Geometric, Algebraic and Topological Methods for Quantum Field Theory. Geometry of closed strings, A and B side of Witten. Part II: B side and mirror symmetry

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1 Introduction

The ***mirror symmetry*** postulate that there are two viewpoints to represent the physics of fermionic strings. The relevance comes from the fact that some problems not solvable from a certain point of view are on the other. Mirror symmetry is very accomplished in the form of the ***T-duality*** appeared in bosonic theory which states that the partition function remains unchanged in the change $R \leftrightarrow \dfrac{1}{R}$, where R denotes the radius of compactification of extra dimension. In a bosonic field theory, where the source space is the cylinder and target a torus, one can notice that the *T-duality* exchange symplectic structure deformation (area) and deformation of complex structure. In this "survey", we will introduce tools from physics and mathematics for understanding some aspects of mirror symmetry. Especially we construct the ***mirror map*** locally as historically Morrison has presented this theory for closed strings. In addition, we briefly explain the implications in enumerative geometry.

2 Landau Ginzburg model and complex manifolds

At the A-side of the mirror , The correlation functions were calculated from the instantons : Holomorphic curves in symplectic tools, in this context, they will be calculated using the tools of complex geometry, the key point

is the statistical physics and the Russian school of Arnold and his theory of singularity.

2.1 φ^4 theory and complex geometry

The model of **Ginzburg Landau** plays a decisive role on this side of the mirror, where we will look at the admissible deformations (distortions marginal) preserving certain symmetries. Starting from the Lagrangian in φ^4 :

$\mathcal{L}_{LG} = \partial_\mu \phi \partial_\mu \phi - V(T, \phi)$ where $V(T, \phi) = \frac{1}{4!}\lambda(t)\phi^4 + \frac{1}{2!}\mu^2(t)\phi^2$ At the critical temperature T_c "mass", $\mu^2(T_c) = 0$, so the correlation length (inverse mass) is infinite. At this temperature the field ϕ_0, solution of $\frac{\partial}{\partial \phi}V(T, \phi)$ is zero three times degenerated. A small perturbation $V(T_c, \phi) \to V(T_c, \phi) + \delta \mu^2(T)\phi^2$, **solves** the singularity and the is **"symmetry breaking"**. The challenge is to find ways of **a marginally perturb**potential theory in order to preserve the symmetry and defined by the fact a critical family of superpotentials.

2.2 Theory of singularities, marginal deformations

The superpotential is an holomorphic function $\mathcal{W} : \mathbb{C}^M \to \mathbb{C}$ is chosen as a potential

$V(x) = \sum_1^N |\partial_i \mathcal{W}(X)|^2 = \sum_1^N \partial_i \mathcal{W}(X)\partial_i \mathcal{W}(X)^*$

if we consider only one field, one can consider the function $\mathcal{W}(X) = \frac{1}{(n+1)!}X^{n+1}$
The bosonic part of the supersymmetric Lagrangian is written then :

$\mathcal{L}_{N=2}^{LG} = -\partial_+ X^* \partial_- X + \partial_- X^* \partial_+ X + V(X)$

There is : $V(X) = 0 \Leftrightarrow \partial_i \mathcal{W}(X_0) = 0$, so it is relevant to define the textbf*ring* Chiral $\mathcal{R}_\mathcal{W} = \frac{\mathbb{C}[X]}{\partial \mathcal{W}(X)}$ where the ratio is proportional to the polynomials of $\partial_i \mathcal{W}(X)$: $P(X) = P^i(X)\partial_i \mathcal{W}(X)$ deformations respecting the Chiral ring are given by : $\mathcal{W}_{def}(X) = \mathcal{W}(X) + \sum_{P \in \mathcal{R}_\mathcal{W}} t_P P(X)$

If we choose $\mathcal{W}(X, Y, Z) = \frac{1}{3}(X^3 + Y^3 + Z^3)$, the deformed potential is given by $\mathcal{W}_{def}(X, Y, Z) = \mathcal{W}(X, Y, Z) + t_0 + t_1 X + t_2 Y + t_3 Z + t_4 XY + t_5 YZ + t_6 ZX + t_7 XYZ$

