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Samir Adly, Alberto Seeger

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Abstract We study several variants of a nonsmooth Newton-type algorithm for solving an eigenvalue problem of the form

$$K \ni x \perp (Ax - \lambda Bx) \in K^+.$$

Such an eigenvalue problem arises in mechanics and in other areas of applied mathematics. The symbol K refers to a closed convex cone in the Euclidean space $\mathbb{R}^n$ and (A, B) is a pair of possibly asymmetric matrices of order n. Special attention is paid to the case in which K is the nonnegative orthant of $\mathbb{R}^n$. The more general case of a possibly unpointed polyhedral convex cone is also discussed in detail.

Keywords Complementarity problem · Cone-constrained eigenvalue problem · Semismooth Newton method · Polyhedral convex cone · Lorentz cone · Matrix pencil

1 Introduction

1.1 The cone-constrained eigenvalue problem

The Euclidean space $\mathbb{R}^n$ is equipped with the standard inner product $\langle y, x \rangle = y^{\mathrm{T}} x$ and the associated norm. The data

$$A, B \in \mathbb{M}_n \equiv \text{space of real matrices of order } n,$$

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$$K \in \Xi(\mathbb{R}^n) \equiv \text{ set of closed convex cones in } \mathbb{R}^n$$

is considered as general as possible, but one always assumes that

$$\langle x, Bx\rangle \neq 0 \quad \text{for all } x \in K\backslash\{0\}.$$

If one chooses B as the identity matrix, as we shall do in all our numerical experiments, then this assumption is automatically satisfied.

A fundamental problem arising in mechanics (cf. [7, 8, 25]) and in other areas of applied mathematics is that of finding $\lambda \in \mathbb{R}$ and a nonzero vector $x \in \mathbb{R}^n$ such that

$$K \ni x \perp (Ax - \lambda Bx) \in K^+ \tag{1}$$

with $\perp$ indicating orthogonality and K^+ standing for the positive dual cone of K. This is what one calls a cone-constrained eigenvalue problem. One refers to x as a K-eigenvector of (A, B) and to λ as a K-eigenvalue of (A, B). The set of all K-eigenvalues, denoted by $\sigma(A, B, K)$, is called the K-spectrum of (A, B). The mechanical or physical interpretation of a K-eigenvalue depends of course on the specific context. The orthogonality condition in (1) implies that

$$\lambda = \frac{\langle x, Ax\rangle}{\langle x, Bx\rangle},$$

as in a classical generalized eigenvalue problem, but $Ax - \lambda Bx$ may be different from zero.

The mathematical theory of K-spectra is nowadays well developed [12, 19, 26–28], but the design of efficient algorithms is still in its infancy. In [13, 15] it is assumed that both A and B are symmetric, but we are not interested in such a framework. We focus the attention to the more complicated case in which A is asymmetric. The Scaling-and-Projection Algorithm and the Power Iteration Method have been introduced and studied in [22, 23]. Global optimization and Branch-and-Bound techniques have been explored by Júdice et al. [14]. In the present paper we discuss a solution technique that is very different in spirit from those just mentioned. The basic idea consists in reformulating (1) as a system

$$\Phi(z) = 0$$

of nonsmooth equations and then applying a Newton-type algorithm. Such an approach makes sense only if the vector function $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ satisfies a minimal set of assumptions. The details are explained next.

1.2 Brief description of the semismooth Newton method

To start with, one asks $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ to be locally Lipschitz. Rademacher's theorem ensures then the existence of the Jacobian matrix $J_\Phi(z)$ for almost all $z \in \mathbb{R}^d$. Hence, at any reference point z, the Clarke generalized Jacobian

$$\partial\Phi(z) = \text{co}\big\{M \in \mathbb{M}_d : M = \lim_{k\to\infty} J_\Phi(z^k) \text{ for some } \{z^k\}_{k\in\mathbb{N}} \to z \text{ with } z^k \in D_\Phi\big\}$$

is a nonempty compact convex set. Here, D_Φ stands for the set of differentiability points of Φ and "co" refers to the convex hull operation.

The standard formulation of the **S**emismooth **N**ewton **M**ethod (SNM) reads as follows:

- *Initialization*. Choose an initial point z^0 and set $t = 0$.
- *Iteration*. One has a current point z^t. Choose $M^t \in \partial\Phi(z^t)$ and compute h^t by solving the linear system

$$M^t h^t = -\Phi(z^t). \tag{2}$$

  Set $z^{t+1} = z^t + h^t$ and increment t by one.

This algorithm has been studied in depth by many authors. Some additional assumptions on Φ are needed for ensuring that (2) admits a unique solution and that $\{z^t\}_{t\in\mathbb{N}}$ converges. The theorem below is taken from Qi and Sun [24, Sect. 3], but other results in the same vein can be found in [5, 17, 21].

Theorem 1 *Let $\bar{z}$ be a zero of the locally Lipschitz function $\Phi : \mathbb{R}^d \to \mathbb{R}^d$. Suppose that*

$$\Phi \text{ is semismooth at } \bar{z}, \quad \text{and} \tag{3}$$

$$\text{all matrices in } \partial\Phi(\bar{z}) \text{ are nonsingular.} \tag{4}$$

Then there exists a neighborhood V of $\bar{z}$ such that the SNM initialized at any $z^0 \in V$ generates a sequence $\{z^t\}_{t\in\mathbb{N}}$ that converges superlinearly to $\bar{z}$.

We opted for a formulation of Theorem 1 that is as simple as possible, but the specialized literature in the area provides also conditions guaranteeing a quadratic rate of convergence of the sequence $\{z^t\}_{t\in\mathbb{N}}$. The technical assumption (3) refers to the existence of the limit

$$\Phi^\diamond(\bar{z}; h) := \lim_{\substack{M\in\partial\Phi(\bar{z}+tw)\\ w\to h,\, t\to 0^+}} Mw \tag{5}$$

for all direction $h \in \mathbb{R}^d$. This is equivalent to saying that

$$\sup_{M\in\partial\Phi(\bar{z}+v)} \|\Phi(\bar{z}+v) - \Phi(\bar{z}) - Mv\| = o(\|v\|) \quad \text{as } v \to 0.$$

Despite the unpleasant appearance of the expression (5), there are plenty of helpful rules for checking semismoothness. While applying the SNM to cone-constrained eigenvalue problems, the main source of annoyances is the nonsingularity hypothesis (4). We shall comment on this issue in due course.

