}), \mathfrak{g}(\mathfrak{h}))$ is in $Z'$ for all $g$ in $B$. Since $\text{Gr}_r(\mathfrak{b})$ is a projective variety, the images of $Z$ by the projections $(V, V') \mapsto V$ and $(V, V') \mapsto V'$ are closed in $\text{Gr}_r(\mathfrak{b})$ and they equal $X$ and $\overline{B' \cdot \mathfrak{h}}$ respectively. Furthermore, $\overline{B' \cdot \mathfrak{h}}$ is contained in $X^s$.

Let $V$ be in $X^s$. For some $V' \subseteq \text{Gr}_r(\mathfrak{b})$, $(V, V')$ is in $Z$. Since

$$V \subset \mathfrak{g}^s, \quad V' \subset \mathfrak{g}^s, \quad V \subset V' \oplus \mathfrak{p}_u$$

$V = V'$ so that $V$ is in $\overline{B' \cdot \mathfrak{h}}$, whence the assertion.

(ii) Since $\mathfrak{z}$ is contained in $\mathfrak{h}$, all element of $\overline{G \cdot \mathfrak{h}}$ is an element of $G.X$ containing $\mathfrak{z}$. Let $V$ be in $G.X$, containing $\mathfrak{z}$. Since $V$ is a commutative subalgebra of $\mathfrak{g}^s$ and since $\mathfrak{g}^s \cap \mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}^s$, for some $g$ in $G^s$, $g(V)$ is contained in $\mathfrak{b} \cap \mathfrak{g}^s$. So, one can suppose that $V$ is contained in $\mathfrak{b}$. According to the Bruhat decomposition of $G$, since $X$ is $B$-invariant, for some $\mathfrak{b}$ in $U$ and for some $w$ in $W(\mathbb{R})$, $V$ is in $\mathfrak{b}^w.X$. Let set:

$$R_{+}^w := \{ \alpha \in R_{+} \mid w(\alpha) \in R_{+} \} \quad R_{+}^w := \{ \alpha \in R_{+} \mid w(\alpha) \notin R_{+} \}$$

$$u_1 := \bigoplus_{\alpha \in R_{+}^w \setminus \mathfrak{h}} g^w(\alpha) \quad u_2 := \bigoplus_{\alpha \in -R_{+}^w \setminus \mathfrak{h}} g^w(\alpha) \quad u_3 := \bigoplus_{\alpha \in R_{+}^w \setminus \mathfrak{h}} g^w(\alpha)$$

$$B^w := wBw^{-1} \quad \mathfrak{h}^w := \mathfrak{h} \oplus u_1 \oplus u_3$$

so that $\text{ad} \mathfrak{b}^w$ is the Lie algebra of $B^w$ and $\mathfrak{b}^w.X$ is the closure in $\text{Gr}_r(\mathfrak{g})$ of the orbit of $\mathfrak{h}$ under $B^w$. Moreover, $u$ is the direct sum of $u_1$ and $u_2$. For $i = 1, 2$, let denote by $U_i$ the closed subgroup of $U$ whose Lie algebra is $\text{ad} u_i$. Then $U = U_2U_1$ and $b = b_2b_1$ with $b_1$ in $U_i$ for $i = 1, 2$. Since $w^{-1}(u_1)$ is contained in $u$ and since $X$ is invariant under $B$, $b_2b_1w.X = b_2w.X$. Since $b_2^{-1}(V)$ is in $\mathfrak{b}^w.X$ and since $V$ is contained in $\mathfrak{b}$,

$$b_2^{-1}(V) \subset \mathfrak{b} \cap \mathfrak{b}^w = \mathfrak{b} \oplus u_1$$

Let set:

$$u_{2,1} := u_2 \cap \mathfrak{g}^s \quad u_{2,2} := u_2 \cap \mathfrak{p}_u$$
and for $i = 1, 2$, let denote by $U_{2,i}$ the closed subgroup of $U_2$ whose Lie algebra is $\text{ad}u_{2,i}$. Then $u_2$ is the direct sum of $u_{2,1}$ and $u_{2,2}$ and $U_2 = U_{2,1}U_{2,2}$ so that $b_2 = b_{2,1}b_{2,2}$ with $b_{2,i}$ in $U_{2,i}$ for $i = 1, 2$. As a result, $\beta$ is contained in $b_{2,1}^{-1}(V)$ and $b_{2,2}^{-1}(\beta)$ is contained in $\mathfrak{h} \oplus u_{1,2}$. Hence $b_{2,2}^{-1}(\beta) = 3$ since $u_1 \cap u_{2,2} = \{0\}$.

Let suppose $b_{2,2} \neq 1$. One expects a contradiction. For some $x$ in $u_{2,2}$, $b_{2,2} = \exp(\text{ad} x)$. The space $u_{2,2}$ is a direct sum of root spaces since $u_{2,1}$ and $u_{2,2}$ are too. Let $\alpha_1, \ldots, \alpha_m$ be the positive roots such that the corresponding root spaces are contained in $u_{2,2}$. They are ordered so that for $i \leq j$, $\alpha_j - \alpha_i$ is a positive root if it is a root. For $i = 1, \ldots, m$, let $c_i$ be the coordinate of $x$ at $x_{a_i}$ and let $i_0$ be the smallest integer such that $c_{i_0} \neq 0$. For all $z$ in $\beta$,

$$b_{2,2}^{-1}(\beta) - z - c_{i_0} \alpha_{i_0}(z) x_{a_{i_0}} \in \bigoplus_{j > i_0} \mathfrak{g}^{\alpha_j}$$

whence the contradiction since for some $z$ in $\beta$, $\alpha_{i_0}(z) \neq 0$. As a result, $b_{2,2}^{-1}(V)$ is an element of $\mathfrak{w} X = \overline{B^+ \mathfrak{h}}$, contained in $\mathfrak{g}^\ell$. So, by (i), $b_{2,2}^{-1}(V)$ and $V$ are in $\overline{G^\ell \mathfrak{h}}$, whence the assertion. \hfill \Box

5.3. For $x$ in $\mathfrak{g}$, let denote by $R_x$ the subset of stabilizers of regular linear forms on $\mathfrak{g}^\ell$ under the coadjoint action. According to [Y06a] and [deG08], all element of $R_x$ is a commutative subalgebra of dimension $\ell$ of $\mathfrak{g}$. For $x$ and $v$ in $\mathfrak{g}$, the stabilizer of the linear form $w \mapsto \langle v, w \rangle$ on $\mathfrak{g}^\ell$ under the coadjoint action is denoted by $(\mathfrak{g}^\ell)^\prime$.

Lemma 5.5. Let $x$ be in $\mathfrak{g}$.

(i) For all $v$ in $\mathfrak{g}$, there exists a positive integer $d$ and a regular map $\beta_x$ from $\mathbb{P}^1(\mathbb{k})$ to $\text{Gr}_d(\mathfrak{g})$ such that $\beta_x(t) = g^{tx + v}$ for all $t$ in a dense open subset of $\mathbb{k}$.

(ii) For all $v$ in a dense open subset $\Omega$ of $\mathfrak{g}$, $x + v$ is regular semisimple and the linear form $w \mapsto \langle v, w \rangle$ on $\mathfrak{g}^\ell$ is regular.

Proof. (i) Let $d$ be the minimal dimension of the $g^{tx + v}$’s, $t \in \mathbb{k}$. Then for all $t$ in a dense open subset $\Omega_v$ of $\mathbb{k}$, the map

$$\Omega_v \longrightarrow \text{Gr}_d(\mathfrak{g}) \quad t \longmapsto g^{tx + v}$$

is regular, whence the assertion by [Sh94, Ch. VI, Theorem 1].

Let $E$ be a complement of $\beta_v(\infty)$ in $\mathfrak{g}$ and let $\Omega_E$ be the set of complements of $E$ in $\mathfrak{g}$. Then $\Omega_E$ is an open neighborhood of $\beta_v(\infty)$ in $\text{Gr}_d(\mathfrak{g})$ and the map

$$\text{Hom}_E(\beta_v(\infty), E) \longrightarrow \Omega_E \quad \varphi \longmapsto \text{span}([w + \varphi(w) \mid v \in E_{\mathfrak{w}(\infty))})$$

is an isomorphism. Let denote by $\chi$ this isomorphism. For all $t$ in a nonempty open subset $T$ of $\mathbb{k}^\ast$, there exists a unique $\varphi_t$ in $\text{Hom}_E(\beta_v(\infty), E)$ such that $\chi(\varphi_t) = g^{tx + v}$. Then

$$\lim_{t \rightarrow \infty} \varphi_t = 0$$

and for all $(w, t)$ in $\beta_v(\infty) \times T$, one has

$$0 = [w + \varphi_t(w), tx + v] = [w + \varphi_t(w), x + \frac{1}{t} v]$$

whence $\beta_v(\infty) \subset g^\ell$. Moreover, for all all $w'$ in $g^\ell$,

$$0 = \langle w', [w + \varphi_t(w), tx + v] \rangle = -\langle w + \varphi_t(w), [w', v] \rangle$$
whence $\beta_\ell(\infty) \subset (g^\mathbb{R})^\vee$.

(ii) Since $g_{\text{reg,ss}}$ is a dense open subset of $g$, for all $v$ in a dense open subset, $x + v$ is regular semisimple. Since the map

$$g \rightarrow (g^\mathbb{R})^\vee \quad v \mapsto (w \mapsto \langle v, w \rangle)$$

is a dominant morphism, for all $v$ in a dense open subset of $g$, the linear form $w \mapsto \langle v, w \rangle$ on $g^\mathbb{R}$ is regular, whence the assertion. \qed

**Corollary 5.6.** For $x$ in $g$, $R_x$ is contained in $G.X$.

**Proof.** Let $v$ be in the open subset $\Omega$ of Lemma 5.5.(ii). Then $g^{t + v}$ is a Cartan subalgebra of $g$ for all $t$ in a dense open subset of $k$. So $\beta_\ell(\infty)$ is in $G.X$ and by Lemma 5.5.(i), $\beta_\ell(\infty) = (g^\mathbb{R})^\vee$ since the index of $g^\mathbb{R}$ equals $\ell$. Denoting by $(g^\mathbb{R})^\vee_{\text{reg}}$ the set of regular linear forms on $g^\mathbb{R}$, the map

$$(g^\mathbb{R})^\vee_{\text{reg}} \rightarrow \text{Gr}_\ell(g) \quad v \mapsto (g^\mathbb{R})^\vee$$

is regular. Hence $R_x$ is contained in $G.X$ since the projection of $\Omega$ to $(g^\mathbb{R})^\vee$ is dense in $(g^\mathbb{R})^\vee_{\text{reg}}$ and since $G.X$ is closed in $\text{Gr}_\ell(g)$. \qed

For $E$ a subspace of $g$ of even dimension $2m$ and for $e = e_1, \ldots, e_{2m}$ a basis of $E$, let set:

$$p_{E,E} := \det \begin{bmatrix} [e_1, e_1] & \cdots & [e_1, e_{2m}] \\ \vdots & \ddots & \vdots \\ [e_{2m}, e_1] & \cdots & [e_{2m}, e_{2m}] \end{bmatrix}$$

The element $p_{E,E}$ of $S(g)$, up to a multiplicative scalar, does not depend on the basis $e$. So, when $p_{E,E}$ is different from zero, one will say that $p_E$ is different from zero. Otherwise, one will say $p_E = 0$.

**Lemma 5.7.** Let $x$ be in $g$.

(i) For $V$ in $\text{Gr}_\ell(g^{\mathbb{R}})$, $V$ is in $R_x$ if and only if for all complement $E$ of $V$ in $g^\mathbb{R}$, $p_E$ is different from zero.

(ii) For $V$ in $\text{Gr}_\ell(g)$, $V$ is in $G.X$ if and only if for all complement $E$ of $V$ in $g$, $p_E$ is different from zero.

(iii) For $E$ in $\text{Gr}_{\dim g^{\mathbb{R}} - \ell}(g^{\mathbb{R}})$ such that $p_E \neq 0$, $p_{E \oplus F} \neq 0$ for all complement $F$ of $g^{\mathbb{R}}$ in $g$.

**Proof.** (i) and (ii) Let denote by $a$ the Lie algebra $g$ or $g^\mathbb{R}$. For $v$ in $g$, let denote by $a^v$ the stabilizer of the linear form $w \mapsto \langle v, w \rangle$ on $a$ and let set:

$$Z_0 := \begin{cases} G.X & \text{if } a = g \\ R_x & \text{if } a = g^\mathbb{R} \end{cases}$$

Let $V$ be in $\text{Gr}_\ell(a)$. For all complement $E$ of $V$ in $a$, $E$ has even dimension. Let suppose $p_E \neq 0$ for all complement $E$ of $V$ in $a$. Let $E_1, \ldots, E_m$ be some complements of $V$ in $a$. Then for all $v$ in a dense open subset of $g$, $v$ is not a zero of $p_{E_1}, \ldots, p_{E_m}$ and the linear form $w \mapsto \langle v, w \rangle$ on $a$ is regular. Hence for $i = 1, \ldots, m$, $a^v$ is a complement of $E_i$ in $a$. As a result, $V$ is in $Z_0$. Conversely, let suppose that $V$ is in $Z_0$ and let $E$ be a complement of $V$ in $a$. Then for some $v$ in $g$, $a^v$ is a complement of $E$ in $a$ so that $v$ is not a zero of $p_E$.

(iii) Let $v$ be in $g$ such that $p_E(v) \neq 0$ and such that the linear form $w \mapsto \langle v, w \rangle$ on $g^\mathbb{R}$ is regular. Then $(g^\mathbb{R})^v$ is a complement of $E$ in $g^\mathbb{R}$. According to Lemma 5.5.(iii), $(g^\mathbb{R})^v$ is in $G.X$. For all complement $F$ of $g^\mathbb{R}$ in $g$, $E \oplus F$ is a complement of $(g^\mathbb{R})^v$ in $g$, whence the assertion by (ii). \qed
5.4. Let call a torus of \( g \) a commutative algebraic subalgebra of \( g \) whose all elements are semisimple. For \( x \) in \( g \), let denote by \( Z_x \) the subset of elements of \( G.X \) containing \( x \) and let denote by \( (G^x)_0 \) the identity component of \( G^x \).

Lemma 5.8. Let \( x \) be in \( \mathfrak{g}_s \) and let \( Z \) be an irreducible component of \( Z_x \). Let suppose that some element of \( Z \) is not contained in \( \mathfrak{g}_s \).

(i) For some torus \( s \) of \( g^s \), all element of a dense open subset of \( Z \) contains a conjugate of \( s \) under \( (G^s)_0 \).

(ii) For some \( s \) in \( s \) and for some irreducible component \( Z_1 \) of \( Z_{s+x} \), \( Z \) is the closure in \( \text{Gr}_t(\mathfrak{g}) \) of \( (G^s)_0.Z_1 \).

(iii) If \( Z_1 \) has dimension smaller than \( \dim g^{s+s} - \ell \), then \( Z \) has dimension smaller than \( \dim g^s - \ell \).

Proof. (i) After some conjugation by an element of \( G \), one can suppose that \( g^s \cap h \) and \( g^s \cap b \) are a Borel subalgebra and a maximal torus of \( g^s \) respectively. Let \( Z_0 \) be the subset of elements of \( Z \) contained in \( h \) and let \( (B^s)_0 \) be the identity component of \( B^s \). Since \( Z \) is an irreducible component of \( Z_s \), \( Z \) is invariant under \( (G^s)_0 \) and \( Z = (G^s)_0.Z_0 \). Since \( (G^s)_0/(B^s)_0 \) is a projective variety, according to the proof of Lemma 1.4, \( (G^s)_0.Z_0 \) is a closed subset of \( Z \) for all closed subset \( Z_0 \) of \( Z \). Hence for some irreducible component \( Z_0 \) of \( Z_0, Z = (G^s)_0.Z_0 \). According to Corollary 5.3, (ii), for all \( V \) in \( Z_0 \), there exists a torus \( s \), contained in \( g^s \cap b \) and verifying the following two conditions:

\begin{enumerate}
  
  \item \( V \) is contained in \( s + (g^s \cap h) \),
  
  \item \( V \) contains a conjugate of \( s \) under \( (B^s)_0 \).
\end{enumerate}

Let \( s \) be a torus of maximal dimension verifying Conditions (1) and (2) for some \( V \) in \( Z_0 \). By hypothesis, \( s \) has positive dimension. Let \( Z_s \) be the subset of elements of \( Z \) verifying Conditions (1) and (2) with respect to \( s \). By maximality of \( \dim s \), for \( V \) in \( Z_s \), \( \dim V \cap h > \ell - \dim s \) or \( \dim V \cap h = \ell - \dim s \) and \( V \) is contained in \( s' \cap h \) for some torus of dimension \( \dim s \), different from \( s \). By rigidity of tori, \( s \) is not in the closure in \( \text{Gr}_{\dim s}(h) \) of the set of tori different from \( s \). Hence \( Z_s \) is a closed subset of \( Z \), since for all \( V \) in \( Z_s \), \( \dim V \cap h \) has dimension at least \( \ell - \dim s \). As a result, \( (G^s)_0.Z_s \) contains a dense open subset whose all elements contain a conjugate of \( s \) under \( (G^s)_0 \).

(ii) For some \( s \) in \( s \), \( g^s \) is the centralizer of \( s \) in \( g \). Let \( Z^s \) be the subset of elements of \( Z \) containing \( s \). Then \( Z^s \) is contained in \( Z_{s+x} \) and according to Corollary 5.3, (i), \( Z^s \) is the subset of elements of \( Z \), containing \( s \). By (i), for some irreducible component \( Z'_1 \) of \( Z^s \), \( (G^s)_0.Z'_1 \) is dense in \( Z \). Let \( Z_1 \) be an irreducible component of \( Z_{s+x} \), containing \( Z'_1 \). According to Corollary 5.3, (ii), \( Z_1 \) is contained in \( Z_s \) since \( x \) is the nilpotent component of \( s + x \). So \( Z_1 = Z'_1 \) and \( (G^s)_0.Z_1 \) is dense in \( Z \).

(iii) Since \( Z_1 \) is an irreducible component of \( Z_{s+x} \), \( Z_1 \) is invariant under the identity component of \( G^{s+x} \). Moreover, \( G^{s+x} \) is contained in \( G^s \) since \( x \) is the nilpotent component of \( s + x \). As a result, by (ii),

\[
\dim Z \leq \dim g^s - \dim g^{s+x} + \dim Z_1
\]

whence the assertion.

Let denote by \( C_h \) the \( G \)-invariant closed cone generated by \( h \).

Lemma 5.9. Let \( g \) semisimple. Let \( \Gamma \) be the closure in \( g \times \text{Gr}_t(\mathfrak{g}) \) of the image of the map

\[
\mathfrak{sl}^* \times G \longrightarrow g \times \text{Gr}_t(\mathfrak{g}) \quad (t, g) \longmapsto (tg(h), g(b))
\]
and let $\Gamma_0$ be the inverse image of the nilpotent cone by the first projection.

(i) The subvariety $\Gamma$ of $g \times \Gr_{r}(g)$ has dimension $2n + 1$. Moreover, $\Gamma$ is contained in $\Delta$.

(ii) The varieties $C_h$ and $G.X$ are the images of $\Gamma$ by the first and second projections respectively.

(iii) The subvariety $\Gamma_0$ of $\Gamma$ is equidimensional of codimension 1.

(iv) For $x$ nilpotent in $g$, the subvariety of elements $V$ of $G.X$, containing $x$ and contained in $\overline{G(x)}$, has dimension at most $\dim g^x - \ell$.

Proof. (i) Since the stabilizer of $(h, h)$ in $k^* \times G$ equals $\{1\} \times H$, $\Gamma$ has dimension $2n + 1$. Since $tg(h)$ is in $g(b)$ for all $(t, g)$ in $k^* \times G$ and since $\Delta$ is a closed subset of $g \times \Gr_r(g)$, $\Gamma$ is contained in $\Delta$.

(ii) Since $\Gr_r(g)$ is a projective variety, the image of $\Gamma$ by the first projection is closed in $g$. So, it equals $C_h$ since it is contained in $C_h$ and since it contains the cone generated by $G.h$. Let $\pi$ be the canonical map from $g \setminus \{0\}$ to the projective space $\mathbb{P}(g)$ and let $\overline{\Gamma}$ be the image of $\pi \cap (g \setminus \{0\}) \times \Gr_r(g)$ by the map $(x, V) \mapsto (\pi(x), V)$. Since $C_h$ is a closed cone, $\overline{\Gamma}$ is a closed subset of $\mathbb{P}(g) \times \Gr_r(g)$. Hence the image of $\Gamma$ by the second projection equals $\overline{\Gamma}$ by the first projection equals $\overline{G.h}$ since it is contained in $\overline{G.h}$ and since it contains $G.h$. As a result, the image of $\Gamma$ by the second projection equals $\overline{G.h}$ since it is contained in $\overline{G.h}$ and since it contains the image of $\Gamma$ by the second projection.

(iii) The subvariety $C_h$ of $g$ has dimension $2n + 1$ and the nullvariety of $p_1$ in $C_h$ is contained in $\mathfrak{R}_g$ since it is the nullvariety in $g$ of the polynomials $p_1, \ldots, p_\ell$. Hence $\mathfrak{R}_g$ is the nullvariety of $p_1$ in $C_h$ and $\Gamma_0$ is the nullvariety in $\Gamma$ of the function $(x, V) \mapsto p_1(x)$. So $\Gamma_0$ is equidimensional of codimension 1 in $\Gamma$.