Only the non vanishing term $\mu = t_7$ preserves the critical situation, it does not break the $\mathbb{Z}_3$ symmetry

$(X, Y, Z) \to (exp(\frac{2ki\pi}{3})X, exp(\frac{2ki\pi}{3})Y, exp(\frac{2ki\pi}{3})Z)$

we just define **continuous** family of **allowed** perturbations :

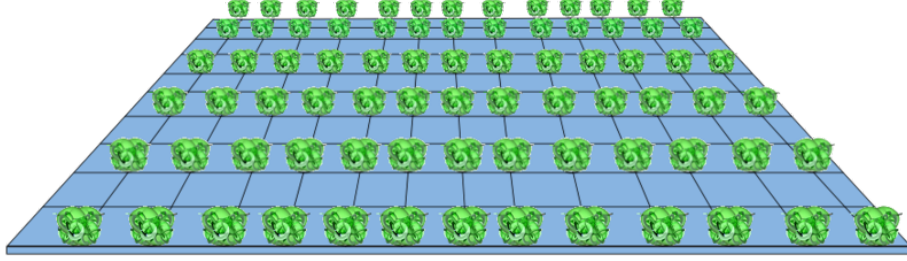$\mathcal{W}_{def}(X, Y, Z, \mu) = \frac{1}{3}(X^3 + Y^3 + Z^3) + \mu XYZ$

3 Calabi-Yau deformation theory

The hypersurface of a complex projective space obtained by canceling $\mathcal{W}_{def}(X, Y, Z, \mu) = \frac{1}{3}(X^3 + Y^3 + Z^3) + \mu XYZ$ is the The simplest example of Calabi-Yau. Is an elliptic curve or complexe torus . **Calabi-Yau variety** is a Kählérienne Ricci flat which is to say the canonical bundle is trivial. There among other $K3$ surfaces involved in branes theory and the quintic threefold for closed strings. The easiest way to realize the Calabi-Yau is to consider an **hypersurface of a complex projective space**. There is a strong constraint between the degree of a hypersurface and the dimension of the ambient space. Subsection Example of a Calabi-Yau Let us write the exact sequence associated to a hypersurface of degree d :

$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-d) \to \mathcal{O}_{\mathbb{P}^{n+1}} \to \mathcal{O}_X \to 0$

With exact long sequence in cohomology, one can calculate the cohomology groups associated.

The Result : $H^n(X, \mathcal{O}_X) = \mathbb{C}^{C_{n-1}^{d-1}} = \mathbb{C}$ (**condition** Calabi-Yau) Necessarily $d = $ n $+2$

- $n = 1$ (d = 3) : **Elliptic Curves**
- $n = 2$ (d = 4) : **K3 Surfaces**
- $n = 3$ (d = 5) : **Quintic threefold**

3.1 Deformations, mirror symmetry

We will focus on two types of **deformations of Calabi-Yau** : the deformation of symplectic structure and those from its complex structure. We can consider the deformation of complex structure (J deformation). She is captured by $H^1(X, TX) \simeq H^{2,1}(X)$, It can also vary the Kähler structure, referenced by $H^{1,1}(X) = H^1(X, \Omega_X^1)$ There are two field theories

(**CFT**) supersymmetric duality in satisfactory : $h^{2,1}(X) = h^{1,1}(MX)$ and $h^{1,1}(X) = h^{2,1}(MX)$: deform the complex structure of M amounts to deform the volume of his mirror.

diamond Hodge plots the Hodge numbers of a complex manifold
Diamond a Calabi-Yau (Quintic Threefold)

$$
\begin{array}{ccccccc}
 & & & h^{0,0}=1 & & & \\
 & & h^{1,0}=0 & & h^{0,1}=0 & & \\
 & h^{2,0}=0 & & \underline{h^{1,1}=1} & & h^{0,2}=0 & \\
h^{3,0}=1 & & \underline{h^{2,1}=101} & & h^{1,2}=101 & & h^{0,3}=1 \\
 & h^{2,0}=0 & & h^{1,1}=1 & & h^{0,2}=0 & \\
 & & h^{1,0}=0 & & h^{0,1}=0 & & \\
 & & & h^{0,0}=1 & & &
\end{array}
$$