2 Eigenvalue problems under nonnegativity constraints

2.1 Theoretical background

The most interesting example of a cone-constrained eigenvalue problem is the one in which the convex cone K is the nonnegative orthant of $\mathbb{R}^n$. In such a case, (1) takes the form

$$x \geq 0, \quad Ax - \lambda Bx \geq 0, \qquad \langle x, Ax - \lambda Bx \rangle = 0, \tag{6}$$

where the notation $x \geq 0$ indicates that each component of $x \in \mathbb{R}^n$ is nonnegative. This is the so-called Pareto eigenvalue problem. One refers to x as a Pareto eigenvector of (A, B) and to λ as a Pareto eigenvalue of (A, B). The model (6) is not as sui generis as it might appear. In fact, a cone-constrained eigenvalue problem associated to a pointed polyhedral convex cone can be brought to the form (6). The details are explained in the next proposition borrowed from [26]. Recall that a convex cone is pointed if it contains no line.

Proposition 2 *Let $K = \{Gu : u \in \mathbb{R}^p_+\}$ for some real matrix G whose columns are positively linearly independent vectors of $\mathbb{R}^n$. Then*

$$\sigma(A, B, K) = \sigma\left(G^{\mathrm{T}}AG, G^{\mathrm{T}}BG, \mathbb{R}^p_+\right), \tag{7}$$

*where the superscript "*T*" indicates transposition.*

The integer p corresponds to the number of generators of K. A pointed polyhedral convex cone in $\mathbb{R}^n$ can always be represented as in Proposition 2, but sometimes the columns of G are not easily available. Needless to say, the evaluation of the right-hand side of (7) could be very expensive if p is large.

As a first step towards a reformulation of the Pareto eigenvalue problem (6) as a system of equations, we write

$$\begin{aligned} x &\geq 0 && \text{primal or eigenvector feasibility,} \\ y &\geq 0 && \text{dual feasibility,} \\ \langle x, y \rangle &= 0 && \text{complementarity slackness,} \\ Ax - \lambda Bx &= y && \text{stationarity or equilibrium law,} \\ \langle \mathbf{1}_n, x \rangle &= 1 && \text{normalization} \end{aligned} \tag{8}$$

with $\mathbf{1}_n \in \mathbb{R}^n$ standing for a vector of one's. The last condition in (8) makes sure that x is a nonzero vector. Normalizing x with the help of the linear function $\langle \mathbf{1}_n, \cdot \rangle$ is just one option among many. One could use instead the normalization condition $\|x\|^2 = 1$ and this would not change substantially the overall presentation of our work. In the sequel, we refer to $y \in \mathbb{R}^n$ as a "dual vector" (or as a vector of dual variables). We are borrowing the terminology of optimization theory, but one should not view (8) as a Karush-Kuhn-Tucker system for an optimization problem. Recall that A and B are not necessarily symmetric.

Next, we observe that the first three conditions in (8) can be written in the succinct form $U_\varphi(x, y) = 0$, where $U_\varphi : \mathbb{R}^{2n} \to \mathbb{R}^n$ is the vector function given by

$$U_\varphi(x, y) = \begin{bmatrix} \varphi(x_1, y_1) \\ \vdots \\ \varphi(x_n, y_n) \end{bmatrix}$$

and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a complementarity function, i.e.,

$$\varphi(a, b) = 0 \quad \Longleftrightarrow \quad a \geq 0, \quad b \geq 0, \quad ab = 0. \tag{9}$$

In short, we are led to solve a system

$$U_\varphi(x, y) = 0, \tag{10}$$

$$Ax - \lambda Bx - y = 0, \tag{11}$$

$$\langle \mathbf{1}_n, x \rangle - 1 = 0 \tag{12}$$

of $2n + 1$ equations involving the same number of variables. The troublesome part of the above nonlinear system is the block (10), and this is because a complementarity function is usually nonsmooth.

The next lemma explains why it is reasonable to apply the SNM to the system (10)–(12). We mention in passing that concept of semismoothness applies to any locally Lipschitz function between finite dimensional vector spaces, even if the output space has different dimension than the input space. The same remark applies to the definition of a Clarke generalized Jacobian.

Lemma 3 *Consider a vector function* $\Phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ *of the form*

$$\Phi(x, y, \lambda) = \begin{bmatrix} U_\varphi(x, y) \\ Ax - \lambda Bx - y \\ \langle \mathbf{1}_n, x \rangle - 1 \end{bmatrix} \tag{13}$$

with φ *being any of the following complementarity functions*:

$$\varphi_{\min}(a, b) = \min\{a, b\}, \tag{14}$$

$$\varphi_{\mathrm{FB}}(a, b) = a + b - \sqrt{a^2 + b^2}, \tag{15}$$

$$\varphi_{\mathrm{EP}}(a, b) = 2ab - (\min\{0, a + b\})^2. \tag{16}$$

Then Φ *is locally Lipschitz and semismooth. Furthermore, its Clarke generalized Jacobian at* $z = (x, y, \lambda)$ *admits the block representation*

$$\partial\Phi(z) = \left\{ \begin{bmatrix} E & F & 0 \\ A - \lambda B & -I_n & -Bx \\ \mathbf{1}_n^{\mathrm{T}} & 0 & 0 \end{bmatrix} : [E, F] \in \partial U_\varphi(x, y) \right\} \tag{17}$$

with I_n *standing for the identity matrix of order* n.

Proof The last $n+1$ components of Φ are continuously differentiable, so they are locally Lipschitz and semismooth. Concerning the first n components of Φ, we recall the following known facts:

- The "lattice" complementarity function (14) is a minimum of two linear functions, therefore it is globally Lipschitz and semismooth.
- The Fischer-Burmeister complementarity function (15) is globally Lipschitz. Furthermore, it is continuously differentiable everywhere except at the origin, where it is semismooth (cf. [5, Sect. 7.4]).
- The Evtushenko-Purtov complementarity function (16) is continuously differentiable because the nonsmooth term $\min\{0, a+b\}$ has been squared. That (16) satisfies the "only if" part of (9) is mentioned for the first time in [4]. The "if" part can be seen by direct inspection. Concerning the name attributed to (16), we follow references [10, 30].

