Bayesian semi-parametric estimation of the long-memory parameter under FEXP-priors

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Abstract

For a Gaussian time series with long-memory behavior, we use the FEXP-model for semi-parametric estimation of the long-memory parameter d. The true spectral density f_o is assumed to have long-memory parameter d_o and a FEXP-expansion of Sobolev-regularity $\beta > 1$. We prove that when k follows a Poisson or geometric prior, or a sieve prior increasing at rate $n^{\frac{1}{1+2\beta}}$, d converges to d_o at a suboptimal rate. When the sieve prior increases at rate $n^{\frac{1}{2\beta}}$ however, the minimax rate is almost obtained. Our results can be seen as a Bayesian equivalent of the result which Moulines and Soulier obtained for some frequentist estimators.

1 Introduction

Let X_t, $t \in \mathbb{Z}$, be a stationary Gaussian time series with zero mean and spectral density $f_o(x)$, $x \in [-\pi, \pi]$, which takes the form

$$|1 - e^{ix}|^{-2d_o} M_o(x), \qquad x \in [-\pi, \pi], \tag{1.1}$$

where $d_o \in (-\frac{1}{2}, \frac{1}{2})$ is called the long-memory parameter, and M is a slowly-varying bounded function that describes the short-memory behavior of the series. If d_o is positive, this makes the autocorrelation function $\rho(h)$ decay polynomially, at rate $h^{-(1-2d_o)}$, and the time series is said to have long-memory. When $d_o = 0$, X_t has short memory, and the case $d_o < 0$ is referred to as intermediate memory. Long memory time series models are used in a wide range of applications, such as hydrological or financial time series; see for example Beran (1994) or Robinson (1994). In parametric approaches, a finite dimensional model is used for the short memory part M_o; the most well known example is the ARFIMA(p,d,q) model. The asymptotic properties of maximum likelihood estimators (Dahlhaus (1989) or Lieberman et al. (2003)) and Bayesian estimators (Philippe and Rousseau (2002)) have been established in such models and these estimators are consistent and asymptotically normal with a convergence rate of order $\sqrt{n}$. However when the model for the short memory part is misspecified, the estimator for d can be inconsistent, calling for semi-parametric methods for the estimation of d. A key feature of semi-parametric estimators of the long-memory parameter is that they converge at a rate which depends on the smoothness of the short-memory part, and apart from the case where M_o is infinitely smooth, the convergence rate is smaller than $\sqrt{n}$. The estimation of the long-memory parameter d can thus be considered as a non-regular semi-parametric problem. In Moulines and Soulier (2003) (p. 274) it is shown that when f_o satisfies (1.4), the minimax rate for d is $n^{-\frac{2\beta-1}{4\beta}}$. There are frequentist estimators for d based on the periodogram that achieve this rate (see Hurvich et al. (2002) and Moulines and Soulier (2003)).

Although Bayesian methods in long-memory models have been widely used (see for instance Ko et al. (2009), Jensen (2004) or Holan and McElroy (2010)), the literature on convergence properties of non- and semi-parametric estimators is sparse. Rousseau et al. (2010) (RCL hereafter) obtain consistency and rates for the L_2-norm of the log-spectral densities (Theorems 3.1 and 3.2), but for d they only show consistency (Corollary 1). No results exist on the posterior concentration rate on d, and thus on the convergence rates of Bayesian semi-parametric estimators of d. In this paper we aim to fill this gap for a specific family of semi-parametric priors.

We study Bayesian estimation of d within the FEXP-model (Beran (1993), Robinson (1995)), that contains densities of the form

$$f_{d,k,\theta}(x) = |1 - e^{ix}|^{-2d} \exp\left\{\sum_{j=0}^{k} \theta_j \cos(jx)\right\}, \tag{1.2}$$

where $d \in (-\frac{1}{2}, \frac{1}{2})$, k is a nonnegative integer and $\theta \in \mathbb{R}^{k+1}$. The factor $\exp\{\sum_{j=0}^{k} \theta_j \cos(jx)\}$ models the function M_o in (1.1). In contrast to the original finite-dimensional FEXP-model (Beran (1993)), where k was supposed to be known, or at least bounded, f_o may have an infinite FEXP-expansion, and we allow k to increase with the number of observations to obtain approximations f that are increasingly close to f_o. Note that the case where the true spectral density satisfies $f_o = f_{d_o,k_o,\theta_o}$, is considered in Holan and McElroy (2010). In this paper we will pursue a fully Bayesian semi-parametric estimation of d, the short memory parameter being considered as an infinite-dimensional nuisance parameter. We obtain results on the convergence rate and asymptotic distribution of the posterior distribution for d, which we summarize below in section 1.2. These are to our knowledge the first of this kind in the Bayesian literature on semi-parametric time series. First we state the most important assumptions.

1.1 Asymptotic framework

For observations $X = (X_1, \ldots, X_n)$ from a Gaussian stationary time series with spectral density f, let $T_n(f)$ denote the associated covariance matrix and $l_n(f)$ denote the log-likelihood

$$l_n(f) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log\det(T_n(f)) - \frac{1}{2}X^t T_n^{-1}(f) X.$$

We consider semi-parametric priors on f based on the FEXP-model defined by (1.2), inducing a parametrization of f in terms of (d,k,θ). Assuming priors π_d for d, and, independent of d, π_k for k and $\pi_{\theta|k}$ for $\theta|k$, we study the (marginal) posterior for d, given by

$$\Pi(d \in D|X) = \frac{\sum_{k=0}^{\infty} \pi_k(k) \int_D \int_{\mathbb{R}^{k+1}} e^{l_n(d,k,\theta)} d\pi_{\theta|k}(\theta) d\pi_d(d)}{\sum_{k=0}^{\infty} \pi_k(k) \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}^{k+1}} e^{l_n(d,k,\theta)} d\pi_{\theta|k}(\theta) d\pi_d(d)}. \tag{1.3}$$

The posterior mean or median can be taken as point-estimates for d, but we will focuss on the posterior $\Pi(d|X)$ itself.

It is assumed that the true spectral density is of the form

$$\begin{aligned} f_o(x) &= |1-e^{ix}|^{-2d_o} \exp\left\{\sum_{j=0}^{\infty} \theta_{o,j} \cos(jx)\right\}, \\ \theta_o \in \Theta(\beta, L_o) &= \{\theta \in l_2(\mathbb{N}) : \sum_{j=0}^{\infty} \theta_j^2 (1+j)^{2\beta} \leq L_o\}, \end{aligned} \tag{1.4}$$

for some known $\beta > 1$.

In particular, we derive bounds on the rate at which $\Pi(d \in D|X)$ concentrates at d_o, together with a Bernstein - von -Mises (BVM) property of this distribution. The posterior concentration rate for d is defined as the fastest sequence α_n converging to zero such that

$$\Pi(|d-d_o| < K\alpha_n|X) \overset{P_o}{\to} 0, \quad \text{for a given fixed K.} \tag{1.5}$$

1.2 Summary of the results

Under the above assumptions we obtain several results for the asymptotic distribution of $\Pi(d \in D|X)$. Our first main result (Theorem 2.1) states that under the sieve prior $k_n \sim (n/\log n)^{1/(2\beta)}$, $\Pi(d \in D|X)$ is asymptotically Gaussian, and we give expressions for the posterior mean and the posterior variance. A consequence (Corollary 2.1) of this result is that the convergence rate for d under this prior is at least $\delta_n = (n/\log n)^{-\frac{2\beta-1}{4\beta}}$, i.e. in (1.5) α_n is bounded by δ_n. Up to a $\log n$ term, this is the minimax rate.