(iv) Let $T$ be the subset of elements $V$ of $G.X$, containing $x$ and contained in $\overline{G(x)}$. Let denote by $\Gamma_T$ the inverse image of $G.T$ by the projection from $\Gamma$ to $G.X$. Then $\Gamma_T$ is contained in $\Gamma_0$. Since $x$ is in all element of $T$ and since $\Gamma_T$ is invariant under $G$, the image of $\Gamma_T$ by the first projection equals $\overline{G(x)}$. Hence

$$\dim \Gamma_T = \dim T + \dim g - \dim g^x$$

Since $\Gamma_T$ is contained in $\Gamma_0$, $\Gamma_T$ has dimension at most $\dim g - \ell$, whence the assertion. $\square$

When $g$ is semisimple, let denote by $(G.X)_u$ the subset of elements of $G.X$ contained in $\mathfrak{R}_g$.

Corollary 5.10. Let suppose $g$ semisimple. Let $x$ be in $\mathfrak{R}_g$.

(i) The variety $(G.X)_u$ has dimension at most $2n - \ell$.

(ii) The variety $Z_x \cap (G.X)_u$ has dimension at most $\dim g^x - \ell$.

Proof. (i) Let $T$ be an irreducible component of $(G.X)_u$ and let $\Delta_T$ be its inverse image by the canonical projection from $\Delta$ to $G.X$. Then $\Delta_T$ is a vector bundle of rank $\ell$ over $T$. So it has dimension $\dim T + \ell$. Let $Y$ be the projection of $\Delta_T$ onto $g$. Since $T$ is an irreducible projective variety, $Y$ is an irreducible closed subvariety of $g$ contained in $\mathfrak{R}_g$. The subvariety $(G.X)_u$ of $G.X$ is invariant under $G$ since it is so for $\mathfrak{R}_g$. Hence $\Delta_T$ and $Y$ are $G$-invariant and for some $y$ in $\mathfrak{R}_g$, $Y = \overline{G(y)}$. Denoting by $F_y$ the fiber at $y$ of the projection $\Delta_T \to Y$, $V$ is contained in $\overline{G(y)}$ and contains $y$ for all $V$ in $F_y$. So, by Lemma 5.9,(iv),

$$\dim F_y \leq \dim g^y - \ell$$

Since the projection is $G$-equivariant, this inequality holds for the fibers at the elements of $G(y)$. Hence,

$$\dim \Delta_T \leq \dim g - \ell \text{ and } \dim T \leq 2n - \ell$$
(ii) Let $Z$ be an irreducible component of $Z_\ell \cap (G.X)_0$ and let $T$ be an irreducible component of $(G.X)_0$, containing $Z$. Let $\Delta_T$ and $Y$ be as in (i). Then $G(x)$ is contained in $Y$ and the inverse image of $G(x)$ in $\Delta_T$ has dimension at least $\dim G(x) + \dim Z$. So, by (i),

$$\dim G(x) + \dim Z \leq \dim g - \ell$$

whence the assertion. \qed

**Theorem 5.11.** For $x$ in $g$, the variety of elements of $G.X$, containing $x$, has dimension at most $\dim g^+ - \ell$.

**Proof.** Let prove the theorem by induction on $\dim g$. If $g$ is commutative, $G.X = \{g\}$. If the derived Lie algebra of $g$ is simple of dimension 3, $G.X$ has dimension 2 and for $x$ not in the centre of $g$, $g^+$ has dimension $\ell$. Let suppose the theorem true for all reductive Lie algebra of dimension strictly smaller than $\dim g$. Let $x$ be in $g$. Since $G.X$ has dimension $\dim g - \ell$, one can suppose $x$ not in the centre of $g$. If $x$ is not nilpotent, $g^+$ has dimension strictly smaller than $\dim g$ and all element of $G.X$ containing $x$ is contained in $g^+$, by Corollary 5.3,(i), whence the theorem in this case by induction hypothesis. As a result, by Lemma 5.8, for all $x$ in $g$, all irreducible component of $Z_\ell$, containing an element not contained in $\mathfrak{g}_g$, has dimension at most $\dim g^+ - \ell$.

Let $\mathfrak{z}_g$ be the centre of $g$ and let $x$ be a nilpotent element of $g$. Denoting by $Z'_x$ the subset of elements of $G.(\mathfrak{h} \cap [g,g])$ containing $x$, $Z_x$ is the image of $Z'_x$ by the map $V \mapsto V + \mathfrak{z}_g$, whence the theorem by Corollary 5.10. \qed

**5.5.** Let $s$ be in $\mathfrak{h} \setminus \{0\}$. Let set $\mathfrak{p} := g^+ + b$ and let denote by $\mathfrak{p}_u$ the nilpotent radical of $\mathfrak{p}$. Let $P$ be the normalizer of $\mathfrak{p}$ and let $P_u$ be its unipotent radical. For a nilpotent orbit $\Omega$ of $G^+$ in $g^+$, let denote by $\Omega^\mathfrak{g}$ the induced orbit by $\Omega$ from $g^+$ to $g$.

**Lemma 5.12.** Let $Y$ be a $G$-invariant irreducible closed subset of $g$ and let $Y'$ be the union of $G$-orbits of maximal dimension in $Y$. Let suppose that $s$ is the semisimple component of an element $x$ of $Y'$. Let denote by $\Omega$ the orbit of $x_u$ under $G^+$ and let set $Y_1 := \mathfrak{z} + \Omega + \mathfrak{p}_u$.

(i) The subset $Y_1$ of $\mathfrak{p}$ is closed and invariant under $P$.

(ii) The subset $G(Y_1)$ of $\mathfrak{g}$ is a closed subset of dimension $\dim \mathfrak{z} + \dim G(x)$.

(iii) For some nonempty open subset $Y''$ of $Y'$, the conjugacy class of $g^+$ under $G$ does not depend on the element $y$ of $Y''$.

(iv) For a good choice of $x$ in $Y''$, $Y$ is contained in $G(Y_1)$.

**Proof.** (i) By [Ko63, §3.2, Lemma 5], $G^+$ is connected and $P = P_uG^+$. For all $y$ in $\mathfrak{p}$ and for all $g$ in $P_u$, $g(y)$ is in $y + \mathfrak{p}_u$. Hence $Y_1$ is invariant under $P$ since it is invariant under $G^+$. Moreover, it is a closed subset of $\mathfrak{p}$ since $\mathfrak{z} + \Omega$ is a closed subset of $g^+$.

(ii) According to (i) and Lemma 1.4, $G(Y_1)$ is a closed subset of $\mathfrak{g}$. According to [CMa93, Theorem 7.1.1], $\Omega^\mathfrak{g} \cap (\Omega + \mathfrak{p}_u)$ is a $P$-orbit and the centralizers in $\mathfrak{g}$ of its elements are contained in $\mathfrak{p}$. So, for all $y$ in $\Omega^\mathfrak{g} \cap (\Omega + \mathfrak{p}_u)$, the subset of elements $g$ of $G$ such that $g(y)$ is in $Y_1$ has dimension $\dim \mathfrak{p}$ since $g(y)$ is in $\Omega + \mathfrak{p}_u$. As a result,

$$\dim G(Y_1) = \dim G \times_P Y_1 = \dim \mathfrak{p}_u + \dim Y_1$$
Since \( \dim g^s = \dim g^x - \dim \Omega \),
\[
\dim Y_1 = \dim 3 + \dim \nu_u + \dim g^x - \dim g^x
\]
\[
\dim G(Y_1) = \dim 3 + 2\dim \nu_u + \dim g^x - \dim g^x
\]
\[= \dim 3 + \dim G(x) \]

(iii) Let \( \tau \) be the canonical morphism from \( g \) to its categorical quotient \( g/G \) under \( G \) and let \( Z \) be the closure in \( g/G \) of \( \tau(Y) \). Since \( Y \) is irreducible, \( Z \) is irreducible and there exists an irreducible component \( \bar{Z} \) of the preimage of \( Z \) in \( h \) whose image in \( g/G \) equals \( Z \). Since the set of conjugacy classes under \( G \) of the centralizers of the elements of \( h \) in \( g \) is finite, for some nonempty open subset \( Z'' \) of \( \bar{Z} \), the centralizers of its elements are conjugate under \( G \). The image of \( Z'' \) in \( g/G \) contains a dense open subset \( Z' \) of \( Z \). Let \( Y'' \) be the inverse image of \( Z' \) by the restriction of \( \tau \) to \( Y' \). Then \( Y'' \) is a dense open subset of \( Y \) and the centralizers in \( g \) of the semisimple components of its elements are conjugate under \( G \).

(iv) Let suppose that \( x \) is in \( Y'' \). Let \( Z_Y \) be the set of elements \( y \) of \( Y'' \) such that \( g^y = g^x \). Then \( G.Z_Y = Y'' \). For all nilpotent orbit \( \Omega \) of \( G^x \) in \( g^x \), let set:
\[
\Omega = 3 + \overline{\Omega} + \nu_u
\]
Then \( Z_Y \) is contained in the union of the \( \Omega \)'s. Hence \( Y'' \) is contained in the union of the \( G(Y(\Omega)) \)'s. According to (ii), \( G(Y(\Omega)) \) is a closed subset of \( g \). Hence \( Y \) is contained in the union of the \( G(Y(\Omega)) \)'s since \( Y'' \) is dense in \( Y \). Then \( Y \) is contained in \( G(Y(\Omega)) \) for some \( \Omega \) since \( Y \) is irreducible and since there are finitely many nilpotent orbits in \( g^x \), whence the assertion. \( \square \)

**Theorem 5.13.** (i) The variety \( G.X \) is the union of \( G.h \) and the \( G.X_\beta \)'s, \( \beta \in \Pi \).

(ii) The variety \( X \) is the union of \( U.h \) and the \( X_\alpha \)'s, \( \alpha \in \mathcal{R}_+ \).

**Proof.** Let \( z_h \) be the centre of \( g \) and let \( \mu \) be the map
\[
\text{Gr}_\ell([g, g]) \to \text{Gr}_\ell(g), \quad V \mapsto z_h + V
\]
and let set:
\[
X_d := B.(h \cap [g, g]) \quad X_{a,d} := B.(V_a \cap [g, g])
\]
for \( \alpha \) in \( \mathcal{R}_+ \). Then, \( X, G.X, X_\alpha, G.X_\alpha \) are the images of \( X_d, G.X_d, X_{a,d}, G.X_{a,d} \) by \( \mu \) respectively. So one can suppose \( g \) semisimple.

(i) For \( \ell = 1 \), \( g \) is simple of dimension 3. In this case, \( G.X \) is the union of \( G.h \) and \( G.g^c \). So, one can suppose \( \ell \geq 2 \). According to Lemma 5.1,(iii), for \( \alpha \) in \( \mathcal{R}_+ \), \( G.X_\alpha \) is an irreducible component of \( G.X \setminus G.h \). Moreover, for all \( \beta \) in \( \Pi \cap W(\mathcal{R})(\alpha) \), \( G.X_\alpha = G.X_\beta \) since \( V_\alpha \) and \( V_\beta \) are conjugate under \( N_G(h) \).

Let \( T \) be an irreducible component of \( G.X \setminus G.h \). Let set:
\[
\Delta_T := \Delta \cap g \times T
\]
and let denote by \( Y \) the image of \( \Delta_T \) by the first projection. Then \( Y \) is closed in \( g \) since \( \text{Gr}_\ell(g) \) is a projective variety. Since \( \Delta_T \) is a vector bundle over \( T \) and since \( T \) is irreducible, \( \Delta_T \) is irreducible and \( Y \) is too. Since \( T \) is an irreducible component of \( G.X \setminus G.h \), \( T, \Delta_T \) and \( Y \) are \( G \)-invariant. According to Lemma 5.1,(iii), \( T \) has codimension 1 in \( G.X \). Hence, by Corollary 5.10,(i) \( Y \) is not contained in the
According to Lemma 5.12, (ii) and (iv), for \( x \) in a \( G \)-invariant dense subset \( Y'' \) of \( Y' \),

\[
\dim Y \leq \dim \mathcal{G}(x) + \dim \mathfrak{g}
\]

with \( \mathfrak{g} \) the centre of \( \mathfrak{g}^\natural \) and according to Theorem 5.11,

\[
\dim \Delta_T \leq \dim \mathcal{G}(x) + \dim \mathfrak{g} + \dim \mathfrak{g}^\natural - \ell = \dim \mathfrak{g} + \dim \mathfrak{g}^\natural - \ell
\]

Hence \( \Delta_T \) has dimension at most \( 2n + \dim \mathfrak{g} \) and \( \dim \mathfrak{g}^\natural = \ell - 1 \) since \( T \) has codimension 1 in \( G.X \). Let \( x \) be in \( Y'''' \) such that \( x_\alpha \) is in \( \mathfrak{h} \). Then \( x_\alpha \) is subregular and \( \mathfrak{g}^\natural \) is the kernel of a positive root \( \alpha \). Denoting by \( \mathfrak{s}_\alpha \) the subalgebra of \( \mathfrak{g} \) generated by \( \mathfrak{g}^\alpha \) and \( \mathfrak{g}^{-\alpha} \), \( \mathfrak{g}^\alpha \) is the direct sum of \( \mathfrak{h}_\alpha \) and \( \mathfrak{s}_\alpha \). Since the maximal commutative subalgebras of \( \mathfrak{s}_\alpha \) have dimension 1, a commutative subalgebra of dimension \( \ell \) of \( \mathfrak{g}^\natural \) is either a Cartan subalgebra of \( \mathfrak{g} \) or conjugate to \( \mathfrak{g}_\alpha \) under the adjoint group of \( \mathfrak{g}^\natural \). As a result, \( \mathfrak{g}_\alpha \) is in \( T \) and \( \mathcal{G}_\alpha = \mathcal{G}.X_\alpha \) since \( T \) is \( G \)-invariant, whence the assertion.

(ii) According to Lemma 5.1, (ii), for \( \alpha \in \mathbb{R}_+ \), \( X_\alpha \) is an irreducible component of \( X \setminus B.\mathfrak{h} \). Let \( g_1, \ldots, g_m \) be its simple factors. For \( j = 1, \ldots, m \), let denote by \( X_j \) the closure in \( G.G(T_{g_j}) \) of the orbit of \( \mathfrak{h} \cap \mathfrak{g}_j \). Then \( X = X_1 \times \cdots \times X_m \) and the complement of \( B.\mathfrak{h} \) in \( X \) is the union of the \( X_1 \times \cdots \times X_{j-1} \times (X_j \setminus B.(\mathfrak{h} \cap \mathfrak{g}_j)) \times X_{j+1} \times \cdots \times X_m \). So, one can suppose \( \mathfrak{g} \) simple. Let consider

\[
b = p_0 \subset \cdots \subset p_\ell = \mathfrak{g}
\]

an increasing sequence of parabolic subalgebras verifying the following condition: for \( i = 0, \ldots, \ell - 1 \), there is no parabolic subalgebra \( \mathfrak{q} \) of \( \mathfrak{g} \) such that

\[
p_i \subset \mathfrak{q} \subset p_{i+1}
\]

For \( i = 0, \ldots, \ell \), let \( P_i \) be the normalizer of \( p_i \) in \( G \), let \( p_{i,u} \) be the nilpotent radical of \( p_i \) and let \( P_{i,u} \) be the unipotent radical of \( P_i \). For \( i = 0, \ldots, \ell \) and for \( \alpha \in \mathbb{R}_+ \), let set \( X_i := X_i.X \) and \( X_{i,\alpha} := P_i.X_i \). Let prove by induction on \( \ell - i \) that for all sequence of parabolic subalgebras verifying the above condition, the \( X_{i,\alpha} \)’s, \( \alpha \in \mathbb{R}_+ \), are the irreducible components of \( X_i \setminus P_i.\mathfrak{h} \).

For \( i = \ell \), it results from (i). Let suppose that it is true for \( i + 1 \). According to Lemma 5.1, (iii), the \( X_{i,\alpha} \)’s are irreducible components of \( X_i \setminus P_i.\mathfrak{h} \).

**Claim** 5.14. Let \( T \) be an irreducible component of \( X_{i} \setminus P_i.\mathfrak{h} \) such that \( P_i \) is its stabilizer in \( P_{i+1} \). Then \( T = X_{i,\alpha} \) for some \( \alpha \in \mathbb{R}_+ \).

**Proof.** According to the induction hypothesis, \( T \) is contained in \( X_{i+1,\alpha} \) for some \( \alpha \in \mathbb{R}_+ \). According to Lemma 5.1, (iv), \( T \) has codimension 1 in \( X_i \) so that \( P_{i+1}.T \) and \( X_{i+1,\alpha} \) have the same dimension. Then they are equal and \( T \) contains \( \mathfrak{g}^\alpha \) for some \( x \) in \( \mathfrak{b}_{\text{reg}} \) such that \( x_\alpha \) is a subregular element belonging to \( \mathfrak{h} \). Denoting by \( \alpha' \) the positive root such that \( \alpha'(x_\alpha) = 0 \), \( \mathfrak{g}^\alpha = V_{\alpha'} \) since \( V_{\alpha'} \) is the commutative subalgebra contained in \( \mathfrak{b} \) and containing \( \mathfrak{h}_{\alpha'} \), which is not Cartan, so that \( T = X_{i,\alpha'} \). 

Let suppose that \( X_i \setminus P_i.\mathfrak{h} \) is not the union of the \( X_{i,\alpha} \)’s, \( \alpha \in \mathbb{R}_+ \). One expects a contradiction. Let \( T \) be an irreducible component of \( X_i \setminus P_i.\mathfrak{h} \), different from \( X_i,\alpha \) for all \( \alpha \). According to Claim 5.14 and according to the condition verified by the sequence, \( T \) is invariant under \( P_{i+1} \). Moreover, according to Claim 5.14,
it is so for all sequence $p'_0, \ldots, p'_\ell$ of parabolic subalgebras verifying the above condition and such that $p'_j = p_j$ for $j = 0, \ldots, i$. As a result, for all simple root $\beta$ such that $\beta^{-\beta}$ is not in $p_i$, $T$ is invariant under the one parameter subgroup of $G$ generated by $\text{ad} g^{-\beta}$. Hence $T$ is invariant under $G$. It is impossible since for $x$ in $g \setminus \{0\}$, the orbit $G(x)$ is not contained in $p_i$ since $g$ is simple, whence the assertion. □

5.6. Let $X'$ be the subset of $g^\mathbb{C}$ with $x$ in $b_{\text{reg}}$ such that $x_\alpha$ is regular or subregular. For $\alpha$ in $\mathbb{R}_+$, let denote by $\theta_\alpha$ the map

$$\mathbb{k} \rightarrow X \quad t \mapsto \exp(t \text{ad} x_\alpha).$$

According to [Sh94, Ch. VI, Theorem 1], $\theta_\alpha$ has a regular extension to $\mathbb{P}^1(\mathbb{k})$, also denoted by $\theta_\alpha$. Let set $Z_\alpha := \theta_\alpha(\mathbb{P}^1(\mathbb{k}))$ and $X'_\alpha := B.Z_\alpha$ so that $X'_\alpha = U.h \cup B.V_\alpha$.

Lemma 5.15. Let $\alpha$ be in $\mathbb{R}_+$ and let $V$ be in $X$. Denote by $\overrightarrow{V}$ the image of $V$ by the projection $x \mapsto \overrightarrow{x}$.

(i) For $x$ in $h$, $x$ is subregular if and only if $V_x = h_x$ for some positive root $\alpha$.

(ii) If $\overrightarrow{V} = h_\alpha$, then $\overrightarrow{V_x} = h_\alpha$ for some $x$ in $V$.

(iii) If $\overrightarrow{V} = h_\alpha$, then $V$ is conjugate to $V_\alpha$ under $B$.

Proof. (i) First of all, since $e_1, \ldots, e_d$ are $G$-equivariant maps, $V_x$ is contained in the centre of $g^\mathbb{C}$ for all $x$ in $g$. Then for $x$ in $h$, $V_x$ is the centre of $g^\mathbb{C}$ by Corollary 5.3,(i), whence the assertion.

(ii) Let suppose $\overrightarrow{V} = h_\alpha$. Then $x$ is not regular semisimple for all $x$ in $V$. Let suppose that $x_\alpha$ is not subregular for all $x$ in $V$. One expects a contradiction. Since $x_\alpha$ and $\overrightarrow{x}$ are conjugate under $B$, for all $x$ in $V$, there exists $\gamma$ in $\mathbb{R}_+ \setminus \{\alpha\}$ such that $\gamma(\overrightarrow{x}) = 0$. Hence $\overrightarrow{V}$ is contained in $b_\gamma$ for some $\gamma$ in $\mathbb{R}_+ \setminus \{\alpha\}$ since $\mathbb{R}_+$ is finite, whence the contradiction. Then by (i), for some $x$ in $V$, $V_x = b_\alpha$ since $x_\alpha$ and $\overrightarrow{x}$ are conjugate under $B$.

(iii) Let suppose $\overrightarrow{V} = h_\alpha$. By (ii), $V_\alpha = b_\alpha$ for some $x$ in $V$. Let $b$ be in $B$ such that $b(x_\alpha) = \overrightarrow{x}$. Then $b(V)$ centralizes $h_\alpha$ by Corollary 5.3,(ii). Moreover, $b(V)$ is not a Cartan subalgebra since $\overrightarrow{V}$ does not contain regular semisimple element. The centralizer of $h_\alpha$ in $b$ equals $h + g^\alpha$ and $V_\alpha$ is the commutative algebra of dimension $\ell$ contained in $h + g^\alpha$ which is not a Cartan subalgebra, whence the assertion. □

Corollary 5.16. Let $\alpha$ be a positive root.