This takes care of the announced behavior of Φ and the representation formula (17). □

For any of the complementarity functions listed in Lemma 3, the computation of the Clarke generalized Jacobian $\partial U_\varphi(x, y)$ offers no difficulty. For saving space, we omit writing down the explicit formulas. Many other complementarity functions could also be considered (cf. [3, 16]), but there is no need of multiplying the examples. For notational convenience, we write

$$U_{\min} = U_{\varphi_{\min}}, \qquad U_{\mathrm{FB}} = U_{\varphi_{\mathrm{FB}}}, \qquad U_{\mathrm{EP}} = U_{\varphi_{\mathrm{EP}}},$$

and define the functions $\Phi_{\min}$, Φ_{FB}, Φ_{EP} according to (13).

Of course, the SNM specialized to the system (10)–(12) takes a particular form. It reads as follows:

- *Initialization*. Choose an initial point $z^0 = (x^0, y^0, \lambda^0)$ and set $t = 0$.
- *Iteration.*[1] One has a current point $z^t = (x^t, y^t, \lambda^t)$. Choose $(E^t, F^t) \in \partial U_\varphi(x^t, y^t)$ and compute $z^{t+1} = (x^{t+1}, y^{t+1}, \lambda^{t+1})$ by solving the linear system

$$\begin{aligned} E^t x^{t+1} + F^t y^{t+1} &= E^t x^t + F^t y^t - U_\varphi(x^t, y^t), \\ (A - \lambda^t B)x^{t+1} - y^{t+1} - \lambda^{t+1} B x^t &= -\lambda^t B x^t, \\ \langle \mathbf{1}_n, x^{t+1} \rangle &= 1. \end{aligned}$$

There are no theoretical results on the best way of selecting the initial point. Guided just by common sense, we apply a rule of the sort:

$$\text{generate a random vector } \xi \in \mathbb{R}^n \text{ with uniform distribution on } [-1, 1]^n, \tag{18}$$

$$\text{and set } x^0 = \xi / \langle \mathbf{1}_n, \xi \rangle, \quad \lambda^0 = \langle x^0, Ax^0 \rangle / \langle x^0, Bx^0 \rangle, \quad y^0 = Ax^0 - \lambda^0 B x^0. \tag{19}$$

Generating an n-dimensional Gaussian vector ξ is another option, but we stick to the uniform distribution rule. A question raised by one of the referees is why we

[1] As stopping criterion we use $\|\Phi(z^t)\| < 10^{-8}$.

do not generate ξ uniformly on $[0, 1]^n$. After all, the sequence $\{x^t\}_{t\in\mathbb{N}}$ is expected to converge to a vector with nonnegative components. This observation is very pertinent, but it happens in practice that an overwhelming majority of Pareto eigenvectors lie on the boundary of the cone $\mathbb{R}^n_+$. Now, a boundary type Pareto eigenvector can equally well be approximated from the interior or from the exterior of the cone. Anyhow, we carried out some numerical experiments with the referee's choice and we did not observe a substantial difference with respect to (18).

An important question now is this: if $\bar{z} = (\bar{x}, \bar{y}, \bar{\lambda})$ is a solution to (10)–(12), are we sure that every matrix in $\partial\Phi(\bar{z})$ is nonsingular? The answer to this question is not always yes. It depends not only on the matrices A and B, but also on the solution $\bar{z}$ under consideration.

Example 4 Consider the Pareto eigenvalue problem for the pair

$$A = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the lattice complementarity function (14). Although $\Phi_{\min}$ is differentiable at both solutions

$$z^{\text{bad}} = ((1/3, 2/3), (0, 0), 1),$$
$$z^{\text{good}} = ((1, 0), (0, 4), 3),$$

the corresponding Jacobian matrices

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 & -1/3 \\ 4 & -2 & 0 & -1 & -2/3 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 4 & -4 & 0 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

are somewhat different in nature. The former is singular and the latter is nonsingular.

The situation described by Example 4 warns us from being overly optimistic, but it does not impede us from using the SNM. What happens in practice is that some solutions could remain undetected. From a theoretical point of view, it is interesting to distinguish between different types of solutions to the Pareto eigenvalue problem (6).

Definition 5 Let $\bar{x}$ be a Pareto eigenvector of (A, B) and write

$$\bar{\lambda} = \langle \bar{x}, A\bar{x}\rangle / \langle \bar{x}, B\bar{x}\rangle, \qquad \bar{y} = A\bar{x} - \bar{\lambda}\bar{x}, \qquad \bar{z} = (\bar{x}, \bar{y}, \bar{\lambda}).$$

The Pareto eigenvector $\bar{x}$ is called:

(i) Strict if $\bar{x}_i + \bar{y}_i > 0$ for all $i \in \{1, \ldots, n\}$.
(ii) Nondegenerate relative to φ if the function (13) is such that every matrix in $\partial\Phi(\bar{z})$ is nonsingular.

Strictness simply means that the variables $\bar{x}_i$ and $\bar{y}_i$ cannot vanish at the same time. Such a requirement has a clear interpretation in mechanics and, more generally, in any mathematical model involving complementarity slackness. The analytical benefit derived from strictness is differentiability:

Proposition 6 *Under the notation of Definition* 5, *the following statements are equivalent*:

(a) *The Pareto eigenvector* $\bar{x}$ *is strict.*
(b) $U_{\min}$ *is continuously differentiable on a neighborhood of* $(\bar{x}, \bar{y})$.
(c) U_{FB} *is continuously differentiable on a neighborhood of* $(\bar{x}, \bar{y})$.

Proof For a Pareto eigenvector $\bar{x}$, strictness can be expressed in the equivalent form

$$\bar{x}_i \neq \bar{y}_i \quad \text{for all } i \in \{1, \ldots, n\}.$$

The proposition is a consequence of this observation and the following two facts: firstly, $\varphi_{\min}$ is continuously differentiable in a neighborhood of $(\bar{a}, \bar{b})$ if and only if $\bar{a} \neq \bar{b}$, and, secondly, φ_{FB} is continuously differentiable everywhere except at the origin. □

Unfortunately, strictness does not prevent degeneracy. The risk of degeneracy persists even in a smooth context and simply cannot be avoided. Anyway, it is good to have the following result in mind.

Proposition 7 *Let* $\bar{x}$ *be a strict Pareto eigenvector of* (A, B). *Then the following statements are equivalent*:

(a) $\bar{x}$ *is nondegenerate with respect to* $\varphi_{\min}$.
(b) $\bar{x}$ *is nondegenerate with respect to* φ_{FB}.
(c) $\bar{x}$ *is nondegenerate with respect to* φ_{EP}.