By our second main result (Theorem 2.2), the rate for d is suboptimal when k is given a a Poisson or a Geometric distribution, or a sieve prior $k_n' \sim (n/\log n)^{\frac{1}{1+2\beta}}$. More precisely, there exists f_o such that the posterior concentration rate α_n is greater than $n^{-(\beta-1/2)/(2\beta+1)}$, and thus suboptimal. Consequently, despite having good frequentist properties for the estimation of the spectral density f itself (see RCL), these priors are much less suitable for the estimation of d. This is not a unique phenomenon in (Bayesian) semi-parametric estimation and is encountered for instance in the estimation of a linear functional of the signal in white-noise models, see Li and Zhao (2002) or Arbel (2010).

The BVM property means that asymptotically the posterior distribution of d behaves like $\alpha_n^{-1}(d-\hat{d}) \sim \mathcal{N}(0,1)$, where $\hat{d}$ is an estimate whose frequentist distribution (associated to the parameter d) is $\mathcal{N}(d_o, \alpha_n^2)$. We prove such a property on the posterior distribution of d given $k = k_n$. In regular parametric long-memory models, the BVM property has been established by Philippe and Rousseau (2002). It is however much more difficult to establish BVM theorems in infinite dimensional setups, even for independent and identically distributed models; see for instance Freedman (1999), Castillo (2010) and Rivoirard and Rousseau (2010). In particular it has been proved that the BVM property may not be valid, even for reasonable priors. The BVM property is however very useful since it induces a strong connection between frequentist and Bayesian methods. In particular, it implies that Bayesian credible regions are asymptotically also frequentist confidence regions with the same nominal level. In section 2 we discuss this issue in more detail.

1.3 Overview of the paper

In section 2, we present three families of priors based on the sieve model defined by (1.2) with either k increasing at the rate $(n/\log n)^{1/(2\beta)}$, k increasing at the rate $(n/\log n)^{1/(2\beta+1)}$ or with random k. We study the behavior of the posterior distribution of d in each case and prove that the former leads to optimal frequentist procedures while the latter two lead to suboptimal procedures. In section 3 we give a decomposition of $\Pi(d \in D|X)$ defined in (1.3), and obtain bounds for the terms in this decomposition in sections 3.2 and 3.3. Using these results we prove Theorems 2.1 and 2.2 in respectively sections 4 and 5. Conclusions are given in section 6. In the appendices we give the proofs of the lemmas in section 3, as well as some additional results on the derivatives of the log-likelihood. The proofs of various technical results can be found in the supplementary material. We conclude this introduction with an overview of the notation.

1.4 Notation

The m-dimensional identity matrix is denoted I_m. We write $|A|$ for the Frobenius or Hilbert-Schmidt norm of a matrix A, i.e. $|A| = \sqrt{\mathrm{tr} A A^t}$, where A^t denotes the transpose of A. The operator or spectral norm is denoted $\|A\|^2 = \sup_{\|x\|=1} x^t A^t A x$. We also use $\|\cdot\|$ for the Euclidean norm on $\mathbb{R}^k$ or $l^2(\mathbb{N})$. The inner-product is denoted $|\cdot|$. We make frequent use of the relations

$$\begin{aligned} &|AB| = |BA| \leq \|A\| \cdot |B|, \quad \|AB\| \leq \|A\| \cdot \|B\|, \quad \|A\| \leq |A| \leq \sqrt{n}\|A\|, \\ &|\mathrm{tr}(AB)| = |\mathrm{tr}(BA)| \leq |A| \cdot |B|, \quad |x^t A x| \leq x^t x \|A\|, \end{aligned} \tag{1.6}$$

see Dahlhaus (1989), p. 1754. For any function $h \in L_1([-\pi,\pi])$, $T_n(h)$ is the matrix with entries $\int_{-\pi}^{\pi} e^{i|l-m|x} h(x)dx$, $l,m = 1,\dots,n$. For example, $T_n(f)$ is the covariance matrix of observations $X = (X_1,\dots,X_n)$ from a time series with spectral density f. If h is square integrable on $[-\pi, \pi]$ we note

$$\|h\|_2 = \int_{-\pi}^{\pi} h^2(x)dx.$$

The norm l between spectral densities f and g is defined as

$$l(f, g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\log f(x) - \log g(x))^2 dx.$$

Unless stated otherwise, all expectations and probabilities are with respect to P_o, the law associated with the true spectral density f_o. To avoid ambiguous notation (e.g. θ_0 versus $\theta_{0,0}$) we write θ_o instead of θ_0. Related quantities such as f_o and d_o are also denoted with the o-subscript.

The symbols o_P and O_P have their usual meaning. We use boldface when they are uniform over a certain parameter range. Given a probability law P, a family of random variables $\{W_d\}_{d\in A}$ and a positive sequence a_n, $W_d = \mathbf{o_P}(a_n, A)$ means that

$$P\left(\sup_{d\in A} |W_d|/a_n > \epsilon\right) \to 0, (n \to \infty).$$

When the parameter set is clear from the context we simply write $\mathbf{o_P}(a_n)$. In a similar fashion, we write $\mathbf{o}(a_n)$ when the sequence is deterministic. In conjunction with the o_P and O_P notation we use the letters δ and ϵ as follows. When, for some $\tau > 0$ and a probability P we write $Z = O_P(n^{\tau-\epsilon})$, this means that $Z = O(n^{\tau+\epsilon})$ *for all* $\epsilon > 0$. When, on the other hand, $Z = O_P(n^{\tau-\delta})$, we mean that this is true for *some* $\delta > 0$. If the value of δ is of importance it is given a name, for example δ_1 in Lemma 3.4.

The true spectral density of the process is denoted f_o. We denote k-dimensional Sobolev-balls by

$$\Theta_k(\beta, L) \quad = \quad \left\{\theta \in \mathbb{R}^{k+1} : \sum_{j=0}^{k} \theta_j^2(1 + j)^{2\beta} \leq L\right\} \subset \mathbb{R}^{k+1}. \tag{1.7}$$

For any real number x, let x_+ denote $\max(0, x)$. The number r_k denotes the sum $\sum_{j\geq k+1} j^{-2}$. Let η be the sequence defined by $\eta_j = -2/j$, $j \geq 1$ and $\eta_0 = 0$. For an infinite sequence $u = (u_j)_{j\geq 0}$, let $u_{[k]}$ denote the vector of the first $k+1$ elements. In particular, $\eta_{[k]} = (\eta_0, \ldots, \eta_k)$. The letter C denotes any generic constant independent of L_o and L, which are the constants appearing in the assumptions on f_o and the definition of the prior.

2 Main results

Before stating Theorems 2.1 and 2.2 in section 2.3, we state the assumptions on f_o and the prior, and give examples of priors satisfying these assumptions.

2.1 Assumptions on the prior and the true spectral density

We assume observations $X = (X_1, \dots, X_n)$ from a stationary Gaussian time series with law P_o, which is a zero mean Gaussian distribution, whose covariance structure is defined by a spectral density f_o satisfying (1.4), for known $\beta > 1$. It is assumed that for a small constant $t > 0$, $d_o \in [-\frac{1}{2} + t, \frac{1}{2} - t]$.

Assumptions on Π. We consider different priors, and first state the assumptions that are common to all these priors. The prior on the space of spectral densities consists of independent priors π_d, π_k and, conditional on k, $\pi_{\theta|k}$. The prior for d has density π_d which is strictly positive on $[-\frac{1}{2} + t, \frac{1}{2} - t]$, the interval which is assumed to contain d_o, and zero elsewhere. The prior for θ given k has a density $\pi_{\theta|k}$ with respect to Lebesgue measure. This density satisfies condition $\mathrm{Hyp}(\mathcal{K}, \mathrm{c}_0, \beta, \mathrm{L}_o)$, by which we mean that for a subset $\mathcal{K}$ of $\mathbb{N}$,

$$\min_{k \in \mathcal{K}} \inf_{\theta \in \Theta_k(\beta, L_o)} e^{c_0 k \log k} \pi_{\theta|k}(\theta) > 1,$$

where L_o is as in (1.4). The choice of $\mathcal{K}$ depends on the prior for k and $\theta|k$. We consider the following classes of priors.