3.2   "B side" Origin of Physics

On the A Side, supersymmetric constraints lead to what the action does ***depends only on the the Kähler form*** ; instantons are holomorphic curves. The calculation of correlation functions is difficult because it takes into account correction on the degree curves (***invariants of Gromov-Witten***) On the side "B", $BRST$ Formalism explained in the other side of the mirror applies here : instantons are ***constants maps*** from the worldsheet Σ on a point of target space X. The correlation functions are simpler to calculate : they require no ***instanton correction***.
If X is a Calabi-Yau 3, the 3 points correlation function is :

$$\langle W_A W_B W_C \rangle = \int_X \Omega^{jkl} A_j \wedge B_k \wedge C_l \wedge \Omega$$

A, B, C belong to $H^1(X, TX)$ and ***depend on the complex structure***, Ω is (3.0) top-form ***holomorphic***. The two numbers $\underline{h^{1,1}(X)=1}$ and $\underline{h^{2,1}(X)=101}$, count the number of deformation structures respectively Kähler and complex. The principle of mirror symmetry, gives $\underline{h^{1,1}(MX)=101}$ and $\underline{h^{2,1}(MX)=1}$. He said in addition that ***correlation functions*** calculated from both sides of the mirror are ***identical*** . The Mirror map associated parameter of deformation of Kälher structure with parameter of deformation of complex structure. If a problem is difficult at the A side, we can try to solve it at the B side. ***In mathematics*** passing through the mirror application can solves ***so important*** old problem of ***geometry Enumerative***

4 The Quintic and its mirror

Recall that the **homogeneous quintic** in $\mathbb{P}^4$, is obtained by canceling the superpotential : $\mathcal{W} = \frac{1}{5}(X_0^5 + ... + X_4^5)$

A **marginal deformation** of this superpotential is **almost** the variety expected the **variety** mirror quintic is associated with the emph**crepant** resolution of :

$$\{(X_0, ..., X_4) \in \mathbb{P}^4 / \tfrac{1}{5}(X_0^5 + ... + X_4^5) - \mu X_0...X_4 = 0\}_{/G}$$

with $G = \{(a_0, ..., a_4) \in \mathbb{Z}_{/5} / \sum a_i = 0\}_{/\mathbb{Z}_{/5} = \{(a,a,a,a,a)\}} \simeq (\mathbb{Z}_{/5})^3$

4.1 Program

> **localizaion** The construction of the **mirror map** is difficult globally . Should be localize and built the map at the **neighborhood of a point**.

> **Plan of the study**
> X denote the Calabi-Yau, MX its mirror .
> – We must first calculate the **Yukawa coupling** $H^1(MX, TMX)$
> – Identify by the mirror map.
> – Deduce the Yukawa couplings of $H^{1,1}(X)$

One can deduce predictions about the number of rational curves in X We will briefly describe in the following emph textbf useful mathematics for the B-side the Quintic.

4.2 Mirror-map

The principle of mirror symmetry says that $< H, H, H >=< \theta, \theta, \theta >$, if tH denotes an **curve parameter** in the module of Kähler X we set $H = \frac{d}{dt}$ $= 2\pi i q \frac{d}{dq}$ its tangent vector $(q = exp(2\pi i t)$ Local coordinates for this module.

The problem is to produce an image $q(x)$ in emph **module** complex deformations was : $q = q(x)$, $\frac{d}{dq} \to \frac{dx}{dq} \frac{d}{dx}$

Then we can write $H = 2\pi i q \frac{d}{dq}) \leftrightarrow \theta = 2\pi i q \frac{dx}{dq} \frac{d}{dx}$

$< H, H, H >= (2\pi i q \frac{d}{dq})^3 < \frac{d}{dx}, \frac{d}{dx}, \frac{d}{dx} >= (2\pi i \frac{q}{x} \frac{dx}{dq})^3 < x \frac{d}{dx}, x \frac{d}{dx}, x \frac{d}{dx} >$

4.3 Mathematical Tools

The simplest examples of Calabi-Yau are elliptic curves. They will guide us to understand the techniques identified below and set the mirror application. **Monodromy** : For a flat bundle, go around a singular object centred in $t = 0$ at constant distance $|t|$, (t :deformation parameter of a smooth family) is namely ***monodromy*** in mathematics. In physics we talk about ***loop Wilson***. ***Residue map*** : We can generalize the formula for residues of a complex variables function around $z = 0$ by replacing function ***differential forms*** and point by ***hypersurface***. This will be very useful for calculating periods.