Proof The complementarity functions $\varphi_{\min}$ and φ_{FB} are calibrated so that

$$\frac{\partial \varphi}{\partial a}(\bar{a}, 0) = 0, \qquad \frac{\partial \varphi}{\partial b}(\bar{a}, 0) = 1, \quad \text{for all } \bar{a} > 0, \tag{20}$$

$$\frac{\partial \varphi}{\partial a}(0, \bar{b}) = 1, \qquad \frac{\partial \varphi}{\partial b}(0, \bar{b}) = 0, \quad \text{for all } \bar{b} > 0. \tag{21}$$

Hence, $U_{\min}$ and U_{FB} have the same Jacobian matrix at the reference point $(\bar{x}, \bar{y})$. Consequently, $\Phi_{\min}$ and Φ_{FB} have the same Jacobian matrix at $\bar{z}$. This is a lot more than merely saying that (a) and (b) are equivalent. The partial derivatives of φ_{EP} are not exactly as in (20)–(21), but they satisfy a calibration condition that is somewhat similar. Indeed, one has

$$\frac{\partial \varphi_{\mathrm{EP}}}{\partial a}(\bar{a}, 0) = 0, \qquad \frac{\partial \varphi_{\mathrm{EP}}}{\partial b}(\bar{a}, 0) \neq 0, \quad \text{for all } \bar{a} > 0,$$

$$\frac{\partial \varphi_{\mathrm{EP}}}{\partial a}(0, \bar{b}) \neq 0, \qquad \frac{\partial \varphi_{\mathrm{EP}}}{\partial b}(0, \bar{b}) = 0, \quad \text{for all } \bar{b} > 0.$$

The Jacobian matrices $J_{\Phi_{\min}}(\bar{z})$ and $J_{\Phi_{\mathrm{EP}}}(\bar{z})$ are different in general. However, we claim that

$$\det\left[J_{\Phi_{\mathrm{EP}}}(\bar{z})\right] = \mu \det\left[J_{\Phi_{\min}}(\bar{z})\right] \tag{22}$$

for some scalar $\mu \neq 0$. To see this, suppose for instance that

$$\begin{cases} \bar{x}_j > 0 & \text{for } j = 1, \ldots, r, \\ \bar{x}_j = 0 & \text{for } j = r+1, \ldots, n \end{cases}$$

with $r \in \{1, \ldots, n\}$. Any other sign configuration of $\bar{x}$ can be treated in the same way. The strictness assumption implies that

$$\begin{cases} \bar{y}_j = 0 & \text{for } j = 1, \ldots, r, \\ \bar{y}_j > 0 & \text{for } j = r+1, \ldots, n. \end{cases}$$

Hence,

$$J_{\Phi_{\min}}(\bar{z}) = \begin{bmatrix} E & F & 0 \\ A - \bar{\lambda} B & -I_n & -B\bar{x} \\ \mathbf{1}_n^{\mathrm{T}} & 0 & 0 \end{bmatrix} \tag{23}$$

with

$$E = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \qquad F = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

The Jacobian matrix $J_{\Phi_{\mathrm{EB}}}(\bar{z})$ has the same structure as (23), except that E and F are slightly different: I_r must be changed by a diagonal matrix with nonzero entries

$$\frac{\partial \varphi_{\mathrm{EP}}}{\partial b}(\bar{x}_1, 0), \quad \ldots, \quad \frac{\partial \varphi_{\mathrm{EP}}}{\partial b}(\bar{x}_r, 0), \tag{24}$$

and I_{n-r} must be changed by a diagonal matrix with nonzero entries

$$\frac{\partial \varphi_{\mathrm{EP}}}{\partial a}(0, \bar{y}_{r+1}), \quad \ldots, \quad \frac{\partial \varphi_{\mathrm{EP}}}{\partial a}(0, \bar{y}_n). \tag{25}$$

Hence, the equality (22) holds if one defines μ as the product of all the terms in (24)–(25). This proves that (a) and (c) are equivalent. □

2.2 Numerical experiments

In all our tests[2], $B = I_n$ and A is an arbitrary matrix, not necessarily symmetric. By a Pareto eigenvalue of A one understands a scalar $\lambda \in \mathbb{R}$ such that the complementarity system

$$x \geq 0, \quad Ax - \lambda x \geq 0, \qquad \langle x, Ax - \lambda x \rangle = 0 \tag{26}$$

admits a nonzero solution $x \in \mathbb{R}^n$. Such an x is called a Pareto eigenvector of A.

[2]The numerical experiments are carried out in a Powerbook Mac OS 10.5.8 with a processor 2.33 GHz Intel Core 2 Duo and 2Go memory. The software used is Matlab 7.7.0.471 (R2008b).

It is worthwhile mentioning that the integer

$$\pi_n = \max_{A \in \mathbb{M}_n} \text{card}[\sigma(A, I_n, \mathbb{R}^n_+)]$$

is finite, but grows exponentially with n. By definition, π_n is the largest number of Pareto eigenvalues that can be found in a matrix of order n. As shown in [23], one has

$$2^n - 1 \leq \pi_n \leq n2^{n-1} - (n-1).$$

Note, for instance, that a matrix of order 40 could have more than 10^{12} Pareto eigenvalues! Finding all of them is simply not realistic, so we aim at detecting just a bunch of them.

2.2.1 Testing on a matrix of order 3 *with* 9 *Pareto eigenvalues*

Our first numerical experiment concerns a matrix of order 3 that is known to have 9 Pareto eigenvalues:

$$A = \begin{bmatrix} 8 & -1 & 4 \\ 3 & 4 & 1/2 \\ 2 & -1/2 & 6 \end{bmatrix}. \tag{27}$$

As seen in the Table 1, the SNM was able to detect all of them.[3] We considered 10^4 random initial points, but we could have used a smaller sample. What Table 1 essentially says is that some Pareto eigenvalues are more likely to be found than others. For easy of visualization, proportions presented as percentages are given without decimals. As one can see, the percentage of convergence towards each particular solution depends on the choice of complementarity function.

Table 1 SNM applied to the matrix (27) with a sample of 10^4 random initial points. The figures on the last three columns refer to percentages of convergence to each particular solution. The cases of divergence are not counted, so the percentages in each column add up to 100%

Solution	% of convergence to each solution		
	$\varphi_{\min}$	φ_{FB}	φ_{EP}
$\lambda_1 = 4.1340$	24	31	22
$\lambda_2 = 4.6021$	13	7	14
$\lambda_3 = 5.0000$	8	6	7
$\lambda_4 = 5.8660$	13	10	8
$\lambda_5 = 6.0000$	13	13	10
$\lambda_6 = 7.0000$	9	5	6
$\lambda_7 = 8.0000$	15	15	13
$\lambda_8 = 9.3979$	2	6	10
$\lambda_9 = 10.0000$	3	7	10

[3] In [22], the Power Iteration Method implemented with 10^8 random initial points detected only four Pareto eigenvalues of (27).