- **Prior A:** k is deterministic and increasing at rate

  $$k_n = \lfloor k_A (n/\log n)^{\frac{1}{2\beta}} \rfloor, \tag{2.1}$$

  for a constant $k_A > 0$. The prior density for $\theta|k$ satisfies $\mathrm{Hyp}(\{\mathrm{k_n}\}, \mathrm{c}_0, \beta - \frac{1}{2}, \mathrm{L}_o)$ for some $c_0 > 0$ and has support $\Theta_k(\beta - \frac{1}{2}, L)$. In addition, for all $\theta, \theta' \in \Theta_k(\beta - \frac{1}{2}, L)$ such that $\|\theta - \theta'\| \le L(n/\log n)^{-\frac{2\beta-1}{4\beta}}$,

  $$\log \pi_{\theta|k}(\theta) - \log \pi_{\theta|k}(\theta') = h_k^t(\theta - \theta') + o(1), \tag{2.2}$$

  for constants $C, \rho_0 > 0$ and vectors h_k satisfying $\|h_k\| \le C(n/k)^{1-\rho_0}$. Finally, it is assumed that L is sufficiently large compared to L_o.

- **Prior B:** k is deterministic and increasing at rate

  $$k_n' = \lfloor k_B (n/\log n)^{\frac{1}{1+2\beta}} \rfloor,$$

  where k_B is such that $k_n' < k_n$ for all n. The prior for $\theta|k$ has density $\pi_{\theta|k}$ with respect to Lebesgue measure which satisfies condition $\mathrm{Hyp}(\{\mathrm{k_n'}\}, \mathrm{c}_0, \beta, \mathrm{L}_o)$ for some $c_0 > 0$ and is assumed to have support $\Theta_k(\beta, L)$. The density also satisfies

  $$\log \pi_{\theta|k}(\theta) - \log \pi_{\theta|k}(\theta') = o(1),$$

  for all $\theta, \theta' \in \Theta_k(\beta, L)$ such that $\|\theta - \theta'\| \le L(n/\log n)^{-\frac{\beta}{2\beta+1}}$. This condition is similar to (2.2), but with $h_k = 0$, and support $\Theta_k(\beta, L)$.

- **Prior C:** $k \sim \pi_k$ on $\mathbb{N}$ with $e^{-c_1 k \log k} \le \pi_k(k) \le e^{-c_2 k \log k}$ for k large enough, where $0 < c_1 < c_2 < +\infty$. There exists $\beta_s > 1$ such that for all

$\beta \geq \beta_s$, the prior for $\theta|k$ has density $\pi_{\theta|k}$ with respect to Lebesgue measure which satisfies condition $\mathrm{Hyp}(\{\mathrm{k} \leq \mathrm{k_0}(\mathrm{n}/\log \mathrm{n})^{1/(2\beta+1)}\}, \mathrm{c_0}, \beta, \mathrm{L_o})$, for all $k_0 > 0$ and some $c_0 > 0$, as soon as n is large enough. It has support included in $\Theta_k(\beta, L)$ and satisfies

$$\log \pi_{\theta|k}(\theta) - \log \pi_{\theta|k}(\theta') = o(1),$$

for all $\theta, \theta' \in \Theta_k(\beta, L)$ such that $\|\theta - \theta'\| \leq L(n/\log n)^{-\frac{\beta}{2\beta+1}}$.

Note that **prior A** is obtained when we take $\beta' = \beta - \frac{1}{2}$ in prior B.

2.2 Examples of priors

The Lipschitz conditions on $\log \pi_{\theta|k}$ considered for the three types of priors are satisfied for instance for the uniform prior on $\Theta_k(\beta - \frac{1}{2}, L)$ (resp. $\Theta_k(\beta, L)$), and for the truncated Gaussian prior, where, for some constants A and $\alpha > 0$,

$$\pi_{\theta|k}(\theta) \propto \mathbb{I}_{\Theta_k(\beta-\frac{1}{2},L)}(\theta) \exp\left(-A \sum_{j=0}^{k} j^\alpha \theta_j^2\right).$$

In the case of **Prior A**, the conditions on $\log \pi_{\theta|k}$ and h_k in (2.2) are satisfied for $\alpha < 4\beta - 2$. To see this, note that for all $\theta, \theta^{'} \in \Theta_h(\beta - 1/2, L)$,

$$\sum_{j=0}^{k} j^\alpha |\theta_j^2 - (\theta_j^{'})^2| \leq L^{1/2} \|\theta - \theta'\| k^{\alpha-\beta+1/2} = o((n/k)^{1-\delta}).$$

In the case of **Prior B** and and **Prior C** we may choose $\alpha < 2\beta$, since for some positive k_0

$$\sum_{j=0}^{k} j^\alpha |\theta_j^2 - (\theta_j^{'})^2| \leq L^{1/2} \|\theta - \theta^{'}\| k^{\alpha-\beta} = o(1),$$

for all $k \leq k_0(n/\log n)^{1/(2\beta+1)}$ and all $\theta, \theta^{'} \in \Theta_k(\beta, L)$ such that $\|\theta - \theta^{'}\| \leq (n/\log n)^{-\beta/(2\beta+1)}$.

Also a truncated Laplace distribution is possible, in which case

$$\pi_{\theta|k}(\theta) \propto \mathbb{I}_{\Theta_k(\beta-\frac{1}{2},L)}(\theta) \exp\left(-a \sum_{j=0}^{k} |\theta_j|\right).$$

The condition on π_k in **Prior C** is satisfied for instance by Poisson distributions.

The restriction of the prior to Sobolev balls is required to obtain a proper concentration rate or even consistency of the posterior of the spectral density f itself, which is a necessary step in the proof of our results. This is discussed in more detail in section 3.1.

2.3 Convergence rates and BVM-results under different priors

Assuming a Poisson prior for k, RCL (Theorem 4.2) obtain a near-optimal convergence rate for $l(f, f_o)$. In Corollary 3.1 below, we show that the optimal rate for l implies that we have at least a suboptimal rate for $|d - d_o|$. Whether this can be improved to the optimal rate critically depends on the prior on k. By our first main result the answer is positive under **prior A**. The proof is given in section 4.

Theorem 2.1. *Under **prior A**, the posterior distribution has the asymptotic expansion*

$$\Pi\left[\sqrt{\frac{nr_{k_n}}{2}}(d - d_o - b_n(d_o)) \leq z|X\right] = \Phi(z) + o_{P_o}(1), \tag{2.3}$$

where, for $r_{k_n} = \sum_{j \geq k_n+1} \eta_j^2$ and some small enough $\delta > 0$,

$$b_n(d_o) = \frac{1}{r_{k_n}} \sum_{j=k_n+1}^{\infty} \eta_j \theta_{o,j} + Y_n + o(n^{-1/2-\delta} k_n^{1/2}), \quad Y_n = \frac{\sqrt{2}}{\sqrt{nr_{k_n}}} Z_n,$$

Z_n being a sequence of random variables converging weakly to a Gaussian variable with mean zero and variance 1.