(i) The subset $X'_\alpha$ of $X$ is open.

(ii) The subset $X'$ of $X$ is open. Moreover, $G.X'_\alpha$ and $G.X'$ are open subsets of $G.X$.

Proof. (i) Since $X'_\alpha$ is $B$-invariant and since $U.h$ is an open subset of $X$, contained in $X'_\alpha$, it suffices to prove that $X'_\alpha$ is a neighborhood of $V_\alpha$ in $X$. Let denote by $H_\alpha$ the coroot of $\alpha$ and let set:

$$E' := \bigoplus_{\gamma \in \mathbb{R}_+ \setminus \{\alpha\}} h^\gamma \quad E := k.H_\alpha \oplus E'.$$

Let $\Omega_E$ be the set of subspaces $V$ of $b$ such that $E$ is a complement of $V$ in $b$ and let $\Omega'_E$ be the complement in $X \cap \Omega_E$ of the union of the $X'_\gamma$’s, $\gamma \in \mathbb{R}_+ \setminus \{\alpha\}$. Then $\Omega'_E$ is an open neighborhood of $V_\alpha$ in $X$. Let $V$ be in $\Omega'_E$ such that $V$ is not a Cartan subalgebra and let denote by $\overrightarrow{V}$ its image by the projection $x \mapsto \overrightarrow{x}$. Then $V$ is contained in $\overrightarrow{V} + u$ so that $\mathbb{h} = k.H_\alpha + \overrightarrow{V}$. Since $V$ is not a Cartan subalgebra and since it is commutative, $\overrightarrow{V} \cap b_{\text{reg}}$ is empty. Hence $\overrightarrow{V} = h_\gamma$ for some positive root $\gamma$. According to Lemma 5.15,(iii), $V$ is conjugate to $V_\gamma$ under $B$. Then $\alpha = \gamma$ and $V$ is in $X'_\alpha$ since $V$ is not in $X_\delta$ for all positive root $\delta$ different from $\alpha$. As a result, $\Omega'_E$ is contained in $X'_\alpha$, whence the assertion.
(ii) By definition, $X'$ is the union the $X'_{\alpha}$'s, $\alpha \in R_+$. Hence $X'$ is an open subset of $X$ by (i). Since $X'_{\alpha}$ is invariant under $B$, $X \setminus X'_{\alpha}$ is a $B$-invariant closed subset of $X$. Hence $G.(X \setminus X'_{\alpha})$ is a closed subset of $G.X$ by Lemma 1.4. Moreover, $G.X'_{\alpha}$ is the complement of $G.(X \setminus X'_{\alpha})$ in $G.X$. Hence $G.X'_{\alpha}$ and $G.X'$ are open subsets of $G.X$. \hfill $\square$

For $\beta$ in $\Pi$, let set:

$$u_\beta := \bigoplus_{\alpha \in R_+ \setminus \{\beta\}} \mathfrak{g}^{\alpha} \quad U_\beta := \exp(ad \cdot u_\beta)$$

Let $Y$ be the subvariety of elements $(V, V')$ of $\text{Gr}_\ell(g) \times \text{Gr}_{\ell-1}(g)$ such that $V'$ is contained in $V$.

**Lemma 5.17.** Let $\beta$ be in $\Pi$ and let set: $Y_\beta := Y \cap (X'_{\beta} \times B.h_{\beta})$.

(i) The variety $Y_\beta$ is a smooth open subset of $Y \times X \times B.h_{\beta}$.

(ii) The variety $X'_{\beta}$ is smooth.

(iii) The subset $G.Y_\beta$ of $Y$ is the intersection of $Y$ and $G.X'_{\beta} \times G.h_{\beta}$. Moreover, the restriction to $G.Y_\beta$ of the first projection has finite fibers.

(iv) The canonical projection from $G.Y_\beta$ to $G.X'_{\beta}$ is a finite surjective morphism.

(v) The variety $G.Y_\beta$ is smooth.

**Proof.** (i) According to Corollary 5.16, $X'_{\beta}$ is an open subset of $X$. Hence $Y_\beta$ is an open subset of $Y \cap X \times B.h_{\beta}$. By definition, $X'_{\beta} = B.Z_{\beta}$. For $(g, g')$ in $B \times B$ and for $V$ in $Z_{\beta}$, $(g(V), g'(h_{\beta}))$ is in $Y$ if and only if $h_{\beta}$ is contained in $(g')^{-1}g(V)$. Since the centralizer of $h_{\beta}$ in $b$ equals $g^{\beta} + b$ and since $V$ is a commutative algebra, $(g')^{-1}g(V)$ is in $Z_{\beta}$ in this case. Hence $Y_\beta = B.(Z_{\beta} \times \{h_{\beta}\})$

Let $T_{\beta}$ be the normalizer of $h_{\beta}$ in $B$. Since $B = U_{\beta}T_{\beta}$, the map $g \mapsto g(h_{\beta})$ from $U_{\beta}$ to $B.h_{\beta}$ is an isomorphism. Hence the map

$$U_{\beta} \times Z_{\beta} \longrightarrow Y_{\beta} \quad (g, V) \longmapsto (g(V), g(h_{\beta}))$$

is an isomorphism so that $Y_\beta$ is smooth since $Z_{\beta}$ is too.

(ii) Since $X'_{\beta} = B.Z_{\beta}$ and since $B.b$ is a smooth open subset of $X'_{\beta}$, it suffices to prove that $V_{\beta}$ is a smooth point of $X'_{\beta}$. Let set:

$$F := u_\beta \oplus kH_{\beta}$$

and let denote by $\Omega_F$ the set of complements of $F$ in $b$. Then $\Omega_F$ is an affine open subset of $\text{Gr}_\ell(b)$, containing $V_{\beta}$, and the map

$$\text{Hom}_k(V_{\beta}, F) \longrightarrow \Omega_F \quad \varphi \longmapsto \text{span}(\{v + \varphi(v) \mid v \in V_{\beta}\}$$

is an isomorphism. Let denote it by $\chi$.

**Claim 5.18.** Let $v_1, \ldots, v_{\ell-1}$ be a basis of $h_{\beta}$. Let denote by $\tilde{\chi}$ the map

$$\text{Hom}_k(V_{\beta}, F) \times k^{\ell-1} \longrightarrow \Omega_F \times \text{Gr}_{\ell-1}(b)$$

$$(\varphi, a_1, \ldots, a_{\ell-1}) \longmapsto (\text{span}(\{v + \varphi(v) \mid v \in V_{\beta}\}), \text{span}(\{v_i + a_i x_{\beta} + \varphi(v_i + a_i x_{\beta}) \mid i = 1, \ldots, \ell - 1\}))$$

Then $\tilde{\chi}$ is an isomorphism onto an open neighborhood $\Omega'_F$ of $(V_{\beta}, h_{\beta})$ in $Y$.\hfill $\square$
Proof. Let \( F' \) be the subspace of \( b \) generated by \( F \) and \( x_\beta \) and let \( \Omega_{F'} \) be the set of complements of \( F' \) in \( b \). Then \( \Omega_F \times \Omega_{F'} \) is an open neighborhood of \( (V_\beta, b_\beta) \) in \( \text{Gr}_f(b) \times \text{Gr}_{f-1}(b) \) and the map
\[
\text{Hom}_k(V_\beta, F) \times \text{Hom}_k(b_\beta, F') \longrightarrow \Omega_F \times \Omega_{F'}
\]
\[\varphi, \psi \mapsto (\text{span}(v + \varphi(v) \mid v \in V_\beta), \text{span}(v + \psi(v) \mid v \in b_\beta))\]
is an isomorphism. For \((\varphi, a_1, \ldots, a_{\ell-1}) \in \text{Hom}_k(V_\beta, F) \times \mathbb{A}^{\ell-1}, \tilde{\chi}(\varphi, a_1, \ldots, a_{\ell-1}) \) is the image of \((\varphi, \psi) \) with \( \psi \) in \( \text{Hom}_k(b_\beta, F') \) defined by \( \psi(v_i) = \varphi(v_i + a_i x_\beta) + a_i x_\beta \) for \( i = 1, \ldots, \ell - 1 \). Conversely, let \((\varphi, \psi) \) be such that its image is in \( Y \). Then for \( i = 1, \ldots, \ell - 1, \)
\[
v_i + \psi(v_i) = \sum_{j=1}^\ell a_{i,j}(v_j + \varphi(v_j)) + a_i x_\beta + \psi(v_i)
\]
with \( a_{i,1}, \ldots, a_{i,\ell} \) in \( \mathbb{A} \) so that
\[
\varphi(v_i) = a_{i,\ell}(x_\beta + \varphi(x_\beta)) + \psi(v_i)
\]
whence the claim since the map
\[
\text{Hom}_k(V_\beta, F) \times \mathbb{A}^{\ell-1} \longrightarrow \text{Hom}_k(V_\beta, F) \times \text{Hom}_k(b_\beta, F')
\]
\[\varphi, a_1, \ldots, a_{\ell-1} \mapsto (\varphi, \psi) \) with \( \psi(v_i) = \varphi(v_i + a_i x_\beta) + a_i x_\beta, \ i = 1, \ldots, \ell - 1 \) is an isomorphism onto a subspace of \( \text{Hom}_k(V_\beta, F) \times \text{Hom}_k(b_\beta, F') \).

Let identify \( \text{Hom}_k(V_\beta, F) \) with \( \text{Hom}_k(V_\beta, b_\beta) \times \mathbb{A}^\ell \) by the isomorphism
\[
\text{Hom}_k(V_\beta, b_\beta) \times \mathbb{A}^\ell \longrightarrow \text{Hom}_k(V_\beta, F)
\]
\[(\varphi, b_1, \ldots, b_\ell) \mapsto (\sum_{j=1}^{\ell-1} t_j v_j + t_\ell x_\beta \mapsto \varphi(\sum_{j=1}^{\ell-1} t_j v_j + t_\ell x_\beta) + (\sum_{j=1}^\ell t_j b_j) H_\beta)\]
Let \( \Sigma \) be the inverse image by \( \chi \) of \( \Omega_F \times X'_\beta \). Then \( \Sigma \) is an irreducible locally closed subset of \( \text{Hom}_k(V_\beta, F) \) since \( \Omega_F \cap X'_\beta \) is an irreducible locally closed subset of \( \text{Gr}_f(b) \). Moreover,
\[
\tilde{\chi}(\Sigma \times \mathbb{A}^{\ell-1}) = \Omega_F \cap Y \times X'_\beta \times \text{Gr}_{f-1}(b)
\]
Let set:
\[
S_0 := \{(\varphi, b_1, \ldots, b_\ell, a_1, \ldots, a_{\ell-1}) \mid (\varphi, b_1, \ldots, b_\ell) \in \Sigma, \ b_i + b_\ell a_i = 0, \ i = 1, \ldots, \ell - 1\}
\]

Claim 5.19. Let \( S \) be the inverse image of \( \Omega_F \cap Y_\beta \) by \( \tilde{\chi} \). Then \( S \) is an irreducible subvariety of \( S_0 \). Moreover, \( \Sigma \) is the image of \( S \) by the canonical projection from \( \text{Hom}_k(V_\beta, F) \times \mathbb{A}^{\ell-1} \) to \( \text{Hom}_k(V_\beta, b_\beta) \times \mathbb{A}^{\ell-1} \).

Proof. Since \( Y_\beta = B(Z_\beta \times [b_\beta]) \), \( Y_\beta \) and \( \Omega_F \cap Y_\beta \) are irreducible varieties. Hence by Claim 5.18, \( S \) is an irreducible variety. Moreover, \( \tilde{\chi}(S) = \Omega_F' \cap Y_\beta \) and \( \Sigma \) is the image of \( S \) by the projection onto \( \text{Hom}_k(V_\beta, F) \) since \( X'_\beta \) is the image of \( Y_\beta \) by the projection from \( Y \) to \( \text{Gr}_f(b) \) and since \( \Sigma = \chi^{-1}(\Omega_F \cap X'_\beta) \).
Let \( (\varphi, b_1, \ldots, b_\ell, a_1, \ldots, a_{\ell-1}) \) be in \( S \). Then \( \chi(\varphi, b_1, \ldots, b_\ell) \) is in \( \Omega_F \cap X'_\beta \) and for \( i = 1, \ldots, \ell - 1 \),
\[
\varphi(v_i + a_i x_\beta) + (b_i + b_\ell a_i) H_\beta + a_i x_\beta \in h_\beta
\]
Hence \( S \) is contained in \( S_0 \). \( \square \)
Let suppose that $S$ is not contained in $\Sigma \times \{0\}$. One expects a contradiction. Since $\Omega_F \cap X'_\beta$ contains Cartan subalgebras and since $\Sigma$ is irreducible, for all $(\varphi, b_1, \ldots, b_\ell)$ in a dense subset $\Sigma'$ of $\Sigma$, $(b_1, \ldots, b_\ell) \neq 0$. Then, by Claim 5.19, $b_\ell \neq 0$. Let set $S' := S \cap \Sigma' \times k^{\ell-1}$ and let $(\varphi, b_1, \ldots, b_\ell, a_1, \ldots, a_{\ell-1})$ be in $S'$ such that $(a_1, \ldots, a_{\ell-1}) \neq 0$. After a permutation of the $a_i$’s, one can suppose $a_1 \neq 0$ so that $b_1 \neq 0$. Then

$$0 = [v_1 + \varphi(v_1) + b_1 H_\beta, x_\beta + \varphi(x_\beta) + b_1 H_\beta] \in 2b_1 x_\beta + u_\beta$$

whence the contradiction. As a result, $S = \Sigma \times \{0\}$. By (i), $S$ is a smooth variety. Hence $\Sigma$ is a smooth variety and $\Omega_F \cap X'_\beta$ is a smooth open subset of $X'_\beta$, containing $V_\beta$, whence the assertion.

(iii) Since $\mathfrak{y}$ is $G$-invariant, $G.Y_\beta$ is contained in $\mathfrak{y} \cap G.X'_\beta \times G.b_\beta$. Let $(V, V')$ be in this intersection. If $V$ is not a Cartan subalgebra, for some $g$ in $G$, $g(V) = V_\beta$ and $g(V') = h_\beta$ since $h_\beta$ is the set of semisimple elements contained in $V_\beta$. Let suppose that $V$ is a Cartan subalgebra, for some $g$ in $G$, $g(V) = \mathfrak{b}$ and $g(V')$ is an element of $G.h_\beta$ contained in $\mathfrak{h}$. In particular, $g(V')$ contains a subregular element. So, $g(V') = h_\beta$ for some positive root $\alpha$. Moreover, $w(\alpha) = \beta$ for some $w$ in $W(\mathbb{R})$ since $g(V')$ is in $G.h_\beta$, whence $w(g(V')) = h_\beta$ and $(V, V')$ is in $G.Y_\beta$, whence the assertion.

(iv) According to (iii), it suffices to prove

\[ \overline{G.Y_\beta} \cap G.X'_\beta \times \text{Gr}_t(\mathfrak{g}) \subseteq G.X'_\beta \times G.h_\beta \]

since $\overline{G.Y_\beta}$ is a projective variety. According to (i) and Lemma 1.4, $\overline{G.Y_\beta} = G.B.(Z_\beta \times \{b_\beta\})$. Let $(V, V')$ be in $\overline{G.Y_\beta}$ such that $V$ is in $X'_\beta$. Then, for some $g$ in $G$, $(g(V), g(V'))$ is in $B.(Z_\beta \times \{b_\beta\})$ so that $g(V')$ is contained in $h_\beta + u_\beta$. According to (i), for some $b$ in $B$, $b.g(V)$ is in $Z_\beta$ and $b.g(V')$ is contained in $(h_\beta + u_\beta) \cap (\mathfrak{h} + g.b)$. Hence $b.g(V') = h_\beta$ and $V'$ is in $G.h_\beta$, whence the assertion.

(v) Let denote by $s_\mathfrak{g}$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^\beta$ and $\mathfrak{g}^{-\beta}$. Let $T'_\beta$ be the normalizer of $h_\beta$ in $G$ and let $Z'_\beta$ be the closure in $\text{Gr}_t(\mathfrak{g})$ of the orbit of $h$ under $T'_\beta$. Since the normalizer of $h_\beta$ in $\mathfrak{g}$ equals $\mathfrak{b} + s_\mathfrak{g}$, $Z'_\beta$ is the set of subspaces of $\mathfrak{g}$ generated by $h_\beta$ and an element of $s_\mathfrak{g} \setminus \{0\}$ so that $Z'_\beta$ is isomorphic to $\mathbb{P}^2(\mathbb{K})$. Moreover, $G.Y_\beta$ equals $G.(Z'_\beta \times \{b_\beta\})$ since $G.Y_\beta = G.(Z_\beta \times \{b_\beta\})$ by (i), and one has a commutative diagram

\[ \begin{array}{c}
G \times_{T'_\beta} (Z'_\beta \times \{b_\beta\}) \\
\downarrow \phi \\\
G/T'_\beta \times G.Y_\beta
\end{array} \]

The canonical projection $G.Y_\beta \to G.h_\beta$ gives a morphism $G.Y_\beta \to G/T'_\beta$ whence an inverse $\varphi$ of the diagonal arrow. Hence $G.Y_\beta$ is isomorphic to $G \times_{T'_\beta} (Z'_\beta \times \{b_\beta\})$ so that $G.Y_\beta$ is smooth since $G/T'_\beta$ and $Z'_\beta \times \{b_\beta\}$ are smooth, whence the assertion. \hfill $\Box$

Let denote by $X_n$ and $(G.X)_n$ the normalizations $X$ of $G.X$ and let denote by $\theta_0$ and $\theta$ the normalization morphisms $X_n \to X$ and $(G.X)_n \to G.X$ respectively.

**Proposition 5.20.** (i) The open subset $\theta^{-1}(G.X')$ of $(G.X)_n$ is smooth and the restriction of $\theta$ to $\theta^{-1}(G.X')$ is a homeomorphism onto $G.X'$.

(ii) The open subset $\theta_0^{-1}(X')$ of $X_n$ is smooth and the restriction of $\theta_0$ to $\theta_0^{-1}(X')$ is a homeomorphism onto $X'$. 
Proof. (i) By definition, $X'$ is the union of the $X'_{\alpha}$'s, $\alpha \in \mathcal{R}_+$. Then, since all orbit of $W(\mathcal{R})$ in $\mathcal{R}$ has a nonempty intersection with $\Pi$, $G.X'$ is the union of the $G.X'_{\beta}$'s, $\beta \in \Pi$. So, it suffices to prove that for $\beta$ in $\Pi$, $0^{-1}(G.X'_{\beta})$ is smooth and the restriction of $0$ to $0^{-1}(G.X'_{\beta})$ is injective since $0$ is closed and surjective as a finite dominant morphism.

Since $G.\mathfrak{h}$ is a smooth open subset of $G.X$, the restriction of $0$ to $0^{-1}(G.\mathfrak{h})$ is an isomorphism onto $G.\mathfrak{h}$. The variety $G.V_{\beta}$ is an hypersurface of $G.X'_{\beta}$. Hence $0^{-1}(G.V_{\beta})$ is an hypersurface of the normal variety $0^{-1}(G.X'_{\beta})$ and its elements are smooth points of $0^{-1}(G.X'_{\beta})$ since $0^{-1}(G.X'_{\beta})$ is a $G$-variety and since a normal variety is smooth in codimension $1$. As a result, $0^{-1}(G.X'_{\beta})$ is smooth since $G.X'_{\beta}$ is the union of $G.\mathfrak{h}$ and $G.V_{\beta}$. Let $x_1$ and $x_2$ be in $0^{-1}(G.X'_{\beta})$ such that $0(x_1) = 0(x_2) = V_{\beta}$. According to Lemma 5.17(iv) and (v), the canonical projection from $G.Y_{\beta}$ to $G.X'_{\beta}$ factorizes through the restriction of $0$ to $0^{-1}(G.X'_{\beta})$ since $0^{-1}(G.X'_{\beta})$ is the normalization of $G.X'_{\beta}$, whence a commutative digram

$$
\begin{array}{ccc}
G.Y_{\beta} & \xrightarrow{0_{\alpha}} & 0^{-1}(G.X'_{\beta}) \\
\downarrow & & \downarrow 0 \\
G.X'_{\beta} & & 
\end{array}
$$

with $0_{\alpha}$ finite and surjective. Let $y_1$ and $y_2$ be in $G.Y_{\beta}$ such that $0_{\alpha}(y_j) = x_j$ for $j = 1, 2$. Since $V_{\beta}$ is the image of $y_1$ and $y_2$ by the canonical projection onto $G.X'_{\beta}$ and since $\mathfrak{h}_{\beta}$ is the set of semisimple elements contained in $V_{\beta}$, $y_1 = y_2$ and $x_1 = x_2$. Hence the restriction of $0$ to $0^{-1}(G.X'_{\beta})$ is injective since $0$ is $G$-equivariant and since $G.X'_{\beta}$ is the union of $G.\mathfrak{h}$ and $G.V_{\beta}$.

(ii) According to Corollary 5.16(ii), $0_{\alpha}^{-1}(X')$ is an open subset of $X_{\alpha}$. Since $X'$ is the union of the $X'_{\alpha}$'s, $\alpha \in \mathcal{R}_+$, it suffices to prove that for $\alpha$ in $\mathcal{R}_+$, $0_{\alpha}^{-1}(X'_{\alpha})$ is smooth and the restriction of $0_{\alpha}$ to $0_{\alpha}^{-1}(X'_{\alpha})$ is injective since $0_{\alpha}$ is closed and surjective as a finite dominant morphism.