2.2.2 *Testing on a matrix of order* 4 *with* 23 *Pareto eigenvalues*

An example of a matrix of order 4 with 23 Pareto eigenvalues was kindly communicated to us by Victor Qi, a student at Tongji University, Shanghai. Qi's example is this:

$$A = \begin{bmatrix} 100 & 106 & -18 & -81 \\ 92 & 158 & -24 & -101 \\ 2 & 44 & 37 & -7 \\ 21 & 38 & 0 & 2 \end{bmatrix}. \tag{28}$$

We tested the SNM on (28) by using the complementarity function φ_{FB} and a sample of 10^3 random initial points. Table 2 displays what we got. The matrix (28) possesses exactly 23 Pareto eigenvalues, so the SNM found all of them. The Pareto eigenvectors are all strict in the sense of Definition 5. Some faces of the cone $\mathbb{R}^4_+$ contain several Pareto eigenvectors, but other faces are free of them.

Table 2 SNM applied to the matrix (28) with a sample of 10^3 random initial points. Pareto eigenvectors are found in various faces of the Pareto cone $\mathbb{R}^4_+$

Pareto	Pareto eigenvector				Dual vector			
eigenvalue	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4
$\lambda_1 = 26.2823$	0.4314	0.0762	0	0.4924	0	0	0.7693	0
$\lambda_2 = 26.4149$	0.4558	0.0368	0.0581	0.4493	0	0	0	0
$\lambda_3 = 28.7114$	0.4527	0	0.1913	0.3559	0	1.1100	0	0
$\lambda_4 = 29.1341$	0.2266	0.2491	0	0.5243	0	0	7.7457	0
$\lambda_5 = 32.6080$	0	0.4461	0	0.5539	2.4261	0	15.7526	0
$\lambda_6 = 32.8635$	0.4258	0	0.2844	0.2897	0	3.0863	0	0
$\lambda_7 = 37.5767$	0.2238	0	0.7762	0	0	1.9626	0	4.7001
$\lambda_8 = 41.0162$	0.1241	0.0681	0.8078	0	0	0	0	5.1944
$\lambda_9 = 46.4681$	0	0.1771	0.8229	0	3.9579	0	0	6.7290
$\lambda_{10} = 49.1435$	0.1561	0.1589	0.4874	0.1976	0	0	0	0
$\lambda_{11} = 66.9700$	0	0.3429	0.4566	0.2005	11.8334	0	0	0
$\lambda_{12} = 77.4251$	0.7814	0	0.0010	0.2176	0	49.8944	0	0
$\lambda_{13} = 77.4575$	0.7823	0	0	0.2177	0	49.9815	0.0406	0
$\lambda_{14} = 99.4233$	0.9690	0	0.0310	0	0	88.3988	0	20.3481
$\lambda_{15} = 100.0000$	1	0	0	0	0	92.0000	2.0000	21.0000
$\lambda_{16} = 107.5010$	0	0.5132	0.3019	0.1849	33.9922	0	0	0
$\lambda_{17} = 127.3920$	0	0.7674	0	0.2326	62.5095	0	32.1390	0
$\lambda_{18} = 148.5319$	0	0.7171	0.2829	0	70.9204	0	0	27.2498
$\lambda_{19} = 158.0000$	0	1	0	0	106.0000	0	44.0000	38.0000
$\lambda_{20} = 197.1730$	0.3415	0.4238	0.1155	0.1193	0	0	0	0
$\lambda_{21} = 204.5836$	0.3874	0.4820	0	0.1306	0	0	21.0694	0
$\lambda_{22} = 226.2813$	0.3935	0.4888	0.1178	0	0	0	0	26.8356
$\lambda_{23} = 231.9223$	0.4455	0.5545	0	0	0	0	25.2880	30.4261

Table 3 SNM applied to (A, I_n) with A following a uniform distribution on $[0, 1]^{n\times n}$. The percentages of convergence are estimated by using a sample of 10^4 randomly generated pairs (A, ξ). The best performances for each n are indicated in bold. The last three columns indicate average numbers of iterations in case of convergence

n	% of convergence			♯ iterations if convergence		
	$\varphi_{\min}$	φ_{FB}	φ_{EP}	$\varphi_{\min}$	φ_{FB}	φ_{EP}
10	**100**	**100**	37	7	10	24
20	99	**100**	24	8	13	19
30	99	**100**	17	9	18	16
40	98	**100**	11	10	23	14
50	97	**99**	9	10	26	13
100	94	**97**	4	11	33	9
200	**93**	**93**	2	12	38	7

2.2.3 Testing on random matrices of various dimensions

Table 3 has been produced as follows. For each $n \in \{10, 20, \ldots, 50\} \cup \{100, 200\}$, one generates a sample of 10^4 random pairs (A, ξ) with ξ as in (18) and A following a uniform distribution on the unit hypercube $[0, 1]^{n\times n}$. One applies the SNM to (A, I_n) with initial point z^0 constructed from ξ as explained in (19). One counts the number of times in which convergence to a solution occurs within 100 iterations. More than 100 iterations, or the occurrence of a badly conditioned Jacobian matrix at current point, is declared a failure.

Some of the conclusions that can be drawn from Table 3 are:

- The overall performance of the SNM diminishes as the dimension n increases. This is something to be expected from any numerical method. We saw such a deterioration phenomenon already with the Scaling-and-Projection Algorithm [23].
- The highest percentages of convergence are obtained with the complementarity function φ_{FB}. The performance of $\varphi_{\min}$ is almost as good as φ_{FB}. Despite being continuously differentiable, φ_{EB} yields very disappointing percentages of convergence. Said in a provocative way, a nonsmooth complementarity function may perfectly well beat a smooth one.
- Concerning the average number of iterations, the quality of φ_{FB} deteriorates as n increases, but the quality of $\varphi_{\min}$ remains rather stable.

Remark 8 Similar numerical tests with A following a uniform distribution on $[-1, 1]^{n\times n}$ lead essentially to the same conclusions. However, the percentages of convergence are weaker, regardless of the choice of complementarity function.