4.4 Elliptic curves

The ***elliptic curve*** $E_\tau = \mathbb{C}/(1, \tau)$ is a Calabi-Yau ***dimension*** 1. The volume form is given by $\Omega = dz$
Emph textbf Numbers Hodge : Hodge diamond is :

$$
\begin{array}{ccccc}
 & h_{00} & & 1 & \\
h_{10} & & h_{01} & = \quad 1 & 1 \\
 & h_{11} & & 1 &
\end{array}
$$

The one parameter family of deformations of an elliptic curve is :

$$X^3 + Y^3 + Z^3 - 3\psi XYZ = 0$$

If α and β are ***homology cycles*** , they depend then ψ, we can find τ to from ***ratio periods*** : $\int_\alpha \Omega$, $\int_\beta \Omega$.
Solving a ***differential equation*** called ***Picard-Fuchs***

4.5 Family of elliptic curves, monodromy

An important concept is the notion of local system we consider one parameter of family of curve which degenerate at $t = 0$:

Degeneration of a family of curves

$$
\begin{array}{ccccc}
\mathcal{X} & \supset & X_t & t \neq t' & X_t \simeq X_t' \\
\pi \downarrow & & \downarrow & & \\
D^2 & \ni & t & t = 0 & X_0 : sing.
\end{array}
$$

Monodromy theorem :

Either X_t where t varies along a loop in $\pi_1(D^2 - \{0\}, t_0)$ around 0, (emphloop Wilson), all elements of this family are ***diffeomorphic*** .This induces an automorphism on homology : $\varphi_\star \in \mathcal{A}ut(H_n(X_{t_0}), \mathbb{Z}))$ Pratical Example Let's illustrate by taking a one parameter family of ***elliptic curve*** $C_t = \{(Y^2 Z = X^3 + X^2 Z - tZ^3\} \subset \mathbb{CP}^2$ which is expressed in affine coordinates : $C_t : y^2 = x^3 + x^2 - t$: elliptic curve defined by an algebraic equation.

The parameter t of the ***elliptic curve*** is the signature of the variation of complex structure, the geometric expression is $E_\tau = \mathbb{C}/(1, \tau(t))$ when t revolves around the origin $\tau(t) \to \tau(t) + 1$ with τ function of : $t : \tau(t) = \frac{lnt}{2\pi i}$
At the level of group automorphism

$$\begin{pmatrix} \tau(t) \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau(t) \\ 1 \end{pmatrix} = \begin{pmatrix} \tau(t) + 1 \\ 1 \end{pmatrix}$$

We find that the complex structure varies but the symplectic structure is unchanged.

We justify the choice of new coordinate $q(t) = e^{2\pi i \tau(t)}$, because a passage to the limit gives : when $t \to 0$, $Im\tau(t) \to +\infty$, d'où $q(t)$, where $q(t)$ is an holomorphic function of t which also tends to 0 ; so it is a local coordinate for this family of elliptic curves. We spoke in physics : ***large complex structure limit(WSCC)*** in "bijection" with ***large volume limit (LVL)*** on the A side

4.6 Application to the quintic

Is the ***quintic*** $\sum_{i=0}^{4} x_i^5 - \psi x_0 x_1 x_2 x_3 x_4 = 0$. As with elliptic curves, Morrison showed by standing near $x = (\frac{1}{\psi})^{-5} = 0$ we could find t (side strain Kähler) from of ψ (or x).