Let $\alpha$ be in $\mathcal{R}_+$ and let $\beta$ in $\Pi$ such that $\beta$ is in the orbit of $\alpha$ under $W(\mathcal{R})$. Since $B.\mathfrak{h}$ is a smooth open subset of $B.X$, the restriction of $0_{\alpha}$ to $0_{\alpha}^{-1}(B.\mathfrak{h})$ is an isomorphism onto $B.\mathfrak{h}$. The variety $B.V_{\alpha}$ is an hypersurface of $X'_{\alpha}$. Hence $0_{\alpha}^{-1}(B.V_{\alpha})$ is an hypersurface of the normal variety $0_{\alpha}^{-1}(X'_{\alpha})$ and its elements are smooth points of $0_{\alpha}^{-1}(X'_{\alpha})$ since $0_{\alpha}^{-1}(X'_{\alpha})$ is a $B$-variety and since a normal variety is smooth in codimension $1$. As a result, $0_{\alpha}^{-1}(X'_{\alpha})$ is smooth since $X'_{\alpha}$ is the union of $B.\mathfrak{h}$ and $B.V_{\alpha}$. Since $\beta$ is in the orbit of $\alpha$ under $W(\mathcal{R})$, $G.X'_{\alpha} = G.X'_{\beta}$. Moreover, the varieties $G \times_B 0_{\alpha}^{-1}(X'_{\beta})$ and $G \times_B 0_{\alpha}^{-1}(X'_{\alpha})$ are smooth as fiber bundles over a smooth variety with smooth fibers, whence a commutative diagram

$$
\begin{array}{ccc}
G \times_B 0_{\alpha}^{-1}(X'_{\beta}) & \xrightarrow{0} & 0^{-1}(G.X'_{\beta}) \\
\downarrow & & \downarrow \\
G \times_B X'_{\beta} & \xrightarrow{0} & G \times_B X'_{\alpha}
\end{array}
$$

by [H77, Ch. II, Proposition 4.1]. By Lemma 1.4, the horizontal arrows are projective morphisms. Indeed, since a regular element is contained in finitely many Borel subalgebras, their fibers are finite so that they are finite. Since $B.\mathfrak{h}$ is an open subset of $X'_{\alpha}$ and $X'_{\beta}$, $G \times_B 0_{\alpha}^{-1}(X'_{\beta})$ and $G \times_B 0_{\alpha}^{-1}(X'_{\alpha})$ have the same field of rational functions. As a result, since these two varieties are normal, there exists a $G$-equivariant
isomorphism from $G \times_B \theta_0^{-1}(X'_p)$ onto $G \times_B \theta_0^{-1}(X'_a)$ by [H77, Ch. II, Proposition 4.1]. According to Lemma 5.17,(ii), the restriction of $\theta_0$ to $\theta_0^{-1}(X'_p)$ is an isomorphism so that the first down arrow in the above diagram is an isomorphism. Moreover, the restriction to all fiber of $G \times_B \theta_0^{-1}(X'_p)$ of the morphism

$$G \times_B \theta_0^{-1}(X'_p) \to 0^{-1}(G.X'_p)$$

is injective. Hence the restriction of $\theta_0$ to $\theta_0^{-1}(X'_p)$ is injective since the restriction of $\theta$ to $0^{-1}(G.X'_p)$ is too by (ii), whence the assertion. 


Let $k \geq 2$ be an integer. Let denote by $\mathfrak{c}_n^{(k)}$ the closure of $G.b^k$ in $\mathfrak{g}^k$ with respect to the diagonal action of $G$ in $\mathfrak{g}$ and let set $\mathfrak{c}_{\eta}^{(k)} := \eta^{-1}(\mathfrak{c}_n^{(k)})$. The varieties $\mathfrak{c}_{\eta}^{(k)}$ and $\mathfrak{c}_n^{(k)}$ are called generalized commuting variety and generalized isospectral commuting variety respectively. For $k = 2$, $\mathfrak{c}_{\eta}^{(k)}$ is the isospectral commuting variety considered by M. Haiman in [Ha99, §8] and [Ha02, §7.2].

6.1. Let set:

$$E^{(k)} := \{(u, x_1, \ldots, x_k) \in X \times b^k \mid u \ni x_1, \ldots, u \ni x_k\}$$

Lemma 6.1. Let denote by $E^{(k, x)}$ the intersection of $E^{(k)}$ and $U.\mathfrak{h} \times (\theta_{\text{reg,ss}} \cap b)^k$ and for $w$ in $W(\mathcal{R})$, let denote by $\theta_w$ the map

$$E^{(k)} \to b^k \times b^k \quad (u, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, w(x_1), \ldots, w(x_k))$$

(i) Denoting by $\mathfrak{x}_{0,k}$ the image of $E^{(k)}$ by the projection $(u, x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k)$, $\mathfrak{x}_{0,k}$ is the closure of $B.b^k$ in $b^k$ and $\mathfrak{c}_n^{(k)}$ is the image of $G \times \mathfrak{x}_{0,k}$ by the map $(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$.

(ii) For all $w$ in $W(\mathcal{R})$, $\theta_w(E^{(k, x)})$ is dense in $\theta_w(E^{(k)})$.

Proof. (i) Since $X$ is a projective variety, $\mathfrak{x}_{0,k}$ is a closed subset of $b^k$. The variety $E^{(k)}$ is irreducible of dimension $n + k\ell$ as a vector bundle of rank $k\ell$ over the irreducible variety $X$. So, $B.\{\mathfrak{h} \times b^k\}$ is dense in $E^{(k)}$ and $\mathfrak{x}_{0,k}$ is the closure of $B.b^k$ in $b^k$, whence the assertion by Lemma 1.4.

(ii) Since $U.\mathfrak{h} \times (\theta_{\text{reg,ss}} \cap b)^k$ is an open subset of $X \times b^k$, $E^{(k, x)}$ is an open subset of $E^{(k)}$. Moreover, it is a dense open subset since $E^{(k)}$ is irreducible as a vector bundle over the irreducible variety $X$, whence the assertion since $\theta_w$ is a morphism of algebraic varieties.

6.2. Let $s$ be in $\mathfrak{h}$ and let $G^s$ be the centralizer of $s$ in $G$. According to [Ko63, §3.2, Lemma 5], $G^s$ is connected. Let denote by $\mathcal{R}_s$ the set of roots whose kernel contains $s$ and let denote by $W(\mathcal{R}_s)$ the Weyl group of $\mathcal{R}_s$. Let $\mathfrak{z}_s$ be the centre of $\mathfrak{g}^s$.

Lemma 6.2. Let $x = (x_1, \ldots, x_k)$ be in $\mathfrak{c}^{(k)}$ verifying the following conditions:

1. $s$ is the semisimple component of $x_1$,
2. for $z$ in $P_s$, the centralizer in $\mathfrak{g}$ of the semisimple component of $z$ has dimension at least $\dim \mathfrak{g}^s$.

Then for $i = 1, \ldots, k$, the semisimple component of $x_i$ is contained in $\mathfrak{z}_s$. 

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Proof. Since $x$ is in $\mathfrak{c}^{(k)}$, $[x_i, x_j] = 0$ for all $(i, j)$. Let suppose that for some $i$, the semisimple component $x_{i,s}$ of $x_i$ is not in $\mathfrak{z}_s$. One expects a contradiction. Since $[x_1, x_i] = 0$, for all $t$ in $\mathfrak{k}$, $s + tx_{1,s}$ is the semisimple component of $x_1 + tx_i$. Moreover, after conjugation by an element of $G^\circ$, one can suppose that $x_{i,s}$ is in $\mathfrak{h}$. Since $R$ is finite, there exists $t$ in $\mathfrak{k}$ such that the subset of roots whose kernel contains $s + tx_{1,s}$ is contained in $R_s$. Since $x_{i,s}$ is not in $\mathfrak{z}_s$, for some $\alpha$ in $R_s$, $\alpha(s + tx_{1,s}) \neq 0$ that is $\mathfrak{g}^{s + tx_{1,s}}$ is strictly contained in $\mathfrak{g}^s$, whence the contradiction. \hfill \Box

For $w$ in $W(R)$, let set:

$$C_w := G^wB/B \quad B^w := wBw^{-1}$$

The following lemma results from [Hu95, §6.17, Lemma].

Lemma 6.3. Let $\mathfrak{B}$ be the set of Borel subalgebras of $\mathfrak{g}$ and let $\mathfrak{B}_s$ be the set of Borel subalgebras of $\mathfrak{g}$ containing $s$.

(i) For all $w$ in $W(R)$, $C_w$ is a connected component of $\mathfrak{B}_s$.

(ii) For $(w, w')$ in $W(R) \times W(R)$, $C_w = C_{w'}$ if and only if $w'w^{-1}$ is in $W(R_s)$.

(iii) The variety $C_w$ is isomorphic to $G^s/(G^s \cap B^w)$.

For $x$ in $\mathfrak{B}^{(k)}$, let denote by $\mathfrak{B}_s$ the subset of Borel subalgebras containing $P_x$.

Corollary 6.4. Let $x = (x_1, \ldots, x_k)$ be in $\mathfrak{c}^{(k)}$. Let suppose that $x$ verifies Conditions (1) and (2) of Lemma 6.2. Then the $C_w \cap \mathfrak{B}_s$, $w$ in $W(R)$, are the connected components of $\mathfrak{B}_s$.

Proof. Since a Borel subalgebra contains the semisimple component of its elements and since $s$ is the semisimple component of $x_1$, $\mathfrak{B}_s$ is contained in $\mathfrak{B}_s$. As a result, according to Lemma 6.3, every connected component of $\mathfrak{B}_s$ is contained in $C_w$ for some $w$ in $W(R)$. Let set $x_0 := (x_{1,1}, \ldots, x_{k,n})$. Since $[x_i, x_j] = 0$ for all $(i, j)$, $P_x$ is contained in $\mathfrak{g}^s$. Let $\mathfrak{B}_x$ be the set of Borel subalgebras of $\mathfrak{g}^s$ and for $y$ in $(\mathfrak{g}^s)^k$, let $\mathfrak{B}_x(y)$ be the set of Borel subalgebras of $\mathfrak{g}^s$ containing $P_y$. According to [Hu95, Theorem 6.15], $\mathfrak{B}_x(y)$ is connected. Moreover, according to Lemma 6.2, the semisimple components of $x_1, \ldots, x_k$ are contained in $\mathfrak{z}_s$ so that $\mathfrak{B}_s \cap \mathfrak{B}_s = \mathfrak{B}_s$. Let $w$ be in $W(R)$. According to Lemma 6.3, there is an isomorphism from $\mathfrak{B}_s$ to $C_w$. Moreover, the image of $\mathfrak{B}_s$ by this isomorphism equals $C_w \cap \mathfrak{B}_s$, whence the corollary. \hfill \Box

Corollary 6.5. Let $x = (x_1, \ldots, x_k)$ be in $\mathfrak{c}^{(k)}$ verifying Conditions (1) and (2) of Lemma 6.2. Then $\eta^{-1}(x)$ is contained in the set of the $(x_1, \ldots, x_k, w(x_{1,s}), \ldots, w(x_{k,s}))$'s with $w$ in $W(R)$.

Proof. Since $y = \eta\gamma_0$, $\eta^{-1}(x)$ is the image of $\gamma^{-1}(x)$ by $\gamma_0$. Furthermore, $\gamma_0$ is constant on the connected components of $\gamma^{-1}(x)$ since $\eta^{-1}(x)$ is finite. Let $C$ be a connected component of $\gamma^{-1}(x)$. Identifying $G \times_B t^k$ with the subvariety of elements $(u, x)$ of $\mathfrak{B} \times \mathfrak{g}^s$ such that $P_x$ is contained in $u$, $C$ identifies with $C_w \cap \mathfrak{B}_s \times \{x\}$ for some $w$ in $W(R)$ by Corollary 6.4. Then for some $g$ in $G^s$ and for some representative $g_w$ of $w$ in $N_G(h)$, $gg_w(b)$ contains $P_x$ so that

$$\gamma_0(C) = \{(x_1, \ldots, x_k, (gg_w)^{-1}(x_1), \ldots, (gg_w)^{-1}(x_k))\}$$

By Lemma 6.2, $x_{1,s}, \ldots, x_{k,s}$ are in $\mathfrak{z}_s$ so that $w^{-1}(x_{i,s})$ is the semisimple component of $(gg_w)^{-1}(x_i)$ for $i = 1, \ldots, k$. Hence

$$\gamma_0(C) = \{(x_1, \ldots, x_k, w^{-1}(x_{1,s}), \ldots, w^{-1}(x_{k,s}))\}$$

whence the corollary. \hfill \Box
Proposition 6.6. The variety $\mathfrak{c}_n^{(k)}$ is irreducible and equal to the closure of $G.\mathfrak{t}_n(b^k)$ in $\mathfrak{B}_n^{(k)}$.

Proof. Let denote by $G.\mathfrak{t}_n(b^k)$ the closure of $G.\mathfrak{t}_n(b^k)$ in $\mathfrak{B}_n^{(k)}$. Since $\eta$ is $G$-equivariant, $\eta(G.\mathfrak{t}_n(b^k)) = G.b^k$. Hence $\eta(G.\mathfrak{t}_n(b^k)) = \mathfrak{c}(k)$ since $\eta$ is a finite morphism and since $\mathfrak{c}(k)$ is the closure of $G.b^k$ in $\mathfrak{b}^k$ by definition. Moreover, $G.\mathfrak{t}_n(b^k)$ is irreducible as the closure of an irreducible set. So, it suffices to prove $\mathfrak{c}_n^{(k)} = G.\mathfrak{t}_n(b^k)$. In other words, for all $x$ in $\mathfrak{c}(k)$, $\eta^{-1}(x)$ is contained in $G.\mathfrak{t}_n(b^k)$. According to Lemma 3.9(ii), $\mathfrak{B}_n^{(k)}$ is a $\text{GL}_k(\mathfrak{k})$-variety and $\eta$ is $\text{GL}_k(\mathfrak{k})$-equivariant. As a result, since $\mathfrak{c}(k)$ is invariant under $\text{GL}_k(\mathfrak{k})$, for $x$ in $\mathfrak{c}(k)$, $\eta^{-1}(x')$ is contained in $G.\mathfrak{t}_n(b^k)$ for all $x'$ in $P_n^k$ such that $P_n' = P_n$ if $\eta^{-1}(x)$ is contained in $G.\mathfrak{t}_n(b^k)$. Then, according to Lemma 6.2, since $\eta$ is $G$-equivariant, it suffices to prove that $\eta^{-1}(x)$ is contained in $G.\mathfrak{t}_n(b^k)$ for $x$ in $\mathfrak{c}(k) \cap \mathfrak{b}^k$ verifying Conditions (1) and (2) of Lemma 6.2 for some $s$ in $\mathfrak{b}$.

According to Corollary 6.5,
$$\eta^{-1}(x) \subset \{(x_1, \ldots, x_k, w(x_1), \ldots, w(x_k)) | w \in W(\mathfrak{R}) \} \text{ with } x = (x_1, \ldots, x_k)$$

For $s$ regular, $P_x$ is contained in $\mathfrak{h}$ and $x_i = x_{i,s}$ for $i = 1, \ldots, k$. By definition,
$$(w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k)) \in \mathfrak{t}_n(b^k)$$
and for $g_w$ a representative of $w$ in $N_G(\mathfrak{h})$,
$$g_w^{-1}(w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k)) = (x_1, \ldots, x_k, w(x_1), \ldots, w(x_k))$$
Hence $\eta^{-1}(x)$ is contained in $G.\mathfrak{t}_n(b^k)$. As a result, according to the notations of Lemma 6.1, for all $w$ in $W(\mathfrak{R})$, $\theta_n(E^{(k, s)})$ is contained in $G.\mathfrak{t}_n(b^k)$. Hence, by Lemma 6.1(ii), $\theta_n(E^{(k)})$ is contained in $G.\mathfrak{t}_n(b^k)$, whence the proposition.

6.3. According to Corollary 3.8(ii), the variety $\mathfrak{c}^{-1}(2^{(k)})$ is invariant under the action of $W(\mathfrak{R})^k$ in $\mathfrak{X}^k$ and according to Proposition 3.10, $2^{(k)}_n$ is an irreducible component of $\mathfrak{c}^{-1}(2^{(k)})$ and $\eta$ is the restriction of $\sigma$ to $2^{(k)}_n$.

Lemma 6.7. Let $\Phi$ be the restriction to $S(\mathfrak{b})^{\otimes k}$ of the canonical map from $\mathfrak{k}[2^{(k)}_n]$ to $\mathfrak{k}[\mathfrak{c}^{(k)}_n]$.

(i) The subvariety $\mathfrak{c}^{(k)}_n$ of $\mathfrak{X}^k$ is invariant under the diagonal action of $W(\mathfrak{R})$ in $\mathfrak{X}^k$.

(ii) The map $\Phi$ is an embedding of $S(\mathfrak{b})^{\otimes k}$ into $\mathfrak{k}[\mathfrak{c}^{(k)}_n]$. Moreover, $\Phi(S(\mathfrak{b})^{\otimes k})$ equals $\mathfrak{k}[\mathfrak{c}^{(k)}_n]^G$.

(iii) The image of $(S(\mathfrak{b})^{\otimes k})W(\mathfrak{R})$ by $\Phi$ equals $\mathfrak{k}[\mathfrak{c}^{(k)}_n]^G$.

Proof. (i) For all $w$ in $W(\mathfrak{R})$ and for all representative $g_w$ of $w$ in $W(\mathfrak{R})$,
$$(x_1, \ldots, x_k, w(x_1), \ldots, w(x_k)) = g_w^{-1}(w(x_1), \ldots, w(x_k), w(x_1), \ldots, w(x_k))$$
for all $(x_1, \ldots, x_k)$ in $\mathfrak{b}^k$. As a result, for all $w$ in $W(\mathfrak{R})$, $w.\mathfrak{t}_n(b^k)$ is contained in $G.\mathfrak{t}_n(b^k)$. Hence $G.\mathfrak{t}_n(b^k)$ is invariant under the diagonal action of $W(\mathfrak{R})$ in $\mathfrak{X}^k$ since the actions of $G$ and $W(\mathfrak{R})^k$ in $\mathfrak{X}^k$ commute, whence the assertion.

(ii) According to Corollary 3.12(i), $S(\mathfrak{b})^{\otimes k}$ equals $\mathfrak{k}[\mathfrak{c}^{(k)}_n]^G$. Moreover, for all $P$ in $S(\mathfrak{b})^{\otimes k}$ and for all $x$ in $\mathfrak{b}^k$, $P.\mathfrak{t}_n(x) = P(x)$. Hence $\Phi$ is injective by Proposition 6.6. Since $G$ is reductive, $\mathfrak{k}[\mathfrak{c}^{(k)}_n]^G$ is the image of $\mathfrak{k}[2^{(k)}_n]^G$ by the quotient morphism, whence the assertion.

(iii) Since $G$ is reductive, $\mathfrak{k}[\mathfrak{c}^{(k)}_n]^G$ is the image of $\mathfrak{k}[\mathfrak{B}^{(k)}_n]^G$ by the quotient morphism, whence the assertion since $(S(\mathfrak{b})^{\otimes k})W(\mathfrak{R})$ equals $\mathfrak{k}[\mathfrak{B}^{(k)}_n]^G$ by Corollary 3.12(ii).
Let identify \( S(b)^{\text{reg}} \) to a subalgebra of \( \mathbb{A}[\mathcal{O}^{(k)}_n] \) by \( \Phi \).

**Proposition 6.8.** Let \( \mathcal{C}^{(k)}_n \) and \( \tilde{\mathcal{C}}^{(k)}_n \) be the normalizations of \( \mathcal{C}^{(k)}_n \) and \( \mathcal{C}^{(k)}_n \).

(i) The variety \( \mathcal{C}^{(k)}_n \) is the categorical quotient of \( \mathcal{O}^{(k)}_n \) under the action of \( W(J) \).

(ii) The variety \( \tilde{\mathcal{C}}^{(k)}_n \) is the categorical quotient of \( \mathcal{O}^{(k)}_n \) under the action of \( W(J) \).

**Proof.** (i) According to Proposition 3.10(iii), \( \mathbb{A}[\mathcal{B}^{(k)}_n] \) is generated by \( \mathbb{A}[\mathcal{C}^{(k)}_n] \) and \( S(b)^{\text{reg}} \). Since \( \mathcal{O}^{(k)}_n = \eta^{-1}(\mathcal{C}^{(k)}_n) \) by Proposition 6.6, the image of \( \mathbb{A}[\mathcal{B}^{(k)}_n] \) in \( \mathbb{A}[\mathcal{O}^{(k)}_n] \) by the restriction morphism equals \( \mathbb{A}[\mathcal{C}^{(k)}_n] \). Hence \( \mathbb{A}[\mathcal{C}^{(k)}_n] \) is generated by \( \mathbb{A}[\mathcal{C}^{(k)}_n] \) and \( S(b)^{\text{reg}} \). Then, by Lemma 6.7(iii), \( \mathbb{A}[\mathcal{O}^{(k)}_n]^{W(J)} = \mathbb{A}[\mathcal{C}^{(k)}_n] \).