Corollary 2.1. *Under **prior A**, the convergence rate for d is $\delta_n = (n/\log n)^{-\frac{2\beta-1}{4\beta}}$, i.e.*

$$\lim_{n\to\infty} E_0^n \left[\Pi(d : |d - d_o| > \delta_n|X)\right] = 0.$$

Equation (2.3) is a Bernstein-von Mises type of result: the posterior distribution is asymptotically normal, centered at a point $d_o + b_n(d_o)$, whose distribution is normal with mean d_o and variance $2/(nr_{k_n})$. The expressions for the posterior mean and variance give more insight in how the prior for k affects the posterior rate for d. The standard deviation of the limiting normal distribution (2.3) is $\sqrt{2/(nr_{k_n})} = O(n^{-\frac{2\beta-1}{4\beta}} (\log n)^{\frac{1}{4\beta}})$ and $b_n(d_o)$ equals

$$\frac{1}{r_{k_n}} \sum_{j=k_n+1}^{\infty} \eta_j \theta_{o,j} + O_{P_o}(k_n^{\frac{1}{2}} n^{\frac{1}{2}}) + o(n^{-1/2-\delta_1} k_n^{1/2}).$$

From the definition of η_j, k_n and r_{k_n} and the assumption on θ_o, it follows that

$$\frac{1}{r_{k_n}} \left| \sum_{j=k_n+1}^{\infty} \eta_j \theta_{o,j} \right| \leq \frac{1}{r_{k_n}} \sqrt{\sum_{l>k_n} \theta_{o,l}^2 j^{2\beta}} \sqrt{\sum_{l>k_n} j^{-2\beta-2}} = o(k_n^{-\beta+\frac{1}{2}}). \tag{2.4}$$

See also (1.9) in the supplement. Hence, when the constant k_A in (2.1) is small enough,

$$|b_n(d_o)| \leq \delta_n, \tag{2.5}$$

and we obtain the δ_n-rate of Corollary 2.1. For smaller k, the standard deviation is smaller but the bias $b_n(d_o)$ is larger. In Theorem 2.2 below it is shown that this indeed leads to a suboptimal rate.

An important consequence of the BVM-result is that posterior credible regions for d (HPD or equal-tails for instance) will also be asymptotic frequentist confidence regions. Consider for instance one-sided credible intervals for d defined by $P^\pi(d \le z_n(\alpha)|X) = \alpha$, so that $z_n(\alpha)$ is the α-th quantile of the posterior distribution of d. Equation (2.3) in Theorem 2.1 then implies that

$$z_n(\alpha) = d_o + b_n(d_o) + \sqrt{\frac{2k_n}{n}}\Phi^{-1}(\alpha)(1 + \mathbf{o_{P_o}}(1)).$$

As soon as $\sum_{j\ge k_n} j^{2\beta}\theta_{o,j}^2 = o((\log n)^{-1})$, we have that

$$z_n(\alpha) = d_o + \sqrt{2/(nr_{k_n})}Z_n + \sqrt{2/(nr_{k_n})}\Phi^{-1}(\alpha)(1 + \mathbf{o_{P_o}}(1))$$

and

$$P_o^n\left(d_o \le z_n(\alpha)\right) = P\left(Z_n \le \Phi^{-1}(\alpha)(1 + o(1))\right) = \alpha + o(1).$$

Similar computations can be made on equal - tail credible intervals or HPD regions for d.

Note that in this paper we assume that the smoothness β of f_o is greater than 1 instead of 1/2, as is required in Moulines and Soulier (2003). This condition is used throughout the proof. Actually had we only assumed that $\beta > 3/2$, the proof of Theorem 2.1 would have been greatly simplified as many technicalities in the paper come from controlling terms when $1 < \beta \le 3/2$. We do not believe that it is possible to weaken this constraint to $\beta > 1/2$ in our setup.

Our second main result states that if k is increasing at a slower rate than k_n, the posterior on d concentrates at a suboptimal rate. The proof is given in section 5.

Theorem 2.2. *Given $\beta > 5/2$, there exists $\theta_o \in \Theta(\beta, L_o)$ and a constant $k_v > 0$ such that under prior B and C defined above,*

$$\Pi(|d - d_o| > k_v w_n(\log n)^{-1}|X) \overset{P_o}{\to} 1.$$

with $w_n = C_w(n/\log n)^{-\frac{2\beta-1}{4\beta+2}}$ and $C_w = C_1(L + L_o)^{\frac{1}{4\beta}} l_0^{\frac{2\beta-1}{2\beta}}$.

The constant C_w comes from the suboptimal rate for $|d - d_o|$ derived in Corollary 3.1. Theorem 2.2 is proved by considering the vector θ_o defined by $\theta_{o,j} = c_0 j^{-(\beta+\frac{1}{2})}(\log j)^{-1}$, for $j \ge 2$. This vector is close to the boundary of the Sobolev-ball $\Theta(\beta, L_o)$, in the sense that for all $\beta' > \beta$, $\sum_j j^{2\beta'}\theta_{o,j}^2 = +\infty$. The proof consists in showing that conditionally on k, the posterior distribution is asymptotically normal as in (2.3), with k replacing k_n, and that the posterior distribution concentrates on values of k smaller than $O(n^{1/(2\beta+1)})$, so that the bias $b_n(d_o)$ becomes of order $w_n(\log n)^{-1}$. The constraint $\beta > 5/2$ is used to simplify the computations and is not sharp.

It is interesting to note that similar to the frequentist approach, a key issue is a bias-variance trade-off, which is optimized when $k \sim n^{1/(2\beta)}$. This choice of k depends on the smoothness parameter β, and since it is not of the same order as the *optimal* values of k for the loss $l(f, f')$ on the spectral densities, the adaptive (near) minimax Bayesian nonparametric procedure proposed in Rousseau and Kruijer (2011) does not lead to optimal posterior concentration rate for d. While it is quite natural to obtain an adaptive (nearly) minimax Bayesian procedure under the loss $l(.,.)$ by choosing a random k, obtaining an adaptive minimax procedure for d remains an open problem. This dichotomy is found in other semi-parametric Bayesian problems, see for instance Arbel (2010) in the case of the white noise model or Rivoirard and Rousseau (2010) for BVM properties.

3 Decomposing the posterior for d

To prove Theorems 2.1 and 2.2 we need to take a closer look at (1.3), to understand how the integration over Θ_k affects the posterior for d. We develop $\theta \to l_n(d, k, \theta)$ in a point $\bar{\theta}_{d,k}$ defined below and decompose the likelihood as

$$\exp\{l_n(d, k, \theta)\} = \exp\{l_n(d, k)\} \exp\{l_n(d, k, \theta) - l_n(d, k)\},$$

where $l_n(d, k)$ is short-hand notation for $l_n(d, k, \bar{\theta}_{d,k})$. Define

$$I_n(d, k) = \int_{\Theta_k} e^{l_n(d,k,\theta) - l_n(d,k)} d\pi_{\theta|k}(\theta), \tag{3.1}$$

where Θ_k is the generic notation for $\Theta_k(\beta - \frac{1}{2}, L)$ under **prior A** and $\Theta_k(\beta, L)$ for priors B and C. The posterior for d given in (1.3) can be written as

$$\Pi(d \in D|X) = \frac{\sum_{k=0}^{\infty} \pi_k(k) \int_D e^{l_n(d,k) - l_n(d_o,k)} I_n(d, k) d\pi_d(d)}{\sum_{k=0}^{\infty} \pi_k(k) \int_{-\frac{1}{2}+t}^{\frac{1}{2}-t} e^{l_n(d,k) - l_n(d_o,k)} I_n(d, k) d\pi_d(d)}. \tag{3.2}$$

The factor $\exp\{l_n(d, k) - l_n(d_o, k)\}$ is independent of θ, and will under certain conditions dominate the marginal likelihood. In section 3.2 we give a Taylor-approximation which, for given k, allows for a normal approximation to the marginal posterior. However, to obtain the convergence rates in Theorems 2.1 and 2.2, it also needs to be shown that the integrals $I_n(d, k)$ with respect to θ do not vary too much with d. This is the most difficult part of the proof of Theorem 2.1 and the argument is presented in section 3.3. Since Theorem 2.2 is essentially a counter-example and it is not aimed to be as general as Theorem 2.1, as far as the range of β is concerned, we can restrict attention to larger β's, i.e. $\beta > 5/2$, for which controlling $I_n(d, k)$ is much easier.