3 Partially constrained eigenvalue problems

3.1 Coping with unpointedness

In some applications only a group of components of $x \in \mathbb{R}^n$ is cone-constrained. To fix the ideas, suppose that the first m components of x are required to be nonnegative, but the remaining ones are free. For notational convenience we write x in the form

$$x = \begin{bmatrix} x_c \\ x_f \end{bmatrix},$$

where the **c**onstrained block x_c is an m-dimensional vector and the **f**ree block x_f is an $(n-m)$-dimensional vector. Consistently, we write

$$y = \begin{bmatrix} y_c \\ y_f \end{bmatrix}, \qquad A = \begin{bmatrix} A_{c,c} & A_{c,f} \\ A_{f,c} & A_{f,f} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{c,c} & B_{c,f} \\ B_{f,c} & B_{f,f} \end{bmatrix}.$$

We are dealing thus with a generalized eigenvalue problem constrained by a convex cone

$$\mathbb{K}_{m,n-m} = \mathbb{R}^m_+ \times \mathbb{R}^{n-m} \tag{29}$$

that is no longer pointed. Hence, we go beyond the framework of Sect. 2 and of reference [22]. Recall that the lineality of a convex cone K in $\mathbb{R}^n$ is defined as the integer

$$\mathrm{lin}(K) = \dim[K \cap -K].$$

The lineality of (29) is clearly $n-m$. The dual cone $\mathbb{K}^+_{m,n-m} = \mathbb{R}^m_+ \times \{0\}^{n-m}$ has empty interior and the spectral problem (1) becomes

$$\begin{aligned}
&x_c \geq 0, \qquad x_f \text{ free},\\
&y_c \geq 0, \qquad y_f = 0,\\
&\langle x_c, y_c\rangle + \langle x_f, y_f\rangle = 0,\\
&\begin{bmatrix} A_{c,c} & A_{c,f} \\ A_{f,c} & A_{f,f} \end{bmatrix}\begin{bmatrix} x_c \\ x_f \end{bmatrix} - \lambda \begin{bmatrix} B_{c,c} & B_{c,f} \\ B_{f,c} & B_{f,f} \end{bmatrix}\begin{bmatrix} x_c \\ x_f \end{bmatrix} = \begin{bmatrix} y_c \\ y_f \end{bmatrix}.
\end{aligned}$$

After adding the usual normalization condition and simplifying, one ends up with

$$U_\varphi(x_c, y_c) = 0, \tag{30}$$

$$A_{c,c}x_c + A_{c,f}x_f - \lambda(B_{c,c}x_c + B_{c,f}x_f) - y_c = 0, \tag{31}$$

$$A_{f,c}x_c + A_{f,f}x_f - \lambda(B_{f,c}x_c + B_{f,f}x_f) = 0, \tag{32}$$

$$\langle \mathbf{1}_m, x_c\rangle + \langle \mathbf{1}_{n-m}, x_f\rangle - 1 = 0. \tag{33}$$

Note that (30)–(33) is a nonsmooth system of $n+m+1$ equations involving the same number of variables.

Table 4 has been produced as follows. For each dimension $n \in \{20, 40, 60, 80\}$ one generates a sample of 10^4 random pairs (A, ξ) with ξ as in (18) and A following a

Table 4 SNM applied to (A, I_n) with A following a uniform distribution on $[0,1]^{n\times n}$. The percentages of convergence are given without decimals and are estimated by using a sample of 10^4 randomly generated pairs (A, ξ)

Dimension	Number m of constrained variables		
n	$m = n/4$	$m = n/2$	$m = 3n/4$
20	91%	95%	98%
40	93%	96%	98%
60	92%	95%	97%
80	91%	94%	96%

uniform distribution on the hypercube $[0,1]^{n\times n}$. One applies the SNM to (A, I_n) with an initial point z^0 constructed from ξ as explained in (19). One counts the number of times in which convergence to a solution occurs within 100 iterations. The same procedure is carried out for each lineality $n-m$ ranging from 1 to 5. We use φ_{FB} as complementarity function. The main conclusion of Table 4 is that, for each dimension n, the SNM performs slightly better as the number m of constrained variables increases.

3.2 The general polyhedral case

Dealing with a general polyhedral convex cone K in $\mathbb{R}^n$ is not more difficult than dealing with the particular case (29). The first thing one has to do is writing

$$K = \{Gu + Fv : u \in \mathbb{R}^p_+,\ v \in \mathbb{R}^q\}, \tag{34}$$

where G and F are real matrices whose columns are vectors in $\mathbb{R}^n$ satisfying a suitable blend of linear independence and positive linear independence:

$$Gu + Fv = 0, \quad u \in \mathbb{R}^p_+,\ v \in \mathbb{R}^q \quad \Longrightarrow \quad u = 0, \quad v = 0. \tag{35}$$

Finding the matrices G and F is often a cumbersome task, but there are a number of interesting examples for which these matrices are readily available. The result stated below extends Proposition 2.

Proposition 9 *Let K be as in (34) with G, F satisfying the hybrid linear independence condition (35). Then $\sigma(A, B, K) = \sigma(\tilde{A}, \tilde{B}, \mathbb{K}_{p,q})$, where*

$$\tilde{A} = \begin{bmatrix} G^{\mathrm{T}}AG & G^{\mathrm{T}}AF \\ F^{\mathrm{T}}AG & F^{\mathrm{T}}AF \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} G^{\mathrm{T}}BG & G^{\mathrm{T}}BF \\ F^{\mathrm{T}}BG & F^{\mathrm{T}}BF \end{bmatrix}$$

are matrices of order $p+q$.