(ii) Let \( K \) be the fraction field of \( \mathbb{A}[\mathcal{C}^{(k)}_n] \). Since \( \mathcal{O}^{(k)}_n \) is a \( W(J) \)-variety, there is an action of \( W(J) \) on \( K \) and \( K^{W(J)} \) is the fraction field of \( \mathbb{A}[\mathcal{O}^{(k)}_n]^{W(J)} \) since \( W(J) \) is finite. As a result, the integral closure \( \mathbb{A}[\mathcal{O}^{(k)}_n] \) of \( \mathbb{A}[\mathcal{O}^{(k)}_n] \) in \( K \) is invariant under \( W(J) \) and \( \mathbb{A}[\mathcal{C}^{(k)}_n]^{W(J)} \) is contained in \( \mathbb{A}[\mathcal{C}^{(k)}_n] \). Let \( a \) be in \( \mathbb{A}[\mathcal{C}^{(k)}_n]^{W(J)} \). Then \( a \) verifies a dependence integral equation over \( \mathbb{A}[\mathcal{C}^{(k)}_n] \).

\[
d^{m} + a_{m-1}d^{m-1} + \cdots + a_{0} = 0
\]

whence

\[
da^{m} + (\frac{1}{|W(J)|} \sum_{w \in W(J)} w.a_{m-1})d^{m-1} + \cdots + (\frac{1}{|W(J)|} \sum_{w \in W(J)} w.a_{0} = 0
\]

since \( a \) is invariant under \( W(J) \) so that \( a \) is in \( \mathbb{A}[\mathcal{C}^{(k)}_n]^{W(J)} \), whence the assertion.

\[
\Box
\]

7. Desingularization.

Let \( k \geq 2 \) be an integer. Let \( X, X', X_n, 0_0 \) be as in Subsection 5.6. Let denote by \( X'_n \) the inverse image of \( X' \) in \( X_n \). According to Proposition 5.20, \( X'_n \) is a smooth open subset of \( X_n \) and according to [Hir64], there exists a desingularization \( (\Gamma, \pi_n) \) of \( X_n \) such that the restriction of \( \pi_n \) to \( \pi_n^{-1}(X'_n) \) is an isomorphism onto \( X'_n \). Let set \( \pi = 0_{\text{reg}} \pi_n \) so that \( (\Gamma, \pi) \) is a desingularization of \( X \). Recall that \( \mathcal{X}_{0,k} \) is the closure in \( b^k \) of \( B.b^k \) and set \( \mathcal{X}_k := G \times_{b^k} \mathcal{X}_{0,k} \). Then \( \mathcal{X}_k \) is a closed subvariety of \( G \times_{b^k} b^k \).

**Lemma 7.1.** Let \( E \) be the restriction to \( X \) of the tautological vector bundle of rank \( \ell \) over \( \text{Gr}_\ell(b) \) and let \( \tau' \) be the canonical morphism from \( E \) to \( b \).

(i) The morphism \( \tau' \) is projective and birational.

(ii) Let \( \nu \) be the canonical map from \( \pi^*(E) \) to \( E \). Then \( \pi := \tau'^{-1} \) is a \( B \)-equivariant birational projective morphism from \( \pi^*(E) \) to \( b \). In particular, \( \pi^*(E) \) is a desingularization of \( b \).

**Proof.** (i) By definition, \( E \) is the subvariety of elements \( (u, x) \) of \( X \times b \) such that \( x \) is in \( u \) so that \( \tau' \) is the projection from \( b \) to \( E \). Since \( X \) is a projective variety, \( \tau' \) is a projective morphism and \( \tau'(E) \) is closed in \( b \). Moreover, \( \tau'(E) \) is \( B \)-invariant since \( \tau' \) is a \( B \)-equivariant morphism and it contains \( b \) since \( b \) is in \( X \). As a result, \( \tau'(E) = b \). By (i), for \( x \) in \( b_{\text{reg}} \), \( (\tau')^{-1}(x) = \{ (b, x) \} \) since \( g^\times = b \). Hence \( \tau' \) is a birational morphism since \( B.b_{\text{reg}} \) is an open subset of \( b \).

(ii) Since \( E \) is a vector bundle over \( X \) and since \( \pi \) is a projective birational morphism, \( \nu \) is a projective birational morphism. Then \( \tau \) is a projective birational morphism from \( \pi^*(E) \) to \( b \) by (i). It is \( B \)-equivariant since \( \nu \) and \( \tau' \) are too. Moreover, \( \pi^*(E) \) is a desingularization of \( b \) since \( \pi^*(E) \) is smooth as a vector bundle over a smooth variety.
Let denote by $\psi$ the canonical projection from $\pi^*(E)$ to $\Gamma$. Then, according to the above notations, one has the commutative diagram:

$$
\begin{array}{ccc}
\pi^*(E) & \xrightarrow{\psi} & \Gamma \\
\tau & \downarrow & \downarrow \pi \\
E & \xrightarrow{\tau} & X
\end{array}
$$

**Lemma 7.2.** Let $E^{(k)}$ be the fiber product $\pi^*(E) \times_{\phi} \cdots \times_{\phi} \pi^*(E)$ and let $\tau_k$ be the canonical morphism from $E^{(k)}$ to $b^k$.

(i) The vector bundle $E^{(k)}$ over $\Gamma$ is a vector subbundle of the trivial bundle $\Gamma \times b^k$. Moreover, $E^{(k)}$ has dimension $k\ell + n$.

(ii) The morphism $\tau_k$ is a projective birational morphism from $E^{(k)}$ onto $X_{0,k}$. Moreover, $E^{(k)}$ is a desingularization of $X_{0,k}$ in the category of $B$-varieties.

**Proof.** (i) By definition, $E^{(k)}$ is the subvariety of elements $(u, x_1, \ldots, x_k)$ of $\Gamma \times b^k$ such that $x_1, \ldots, x_k$ are in $\pi(u)$. Since $X$ is the closure of $B.b$, $X$ and $\Gamma$ have dimension $n$. Hence $E^{(k)}$ has dimension $k\ell + n$ since $E^{(k)}$ is a vector bundle of rank $k\ell$ over $\Gamma$.

(ii) Since $\Gamma$ is a projective variety, $\tau_k$ is a projective morphism and $\tau_k(E^{(k)}) = X_{0,k}$ by Lemma 6.1(i). For $(x_1, \ldots, x_k)$ in $b^k_{reg,ss}$, $\tau_k^{-1}(x_1, \ldots, x_k) = \{ (g^{(1)}, (x_1, \ldots, x_k)) \}$ since $g^{(1)}$ is a Cartan subalgebra. Hence $\tau_k$ is a birational morphism, whence the assertion since $E^{(k)}$ is a smooth variety as a vector bundle over the smooth variety $\Gamma$. □

Let set $\mathcal{Y} := G \times_B (\Gamma \times b^k)$. The canonical projections from $G \times \Gamma \times b^k$ to $G \times \Gamma$ and $G \times b^k$ define through the quotients morphisms from $\mathcal{Y}$ to $G \times_B \Gamma$ and $G \times_B b^k$. Let denote by $\varsigma$ and $\zeta$ these morphisms. Then one has the following diagram:

$$
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{\zeta} & G \times_B b^k \\
\varsigma & \downarrow & \downarrow \gamma_n \\
G \times_B \Gamma & \xrightarrow{\gamma_n} & b^k_n
\end{array}
$$

The map $(g, x) \mapsto (g, \tau_k(x))$ from $G \times E^{(k)}$ to $G \times b^k$ defines through the quotient a morphism $\overline{\tau_k}$ from $G \times_B E^{(k)}$ to $x_k$.

**Proposition 7.3.** Let set $\xi := \gamma_n \overline{\tau_k}$.

(i) The variety $G \times_B E^{(k)}$ is a closed subvariety of $\mathcal{Y}$.

(ii) The variety $G \times_B E^{(k)}$ is a vector bundle of rank $k\ell$ over $G \times_B \Gamma$. Moreover, $G \times_B \Gamma$ and $G \times_B E^{(k)}$ are smooth varieties.

(iii) The morphism $\xi$ is a projective birational morphism from $G \times_B E^{(k)}$ onto $\mathcal{O}^{(k)}_n$. Moreover $G \times_B E^{(k)}$ is a desingularization of $\mathcal{O}^{(k)}_n$.

**Proof.** (i) According to Lemma 7.2,(i), $E^{(k)}$ is a closed subvariety of $\Gamma \times b^k$, invariant under the diagonal action of $B$. Hence $G \times E^{(k)}$ is a closed subvariety of $G \times \Gamma \times b^k$, invariant under the action of $B$, whence the assertion.
(ii) Since $E^{(k)}$ is a $B$-equivariant vector bundle over $\Gamma$, $G \times_B E^{(k)}$ is a $G$-equivariant vector bundle over $G \times_B \Gamma$. Since $G \times_B \Gamma$ is a fiber bundle over the smooth variety $G/B$ with smooth fibers, $G \times_B \Gamma$ is a smooth variety. As a result, $G \times_B E^{(k)}$ is a smooth variety.

(iii) According to Lemma 7.2,(ii), $\tau_k$ is a projective birational morphism from $G \times_B E^{(k)}$ to $X_k$. Since $X_{0,k}$ is a $B$-invariant closed subvariety of $b^k$, $X_k$ is closed in $G \times_B b^k$. According to Lemma 6.1,(i), $\gamma(X_k) = \mathcal{O}(k)$. Moreover, $\gamma_n(X_k)$ is a closed subvariety of $\mathcal{O}^{(k)}_n$ since $\gamma_n$ is a projective morphism by Lemma 1.4. Hence $\gamma_n(X_k) = \mathcal{O}(k)$ by Proposition 6.6. For all $z \in G \cdot t_k(b_{\text{reg}}^k)$, $|\gamma_n^{-1}(z)| = 1$. Hence the restriction of $\gamma_n$ to $X_k$ is a birational morphism onto $\mathcal{O}(k)$ since $G \cdot t_k(b_{\text{reg}}^k)$ is dense in $\mathcal{O}(k)$. Moreover, this morphism is projective since $\gamma_n$ is projective. As a result, $\mathcal{O}(k)$ is a projective birational morphism from $G \times_B E^{(k)}$ onto $\mathcal{O}(k)$ and $G \times_B E^{(k)}$ is a desingularization of $\mathcal{O}(k)$ by (ii). □

The following corollary results from Lemma 7.2,(ii), Proposition 7.3,(iii) and Lemma 1.1.

**Corollary 7.4.** Let $X_{0,k}$ and $\mathcal{O}(k)$ be the normalizations of $X_{0,k}$ and $\mathcal{O}(k)$ respectively. Then $k[\overline{X_{0,k}}]$ and $k[\overline{\mathcal{O}(k)}]$ are the spaces of global sections of $\mathcal{O}_{E^{(k)}}$ and $\mathcal{O}_{G \times_B E^{(k)}}$ respectively.

### 8. Rational singularities

Let $k \geq 2$ be an integer. Let $X$, $X'$, $X_0$, $X_0'$, $\pi$, $\pi'$, $\tau_k$, $E$, $P$, $\psi$, $\nu$, $\tau_k$ be as in Section 7. One has the commutative diagram:

\[
\begin{array}{ccc}
E^{(k)} & \to & X_{0,k} \\
\downarrow \psi_k & & \downarrow \\
\Gamma & \to & X_0 \\
\downarrow \pi & & \downarrow \pi' \\
X & & X
\end{array}
\]

with $\psi_k$ the canonical projection from $E^{(k)}$ onto $\Gamma$.

#### 8.1. According to the notations of Subsection 5.1, let denote by $S_\alpha$ the closure of $U(b_\alpha)$ in $b$. For $\beta$ in $\Pi$, let set:

$$u_\beta := \bigoplus_{\alpha \in \mathcal{R}_+, \beta} g^{\beta} \quad b_\beta := b_\beta \oplus u_\beta$$

**Lemma 8.1.** For $\alpha$ in $\mathcal{R}_+$, let $b'_\alpha$ be the set of subregular elements belonging to $b_\alpha$.

(i) For $\alpha$ in $\mathcal{R}_+$, $S_\alpha$ is a subvariety of codimension 2 of $b$. Moreover, it is contained in $b \setminus b_{\text{reg}}$.

(ii) For $\beta$ in $\Pi$, $S_\beta = b_\beta$.

(iii) The $S_\alpha$’s, $\alpha \in \mathcal{R}_+$, are the irreducible components of $b \setminus b_{\text{reg}}$.

**Proof.** (i) For $x$ in $b'_\alpha$, $b^x = b + \mathfrak{k} x_\alpha$. Hence $U(b'_\alpha)$ has dimension $n - 1 + \ell - 1$, whence the assertion since $U(b'_\alpha)$ is dense in $S_\alpha$ and since $b'_\alpha$ is contained in $b \setminus b_{\text{reg}}$.

(ii) For $\beta$ in $\Pi$, $U(b'_\beta)$ is contained in $b_\beta$ since $b_\beta$ is an ideal of $b$, whence the assertion by (i).

(iii) According to (i), it suffices to prove that $b \setminus b_{\text{reg}}$ is the union of the $S_\alpha$’s. Let $x$ be in $b \setminus b_{\text{reg}}$. According to [V72], for some $g$ in $G$ and for some $\beta$ in $\Pi$, $x$ is in $g(b_\beta)$. Since $b_\beta$ is an ideal of $b$, by
Bruhat’s decomposition of $G$, for some $b$ in $B$ and for some $w$ in $W(R)$, $b^{-1}(x)$ is in $w(b\beta) \cap b$. By definition, 

\[ w(b\beta) = w(b\beta) \oplus w(\alpha) = b_{w(\beta)} \oplus \bigoplus_{\alpha \in R_+ \setminus \{\beta\}} g^{w(\alpha)} \]

So, 

\[ w(b\beta) \cap b = b_{w(\beta)} \oplus u_0 \text{ with } u_0 := \bigoplus_{\alpha \in R_+ \setminus \{\beta\}, w(\alpha) \in R_+} g^{w(\alpha)} \]

The subspace $u_0$ of $u$ is a subalgebra, not containing $g^{w(\beta)}$. Then, denoting by $U_0$ the closed subgroup of $U$ whose Lie algebra is $\text{ad} u_0$, 

\[ U_0(b_{w(\beta)}) = w(b\beta) \cap b \]

since the left hand side is contained in the right hand side and has the same dimension. As a result, $x$ is in $S_{w(\beta)}$ since $S_{w(\beta)}$ is $B$-invariant, whence the assertion. \qed

Let $\mathfrak{g}'_{\text{reg}}$ be the set of regular elements $x$ such that $x_s$ is regular or subregular and let set $\mathfrak{b}'_{\text{reg}} := \mathfrak{g}'_{\text{reg}} \cap \mathfrak{b}$.

**Lemma 8.2.** (i) The subset $\mathfrak{b}'_{\text{reg}}$ of $\mathfrak{b}$ is a big open subset of $\mathfrak{b}$.

(ii) The subset $\mathfrak{g}'_{\text{reg}}$ of $\mathfrak{g}$ is a big open subset of $\mathfrak{g}$.

**Proof.** Let $x$ be in $\mathfrak{g}'_{\text{reg}} \setminus \mathfrak{g}_{\text{reg},ss}$. Let $W$ be the set of elements $y$ of $\mathfrak{g}'_{x_s}$ such that the restriction of ad $y$ to $[x_s, \mathfrak{g}]$ is injective. Then $W$ is an open subset of $\mathfrak{g}'_{x_s}$, containing $x$, and the map

\[ G \times W \longrightarrow \mathfrak{g} \quad (g, y) \longmapsto g(y) \]

is a submersion. Let $3$ be the centre of $\mathfrak{g}'_{x_s}$ and let set $3' := W \cap 3$. For some open subset $W'$ of $W$, containing $x$, for all $y$ in $W'$, the component of $y$ on $3$ is in $3'$. Since $[\mathfrak{g}'_{x_s}, \mathfrak{g}'_{x_s}]$ is a simple algebra of dimension 3, $W' \cap \mathfrak{g}_{\text{reg}}$ is contained in $\mathfrak{g}'_{\text{reg}}$ and $G(W' \cap \mathfrak{g}_{\text{reg}})$ is an open set, contained in $\mathfrak{g}'_{\text{reg}}$ and containing $x$. As a result, $\mathfrak{g}'_{\text{reg}}$ is an open subset of $\mathfrak{g}$ and $\mathfrak{b}'_{\text{reg}}$ is an open subset of $\mathfrak{b}$.

(i) Let suppose that $\mathfrak{b} \setminus \mathfrak{b}'_{\text{reg}}$ has an irreducible component $\Sigma$ of codimension 1 in $\mathfrak{b}$. One expects a contradiction. Since $\Sigma$ is invariant under $B$, $\Sigma \cap \mathfrak{h}$ is the image of $\Sigma$ by the projection $x \mapsto x$ by Lemma 1.5. Since $\Sigma$ has codimension 1 in $\mathfrak{b}$, $\Sigma \cap \mathfrak{h} = \mathfrak{h}$ or $\Sigma = \Sigma \cap \mathfrak{h} + \mathfrak{u}$. Since $\Sigma$ does not contain regular semisimple element, $\Sigma \cap \mathfrak{h}$ is an irreducible subset of codimension 1 of $\mathfrak{h}$, not containing regular semisimple elements. Hence $\Sigma \cap \mathfrak{h}$ is $\mathfrak{h}$, for some positive root and $\Sigma \cap (\mathfrak{b} + \mathfrak{g}'_{\text{reg}}) \cap \mathfrak{g}_{\text{reg}}$ is not empty, whence the contradiction.

(ii) Since $\mathfrak{b} \setminus \mathfrak{b}'_{\text{reg}}$ is invariant under $B$, $\mathfrak{g} \setminus \mathfrak{g}'_{\text{reg}} = G(\mathfrak{b} \setminus \mathfrak{b}'_{\text{reg}})$ and 

\[ \dim \mathfrak{g} \setminus \mathfrak{g}'_{\text{reg}} \leq n + \dim \mathfrak{b} \setminus \mathfrak{b}'_{\text{reg}} \]

whence the assertion by (i). \qed

Setting $\mathfrak{b}_{\text{reg},0} := \mathfrak{b}_{\text{reg}}$ and $\mathfrak{b}_{\text{reg},1} := \mathfrak{b}'_{\text{reg}}$, let $V_{k,j}$ be the subset of elements $x$ of $X_{0,k}$ such that $P_x \cap \mathfrak{b}_{\text{reg},j}$ is not empty for $j = 0, 1$.

**Proposition 8.3.** For $j = 0, 1$, let $V'_{k,j}$ be the subset of elements $x = (x_1, \ldots, x_k)$ of $X_{0,k}$ such that $x_1$ is in $\mathfrak{b}_{\text{reg},j}$.

(i) For $j = 0, 1$, $V'_{k,j}$ is a smooth open subset of $X_{0,k}$.

(ii) For $j = 0, 1$, $V_{k,j}$ is a smooth open subset of $X_{0,k}$.

(iii) For $j = 0, 1$, $V_{k,j}$ is a big open subset of $X_{0,k}$. 

Proof. (i) By definition, \( V'_{k,j} \) is the intersection of \( X_{0,k} \) and the open subset \( b_{\text{reg},j} \times b^{k-1} \) of \( b^k \). Hence \( V'_{k,j} \) is an open subset of \( X_{0,k} \). For \( x_1 \) in \( b_{\text{reg},0} \) (\( x_1, \ldots, x_k \)) is in \( V'_{k,0} \) if and only if \( x_2, \ldots, x_k \) are in \( g^{x_1} \) by Corollary 5.3,(ii) and Lemma 7.2,(ii) since \( g^{x_1} \) is in \( X \). According to [Ko63, Theorem 9], for \( x \) in \( b_{\text{reg}} \), \( e_1(x), \ldots, e_\ell(x) \) is a basis of \( g^x \). Hence the map

\[
\begin{align*}
b_{\text{reg}} \times M_{k-1,\ell}(k) & \xrightarrow{\theta} V'_{k,0} \\
(x, (a_{i,j}, 1 \leq i \leq k - 1, 1 \leq j \leq \ell)) & \mapsto (x, \sum_{j=1}^\ell a_{1,j}e_j(x), \ldots, \sum_{j=1}^\ell a_{k-1,j}e_j(x))
\end{align*}
\]

is a bijective isomorphism. The open subset \( b_{\text{reg}} \) has a cover by open subsets \( V \) such that for some \( e_1, \ldots, e_n \) in \( b, e_1(x), \ldots, e_\ell(x), e_1, \ldots, e_n \) is a basis for all \( x \) in \( V \). Then there exist regular functions \( \varphi_1, \ldots, \varphi_\ell \) on \( V \times b \) such that

\[
v - \sum_{j=1}^\ell \varphi_j(x, v)e_j(x) \in \text{span}(e_1, \ldots, e_n)
\]

for all \( (x, v) \) in \( V \times b \), so that the restriction of \( \theta \) to \( V \times M_{k-1,\ell}(k) \) is an isomorphism onto \( X_{0,k} \cap V \times b^{k-1} \) whose inverse is

\[
(x_1, \ldots, x_k) \mapsto (x_1, (\varphi_1(x_1, x_1), \ldots, \varphi_\ell(x_1, x_1)), i = 2, \ldots, k)
\]

As a result, \( \theta \) is an isomorphism and \( V'_{k,0} \) is a smooth variety, whence the assertion since \( V'_{k,1} \) is an open subset of \( V'_{k,0} \).

(ii) The subvariety \( X_{0,k} \) of \( b^k \) is invariant under the natural action of \( \text{GL}_k(k) \) in \( b^k \) and \( V_{k,j} = \text{GL}_k(k) \cdot V'_{k,j} \) by Lemma 1.6, whence the assertion by (i).