3.1 Preliminaries

First we define the point $\bar{\theta}_{d,k}$ in which we develop $\theta \to l_n(d, k, \theta)$. Since the function $\log(2 - 2\cos(x))$ has Fourier coefficients against $\cos jx$, $j \in \mathbb{N}$ equal to

$0, 2, \frac{2}{2}, \frac{2}{3}, \ldots$, FEXP-spectral densities can be written as

$$|1 - e^{ix}|^{-2d} \exp\left\{\sum_{j=0}^{\infty} \theta_j \cos(jx)\right\} = \exp\left\{\sum_{j=0}^{\infty} (\theta_j + d\eta_j)\cos(jx)\right\}.$$

Given $f = f_{d,k,\theta}$ and $f' = f_{d',k',\theta'}$ we can therefore express the norm $l(f, f')$ in terms of $(\theta - \theta')$ and $(d - d')$:

$$l(f, f') = \frac{1}{2}\sum_{j=0}^{\infty}((\theta_j - \theta'_j) + \eta_j(d - d'))^2, \tag{3.3}$$

where θ_j and θ'_j are understood to be zero when j is larger than k respectively k'. Equation (3.3) implies that for given d and k, $l(f_o, f_{d,k,\theta})$ is minimized by

$$\bar{\theta}_{d,k} := \text{argmin}_{\theta \in \mathbb{R}^{k+1}} \sum_{j=0}^{\infty}(\theta_j - \theta_{o,j} + (d - d_o)\eta_j)^2 = \theta_{o[k]} + (d_o - d)\eta_{[k]}.$$

In particular, $\theta = \theta_{o[k]}$ minimizes $l(f_o, f_{d,k,\theta})$ only when $d = d_o$; when $d \neq d_o$ we need to add $(d_o - d)\eta_{[k]}$. The following lemma shows that an upper bound on $l(f_o, f_{d,k,\theta})$ leads to upper bounds on $|d - d_o|$ and $\|\theta - \theta_o\|$.

Lemma 3.1. *Suppose that $\theta \in \Theta_k(\gamma, L)$ and $\theta_o \in \Theta_k(\beta, L_o)$, where $\gamma \leq \beta$. Also suppose that for a sequence $\alpha_n \to 0$, $l(f_o, f_{d,k,\theta}) \leq \alpha_n^2$ for all n. Then there are universal constants $C_1, C_2 > 0$ such that for all n,*

$$|d - d_o| \leq C_1(L + L_o)^{\frac{1}{4\gamma}} \alpha_n^{\frac{2\gamma-1}{2\gamma}}, \qquad \|\theta - \theta_o\| \leq C_2(L + L_o)^{\frac{1}{4\gamma}} \alpha_n^{\frac{2\gamma-1}{2\gamma}}.$$

Proof. For all (d, k, θ) such that $l(f_{d,k,\theta}, f_o) \leq \alpha_n$, we have, using (3.3),

$$\begin{aligned} 2\alpha_n^2 \geq 2l(f_{d,k,\theta}, f_o) &= 2(\theta_{o,0} - \theta_0)^2 + \sum_{j\geq 1}\left((\theta_{o,j} - \theta_j) + \eta_j(d_o - d)\right)^2 \\ &\geq \sum_{j\geq 1}(\theta_{o,j} - \theta_j)^2 + (d - d_o)^2 \sum_{j\geq 1}\eta_j^2 - 2|d - d_o|\sqrt{\sum_{j\geq 1}\eta_j^2}\sqrt{\sum_{j\geq 1}(\theta_{o,j} - \theta_j)^2} \\ &= \left(\|\theta - \theta_o\| - |d - d_o|\|\eta\|\right)^2. \end{aligned}$$

The inequalities remain true if we replace all sums over $j \geq 1$ by sums over $j \geq m_n$, for any nondecreasing sequence m_n. Since $\|(\eta_j 1_{j>m_n})_{j\geq 1}\|^2$ is of order m_n^{-1} and $\|(\theta - \theta_o\|_j 1_{j>m_n})_{j\geq 1}\|^2 \leq m_n^{-2\gamma}\sum_{j>m_n}(1+j)^{2\beta}(\theta_j - \theta_{o,j})^2 < 2(L+L_o)m_n^{-2\gamma}$, setting $m_n = \alpha_n^{-\frac{1}{\gamma}}$ gives the desired rate for $|d - d_o|$ as well as for $\|\theta - \theta_o\|$. $\square$

The convergence rate for $l(f_o, f_{d,k,\theta})$ required in Lemma 3.1 can be found in Rousseau and Kruijer (2011). For easy reference we restate it here. Compared to a similar result in RCL, the $\log n$ factor is improved.

Lemma 3.2. *Under **prior A**, there exists a constant l_0 depending only on L_o and k_A (and not on L) such that*

$$\Pi((d,k,\theta): l(f_{d,k,\theta}, f_o) \geq l_0^2 \delta_n^2 | X) \overset{P_o}{\to} 0,$$

*where $\delta_n = (n/\log n)^{-\frac{2\beta-1}{4\beta}}$. Under **priors B** and **C**, this statement holds with $\epsilon_n = (n/\log n)^{-\frac{\beta}{2\beta+1}}$ replacing δ_n.*

In the proof of Theorem 2.1 (resp. 2.2), this result allows us to restrict attention to the set of spectral densities f such that $l(f, f_o) \leq l_0^2\delta_n^2$ (resp. $l_0^2\epsilon_n^2$). In addition, by combination with Lemma 3.1 we can now deduce bounds on $|d - d_o|$ and $\|\theta - \bar{\theta}_{d,k}\|$. These bounds, although suboptimal, will be important in the sequel for obtaining the near-optimal rate in Theorem 2.1.

Corollary 3.1. *Under the result of Lemma 3.2 and **prior A**, we can apply Lemma 3.1 with $\alpha_n^2 = l_0^2\delta_n^2$ and $\gamma = \beta - \frac{1}{2}$, and obtain*

$$\Pi_d(d : |d - d_o| \geq \bar{v}_n | X) \overset{P_o}{\to} 0, \quad \Pi(\|\theta - \bar{\theta}_{d,k}\| \geq 2 l_0 \delta_n | X) \overset{P_o}{\to} 0,$$

*where $\bar{v}_n = C_1(L + L_o)^{\frac{1}{4\beta-2}} l_0^{\frac{2\beta-2}{2\beta-1}} (n/\log n)^{-\frac{\beta-1}{2\beta}}$. Under **priors B** and **C** we have $\gamma = \beta$; the rate for $|d - d_o|$ is then $w_n = C_w(n/\log n)^{-\frac{2\beta-1}{4\beta+2}}$ and the rate for $\|\theta - \bar{\theta}_{d,k}\|$ is $2l_0\epsilon_n$. The constant $C_w = C_1(L+L_o)^{\frac{1}{4\beta}} l_0^{\frac{2\beta-1}{2\beta}}$ is as in Theorem 2.2.*

Proof. The rate for $|d - d_o|$ follows directly from Lemma 3.1. To obtain the rate for $\|\theta - \bar{\theta}_{d,k}\|$, let α_n denote either $l_0\delta_n$ (the rate for $l(f_o, f)$ under prior A) or $l_0\epsilon_n$ (the rate under priors B and C). Although Lemma 3.1 suggests that the Euclidean distance from θ_o to θ (contained in $\Theta_k(\beta, L)$ or $\Theta_k(\beta - \frac{1}{2}, L)$) may be larger than α_n, the distance from θ to $\bar{\theta}_{d,k}$ is certainly of order α_n. To see this, note that Lemma 3.2 implies the existence of d, k, θ in the model with $l(f_o, f_{d,k,\theta}) \leq \alpha_n^2$. From the definition of $\bar{\theta}_{d,k}$ it follows that $l(f_o, f_{d,k,\bar{\theta}_{d,k}}) \leq \alpha_n^2$. The triangle inequality gives $\|\theta - \bar{\theta}_{d,k}\|^2 = l(f_{d,k,\theta}, f_{d,k,\bar{\theta}_{d,k}}) \leq 4\alpha_n^2$. □

The rates $\bar{v}_n$ and w_n obtained in Corollary 3.1 are clearly suboptimal; their importance however lies in the fact that they narrow down the set for which we need to prove Theorems 2.1 and 2.2. To prove Theorem 2.2 for example it suffices to show that the posterior mass on $k_v w_n (\log n)^{-1} < |d - d_o| < w_n$ tends to zero. Note that the lower and the upper bound differ only by a factor $(\log n)$. Hence under priors B and C, the combination of Corollary 3.1 and Theorem 2.2 characterizes the posterior concentration rate (up to a $\log n$ term) for the given θ_o. Another consequence of Corollary 3.1 is that we may neglect the posterior mass on all (d, k, θ) for which $\|\theta - \bar{\theta}_{d,k}\|$ is larger than $2l_0\delta_n$ (under prior A) or $2l_0\epsilon_n$ (under priors B and C).