As with elliptic curves, $t = \dfrac{\int_{\gamma_1} \Omega}{\int_{\gamma_0} \Omega}$, we must calculate the periods $\phi_i(x) = \int_{\gamma_i} \Omega$, $i = 1, 2$. γ_0 is invariant under monodromy around $x = 0$ (***large complex structure limit (WSCC)***), $\psi \to \infty$ $\gamma_1 \to \gamma_0 + \gamma_1$.

Both quantities depend on ψ (or x)
We can compute locally the three forms (*) Ω using a version "differential form" of ***theorem of residue*** (and the theorem of emphimplicit function) and deduce by ***direct calculation*** , the first period.

The equation called emph **Picard Fuchs** calculates the other period. Finally, we find :

$\phi_0(x) = \sum_{n=0}^{\infty} \frac{5n!}{(n!)^5} x^n$,

$\phi_1(x) = \phi_0(x) log(x) + f(x)$, with $f(x) = 5 \sum_{n=0}^{\infty} \frac{5n!}{(n!)^5} (\sum_{j=n+1}^{5n} \frac{1}{j}) x^n$

4.7 Calculation of Yukawa couplings on the B-side

Let $\Theta^{(i)} = (x\frac{d}{dx})^{(i)}$, the equation of Picard-Fuch written :

$$\Theta^{(4)}y + \frac{2.5^5 x}{1+5^5 x}\Theta^{(3)}y + \frac{7.5^4 x}{1+5^5 x}\Theta^{(2)}y + \frac{2.5^4 x}{1+5^5 x}\Theta^{(1)}y + \frac{2.5^5 x}{1+5^5 x}\Theta y = 0$$

It is applied to Ω :
If $Y = \int_{MX} \Omega \wedge \Omega^{(3)}$, $\int_{MX} \Omega \wedge \Omega = \int_{MX} \Omega \wedge \Omega' = \int_{MX} \Omega \wedge \Omega" = 0$, twice differentiating the last equality we have :

$\int_{MX} \Omega \wedge \Omega^{(4)} + 2 \int_{MX} \Omega' \wedge \Omega^{(3)} = 0$

We deduce the differential equation $(x\frac{d}{dx})Y = \frac{-5^5 x}{1+5^5 x}Y$

The solution is $Y = < x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx} >= \frac{c_2}{1+5^5 x}$

We must normalize Y in agreement with (*) and then divide Ω by $\Phi_0(x)$

Thus :

$< x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx} >= \frac{c_2}{(1+5^5 x)\phi_0(x)^2}$ For identification,

$< H, H, H >= (2\pi i\frac{q}{x}\frac{dx}{dq})^3 < x\frac{d}{dx}, x\frac{d}{dx}, x\frac{d}{dx} >= \frac{c_2(2\pi i\frac{q}{x}\frac{dx}{dq})^3}{(1+5^5 x)\phi_0(x)^2}$

5 Conclusions : Application to enumerative geometry

The parameter t Kähler deformation, expressed as a function of the ratio of the first two periods, we get : $q = e^{2i\pi \frac{\phi_1(x)}{\phi_0(x)}}$ Where :
$q = c_1(x - 770x^2 + ...)$ and conversely $x = \frac{q}{c_1} + 770(\frac{q}{c_1})^2 + ...$
We can now calculate everything according to the variable q :

$< H, H, H >= (2\pi i)^3(-c_2 - 575(\frac{c_2}{c_1})q - 19575(\frac{c_2}{c_1^2})q^2 + ...)$

It remains to calculate the constants c_1 and c_2.
We remember that it was not enumerate rational curves of degree n_d is known for first degrees, which calculates c_1 and c_2.

$< H, H, H >= 5 + \sum_{d=1}^{\infty} n_d d^3 \frac{q^d}{1-q^d} = 5 + 2875q + ...$

We deduce $c_2 = \frac{-5}{(2\pi i)^3}$, $c_1 = -1$ Finally, we can enumerate the **_number curve of a rational quintic_** of $\mathbb{P}^4$ $\underline{all}$ degrees

$$< H, H, H >= 5 + \sum_{d=1}^{\infty} n_d \frac{d^3 q^d}{1-q^d} = \mathbf{5} + \mathbf{2875} \frac{q}{1-q} + \mathbf{609250}.2^3 \frac{q^2}{1-q^2} + ...$$

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