(iii) Since \( V_{k,1} \) is contained in \( V_{k,0} \), it suffices to prove the assertion for \( j = 1 \). Let suppose that \( X_{0,k} \setminus V_{k,1} \) has an irreducible component \( \Sigma \) of codimension 1. One expects a contradiction. Since \( X_{0,k} \) and \( V_{k,1} \) are invariant under \( B \) and \( \text{GL}_k(k) \), it is so for \( \Sigma \). Since \( \Sigma \) has codimension 1 in \( X_{0,k} \), \( \tau_k^{-1}(\Sigma) \) has codimension 1 in \( E^{(k)} \). Let \( \Sigma_0 \) be an irreducible component of codimension 1 of \( \tau_k^{-1}(\Sigma) \) and let set \( T := \pi_\psi(\Sigma_0) \). Since \( \Sigma \) is invariant under \( \text{GL}_k(k) \), \( \Sigma_0 \) is invariant under the action of \( \text{GL}_k(k) \) so that the intersection of \( (\pi_\psi)^{-1}(T) \) and the null section of \( E^{(k)} \) is contained in \( \Sigma_0 \). So, \( T \) is a closed irreducible subset of \( X \). Moreover, \( T \) is strictly contained in \( X \). Indeed, if it is not so, for all \( u \) in \( U \), \( [u] \times u^k \cap \Sigma_0 \) has dimension at most \( k(l - 1) \) since \( \Sigma_0 \) is invariant under \( \xi_k \). Then \( T \) has codimension 1 in \( X \) and \( \Sigma_0 = (\pi_\psi)^{-1}(T) \). According to Theorem 5.13,(ii), for some \( u \) in \( T \), \( u \cap b_{\text{reg},1} \) is not empty, whence the contradiction since for all \( x \) in \( \Sigma \), \( P_x \cap b_{\text{reg},1} \) is empty and since \( u^k \) is contained in \( \Sigma \) for all \( u \) in \( T \).

Let \( \widetilde{X}_{0,k} \) be the normalization of \( X_{0,k} \) and let \( \lambda_k \) be the normalization morphism whence a commutative diagram

\[
\begin{array}{ccc}
E^{(k)} & \xrightarrow{\tau_k} & \widetilde{X}_{0,k} \\
\tau_k \downarrow & & \downarrow \lambda_k \\
X_{0,k} & \xrightarrow{\lambda_k} & \widetilde{X}_{0,k}
\end{array}
\]

since \( (E^{(k)}, \tau_k) \) is a desingularization of \( X_{0,k} \).

**Corollary 8.4.** For \( j = 0, 1 \), \( \lambda_k^{-1}(V_{k,j}) \) is a smooth big open subset of \( \widetilde{X}_{0,k} \) and the restriction of \( \tau_k \) to \( \lambda_k^{-1}(V_{k,j}) \) is an isomorphism onto \( \lambda_k^{-1}(V_{k,j}) \).
Proof. According to Proposition 8.3, $V_{k,j}$ is a smooth big open subset of $X_{0,k}$. Hence the restriction of $\lambda_k$ to $\lambda_k^{-1}(V_{k,j})$ is an isomorphism onto $V_{k,j}$. For all $x \in V_{k,j}$, $\tau_k^{-1}(x) = (u, x)$ with $u$ equal to the centralizer of a regular element contained in $P_x$. Hence, by Zariski Main Theorem [Mu88, § 9], the restriction of $\tau_k$ to $\tau_k^{-1}(V_{k,j})$ is an isomorphism onto $V_{k,j}$ since $V_{k,j}$ is smooth, whence the corollary. □

8.2. By definition, the restriction of $\lambda$ to $\tau$ arrow is birational and projective. According to Lemma 7.2,(ii), the diagonal arrow is a birational projective morphism. Hence the horizontal smooth open subset of $X$ homeomorphism from Lemma 8.5.

Moreover, $E$ (i) Since $\pi_n$ to $\pi_n^{-1}(X'_n)$ is an isomorphism onto $X'_n$. Let identify $\pi_n^{-1}(X'_n)$ and $X'_n$ by $\pi_n$. Let denote by $E_k$ the restriction of $E^{(k)}$ to $X'_n$. According to Proposition 5.20,(ii), $\theta_0$ is a homeomorphism from $\theta_0^{-1}(X')$ to $X'$. Moreover, $U.$ identifies with an open subset of $X'_n$ since it is a smooth open subset of $X'$.

Lemma 8.5. Let set $E_n := \theta_0^*(E)$ and let denote by $\nu_n$ the canonical morphism from $E_n$ to $E$.

(i) There exists a well defined projective birational morphism $\tau_n$ from $\pi^*(E)$ to $E_n$ such that $\nu = \nu_n \circ \tau_n$. Moreover, $E_n$ is normal.

(ii) The $\Omega_{\pi^*(E)}$-module $\Omega_{\pi^*(E)}$ is free.

(iii) The variety $E_n$ is Gorenstein and has rational singularities.

Proof. (i) Since $E_n$ is a vector bundle over $X_n$, $E_n$ is a normal variety. Moreover, it is the normalization of $E$ and $\nu_n$ is the normalization morphism, whence the assertion by Lemma 7.1,(ii).

(ii) Let $\omega$ be a volume form on $b$. According to Lemma 7.1,(ii), $\tau^*(\omega)$ is a global section of $\Omega_{\pi^*(E)}$, without zero, whence the assertion since $\Omega_{\pi^*(E)}$ is locally free of rank 1.

(iii) According to (ii), $\Omega_{\pi^*(E)}$ is isomorphic to $\Omega_{\pi^*(E)}$. So, by Grauert-Riemenschneider Theorem [GR70], $R^i(\tau_n)_*(\Omega_{\pi^*(E)}) = 0$ for $i > 0$. Hence $E_n$ has rational singularities by (i). Moreover, $(\tau_n)_*(\Omega_{\pi^*(E)})$ is free of rank 1 by (ii). In other words, the canonical module of $E_n$ is isomorphic to $\Omega_{E_n}$, that is $E_n$ is Gorenstein.

Let $\rho_n$ be the canonical projection from $E_n$ to $X_n$ and let set $E^{(k)}_n := E_n \times_{\rho_n} \cdots \times_{\rho_n} E_n$. k factors

Corollary 8.6. (i) The variety $E^{(k)}_n$ is a desingularization of $E^{(k)}_n$.

(ii) The variety $E^{(k)}_n$ is Gorenstein and has rational singularities.

Proof. (i) Let $\rho$ be the canonical projection from $E$ to $X$ and let set $E^{(k)} := E \times_{\rho} \cdots \times_{\rho} E$. Since $E^{(k)}_n$ is a vector bundle over the normal variety $X_n$, $E^{(k)}_n$ is a normal variety. Moreover, it is the normalization of $\overline{E^{(k)}}$ since $X_n$ is the normalization of $X$, whence a commutative diagram

$$
\begin{array}{c}
E^{(k)} \longrightarrow E^{(k)}_n \\
\downarrow \quad \downarrow \\
\overline{E^{(k)}}
\end{array}
$$

According to Lemma 7.2,(ii), the diagonal arrow is a birational projective morphism. Hence the horizontal arrow is birational and projective.

(ii) The variety $E^{(k)}_n$ is a vector bundle over $E_n$. So, by Lemma 8.5,(iii), $E^{(k)}_n$ is Gorenstein and has rational singularities. □
Theorem 8.7. The normalization $\tilde{X}_{0,k}$ of $X_{0,k}$ has rational singularities.

Proof. By definition, the morphism $\tilde{\tau}_k$ from $E^{(k)}$ to $\tilde{X}_{0,k}$ factorizes through the morphism $E^{(k)} \to E^{(k)}_n$ so that there is a commutative diagram

$$
\begin{array}{ccc}
E^{(k)} & \longrightarrow & E^{(k)}_n \\
\downarrow \tilde{\tau}_k & & \downarrow \\
\tilde{X}_{0,k} & &
\end{array}
$$

Moreover, according to Lemma 7.2, (ii) and Corollary 8.6, (i), all the arrows are projective and birational. According to the previous identifications, $E_k$ is a smooth big open subset of $E^{(k)}_n$ since $X'_n$ is a smooth big open subset of $X_n$. According to Corollary 8.4, the open subset $V_{k,1}$ of $X_{0,k}$ identifies with its inverse images in $\tilde{X}_{0,k}$ and $E_k$. Moreover, $V_{k,1}$ is a big open subset of $\tilde{X}_{0,k}$. For all Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$, contained in $\mathfrak{b}$, $\mathfrak{k} \setminus V_{k,1}$ is contained in $(\mathfrak{c} \setminus \mathfrak{b}_{\text{reg}})^k$ so that it has codimension at least 2 in $\mathfrak{k}$ since $k \geq 2$. As a result, $V_{k,1}$ is a big open subset of $E^{(k)}_n$ since for all $u$ in $X'_n \setminus U, u^k$ is not contained in $V_{k,1}$. Then, according to Corollary 8.6 and Proposition C.2, with $Y = E^{(k)}_n$, $\tilde{X}_{0,k}$ has rational singularities. □

8.3. Let denote by $E^*$ the dual of the vector bundle $\pi^*(E)$ over $\Gamma$. 

Lemma 8.8. Let $\mathcal{E}^*$ be the sheaf of local sections of $E^*$. For $i > 0$ and for $j \geq 0$, $H^i(\Gamma, S^j(\mathcal{E}^*)) = 0$.

Proof. Since $\psi$ is the canonical projection from $\pi^*(E)$ to $\Gamma$, $\mathcal{O}_{\pi^*(E)}$ equals $\psi^*(\mathcal{S}(\mathcal{E}^*))$ so that

$$(\psi)_*(\mathcal{O}_{\pi^*(E)}) = \mathcal{S}(\mathcal{E}^*)$$

As a result, for $i \geq 0$,

$$H^i(\pi^*(E), \mathcal{O}_{\pi^*(E)}) = H^i(\Gamma, S(\mathcal{E}^*)) = \bigoplus_{j \geq i} H^i(\Gamma, S^j(\mathcal{E}^*))$$

According to Lemma 7.1, (ii), $\pi^*(E)$ is a desingularization of the smooth variety $\mathfrak{b}$. Hence by [El78],

$$H^i(\pi^*(E), \mathcal{O}_{\pi^*(E)}) = 0$$

for $i > 0$, whence

$$H^i(\Gamma, S^j(\mathcal{E}^*)) = 0$$

for $i > 0$ and $j \geq 0$. □

According to the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ by the Killing form, $\mathfrak{b}_-$ identifies with $\mathfrak{b}^*$. Let denote by $E_-$ the orthogonal complement of $\pi^*(E)$ in $\Gamma \times \mathfrak{b}_-$ so that $E_-$ is a vector bundle of rank $n$ over $\Gamma$. Let $\mathcal{E}_-$ be the sheaf of local sections of $E_-$. 

Corollary 8.9. Let $\mathfrak{g}_0$ be the ideal of $\mathcal{O}_\Gamma \otimes_\mathcal{S} \mathcal{S}(\mathfrak{b}_-)$ generated by $\mathcal{E}_-$. Then, for $i \geq 0$, $H^i(\Gamma, \mathfrak{g}_0) = 0$ and $H^i(\Gamma, \mathcal{E}_-) = 0$.

Proof. Since $E_-$ is the orthogonal complement of $\pi^*(E)$ in $\Gamma \times \mathfrak{b}_-$, $\mathfrak{g}_0$ is the ideal of definition of $\pi^*(E)$ in $\mathcal{O}_\Gamma \otimes_\mathcal{S} \mathcal{S}(\mathfrak{b}_-)$ whence a short exact sequence

$$0 \to \mathfrak{g}_0 \to \mathcal{O}_\Gamma \otimes_\mathcal{S} \mathcal{S}(\mathfrak{b}_-) \to \mathcal{S}(\mathcal{E}^*) \to 0$$
and whence a cohomology long exact sequence
\[
\cdots \rightarrow H^i(\Gamma, S(\mathcal{E}^*)) \rightarrow H^{i+1}(\Gamma, \mathcal{J}_0) \rightarrow H^{i+1}(\Gamma, \mathcal{O}_\Gamma \otimes_k \mathcal{S}(b_-)) \rightarrow \cdots
\]
Then, by Lemma 8.8, from the equality
\[
\text{Proposition 8.10.}
\]
Let \( l, m \) be nonnegative integers.

(i) For all positive integer \( i \), 
\[
H^i(\Gamma, (\mathcal{E}^*)^\otimes m) = 0.
\]

(ii) For all positive integer \( i \),
\[
H^{i+l}(\Gamma, \mathcal{E}^\otimes_\mathcal{I} \otimes_{\mathcal{O}_\mathcal{I}} (\mathcal{E}^*)^\otimes m) = 0
\]
Proof. (i) According to Lemma 8.8, one can suppose \( m > 1 \). Since \( E^* \) is the dual of the vector bundle \( \pi^* E \) over \( \Gamma \), the fiber product \( E_m^* := E^* \times_{\mathcal{O}_\mathcal{I}} \cdots \times_{\mathcal{O}_\mathcal{I}} E^* \) is the dual of the vector bundle \( E_m^* \) over \( \Gamma \). Let \( \psi_m \) be the canonical projection from \( E_m^* \) to \( \Gamma \) and let \( E_m^* \) be the sheaf of local sections of \( E_m^* \). Then \( \mathcal{O}_{E_m^*} \) equals \( \psi_m^*(S(E_m^*)) \) and since \( E_m^* \) is a vector bundle over \( \Gamma \), for all nonnegative integer \( i \),
\[
H^i(E_m^*, \mathcal{O}_{E_m^*}) = H^i(\Gamma, S(E_m^*)) = \bigoplus_{q \in \mathbb{N}} H^i(\Gamma, S^q(E_m^*))
\]
According to Theorem 8.7, for \( i > 0 \), the left hand side equals 0 since \( E_m^* \) is a desingularization of \( X_{\mathcal{I}, m} \) by Lemma 7.2.(iv). As a result, for \( i > 0 \),
\[
H^i(\Gamma, S^m(E_m^*)) = 0
\]
The decomposition of \( E_m^* \) as a direct sum of \( m \) copies isomorphic to \( \mathcal{E}^* \) induces a multigraduation of \( S(E^*) \). Denoting by \( S_{j_1, \ldots, j_m} \) the subsheaf of multidegree \((j_1, \ldots, j_m)\), one has
\[
S^m(E_m^*) = \bigoplus_{(j_1, \ldots, j_m) \in \mathbb{N}^m} S_{j_1, \ldots, j_m} \text{ and } S_{1, \ldots, 1} = (\mathcal{E}^*)^\otimes m
\]
Hence for \( i > 0 \),
\[
0 = H^i(\Gamma, S^m(E_m^*)) = \bigoplus_{(j_1, \ldots, j_m) \in \mathbb{N}^m} H^i(\Gamma, S_{j_1, \ldots, j_m})
\]
whence the assertion.

(ii) Let \( m \) be a nonnegative integer. Let prove by induction on \( j \) that for \( i > 0 \) and for \( l \geq j \),
\[
H^{i+l}(\Gamma, \mathcal{E}^\otimes_\mathcal{I} \otimes_{\mathcal{O}_\mathcal{I}} (\mathcal{E}^*)^\otimes(m+l-j)) = 0
\]
By (i) it is true for \( j = 0 \). Let suppose \( j > 0 \) and (6) true for \( j - 1 \) and for all \( l \geq j - 1 \). From the short exact sequence of \( O_\Gamma \)-modules

\[
0 \rightarrow E_\subset \rightarrow O_\Gamma \otimes_k b_- \rightarrow E^* \rightarrow 0
\]

one deduces the short exact sequence of \( O_\Gamma \)-modules

\[
0 \rightarrow E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j) \rightarrow b_\subset \otimes_k E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j) \rightarrow E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j+1) \rightarrow 0
\]

From the cohomology long exact sequence deduced from this short exact sequence, one has the exact sequence

\[
H^{i+j-1}(\Gamma, E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j+1)) \rightarrow H^{i+j}(\Gamma, E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j)) \rightarrow H^{i+j}(\Gamma, b_\subset \otimes_k E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j))
\]

for all positive integer \( i \). By induction hypothesis, the first term equals 0 for all \( i > 0 \). Since

\[
H^{i+j}(\Gamma, b_\subset \otimes_k E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j)) = b_\subset \otimes_k H^{i+j}(\Gamma, E^\otimes_j \otimes O_\Gamma (E^*)^\otimes (m+l+j))
\]

the last term of the last exact sequence equals 0 by induction hypothesis again, whence Equality (6) and whence the assertion for \( j = l \).

The following corollary results from Proposition 8.10,(ii) and Proposition B.2.

**Corollary 8.11.** For \( m \) positive integer and for \( l = (l_1, \ldots, l_m) \in \mathbb{N}^m \),

\[
H^{i+j}(\Gamma, \wedge^{l_1}(E_\subset) \otimes \cdots \otimes O_\Gamma \wedge^{l_m}(E_\subset)) = 0
\]

for all positive integer \( i \).

### 8.4

By definition, \( E^{(k)} \) is a closed subvariety of \( \Gamma \times b^k \). Let denote by \( \varphi \) the canonical projection from \( \Gamma \times b^k \) to \( \Gamma \), whence the diagram

\[
\begin{array}{ccc}
E^{(k)} & \leftarrow & \Gamma \times b^k \\
\varphi & \downarrow & \\
& \Gamma & \\
\end{array}
\]

For \( j = 1, \ldots, k \), let denote by \( \mathcal{E}_{j,k} \) the set of injections from \( \{1, \ldots, j\} \) to \( \{1, \ldots, k\} \) and for \( \sigma \) in \( \mathcal{E}_{j,k} \), let set:

\[
\mathcal{K}_\sigma := M_{i} \otimes O_{\Gamma} \cdots \otimes O_{\Gamma} M_{k} \text{ with } M_{i} := \begin{cases} O_{\Gamma} \otimes_k S(b_-) & \text{if } i \notin \sigma((1, \ldots, j)) \\ \mathcal{J}_{0} & \text{if } i \in \sigma((1, \ldots, j)) \end{cases}
\]

For \( j \) in \( \{1, \ldots, k\} \), the direct sum of the \( \mathcal{K}_\sigma \)'s is denoted by \( \mathcal{J}_{j,k} \) and for \( \sigma \) in \( \mathcal{E}_{1,k} \), \( \mathcal{K}_\sigma \) is also denoted by \( \mathcal{K}_{\sigma(1),k} \).

**Lemma 8.12.** Let \( \mathcal{J} \) be the ideal of definition of \( E^{(k)} \) in \( O_{\Gamma \times b^k} \).

(i) The ideal \( \varphi_*(\mathcal{J}) \) of \( O_{\Gamma} \otimes_k S(b^k) \) is the sum of \( \mathcal{K}_{1,k}, \ldots, \mathcal{K}_{k,k} \).

(ii) There is an exact sequence of \( O_\Gamma \)-modules

\[
0 \rightarrow \mathcal{J}_{k,k} \rightarrow \mathcal{J}_{k-1,k} \rightarrow \cdots \rightarrow \mathcal{J}_{1,k} \rightarrow \varphi_*(\mathcal{J}) \rightarrow 0
\]

(iii) For \( i > 0 \), \( H^i(\Gamma \times b^k, \mathcal{J}) = 0 \) if \( H^{i+j}(\Gamma, \mathcal{J}_{0}^\otimes j) = 0 \) for \( j = 1, \ldots, k \).
Proof. (i) Let \( \mathcal{J}_k \) be the sum of \( \mathcal{K}_{1,k}, \ldots, \mathcal{K}_{k,k} \). Since \( \mathcal{J}_0 \) is the ideal of \( \mathcal{O}_\Gamma \otimes_k S(b_-) \) generated by \( \mathcal{E}_- \), \( \mathcal{J}_k \) is a prime ideal of \( \mathcal{O}_\Gamma \otimes_k S(b_-) \). Moreover, \( \mathcal{E}_- \) is the sheaf of local sections of the orthogonal complement of \( E \) in \( \Gamma \times b_- \). Hence \( \mathcal{J}_k \) is the ideal of definition of \( E^{(k)} \) in \( \mathcal{O}_\Gamma \otimes_k S(b_-) \), whence the assertion.

(ii) For a local section of \( \mathcal{J}_{jk} \) and for \( \sigma \) in \( \mathcal{Z}_{jk} \), let denote by \( a_{\sigma(1), \ldots, \sigma(j)} \) the component of \( a \) on \( \mathcal{K}_{\sigma} \). Let \( d \) be the map \( \mathcal{J}_{jk} \rightarrow \mathcal{J}_{j-1,k} \) such that

\[
d a_{i_1, \ldots, i_j} = \sum_{l=1}^j (-1)^{j+1} a_{i_1, \ldots, i_{l-1}, i_{l+1}, \ldots, i_j}
\]

Then by (i), one has an augmented complex

\[
0 \rightarrow \mathcal{J}_{kk} \rightarrow \mathcal{J}_{k-1,k} \rightarrow \cdots \rightarrow \mathcal{J}_{1,k} \rightarrow \mathcal{G}((\mathcal{J})_ k) \rightarrow 0
\]

Let \( J \) the the subbundle of the trivial bundle \( \Gamma \times S(b_-) \) such that the fiber at \( x \) is the ideal of \( S(b_-) \) generated by the fiber \( E_{-x} \) of \( E_- \) at \( x \). Then \( \mathcal{J}_0 \) is the sheaf of local sections of \( J \) and the above augmented complex is the sheaf of local sections of the augmented complex of vector bundles over \( \Gamma \),

\[
0 \rightarrow C^{(k)}(\Gamma \times S(b_-), J) \rightarrow \cdots \rightarrow C^{(k)}(\Gamma \times S(b_-), J) \rightarrow J \rightarrow 0
\]

According to Lemma B.3 and Remark B.4, this complex is acyclic, whence the assertion by Nakayama Lemma since \( J \) and \( S(b_-) \) are graded.