We conclude this section with a result on $\bar{\theta}_{d,k}$ and $\Theta_k(\beta, L)$. In the definition of $\bar{\theta}_{d,k}$ we minimize over $\mathbb{R}^{k+1}$, whereas the support of priors A-C is the Sobolev ball $\Theta_k(\beta, L)$ or $\Theta_k(\beta - \frac{1}{2}, L)$. Under the assumptions of Theorems 2.1 and 2.2 however, $\bar{\theta}_{d,k}$ is contained in $\Theta_k(\beta - \frac{1}{2}, L)$ respectively $\Theta_k(\beta, L)$. Also the l_2-ball of radius $2l_0\delta_n$ (or $2l_0\epsilon_n$) is contained in these Sobolev-balls.

Lemma 3.3. *Under the assumptions of Theorem 2.1, $B_k(\bar{\theta}_{d,k}, 2l_0\delta_n)$ is contained in $\Theta_k(\beta - \frac{1}{2}, L)$ for all $d \in [d_o - \bar{v}_n, d_o + \bar{v}_n]$, if L is large enough. In particular, $\bar{\theta}_{d,k} \in \Theta_k(\beta - \frac{1}{2}, L)$. Similarly, under the assumptions of Theorem 2.2, $B_k(\bar{\theta}_{d,k}, 2l_0\epsilon_n) \subset \Theta_k(\beta, L)$, for all $d \in [d_o - w_n, d_o + w_n]$.*

Proof. Since the constant l_0 is independent of L, $\theta \in B_k(\bar{\theta}_{d,k}, 2l_0\delta_n)$ implies that for n large enough ,

$$\sum_{j=0}^{k} \theta_j^2 (j+1)^{2\beta-1} \leq 2\sum_{j=0}^{k} (\theta - \bar{\theta}_{d,k})_j^2 (j+1)^{2\beta-1} + 2\sum_{j=0}^{k} (\bar{\theta}_{d,k})_j^2 (j+1)^{2\beta-1}$$
$$\leq 8\delta^2(L_o)(n/\log n)^{\frac{2\beta-1}{2\beta}}(k_n+1)^{2\beta-1} + 4\sum_{j=0}^{k_n} \theta_{o,j}^2 (j+1)^{2\beta-1}$$
$$+ 16(d-d_o)^2 \sum_{j=1}^{k_n} j^{2\beta-3}.$$

The first two terms on the right only depend on L_o, and are smaller than $L/4$ when L is chosen sufficiently large. Because $\bar{v}_n = C_1(L+L_o)^{\frac{1}{4\beta-2}} l_0^{\frac{2\beta-2}{2\beta-1}} (n/\log n)^{-\frac{\beta-1}{2\beta}}$, the last term in the preceding display is at most

$$C_1^2(L+L_o)^{\frac{1}{2\beta-1}} l_0^{\frac{4\beta-4}{2\beta-1}} (n/\log n)^{-\frac{\beta-1}{\beta}} k_A^{2\beta-2} (n/\log n)^{\frac{\beta-1}{\beta}},$$

which, since $\beta > 1$, is smaller than $L/2$ when L is large enough. We conclude that $B_k(\bar{\theta}_{d,k}, 2l_0\delta_n)$ is contained in $\Theta_k(\beta - \frac{1}{2}, L)$ provided L is chosen sufficiently large. The second statement can be proved similarly. □

3.2 A Taylor approximation for $l_n(d,k)$

Provided that the integrals $I_n(d,k)$ have negligible impact on the posterior for d, the conditional distribution of d given k will only depend on $\exp\{l_n(d,k) - l_n(d_o,k)\}$. Let $l_n^{(1)}(d,k)$, $l_n^{(2)}(d,k)$ denote the first two derivatives of the map $d \mapsto l_n(d,k)$. There exists a $\bar{d}$ between d and d_o such that

$$l_n(d,k) = l_n(d_o,k) + (d-d_o) l_n^{(1)}(d_o,k) + \frac{(d-d_o)^2}{2} l_n^{(2)}(\bar{d},k). \quad (3.4)$$

Defining

$$b_n(d) = -\frac{l_n^{(1)}(d_o,k)}{l_n^{(2)}(d,k)},$$

which is the b_n used in Theorem 2.1, we can rewrite (3.4) as

$$l_n(d,k) - l_n(d_o,k) = -\frac{1}{2}\frac{(l_n^{(1)}(d_o,k))^2}{l_n^{(2)}(\bar{d},k)} + \frac{1}{2} l_n^{(2)}(\bar{d},k)\left(d - d_o - b_n(\bar{d})\right)^2. \quad (3.5)$$

Note that each derivative $l_n^{(i)}(d,k)$, $i=1,2$, can be decomposed into a centered quadratic form denoted $\mathcal{S}(l_n^{(i)}(d,k))$ and a deterministic term $\mathcal{D}(l_n^{(i)}(d,k))$. In the following lemma we give expressions for $l_n^{(1)}(d_o,k)$, $l_n^{(2)}(d,k)$ and b_n, making explicit their dependence on k and θ_o. Since $k_n^{'} \leq k_n$ and $w_n < \bar{v}_n$ (see Corollary 3.1) the result is valid for all priors under consideration. The proof is given in appendix A.

Lemma 3.4. *Given $\beta > 1$, let $\theta_o \in \Theta(\beta, L_o)$. If $k \leq k_n$ and $|d - d_o| \leq \bar{v}_n$, then there exists $\delta_1 > 0$ such that*

$$\begin{aligned} l_n^{(1)}(d_o,k) &:= \mathcal{S}(l_n^{(1)}(d_o,k)) + \mathcal{D}(l_n^{(1)}(d_o,k)) \\ &= \mathcal{S}(l_n^{(1)}(d_o,k)) + \frac{n}{2}\sum_{j=k+1}^{\infty} \theta_{o,j}\eta_j + o(n^{\epsilon}(k^{-\beta+3/2} + n^{-1/(2\beta)})), \\ l_n^{(2)}(d,k) &= l_n^{(2)}(d_o,k)\left(1 + \frac{k^{1/2}}{n^{1/2+\epsilon}} + \frac{k^{-2\beta+1+\epsilon}}{n}\right) = -\frac{1}{2} n r_k \left(1 + o_{\mathbf{P_o}}(n^{-\delta_1})\right), \end{aligned}$$

where $\mathcal{S}(l_n^{(1)}(d_o,k))$ is a centered quadratic form with variance

$$Var(\mathcal{S}(l_n^{(1)}(d_o,k))) = \frac{n}{2}\sum_{j>k}\eta_j^2(1+o(1)) = \frac{n r_k}{2}(1+o(1)) = O(nk^{-1}).$$

Consequently,

$$\begin{aligned} b_n(d) = -\frac{l_n^{(1)}(d_o,k)}{l_n^{(2)}(d,k)} &= \frac{1}{r_k}\sum_{j=k+1}^{\infty} \theta_{o,j}\eta_j(1 + o_{\mathbf{P_o}}(n^{-\delta})) \\ &+ \frac{2\mathcal{S}(l_n^{(1)}(d_o,k))(1 + o_{\mathbf{P_o}}(n^{-\delta}))}{n r_k} + o_{\mathbf{P_o}}(n^{\epsilon-1}k^{-\beta+5/2} + n^{\epsilon-1}), \end{aligned} \quad (3.6)$$

with

$$\frac{2\mathcal{S}(l_n^{(1)}(d_o,k))}{n r_k} = \mathbf{O}_{\mathbf{P_o}}(n^{-\frac{1}{2}}k^{\frac{1}{2}}).$$

Remark 3.1. *Recall from (2.4) that $r_k^{-1}\sum_{j=k+1}^{\infty}\theta_{o,j}\eta_j$ is $O(k^{-\beta+1/2})$. The term $2\mathcal{S}(l_n^{(1)}(d_o,k))/(nr_k)$ is $O_{P_o}(k^{-\beta+1/2})$ whenever $k \sim n^{1/(2\beta)}$, which is the case under all priors under consideration.*

Substituting the above results on $l_n^{(1)}$, $l_n^{(2)}$ and b_n in (3.5), we can give the following informal argument leading to Theorems 2.1 and Theorem 2.2. If we consider k to be fixed and $I_n(d,k)$ constant in d, then (3.5) implies that the posterior distribution for d is asymptotically normal with mean $d_o + b_n(d_o)$ and variance of order k/n.