Proof The dual cone of (34) is clearly

$$K^+ = \{y \in \mathbb{R}^n : G^{\mathrm{T}}y \geq 0,\ F^{\mathrm{T}}y = 0\}.$$

Plugging $x = Gu + Fv$ into (1) leads to the system

$$\begin{aligned} u \geq 0, \quad v \text{ free}, \\ G^{\mathrm{T}}[A(Gu+Fv) - \lambda B(Gu+Fv)] \geq 0, \\ F^{\mathrm{T}}[A(Gu+Fv) - \lambda B(Gu+Fv)] = 0, \\ \langle Gu+Fv,\ A(Gu+Fv) - \lambda B(Gu+Fv)\rangle = 0, \end{aligned}$$

which can be written in the more compact form

$$\mathbb{K}_{p,q} \ni \begin{bmatrix} u \\ v \end{bmatrix} \perp \left(\begin{bmatrix} G^{\mathrm{T}}AG & G^{\mathrm{T}}AF \\ F^{\mathrm{T}}AG & F^{\mathrm{T}}AF \end{bmatrix} - \lambda \begin{bmatrix} G^{\mathrm{T}}BG & G^{\mathrm{T}}BF \\ F^{\mathrm{T}}BG & F^{\mathrm{T}}BF \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{K}^+_{p,q}. \tag{36}$$

Given the hybrid linear independence condition (35), x is a nonzero solution to (1) if and only if (u, v) is a nonzero solution to (36). □

Example 10 By way of illustration, consider a generalized eigenvalue problem constrained by the downward monotonic cone $K_{\text{down}} = \{x \in \mathbb{R}^n : x_1 \geq \cdots \geq x_n\}$. For this particular case, one can take

$$G = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{bmatrix}.$$

Note that G is formed by $p = n - 1$ columns and F is formed by $q = 1$ column. This is consistent with the fact that $K_{\text{down}} \cap -K_{\text{down}} = \mathbb{R}\mathbf{1}_n$ is a one-dimensional linear subspace. The case of an upward monotonic cone can be handled in a similar way.

4 Extensions

4.1 Beyond a polyhedral setting

The Lorentz eigenvalue problem is the prototype example of a cone-constrained eigenvalue problem involving a nonpolyhedral convex cone. It can be formulated in the succinct form

$$0 \preceq x \perp (Ax - \lambda Bx) \succeq 0, \tag{37}$$

where $\succeq$ stands for the Lorentz ordering relation on $\mathbb{R}^n$. By definition, $u \succeq v$ means that $u - v$ belongs to the Lorentz cone

$$\mathbb{L}_n = \left\{x \in \mathbb{R}^n : [x_1^2 + \cdots + x_{n-1}^2]^{1/2} \leq x_n\right\}.$$

Some authors refer to $\mathbb{L}_n$ as the n-dimensional ice-cream cone [20, 29] or as the second order cone [2]. A nonzero vector x satisfying (37) is called a Lorentz eigenvector of the pair (A, B). The reader should be aware that the Lorentz spectrum $\sigma(A, B, \mathbb{L}_n)$ may contain infinitely many elements, even if one assumes that

$$\langle x, Bx\rangle \neq 0 \quad \text{for all } x \in \mathbb{L}_n \backslash \{0\}.$$

The lack of polyhedrality in $\mathbb{L}_n$ is at the origin of many complications. Anyhow, the SNM can be applied also to (37). Indeed, the Lorentzian eigenvalue problem can be reformulated as a nonsmooth system of equations, namely

$$\mathcal{U}(x, y) = 0, \tag{38}$$

$$Ax - \lambda Bx - y = 0, \tag{39}$$

$$x_n - 1 = 0. \tag{40}$$

The equilibrium law (39) is as in Sect. 2.1. The normalization condition (40) guarantees that x is a nonzero vector. This is the simplest and most natural way of normalizing a Lorentz eigenvector. The function $\mathcal{U} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ appearing in (38) is a vector complementarity function for the Lorentz ordering. This simply means that

$$\mathcal{U}(x, y) = 0 \quad \Longleftrightarrow \quad x \succeq 0,\ y \succeq 0, \quad x \perp y.$$

For making sure that the SNM works properly, one asks $\mathcal{U}$ to be locally Lipschitz and semismooth. A vector complementarity function $\mathcal{U}$ with such properties does exist, the two most popular examples being

$$\mathcal{U}_{\min}(x, y) = x - \Pi_{\mathbb{L}_n}(x - y), \tag{41}$$

$$\mathcal{U}_{\mathrm{FB}}(x, y) = x + y - \sqrt{x \circ x + y \circ y}\,. \tag{42}$$

The symbol $\Pi_{\mathbb{L}_n}$ in (41) stands for the metric projection onto $\mathbb{L}_n$. The product $\circ$ and the square root in (42) are to be understood in the Jordan algebra framework associated to $\mathbb{L}_n$. The details concerning the definition and properties of $\mathcal{U}_{\min}$ and $\mathcal{U}_{\mathrm{FB}}$ can be found in [9, 11, 18].

4.2 Beyond affine matrix pencils

The expression $A - \lambda B$ is a particular instance of a matrix pencil. So, the cone-constrained eigenvalue problem (1) is a particular case of the model

$$K \ni x \perp M(\lambda)x \in K^+ \tag{43}$$

with M standing for the pencil associated to a finite collection $\{A_0, A_1, \ldots, A_r\}$ of real matrices of order n, that is,

$$M(\lambda) = \sum_{k=0}^{r} \lambda^k A_k. \tag{44}$$

The way we handle an eigenvalue problem like (43) is as follows. To fix the ideas, consider for instance the Paretian version

$$x \geq 0, \quad M(\lambda)x \geq 0, \qquad \langle x, M(\lambda)x\rangle = 0. \tag{45}$$

It should be clear that solving (45) is a matter of finding a zero of

$$\Phi(x, y, \lambda) = \begin{bmatrix} U_\varphi(x, y) \\ M(\lambda)x - y \\ \langle \mathbf{1}_n, x\rangle - 1 \end{bmatrix} \tag{46}$$

with φ being any of the complementarity functions listed in Lemma 3. The function (46) is locally Lipschitz and semismooth. Furthermore, its Clarke generalized Jacobian at $z = (x, y, \lambda)$ is given by

$$\partial\Phi(z) = \left\{ \begin{bmatrix} E & F & 0 \\ M(\lambda) & -I_n & M'(\lambda)x \\ \mathbf{1}_n^{\mathrm{T}} & 0 & 0 \end{bmatrix} : [E, F] \in \partial U_\varphi(x, y) \right\}$$

with $M'(\lambda) = \sum_{k=1}^{r} k\lambda^{k-1} A_k$.

In principle, everything is more or less the same as in Sect. 2.1. Of course, the expression "dual vector" refers now to $y = M(\lambda)x$. There is however an important theoretical difference between an affine pencil like $A - \lambda B$ and a polynomial pencil like (44). In the latter case, the existence of Pareto eigenvalues is not automatically guaranteed. We end this work with an existential result for the case of a quadratic pencil

$$M(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2. \tag{47}$$

Cone-constrained eigenvalue problems involving quadratic pencils arise, for instance, in the analysis of mechanical systems with frictional contacts (cf. [7]). An application to electronic circuits with ideal diodes will be discussed in a forthcoming paper of ours.