(iii) Let \( i \) be a positive integer such that \( H^{i+j}(\Gamma, \mathcal{J}_0) = 0 \) for \( j = 1, \ldots, k \). Then for \( j = 1, \ldots, k \) and for \( \sigma \) in \( \mathcal{Z}_{jk} \), \( H^{i+j}(\Gamma, \mathcal{K}_{\sigma}) = 0 \) since \( \mathcal{K}_{\sigma} \) is isomorphic to a sum of copies of \( \mathcal{J}_0 \). Moreover, \( H^i(\Gamma, \mathcal{K}_{jk}) = 0 \) for \( l = 1, \ldots, k \) since \( H^i(\Gamma, \mathcal{J}_0) = 0 \) by Corollary 8.9. Hence by (ii), since \( H^* \) is an exact \( \delta \)-functor, \( H^i(\Gamma, \mathcal{G}(\mathcal{J})) = 0 \), whence the assertion since \( \mathcal{G} \) is an affine morphism.

8.5. For \( m \) positive integer, for \( j \) nonnegative integer and for \( l = (l_1, \ldots, l_m) \) in \( \mathbb{N}^m \), let set:

\[
\mathcal{M}_{jl} := \mathcal{J}^{\otimes j} \otimes \mathcal{O}_\Gamma \wedge (\mathcal{E}_-) \otimes \mathcal{O}_\Gamma \wedge \cdots \otimes \mathcal{O}_\Gamma \wedge \mathcal{E}_-
\]

Lemma 8.13. Let \( j, m \) be positive integers and let \( l \) be in \( \mathbb{N}^m \).

(i) The \( \mathcal{O}_\Gamma \)-module \( \mathcal{J}_0 \) is locally free.

(ii) There is an exact sequence

\[
0 \rightarrow S(b_-) \otimes_k \mathcal{M}_{j-1,(n,l)} \rightarrow S(b_-) \otimes_k \mathcal{M}_{j-1,(n-1,l)} \rightarrow \cdots \rightarrow S(b_-) \otimes_k \mathcal{M}_{j-1,(1,l)} \rightarrow \mathcal{M}_{jl} \rightarrow 0
\]

(iii) For \( i > 0 \), \( H^{i+j+l}(\Gamma, \mathcal{M}_{jl}) = 0 \).

Proof. (i) Let \( x \) be in \( \Gamma \) and let \( E_{-x} \) be the fiber at \( x \) of the vector bundle \( E_- \) over \( \Gamma \). Then \( E_{-x} \) is a subspace of dimension \( n \) of \( b_- \). Let \( M \) be a complement of \( E_{-x} \) in \( b_- \). Since the map \( y \mapsto E_{-y} \) is a regular map from \( \Gamma \) to \( \text{Gr}_n(b_-) \), for all \( y \) in an open neighborhood \( V \) of \( x \) in \( \Gamma \),

\[
b_- = E_{-x} \oplus M
\]

Denoting by \( \mathcal{E}_{-V} \) the restriction of \( \mathcal{E}_- \) to \( V \), one has

\[
\mathcal{O}_V \otimes_k b_- = \mathcal{E}_{-V} \oplus \mathcal{O}_V \otimes_k M
\]
so that
\[ O_V \otimes_k S(b_-) = S(E_{-\cdot}) \otimes_k S(M) \]
whence
\[ \mathcal{J}_0|_V = S_+(E_{-\cdot}) \otimes_k S(M) \]

As a result, \( \mathcal{J}_0 \) is locally free since \( E_{-\cdot} \) is locally free.

(ii) Since \( \mathcal{J}_0 \) is the ideal of \( O_\Gamma \otimes_k S(b_-) \) generated by the locally free module \( E_{-\cdot} \) of rank \( n \) and since \( E_{-\cdot} \) is locally generated by a regular sequence of the algebra \( O_\Gamma \otimes_k S(b_-) \), having \( n \) elements, one has an exact Koszul complex
\[
0 \rightarrow S(b_-) \otimes_k \wedge^n(E_{-\cdot}) \rightarrow \cdots \rightarrow S(b_-) \otimes_k E_{-\cdot} \rightarrow \mathcal{J}_0 \rightarrow 0
\]
whence a complex
\[
0 \rightarrow S(b_-) \otimes_k \wedge^n(E_{-\cdot}) \otimes_{O_\Gamma} M_{j-1,l} \rightarrow \cdots \rightarrow S(b_-) \otimes_k E_{-\cdot} \otimes_{O_\Gamma} M_{j-1,l} \rightarrow \mathcal{J}_0 \otimes_{O_\Gamma} M_{j-1,l} \rightarrow 0
\]

According to (i), \( M_{j-1,l} \) is a locally free module. Hence this complex is acyclic.

(iii) Let prove the assertion by induction on \( j \). According to Corollary 8.11, it is true for \( j = 0 \). Let suppose that it is true for \( j - 1 \). According to the induction hypothesis, for all positive integer \( i \) and for \( p = 1, \ldots, n \),
\[
H^{i+j-1+p+|l|}(\Gamma, S(b_-) \otimes_k M_{j-1,(p,l)}) = S(b_-) \otimes_k H^{i+j-1+p+|l|}(\Gamma, M_{j-1,(p,l)}) = 0
\]
Then, according to (ii), \( H^{i+j+|l|}(\Gamma, M_{j,l}) = 0 \) for all positive integer \( i \) since \( H^* \) is an exact \( \delta \)-functor.

**Proposition 8.14.** The variety \( X_{0,k} \) has rational singularities and its ideal of definition in \( O_{\Gamma \times b^k} \) is the space of global sections of \( \mathcal{J} \).

**Proof.** From the short exact sequence,
\[ 0 \rightarrow \mathcal{J} \rightarrow O_{\Gamma \times b^k} \rightarrow O_{E(k)} \rightarrow 0 \]
one deduces the long exact sequence
\[
\cdots \rightarrow H^i(\Gamma \times b^k, \mathcal{J}) \rightarrow S(b_-)^{\otimes k} \otimes_k H^i(\Gamma, O_\Gamma) \rightarrow H^i(E^{(k)}, O_{E(k)}) \rightarrow H^{i+1}(\Gamma \times b^k, \mathcal{J}) \rightarrow \cdots
\]

According to Lemma 8.8, \( H^i(\Gamma, O_\Gamma) = 0 \) for \( i > 0 \) and according to Lemma 8.12,(iii) and Lemma 8.13,(iii), \( H^i(\Gamma \times b^k, \mathcal{J}) = 0 \) for \( i > 0 \). Hence, \( H^i(E^{(k)}, O_{E(k)}) = 0 \) for \( i > 0 \), whence the short exact sequence
\[
0 \rightarrow H^0(\Gamma \times b^k, \mathcal{J}) \rightarrow S(b_-)^{\otimes k} \rightarrow H^0(E^{(k)}, O_{E(k)}) \rightarrow 0
\]
Since the image of \( S(b_-)^{\otimes k} \) is contained in \( k[X_{0,k}], k[X_{0,k}] = k[\bar{X}_{0,k}] \) by Corollary 7.4, whence the proposition by Theorem 8.7 since \( E^{(k)} \) is a desingularization of \( X_{0,k} \) by Lemma 7.2.(iii).

**Corollary 8.15.** (i) The normalization morphism of \( O^{(k)}_0 \) is a homeomorphism.

(ii) The normalization morphism of \( E^{(k)} \) is a homeomorphism.
Proof. (i) According to Proposition 3.10, one has the commutative diagram

\[
\begin{array}{ccc}
G \times_B X_{0,k} & \longrightarrow & G \times_B b^k \\
\gamma_n & \downarrow & \gamma_n \\
\mathcal{C}_n^{(k)} & \longrightarrow & \mathcal{B}_n^{(k)}
\end{array}
\]

Since \(\mathcal{B}_n^{(k)}\) is a normal variety and since \(G \times_B b^k\) is a desingularization of \(\mathcal{B}_n^{(k)}\) and \(\mathcal{B}_n^{(k)}\), the fibers of \(\gamma_n\) are connected by Zariski Main Theorem [Mu88, S 9]. Then the fibers of the restriction of \(\gamma_n\) to \(G \times_B X_{0,k}\) are too since \(G \times_B X_{0,k}\) is the inverse image of \(\mathcal{C}_n^{(k)}\). According to Proposition 8.14, \(G \times_B X_{0,k}\) is a normal variety. Moreover, the restriction of \(\gamma_n\) to \(G \times_B X_{0,k}\) is projective and birational, whence the commutative diagram

\[
\begin{array}{ccc}
G \times_B X_{0,k} & \longrightarrow & \mathcal{C}_n^{(k)} \\
\gamma_n & \downarrow & \mu \\
\mathcal{C}_n^{(k)} & \longrightarrow & \mathcal{C}_n^{(k)}
\end{array}
\]

with \(\mu\) the normalization morphism. For \(x\) in \(\mathcal{C}_n^{(k)}\), \(\mu^{-1}(x) = \gamma_n(\gamma_n^{-1}(x))\). Hence \(\mu\) is injective since the fibers of \(\gamma_n\) are connected, whence the assertion since \(\mu\) is closed as a finite morphism.

(ii) One has a commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}_n^{(k)} & \longrightarrow & \mathcal{C}_n^{(k)} \\
\eta & \downarrow & \eta \\
\mathcal{(k)} & \longrightarrow & \mathcal{(k)}
\end{array}
\]

with \(\mu_0\) the normalization morphism. According to Proposition 6.8, all fiber of \(\eta\) or \(\eta\) is one single \(W(\mathbb{R})\)-orbit and by (i), \(\mu\) is injective. Hence \(\mu_0\) is injective, whence the assertion since \(\mu_0\) is closed as a finite morphism. \(\square\)

8.6. In this subsection \(k = 2\). The open subset \(E_2\) of \(E^{(2)}\) identifies with an open subset of \(E_n^{(2)}\) and it is \(B\)-invariant so that \(G \times_B E_2\) is an open subset of \(G \times_B E^{(2)}\) and \(G \times_B E_n^{(2)}\).

Lemma 8.16. (i) The variety \(G \times_B X_{0,2}\) has rational singularities.

(ii) The set \(G \times_B V_{2,1}\) is a smooth big open subset of \(G \times_B X_{0,2}\).

(iii) The set \(G \times_B V_{2,1}\) is a smooth big open subset of \(E_n^{(2)}\).

(iv) A global section of \(\Omega_{G \times_B V_{2,1}}\) has a regular extension to the smooth locus of \(G \times_B X_{0,2}\).

Proof. (i) According to Proposition 8.14, \(X_{0,2}\) has rational singularities, whence the assertion since \(G \times_B X_{0,2}\) is a fiber bundle over the smooth variety \(G/B\) with fibers isomorphic to \(X_{0,2}\).

(ii) According to Proposition 8.3,(iii), \(V_{2,1}\) is a smooth big open subset of \(X_{0,k}\). Then \(G \times_B V_{2,1}\) is a smooth big open subset of \(G \times_B X_{0,2}\) since \(G/B\) is smooth.

(iii) Since \(\gamma_n^{-1}(G \times_B V_{2,1})\) equals \(G \times_B V_{2,1}\) and since \(\gamma_n\) is projective and birational, \(G \times_B V_{2,1}\) is a big open subset of \(E_n^{(2)}\). Moreover, \(G \times_B V_{2,1}\) is contained in the open subset \(\gamma_n^{-1}(W_2)\) of \(G \times_B b^2\) and the
restriction of $\gamma_n$ to $\gamma_n^{-1}(W_2)$ is an isomorphism onto $W_2$ by Proposition 3.10,(iv) so that the restriction of $\gamma_n$ to $G \times_B V_{2,1}$ is an isomorphism onto $G.t_n(V_{2,1})$, whence the assertion.

(iv) The assertion results from (iii) and Lemma C.1,(v). \hfill \Box

**Corollary 8.17.** The varieties $\overline{C_n}^{(2)}$ and $\overline{C}^{(2)}$ have rational singularities.

**Proof.** According to the proof of Corollary 8.15, one has the following commutative diagram:

$G \times_B X_{0,2} \xrightarrow{\gamma_n} \overline{C_n}^{(2)} \xrightarrow{\mu} \overline{C}^{(2)}$

with $\mu$ the normalization morphism. Moreover, $\overline{\gamma}_n$ is a projective and birational morphism. By Lemma 8.16,(iii), $\mu^{-1}(G.t_n(V_{2,1}))$ is a smooth big open subset of $\overline{C_n}^{(2)}$ and the restriction of $\mu$ to $\mu^{-1}(G.t_n(V_{2,1}))$ is an isomorphism onto $G.t_n(V_{2,1})$. So, by Lemma 8.16,(iv), all global section of $\Omega_{\mu^{-1}(G.t_n(V_{2,1}))}$ has a regular extension to the smooth locus of $G \times_B X_{0,2}$. According to Proposition 7.3,(ii), $G \times_B E^{(2)}$ is a desingularization of $\overline{C_n}^{(2)}$ and $E^{(2)}$ is a desingularization of $X_{0,2}$ with a $B$-equivariant desingularization morphism by Lemma 7.2,(ii). Hence $G \times_B E^{(2)}$ is a desingularization of $\overline{C_n}^{(2)}$ and $G \times_B X_{0,2}$. As a result by Lemma 8.16,(i) and [KK73, p.50], all global section of $\Omega_{\mu^{-1}(G.t_n(V_{2,1}))}$ has a regular extension to $G \times_B E^{(2)}$. According to Proposition 6.6, $\overline{C_n}^{(2)}$ is the normalization of the isospectral commuting variety and according to [Gi11, Theorem 1.3.4], $\overline{C_n}^{(2)}$ is Gorenstein. Hence by [KK73, p.50], $\overline{C}^{(2)}$ has rational singularities. By Proposition 6.8,(ii), $\overline{C}^{(2)}$ is the categorical quotient of $\overline{C_n}^{(2)}$ under the action of $W(\mathbb{R})$. So, by [El81, Lemme 1], $\overline{C}^{(2)}$ has rational singularities. \hfill \Box

**APPENDIX A. NOTATIONS.**

In this appendix, $V$ is a finite dimensional vector space. Let denote by $S(V)$ and $\wedge(V)$ the symmetric and exterior algebras of $V$ respectively. For all integer $i$, $S^i(V)$ and $\wedge^i(V)$ are the subspaces of degree $i$ for the usual gradation of $S(V)$ and $\wedge(V)$ respectively. In particular, $S^i(V)$ and $\wedge^i(V)$ are equal to zero for $i$ negative.

- For $l$ positive integer, let denote by $\mathfrak{S}_l$ the group of permutations of $l$ elements.
- For $m$ positive integer and for $l = (l_1, \ldots, l_m)$ in $\mathbb{N}^m$, let set:

$$|l| := l_1 + \cdots + l_m$$

$$S^l(V) := S^{l_1}(V) \otimes \cdots \otimes S^{l_m}(V)$$

$$\wedge^l(V) := \wedge^{l_1}(V) \otimes \cdots \otimes \wedge^{l_m}(V)$$

- For $k$ positive integer and for $l = (l_1, \ldots, l_m)$ in $\mathbb{N}^m$ such that $1 \leq |l| \leq k$, let denote by $V^{\otimes k}$ the $k$-th tensor power of $V$ and let denote by $\mathfrak{S}_l$ the direct product $\mathfrak{S}_{l_1} \times \cdots \times \mathfrak{S}_{l_m}$. The group $\mathfrak{S}_l$ has a natural action
on $V^{\otimes k}$ given by

$$(\sigma_1, \ldots, \sigma_m). (v_1 \otimes \cdots \otimes v_k) = \sum_{l=1}^{m} \cdots \sum_{r \in [v]} \sigma_i a.$$

The map

$$a \mapsto \pi_{k,j}(a) := \prod_{j=1}^{m} \sum_{\sigma \in \Sigma_j} \sigma \cdot a$$

is a projection of $V^{\otimes k}$ onto $(V^{\otimes k})^{\Sigma_j}$. Moreover, the restriction to $(V^{\otimes k})^{\Sigma_j}$ of the canonical map from $V^{\otimes k}$ to $S^l(V) \otimes_k V^{\otimes (k-|l|)}$ is an isomorphism of vector spaces.

**Appendix B. Some complexes.**

Let $X$ be a smooth algebraic variety. For $M$ a coherent $O_X$-module and for $k$ positive integer, let denote by $M^{\otimes k}$ the $k$-th tensor power of $M$. According to Notations A, for all $l$ in $\mathbb{N}^{m}$ such that $|l| \leq k$, there is an action of $\Sigma_j$ on $M^{\otimes k}$. Moreover, $S^i(M)$ and $\bigwedge^i(M)$ are coherent modules defined by the same formulas as in Notations A.

**B.1.** Let $D(V)$ be the algebra $S(V) \otimes_k \bigwedge(V)$ and let $d$ be the $\bigwedge(V)$-derivation of $D(V)$ such that $dv \cdot a = 1 \cdot (v \wedge a)$ for all $(v, a)$ in $V \times \bigwedge(V)$. The gradation of $\bigwedge(V)$ induces on $D(V)$ a gradation so that $D(V)$ is a graded cohomology complex denoted by $D^*(V)$. For $k$ positive integer, let denote by $D^k_k(V)$ the graded subcomplex of $D^*(V)$ whose space of degree $i$ is $S^{k-i}(V) \otimes_k \bigwedge^i(V)$:

$$D^k_k(V) := \bigoplus_{i=0}^{k} D^i_k(V) = \bigoplus_{i=0}^{k} S^{k-i}(V) \otimes_k \bigwedge^i(V)$$

**Lemma B.1.** Let $k$ be a positive integer.

(i) The cohomology of $D^*(V)$ equals $k$.

(ii) For $k$ positive, the subcomplex $D^k_k(V)$ of $D^*(V)$ is acyclic.

**Proof.** (i) We prove the assertion by induction on $\dim V$. Let denote by $d$ the differential of $D^*(V)$. The cohomology in degree $0$ of $D^*(V)$ equals $k$. For $\dim V = 1$, $D^*(V)$ has no cohomology in positive degree since $dv \cdot a = m^{i=0} \cdot v$ for all $v$ in $V$. Let suppose that it is true for all vector space of dimension at most $\dim V - 1$. Let $a$ be an homogeneous cocycle of positive degree $d$, let $W$ be a subspace of codimension $1$ of $V$ and let $v$ be in $V \setminus W$. Then $a$ has a unique expansion

$$a = v^m (a'_m + a''_m \wedge v) + \cdots + a'_0 + a''_0 \wedge v,$$

with $a'_i$ and $a''_i$ in $D^i(W)$ and $D^{i-1}(W)$ respectively for $i = 0, \ldots, m$. From the equality

$$da = \sum_{i=0}^{m} v^i (da'_i + (da''_i) \wedge v) + \sum_{i=1}^{m} (-1)^{d+1} v^i a'_i \wedge v$$

one deduces that $a'_m$ and $a''_m$ are cocycles of degree $d$ and $d-1$ respectively of $D^*(V)$ since $a$ is a cocycle. Hence by induction hypothesis, $a'_m = db'_m$ for some element $b'_m$ of $D^{d-1}(W)$. If $d > 1$, by induction
hypothesis again, \( a''_m = db'_m \) for some element \( b'_m \) of \( D^{d-2}(W) \). As a result,

\[
a - dv^m(b'_m + b''_m \wedge v) = (-1)^d m v^{m-1} b'_m \wedge v + \sum_{i=0}^{m-1} v^i(a'_i + a''_m \wedge v).
\]

So by induction on \( m \), \( a \) is a coboundary. Let suppose \( m = 1 \). Since \( a''_m \) is a cocycle, it is in \( k \). Then

\[
a - d(v^m b'_m + \frac{1}{m+1} a''_m v^{m+1}) = -mv^{m-1} b'_m \wedge v + \sum_{i=0}^{m-1} v^i(a'_i + a''_m \wedge v).
\]

So by induction on \( m \), \( a \) is a coboundary, whence the assertion.

(ii) Since \( D^*(V) \) is the direct sum of the subcomplexes \( D^*_i(V), k \in \mathbb{N} \), the assertion results from (i). \( \square \)

**B.2.** Let \( \mathcal{E} \) and \( \mathcal{M} \) be locally free \( \mathcal{O}_X \)-modules.

**Proposition B.2.** Let \( i \) be a positive integer and let suppose that

\[
H^{i+j}(X, \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0
\]

for all nonnegative integers \( j, k \).