3.3 Integration of the short memory parameter

A key ingredient in the proofs of both Theorems 2.1 and 2.2 is the control of the integral $I_n(d,k)$ appearing in (1.3), whose dependence on d should be negligible with respect to $\exp\{l_n(d,k) - l_n(d_o,k)\}$. In Lemma 3.5 below we prove this to be the case under the assumptions of Theorems 2.1 and 2.2. For the case of Theorem 2.2 this is fairly simple: the conditional posterior distribution of θ given (d,k) can be proved to be asymptotically Gaussian by a Laplace-approximation. For smaller β and larger k the control is technically more demanding. In both cases the proof is based on the following Taylor expansion of $l_n(d,k,\theta)$ around $\bar{\theta}_{d,k}$:

$$l_n(d,k,\theta) - l_n(d,k) = \sum_{j=1}^{J} \frac{(\theta - \bar{\theta}_{d,k})^{(j)} \nabla^j l_n(d,k)}{j!} + R_{J+1,d}(\theta), \tag{3.7}$$

where

$$(\theta - \bar{\theta}_{d,k})^{(j)} \nabla^j l_n(d,k) = \sum_{l_1,\dots,l_j=0}^{k} (\theta - \bar{\theta}_{d,k})_{l_1} \dots (\theta - \bar{\theta}_{d,k})_{l_j} \frac{\partial^j l_n(d,k,\bar{\theta}_{d,k})}{\partial \theta_{l_1} \dots \partial \theta_{l_j}},$$

$$R_{J+1,d}(\theta) = \frac{1}{(J+1)!} \sum_{l_1,\dots,l_{J+1}=0}^{k} (\theta - \bar{\theta}_{d,k})_{l_1} \dots (\theta - \bar{\theta}_{d,k})_{l_{J+1}} \frac{\partial^{J+1} l_n(d,k,\tilde{\theta})}{\partial \theta_{l_1} \dots \partial \theta_{l_{J+1}}}. \tag{3.8}$$

The above expressions are used to derive the following lemma, which gives control of the term $I_n(d,k)$.

Lemma 3.5. *Under the conditions of Theorem 2.1, the integral $I_n(d,k)$ defined in* (3.1) *equals*

$$I_n(d_o,k) \exp\left\{ \mathbf{o}_{\mathbf{P_o}}(1) + \mathbf{o}_{\mathbf{P_o}}\left(\frac{|d-d_o| n^{\frac{1}{2}-\delta_2}}{\sqrt{k}} \right) + \mathbf{o}_{\mathbf{P_o}}\left((d-d_o)^2 \frac{n^{1-\delta_2}}{k} \right) \right\},$$

for some $\delta_2 > 0$. Under the conditions of Theorem 2.2,

$$I_n(d,k) = I_n(d_o,k) \exp\left\{ \mathbf{o}_{\mathbf{P_o}}(1) \right\}.$$

The proof is given in Appendix C, and relies on the expressions for the derivatives $\nabla^j l_n$ given in Appendix B. Lemma 3.5 should be seen in relation to Lemma 3.4 and the expressions for $\Pi(d|X)$ and $l_n(d,k) - l_n(d_o,k)$ in equations (3.2) and (3.4). Lemma 3.5 then shows that the dependence on the integrals $I_n(d,k)$ on d is asymptotically negligible with respect to $l_n(d,k) - l_n(d_o,k)$. This is made rigorous in the following section.

4 Proof of Theorem 2.1

By Lemma 3.2 we may assume posterior convergence of $l(f_o, f_{d,k,\theta})$ at rate $l_0^2 \delta_n^2$, and, by Corollary 3.1, also convergence of $|d - d_o|$ at rate $\bar{v}_n$. By Lemma 3.3, we may restrict the integration over θ to $B_k(\bar{\theta}_{d,k}, 2l_0\delta_n)$. Let $\Gamma_n(z) = \{d : \sqrt{\frac{nr_k}{2}}(d-d_o-b_n(d_o)) \leq z\}$. Under **prior A**, it suffices to show that for $k = k_n$,

$$\frac{N_n}{D_n} := \frac{\int_{\Gamma_n(z)} e^{l_n(d,k)-l_n(d_o,k)} \int_{B_k(\bar{\theta}_{d,k},2l_0\delta_n)} e^{l_n(d,k,\theta)-l_n(d,k)} d\pi_{\theta|k}(\theta) d\pi_d(d)}{\int_{|d-d_o|<\bar{v}_n} e^{l_n(d,k)-l_n(d_o,k)} \int_{B_k(\bar{\theta}_{d,k},2l_0\delta_n)} e^{l_n(d,k,\theta)-l_n(d,k)} d\pi_{\theta|k}(\theta) d\pi_d(d)}$$
$$= \frac{\int_{\Gamma_n(z)} \exp\{l_n(d,k)-l_n(d_o,k)+\log I_n(d,k)\} d\pi_d(d)}{\int_{|d-d_o|<\bar{v}_n} \exp\{l_n(d,k)-l_n(d_o,k)+\log I_n(d,k)\} d\pi_d(d)} = \Phi(z) + \mathbf{o_{P_o}}(1). \tag{4.1}$$

Using the results for $l_n(d,k) - l_n(d_o,k)$ and $I_n(d,k)$ given by Lemmas 3.4 and 3.5, we show that for $A_n \subset \mathbb{R}^n$ defined below such that $P_o^n(A_n) \to 1$,

$$\frac{N_n}{D_n} \leq \Phi(z) + o(1), \quad \frac{N_n}{D_n} \geq \Phi(z) + o(1), \quad \forall X \in A_n. \tag{4.2}$$

Since $P_o^n(A_n) \to 1$ this implies the last equality in (4.1).

Note that Lemmas 3.4 and 3.5 also hold for all $\delta_1' < \delta_1$ and $\delta_2' < \delta_2$. In the remainder of the proof, let $0 < \delta \leq \min(\delta_1, \delta_2)$. For notational simplicity, let $\mathcal{D} = \mathcal{D}(l_n^{(1)}(d_o,k)$, the deterministic part of $l_n^{(1)}(d_o,k)$. For a sufficiently large constant C_1 and arbitrary $\epsilon_1 > 0$, let A_n be the set of $X \in \mathbb{R}^n$ such that

$$\left.\begin{array}{l} |\log I_n(d,k) - \log I_n(d_o,k)| \leq \epsilon_1 + (d-d_o)^2 k^{-1} n^{1-\delta} + |d-d_o| k^{-\frac{1}{2}} n^{\frac{1}{2}-\delta} \\ \left| l_n^{(1)}(d_o,k) - \mathcal{D} \right| \leq C_1 n^{\frac{1}{2}} k^{-\frac{1}{2}} \sqrt{\log n}, \qquad \left| l_n^{(2)}(d,k) + \frac{1}{2} n r_k \right| \leq n^{1-\delta} k^{-1} \end{array}\right\}$$

for all $|d-d_o| \leq \bar{v}_n$. Since $k = k_n$ and $\beta > 1$, Lemmas 3.4 and 3.5 imply that $P_o^n(A_n^c) \to 0$. We prove the first inequality in (4.2); the second one can be obtained in the same way. Using (3.4) and the definition of A_n, it follows that for all $X \in A_n$,