Theorem 11 *Let M be the quadratic pencil associated to a triplet (A_0, A_1, A_2) of real matrices of order n such that*

$$\langle x, A_2 x\rangle \neq 0 \quad \textit{for all } x \in \mathbb{R}^n_+ \setminus \{0\},$$
$$\langle x, A_1 x\rangle^2 - 4\langle x, A_2 x\rangle\langle x, A_0 x\rangle \geq 0 \quad \textit{for all } x \in \mathbb{R}^n_+.$$

Then there exist a scalar $\bar{\lambda} \in \mathbb{R}$ and a nonzero vector $\bar{x} \in \mathbb{R}^n$ solving (45).

Proof We rely on a theorem by Ky Fan on the existence of solutions to variational inequalities (cf. [1, Theorem 3.1.1] or the original source [6]). Let $\Omega = \{x \in \mathbb{R}^n_+ : \langle \mathbf{1}_n, x\rangle = 1\}$. Note that Ω is a convex compact set not containing the origin. Both roots

$$\lambda_\pm(x) = \frac{-\langle x, A_1 x\rangle \pm [\langle x, A_1 x\rangle^2 - 4\langle x, A_2 x\rangle\langle x, A_0 x\rangle]^{1/2}}{2\langle x, A_2 x\rangle}$$

of the quadratic equation $\langle x, M(\lambda)x\rangle = 0$ are real and well defined for every $x \in \Omega$. Consider for instance the function $\Psi : \Omega \times \Omega \to \mathbb{R}$ given by

$$\Psi(x, w) = \langle M(\lambda_+(x))x, w\rangle .$$

One readily sees that

$$\begin{cases} \text{for all } w \in \Omega, & \Psi(\,\cdot\,, w) \text{ is continuous,} \\ \text{for all } x \in \Omega, & \Psi(x, \,\cdot\,) \text{ is linear,} \\ \text{for all } x \in \Omega, & \Psi(x, x) = 0. \end{cases}$$

Ky Fan's theorem ensures the existence of $\bar{x} \in \Omega$ such that $\Psi(\bar{x}, w) \geq 0$ for all $w \in \Omega$. If one sets $\bar{\lambda} = \lambda_+(\bar{x})$, then it is clear that $(\bar{\lambda}, \bar{x})$ solves (45) with $M(\lambda)$ as in (47). □

Example 12 The quadratic pencil

$$M(\lambda) = \lambda^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 10 \end{bmatrix} + \lambda \begin{bmatrix} 7 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 20 \end{bmatrix} + \begin{bmatrix} -2 & 6 & 0 \\ 2 & 16 & 3 \\ 0 & 5 & 0 \end{bmatrix} \tag{48}$$

Table 5 SNM applied to the quadratic pencil (48) with a sample of 10^4 random initial points. Pareto eigenvectors were found in various faces of $\mathbb{R}^3_+$

Pareto eigenvalue	Pareto eigenvector			Dual vector		
	x_1	x_2	x_3	y_1	y_2	y_3
$\lambda_1 = -4.3930$	0	1	0	6.0000	0	5.0000
$\lambda_2 = -3.7656$	1	0	0	0	2.0000	0
$\lambda_3 = -3.6524$	0.8712	0.1288	0	0	0	0.6438
$\lambda_4 = -2.0000$	0	0	1	0	3.0000	0
$\lambda_5 = -1.9613$	0	0.1318	0.8682	0.7909	0	0
$\lambda_6 = -1.9580$	0.0954	0.1277	0.7769	0	0	0
$\lambda_7 = -0.7689$	0.3877	0.4006	0.2116	0	0	0
$\lambda_8 = -0.6986$	0.5036	0.4964	0	0	0	2.4820
$\lambda_9 = -0.6820$	0	0.6426	0.3574	3.8554	0	0
$\lambda_{10} = -0.6070$	0	1	0	6.0000	0	5.0000
$\lambda_{11} = 0.0000$	0	0	1	0	3.0000	0
$\lambda_{12} = 0.2656$	1	0	0	0	2.0000	0

satisfies the hypotheses of Theorem 11. In fact, (48) satisfies even stronger assumptions, namely, A_2 is positive definite and

$$\langle x, A_1 x\rangle^2 - 4\langle x, A_2 x\rangle\langle x, A_0 x\rangle > 0 \quad \text{for all } x \in \mathbb{R}^3\backslash\{0\}.$$

The existence of solutions to (45) is then guaranteed. Table 5 displays the solutions we got by implementing the SNM with the complementarity function φ_{FB} and a sample of 10^4 random initial points. We used the initialization rule

$$x^0 = \xi/\langle \mathbf{1}_n, \xi\rangle, \lambda^0 = \lambda_+(x^0), \qquad y^0 = M(\lambda^0)x^0$$

with ξ being a random vector as in (18).

Note that λ_6 and λ_7 are eigenvalues of (48) in the classical sense. Indeed, the corresponding eigenvectors are in the interior of $\mathbb{R}^3_+$ and therefore the corresponding dual vectors are equal to zero. All the other Pareto eigenvalues are not classical, but created by the cone constraint imposed on the pencil.

5 By way of conclusion

The SNM is well suited for solving the cone-constrained eigenvalue problem (1), regardless of whether the matrices A and B are symmetric or not. The SNM is also successful when it comes to solve the more general problem (43) involving a polynomial matrix pencil.

More important than the structure of A and B, what really makes a practical difference is the complexity of the cone K. If this cone is polyhedral, then (1) can be

converted into a system of equations like in (30)–(33), and the SNM becomes directly applicable. As reasonable to expect, the nonpolyhedral case is more involved. One has to worry about finding a suitable vector complementarity function for the cone. Such function must be semismooth and, at the same time, it must have an easily computable Clarke generalized Jacobian.

It is hard to compare the SNM with other methods for solving (1). For instance, the numerical results reported in [14] are different in spirit from ours and focus on different issues. What is safe to say, at least, is this:

– While working with some particular examples, like the one of Sect. 2.2.1, the SNM was able to find the solutions that were left undetected by the Scaling-and-Projection Algorithm [23] and by the Power Iteration Method [22, 23].

This does not mean that the SMN is systematically better than the methods suggested in [22, 23]. As said before, implementing the SNM could be a headache if the cone K is not polyhedral. By contrast, implementing the Scaling-and-Projection Algorithm offers no difficulty if one has to deal with a revolution cone of arbitrary half-aperture angle [22].

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