(i) For all positive integers \( m \) and \( k \) and for all \( l \) in \( \mathbb{N}^m \) such that \( |l| \leq k \),

\[
H^i(X, S^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l)} \otimes_{\mathcal{O}_X} \mathcal{M}) = 0
\]

(ii) For all positive integers \( n_1, n_2 \), \( k \) and for all \( (l, m) \) in \( \mathbb{N}^{n_1} \times \mathbb{N}^{n_2} \) such that \( |l| + |m| \leq k \),

\[
H^i(X, S^l(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{M}) = 0
\]

**Proof.** (i) Let \( \mathcal{U} \) be an affine open cover of \( X \) so that the cohomology of the Čech complexes \( C^*(\mathcal{U}, \mathcal{E}^{\otimes k}) \) and \( C^*(\mathcal{U}, S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l)}) \) are the cohomology of the \( \mathcal{O}_X \)-modules \( \mathcal{E}^{\otimes k} \) and \( S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l)} \) respectively. The action of \( \mathcal{E} \) on \( \mathcal{E}^{\otimes k} \) induces an action on \( C^*(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}) \) commuting with its derivation denoted by \( d \). Let \( \varphi \) be a cocycle of degree \( i \) of

\[
C^*(\mathcal{U}, S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l)} \otimes_{\mathcal{O}_X} \mathcal{M})
\]

and let \( \varphi \) be the representative of \( \varphi \) in \( C^i(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}) \). Then \( \varphi \) is a cocycle of degree \( i \) of \( C^*(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}) \). By hypothesis, for some \( \psi \) in \( C^{i-1}(\mathcal{U}, \mathcal{E}^{\otimes k} \otimes_{\mathcal{O}_X} \mathcal{M}) \), \( \varphi = d\psi \). Then, since \( \varphi \) is invariant under \( \mathcal{E} \), \( \varphi = d\psi \)\# . Hence \( \varphi \) is the coboundary of the image of \( \psi \) in \( C^{i-1}(\mathcal{U}, S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l)} \otimes_{\mathcal{O}_X} \mathcal{M}) \), whence the assertion.

(ii) Let suppose \( n_2 = 1 \) and let prove the assertion by induction on \( m \). Since \( \mathcal{E} \) is a locally free module, according to Lemma B.1(ii), one has a long exact sequence of \( \mathcal{O}_X \)-modules,

\[
0 \longrightarrow S^m(\mathcal{E}) \longrightarrow S^{m-1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \longrightarrow \cdots \longrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots
\]

whence an exact sequence

\[
0 \longrightarrow S^m(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow S^{m-1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots
\]

\[
S^i(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow S^{i+1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow \cdots
\]

\[
\longrightarrow S^1(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-l-m)} \otimes_{\mathcal{O}_X} \mathcal{M} \longrightarrow 0
\]
with $k^j = (l, j)$ in $\mathbb{N}^{m+1}$ for all $j$ in $\mathbb{N}$, since $S^j(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-j)-m}$ and $\mathcal{M}$ are locally free modules. According to the induction hypothesis for $j > 0$, $H^{i+j-1}(X, S^{k^j}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-j)-m}) = 0$

Then, $H^i(X, S^j(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-j)-m}) = 0$

since $H^*$ is an exact $\delta$-functor.

Let suppose the assertion true for $n_2 - 1$, and let prove the assertion by induction on $m_{n_2}$. According to the induction hypothesis, it is true for $m_{n_2} = 0$. According to Lemma B.1(ii), one has a long exact sequence of $\mathcal{O}_X$-modules,

$0 \to S^{m_2}(\mathcal{E}) \to S^{m_2-1}(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E} \to \cdots \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-j)-1} \to \mathcal{E}^{\otimes (k-j)-2} \to 0$

Tensoring this sequence by the locally free module $S^j(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-j)-m}$ with $m' = (m_1, \ldots, m_{n_2-1})$ and arguing as before, we deduce the equality $H^i(X, S^j(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{E}^{\otimes (k-j)-m}) = 0$

from the induction hypothesis, whence the assertion. \hfill \Box

**B.3.** Let $W$ be a subspace of $V$ and let set $E := V/W$. Let $C^{(n)}_*(V, W)$, $n = 1, 2, \ldots$ be the sequence of graded spaces over $\mathbb{N}$ defined by the induction relations:

$C^{(1)}_0(V, W) := V \quad C^{(1)}_i(V, W) := W \quad C^{(1)}_i(V, W) := 0$

$C^{(n)}_0(V, W) := V^{\otimes n} \quad C^{(n)}_j(V, W) := C^{(n-1)}_j(V, W) \otimes_k V \oplus C^{(n-1)}_{j-1}(V, W) \otimes_k W$

for $i \geq 2$ and $j \geq 1$.

**Lemma B.3.** Let $n$ be a positive integer. There exists a graded differential of degree $-1$ on $C^{(n)}_*(V, W)$ such that the complex so defined has no homology in positive degree.

**Proof.** Let prove the lemma by induction on $n$. For $n = 1$, $d$ is given by the canonical injection of $W$ in $V$. Let suppose that $C^{(n-1)}_*(V, W)$ has a differential $d$ verifying the conditions of the lemma. For $j > 0$, let denote by $\delta$ the linear map $C^{(n)}_j(V, W) \to C^{(n)}_{j-1}(V, W)$ $$(a \otimes v, b \otimes w) \mapsto (da \otimes v + (-1)^j b \otimes w, db \otimes w)$$

with $a, b, v, w$ in $C^{(n-1)}_j(V, W), C^{(n-1)}_{j-1}(V, W), V, W$ respectively. Then $\delta$ is a graded differential of degree $-1$. Let $c$ be a cycle of positive degree $j$ of $C^{(n)}_*(V, W)$. Then $c$ has an expansion $c = \left( \sum_{i=1}^{d} a_i \otimes v_i, \sum_{i=1}^{d'} b_i \otimes v_i \right)$.
with \( v_1, \ldots, v_d \) a basis of \( V \) such that \( v_1, \ldots, v_{d'} \) is a basis of \( W \) and with \( a_1, \ldots, a_d \) and \( b_1, \ldots, b_{d'} \) in \( C_j^{(n-1)}(V, W) \) and \( C_j^{(n-1)}(V, W) \) respectively. Since \( c \) is a cycle,
\[
\sum_{i=1}^{d} \delta_i a_i \otimes v_i + (-1)^j \sum_{i=1}^{d'} b_i \otimes v_i = 0
\]
Hence \( b_i = (-1)^{j+1} \delta_i a_i \) for \( i = 1, \ldots, d' \) so that
\[
c + \delta \left( \sum_{i=1}^{d} (-1)^j \delta_i a_i \otimes v_i, 0 \right) = \left( \sum_{i=1}^{d} \delta_i a_i \otimes v_i + \sum_{i=1}^{d'} a_i \otimes v_i, \sum_{i=1}^{d'} (b_i \otimes v_i + (-1)^j \delta_i a_i \otimes v_i) \right) = (\sum_{i=1}^{d} a_i \otimes v_i + \sum_{i=1}^{d'} a_i \otimes v_i, 0)
\]
So one can suppose \( b_1, \ldots, b_{d'} \) all equal to 0. Then \( a_1, \ldots, a_d \) are cycles of degree \( j \) of \( C_j^{(n-1)}(V, W) \). By induction hypothesis, they are boundaries of \( C_j^{(n-1)}(V, W) \) so that \( c \) is a boundary of \( C_j^{(n)}(V, W) \), whence the lemma.

\( \square \)

**Remark B.4.** The results of this subsection remain true for \( V \) or \( W \) of infinite dimension since a vector space is an inductive limit of finite dimensional vector spaces.

**Appendix C. Rational Singularities.**

Let \( X \) be an affine irreducible normal variety and let \( X' \) be a smooth big open subset of \( X \).

**Lemma C.1.** Let \( Y \) be an irreducible Gorenstein variety and let \( \pi \) be a projective birational morphism from \( Y \) to \( X \). Let denote by \( K \) the canonical module of \( Y \). Let suppose that the following conditions are verified:

1. the open subset \( \pi^{-1}(X') \) of \( Y \) is big,
2. the restriction of \( \pi \) to \( \pi^{-1}(X') \) is an isomorphism onto \( X' \).

Let denote by \( J \) the space of global sections of \( K \) and let \( \mathfrak{J} \) be the localization of \( J \) on \( X \).

(i) The algebra \( \mathbb{K}[X] \) is the space of global sections of \( \mathcal{O}_Y \) and \( Y \) is a normal variety.

(ii) For all open subset \( O \) of \( X \) and for all local section \( a \) of \( \mathfrak{J} \) over \( O \cap X' \), \( a \) is the restriction to \( O \cap X' \) of one and only one local section of \( \mathfrak{J} \) over \( O \).

(iii) The \( \mathcal{O}_Y \)-modules \( \pi^*(\mathfrak{J}) \) and \( K \) are equal.

(iv) For all injective \( \mathbb{K}[X] \)-module \( I \), the canonical morphism
\[
J \otimes_{\mathbb{K}[X]} \text{Hom}_{\mathbb{K}[X]}(J, I) \rightarrow I
\]
is an isomorphism.

(v) All regular form of top degree on \( X' \) has a unique regular extension to the smooth locus of \( Y \).

**Proof.** (i) If \( Y' \rightarrow Y \) is a desingularization of \( Y \), \( Y' \rightarrow X \) is a desingularization of \( X \) since \( \pi \) is projective and birational. Moreover, all global section of \( \mathcal{O}_Y \) is a global section of \( \mathcal{O}_{Y'} \), whence, by Lemma 1.1, \( \mathbb{K}[X] \) is the space of global sections of \( \mathcal{O}_Y \) since \( X \) is normal. According to Conditions (1) and (2), \( \pi^{-1}(X') \) is a smooth big open subset of \( Y \). So, by Serre’s normality criterion [Bou98, §1, no 10, Théorème 4], \( Y \) is normal since \( Y \) is Gorenstein.

(ii) Since \( \mathfrak{J} \) is the localization of \( J \) on \( X \), it suffices to prove the assertion for \( O = X \). Let \( a \) be a local section of \( \mathfrak{J} \) over \( X' \). According to (2), \( \pi^*(a) \) is a local section of \( K \) over \( \pi^{-1}(X') \). Since \( Y \) is Gorenstein,
\( \mathcal{K} \) is locally free of rank 1. So, there is an affine open cover \( V_1, \ldots, V_l \) of \( Y \) such that the restriction of \( \mathcal{K} \) to \( V_i \) is a free \( \mathcal{O}_{V_i} \)-module of rank 1. Let \( p_i \) be a generator of this module. Setting \( V'_i := V_i \cap \pi^{-1}(X') \), for some regular function \( a'_i \) on \( V'_i \), \( a'_i p_i \) is the restriction of \( \pi^*(a) \) to \( V'_i \). According to (i), \( a'_i \) has a regular extension to \( V_i \) since \( V'_i \) is a big open subset of \( V_i \) by Condition (1). Let denote by \( a_i \) this extension. Then, for \( 1 \leq i, j \leq l \), the restrictions of \( a_i p_i \) and \( a_j p_j \) to \( V_i \cap V_j \) are two local sections of \( \mathcal{K} \) over \( V_i \cap V_j \) which are equal on \( V'_i \cap V'_j \). Hence \( a_i p_i \) and \( a_j p_j \) have the same restriction to \( V_i \cap V_j \) since \( \mathcal{K} \) is torsion free as a locally free module. As a result, \( \pi^*(a) \) is the restriction to \( \pi^{-1}(X') \) of a unique global section of \( \mathcal{K} \) since \( \mathcal{K} \) is torsion free.

(iii) Let \( a \) be in \( k[V_i] \otimes_{\mathcal{O}[X]} J \). By condition (2), for some regular function \( a' \) on \( V'_i \), \( a' p_i \) is the restriction of \( a \) to \( V'_i \). Since \( V_i \) is normal and since \( V'_i \) is a big open subset of \( V_i \), \( a' \) has a regular extension to \( V_i \) so that \( a \) is in \( \Gamma(V_i, \mathcal{K}) \). Conversely, let \( a \) be in \( \Gamma(V_i, \mathcal{K}) \). Since \( V'_i \) is a big open subset of \( V_i \), for some open subset \( V''_i \) of \( X \), \( V_i \) is contained in \( \pi^{-1}(V''_i) \) and \( X' \cap V''_i \) equals \( \pi(V'_i) \). By Condition (2), for some \( a'' \) in \( \Gamma(\pi(V''_i), \mathcal{J}) \), \( \pi^*(a'(a'')) \) is the restriction of \( a \) to \( V'_i \). According to (ii), \( a' \) is the restriction to \( \pi(V'_i) \) of a unique local section \( a'' \) over \( V''_i \). Then the restriction of \( \pi^*(a'') \) to \( V'_i \) equals \( a \) since \( a \) and \( \pi^*(a'') \) have the same restriction to \( V'_i \) and since \( \mathcal{K} \) is torsion free, whence the assertion.

(iv) Let denote by \( \psi \) the canonical morphism

\[
J \otimes_{\mathcal{O}[X]} \text{Hom}_{\mathcal{O}[X]}(J, I) \rightarrow I \quad a \mapsto \varphi(a)
\]

Let \( x \) be in \( I \) and let \( a \) be in \( J \setminus \{0\} \). Since \( I \) is an injective module, \( I \) is divisible so that \( x = bx' \) for some \( x' \) in \( J \). Denoting by \( \varphi \) the morphism \( c \mapsto cx' \) from \( J \) to \( I \), \( \varphi(bx) = x \). So, \( \psi \) is surjective.

Let denote by \( K \) the kernel of \( \psi \) and let suppose \( K \) different from 0. One expects a contradiction. Let \( \varphi \) be in \( K \). For \( i = 1, \ldots, l \), let set:

\[
J_i := k[V_i] \otimes_{\mathcal{O}[X]} J \quad I_i := k[V_i] \otimes_{\mathcal{O}[X]} I
\]

so that

\[
k[V_i] \otimes_{\mathcal{O}[X]} J \otimes_{\mathcal{O}[X]} \text{Hom}_{\mathcal{O}[X]}(J, I) = J_i \otimes_{k[V_i]} \text{Hom}_{k[V_i]}(J_i, I_i)
\]

and let denote by \( \psi_i \) the canonical morphism

\[
J_i \otimes_{k[V_i]} \text{Hom}_{k[V_i]}(J_i, I_i) \rightarrow I_i
\]

so that the restriction of \( \pi^*(\varphi) \) to \( V_i \) is in the kernel of \( \psi_i \). According to (iii), \( J_i \) is the free \( k[V_i] \)-module generated by \( p_i \) so that the morphism

\[
I_i \rightarrow J_i \otimes_{k[V_i]} \text{Hom}_{k[V_i]}(J_i, I_i) \quad x \mapsto p_i \circ \varphi_x \quad \text{with} \quad \varphi_x(a p_i) = ax
\]

is an isomorphism equals to the inverse of \( \psi_i \). Hence the restriction of \( \pi^*(\varphi) \) to \( V_i \) equals 0. As a result, \( \pi^*(\varphi) = 0 \). Hence \( \pi^*(\mathcal{O}_X \otimes_{\mathcal{O}[X]} K) = 0 \). Since \( K \) is different from 0, \( K \) contains a finitely generated submodule \( K' \), different from 0. Then, for some locally closed subvariety \( X_{K'} \) of \( X \), \( \mathcal{O}_{X_{K'}} \otimes_{\mathcal{O}[X]} K' \) is a free \( \mathcal{O}_{X_{K'}} \)-module different from 0. Denoting, by \( \pi_{K'} \) the restriction of \( \pi \) to \( \pi^{-1}(X_{K'}) \), \( \pi_{K'}^*(\mathcal{O}_{X_{K'}} \otimes_{\mathcal{O}[X]} K') \) is different from zero, whence the contradiction since it is the restriction to \( \pi_{K'}^{-1}(X_{K'}) \) of \( \pi^*(\mathcal{O}_X \otimes_{\mathcal{O}[X]} K') \).

(v) Let \( Y' \) be the smooth locus of \( Y \). According to Condition (2), \( \pi^{-1}(X') \) is a dense open subset of \( Y' \). Moreover, \( \pi^{-1}(X') \) identifies with \( X' \). Let \( \omega \) be a differential form of top degree on \( X' \). Since \( \Omega_{Y'} \) is a locally free module of rank one, there is an affine open cover \( O_1, \ldots, O_k \) on \( Y' \) such that restriction of \( \Omega_{Y'} \) to \( O_i \) is a free \( \mathcal{O}_{O_i} \)-module generated by some section \( \omega_i \). For \( i = 1, \ldots, k \), let set \( O'_i := O_i \cap X' \). Let
ω be a regular form of top degree on X'. For i = 1, . . . , k, for some regular function a_i on O_i′, a_iω_i is the restriction of ω to O_i′. According to Condition (1), O_i′ is a big open subset of O_i. Hence a_i has a regular extension to O_i since O_i is normal. Denoting again by a_i this extension, for 1 ≤ i, j ≤ k, a_iω_i and a_jω_j have the same restriction to O_i′ ∩ O_j and O_i ∩ O_j since Ω_yy′ is torsion free as a locally free module. Let ω′ be the global section of Ω_yy′ extending the a_iω_i’s. Then ω′ is a regular extension of ω to y′ and this extension is unique since X′ is dense in y′ and since Ω_yy′ is torsion free.

**Proposition C.2.** Let suppose that there exist an irreducible Gorenstein variety Y, with rational singularities, and a projective birational morphism π from Y to X verifying Conditions (1) and (2) of Lemma C.1. Then X has rational singularities.

**Proof.** Let Y′ be the smooth locus of Y. According to [Hir64], there exists a desingularization Z of Y, with morphism τ, such that the restriction of τ to τ^{-1}(Y′) is an isomorphism onto Y′. According to Lemma C.1.(v), all regular differential form of top degree on the smooth locus of X has a regular extension to Y′. Since Y has rational singularities and since Z is a desingularization of Y, all regular differential form of top degree on the smooth locus of Y has a regular extension to Z by [KK73, p.50]. Hence all regular differential form of top degree on the smooth locus of X has a regular extension to Z. Since Z is a desingularization of Y and since π is projective and birational, Z is a desingularization of X. So, by [KK73, p.50] again, it remains to prove that X is Cohen-Macaulay.

Since Z, Y, X are varieties over k, one a has the commutative diagrams

\[
\begin{array}{ccc}
Z & \xrightarrow{\tau} & Y \\
\downarrow{p} & & \downarrow{q} \\
\text{Spec}(k) & & \text{Spec}(k)
\end{array}
\quad
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow{q} & & \downarrow{r} \\
\text{Spec}(k) & & \text{Spec}(k)
\end{array}
\]

According to [Hi91, 4.3.(iv)], p^i(k), q^i(k), r^i(k) are dualizing complexes over Z, Y, X respectively. Furthermore, by [Hi91, 4.3,(ii)], p^i(k)[−dim Z] equals Ω_Z and since Y is Gorenstein, the cohomology of q^i(k)[−dim Z] is concentrated in degree 0 and equals the canonical module K of Y. Let set D := r^i(k)[−dim Z] so that p^i(D) = K and (π ∘ τ)^i(D) = Ω_Z by [Hi91, 4.3,(iv)]. Since τ and π are projective morphisms, one has the isomorphisms

\[
\begin{align*}
R(τ)_*(R\mathcal{H}om_Z(Ω_Z, Ω_Z)) & \longrightarrow R\mathcal{H}om_Y(R(τ)_*(Ω_Z), K) \\
R(π)_*(R\mathcal{H}om_Y(K, K)) & \longrightarrow R\mathcal{H}om_X(R(π)_*(K), D)
\end{align*}
\]

by [Hi91, 4.3.(iii)]. Since Ω_Z and K are locally free of rank 1,

\[
\begin{align*}
H^i(R\mathcal{H}om_Z(Ω_Z, Ω_Z)) &= \begin{cases} Ω_Z & \text{if } i = 0 \\
0 & \text{if } i > 0 \end{cases} \\
H^i(R\mathcal{H}om_Y(K, K)) &= \begin{cases} Ω_Y & \text{if } i = 0 \\
0 & \text{if } i > 0 \end{cases}
\end{align*}
\]

the left hand sides can be identified to R(τ)_*(Ω_Z) and R(π)_*(K) respectively, whence an isomorphism

\[
R(π)_*(Ω_Y) \longrightarrow R\mathcal{H}om_X(R(π)_*(K), D)
\]
Let $J$ be the space of global sections of $\mathcal{K}$. According to Grauert-Riemenschneider Theorem [GR70], denoting by $\mathcal{J}$ the localization of $J$ on $X$,

$$R(\tau)_*(\Omega_Z) = \mathcal{K}$$

whence $R(\pi)_*(\mathcal{K}) = \mathcal{J}$ and one has an isomorphism

$$R(\pi)_*(\mathcal{O}_Y) \rightarrow R\mathcal{H}om_X(\mathcal{J}, \mathcal{D})$$

According to Lemma C.1,(iv), there is an isomorphism

$$\mathcal{J} \otimes \mathcal{B} \rightarrow R\mathcal{H}om_X(\mathcal{J}, \mathcal{D}) \rightarrow \mathcal{D}$$

in the derived category $D^+(X)$ of complexes bounded below of $\mathcal{O}_X$-modules, whence an isomorphism

$$\mathcal{J} \otimes \mathcal{B} \rightarrow R(\pi)_*(\mathcal{O}_Y) \rightarrow \mathcal{D}$$

According to Lemma C.1,(iii), $\pi^*(\mathcal{J}) = \mathcal{K}$. Then, since $\mathcal{J} = R(\pi)_*(\mathcal{K})$, one has an isomorphism

$$\mathcal{J} \otimes \mathcal{B} \rightarrow R(\pi)_*(\mathcal{O}_Y) \rightarrow \mathcal{D}$$

by the projection formula [Mebk89, Appendice B]. So, since the right hand side equals $\mathcal{J}$, there is an isomorphism

$$\mathcal{J} \rightarrow \mathcal{D}$$

in $D^+(X)$. As a result, the cohomology of the dualizing complex $\mathcal{D}$ of $X$ is concentrated in degree 0. Hence $X$ is Cohen-Macaulay [El78].

References


[Gi11] V. Ginzburg, *Isospectral commuting variety, the Harish-Chandra $\mathcal{D}$-module, and principal nilpotent pairs* atXiv 1108.5367 [Math.AG].