$$\begin{aligned}
&l_n(d,k) - l_n(d_o,k) + \log I_n(d,k) - \log I_n(d_o,k) \leq \epsilon_1 + (d-d_o)^2 \frac{n^{1-\delta}}{k} \\
&\quad + |d-d_o| \frac{n^{\frac{1}{2}-\delta}}{k^{\frac{1}{2}}} + (d-d_o) l_n^{(1)}(d_o,k) - \frac{nr_k}{4}(d-d_o)^2(1-n^{-\delta}) \\
&\leq 2\epsilon_1 - \frac{nr_k}{4}\left(1-\frac{2}{n^\delta}\right)\left(d-d_o-\frac{2l_n^{(1)}(d_o,k)}{\left(1-\frac{2}{n^\delta}\right) nr_k}\right)^2 + |d-d_o| \frac{n^{\frac{1}{2}-\delta}}{k^{\frac{1}{2}}} + \frac{(l_n^{(1)}(d_o,k))^2}{\left(1-\frac{2}{n^\delta}\right) nr_k} \\
&\leq 3\epsilon_1 - \frac{nr_k}{4}\left(1-\frac{2}{n^\delta}\right)\left(d-d_o-\frac{b_n(d_o,k)}{1-\frac{2}{n^\delta}}\right)^2 \\
&\quad + \left| d-d_o-\frac{b_n(d_o,k)}{1-\frac{2}{n^\delta}} \right| \frac{n^{\frac{1}{2}-\delta}}{k^{\frac{1}{2}}} + \frac{(l_n^{(1)}(d_o,k))^2}{\left(1-\frac{2}{n^\delta}\right) nr_k},
\end{aligned} \tag{4.3}$$

The third inequality follows from (2.5) and Remark 3.1, by which $b_n(d_o) = O(k^{-\beta+\frac{1}{2}}) = O(\delta_n)$. This implies that $|b_n(d_o)| k^{-\frac{1}{2}} n^{\frac{1}{2}-\delta} < \epsilon_1$, again for large enough n. Similar to the preceding display, we have the lower-bound

$$\begin{aligned} l_n(d,k) - l_n(d_o,k) + \log I_n(d,k) - \log I_n(d_o,k) \\ \geq -3\epsilon_1 - \frac{nr_k}{4}(1+2n^{-\delta})\left(d - d_o - \frac{b_n(d_o,k)}{(1+2n^{-\delta})}\right)^2 \\ - \left| d - d_o - \frac{b_n(d_o,k)}{(1+2n^{-\delta})}\right| k^{-\frac{1}{2}} n^{\frac{1}{2}-\delta} + \frac{(l_n^{(1)}(d_o,k))^2}{(1+2n^{-\delta})nr_k}. \end{aligned} \tag{4.4}$$

Note that

$$\exp\left\{\frac{(l_n^{(1)}(d_o,k))^2}{(1-2n^{-\delta})nr_k} - \frac{(l_n^{(1)}(d_o,k))^2}{(1+2n^{-\delta})nr_k}\right\} = \exp\{o(1)\}, \tag{4.5}$$

which follows from the expression for $l_n^{(1)}(d_o,k)$ in Lemma 3.4, the definition of A_n and the assumption that $X \in A_n$. Therefore, substituting (4.3) in N_n and (4.4) in D_n, the terms $\frac{(l_n^{(1)}(d_o,k))^2}{4nr_k}$ cancel out and by (4.5) we can neglect the difference between $\frac{(l_n^{(1)}(d_o,k))^2}{(1\pm 2n^{-\delta})nr_k}$ and $\frac{(l_n^{(1)}(d_o,k))^2}{nr_k}$.

To conclude the proof that $N_n/D_n \leq \Phi(z) + o(1)$ for each $X \in A_n$, we make the change of variables

$$u = \sqrt{\frac{nr_k}{2}(1 \pm 2n^{-\delta})}\left(d - d_o - \frac{b_n(d_o)}{1 \pm 2n^{-\delta}}\right),$$

where we take $+$ in the lower bound for D_n and $-$ in the upper-bound for N_n. Using once more that $b_n(d_o) = O(\delta_n)$, we find that for large enough n, $|u| \leq \frac{\bar{v}_n}{4}\sqrt{nr_k}$ implies $|d - d_o| \leq \bar{v}_n$. Hence we may integrate over $|u| \leq \frac{\bar{v}_n}{4}\sqrt{nr_k}$ in the lower-bound for D_n. In the upper-bound for N_n we may integrate over $u \leq z + \epsilon_1$.

Combining (4.3)-(4.5), it follows that for all ϵ_1 and all $X \in A_n$,

$$\begin{aligned} \frac{N_n}{D_n} &\leq e^{7\epsilon_1}\left(\frac{1+2n^{-\delta}}{1-2n^{-\delta}}\right)^{\frac{1}{2}} \frac{\int_{u<z+\epsilon_1}\exp\{-\frac{1}{2}u^2 + Cn^{-\delta}|u|\}du}{\int_{|u|\leq \frac{\bar{v}_n}{4}\sqrt{nr_k}}\exp\{-\frac{1}{2}u^2 - Cn^{-\delta}|u|\}du} \\ &\leq e^{8\epsilon_1}\frac{\int_{u<z+\epsilon_1}\exp\{-\frac{1}{2}u^2 + Cn^{-\delta}|u|\}du}{\int_{|u|\leq \frac{\bar{v}_n}{4}\sqrt{nr_k}}\exp\{-\frac{1}{2}u^2 - Cn^{-\delta}|u|\}du} \to \Phi(z+\epsilon_1)e^{8\epsilon_1}. \end{aligned}$$

Similarly we prove that for all ϵ_1, $N_n/D_n \geq \Phi(z - \epsilon_1)e^{-8\epsilon_1}$, when n is large enough, which terminates the proof of Theorem 2.1.

5 Proof of Theorem 2.2

Let $\beta > 5/2$ and $\theta_{o,j} = c_0 j^{-(\beta+\frac{1}{2})}(\log j)^{-1}$. When the constant c_0 is chosen small enough, $\theta_o \in \Theta(\beta, L_o)$. In view of Corollary 3.1, the posterior mass on the events $\{(d,k,\theta) : \|\theta - \bar{\theta}_{d,k}\| \geq 2l_0\epsilon_n\}$ and $\{(d,k,\theta) : |d - d_o| \geq w_n\}$ tends to zero in probability, and may be neglected. Moreover Lemma 3.1 implies that with posterior probability going to 1, $\|\theta - \theta_0\| \lesssim (n/\log n)^{-(\beta-1/2)/(2\beta+1)}$. However, within the $(k+1)$-dimensional FEXP-model, $\|\theta - \theta_o\|$ is minimized by setting $\theta_j = \theta_{o,j}$ $(j = 0, \ldots, k)$, and for this choice of θ we have

$$\|\theta - \theta_o\|^2 = \sum_{l>k} \theta_{o,l}^2 \gtrsim k^{-2\beta}(\log k)^{-2}.$$

Consequently, the fact that $\|\theta - \theta_0\| \lesssim (n/\log n)^{-(\beta-1/2)/(2\beta+1)}$ implies that $k > k_n'' := k_l(n/\log n)^{(\beta-1/2)/(\beta(2\beta+1))}(\log n)^{-1/\beta}$, for some constant k_l. We conclude that

$$\Pi\left(k \leq k_n''|X\right) = \mathbf{o}_{\mathbf{P_o}}(1),$$

and we can restrict our attention to $k > k_n''$.

We decompose $\Pi_d(|d - d_o| \leq k_v w_n(\log n)^{-1}, k > k_n''|X)$ as

$$\begin{aligned}&\sum_{m>k_n''} \Pi(|d - d_o| \leq k_v w_n(\log n)^{-1}, k = m|X)\\ &\quad= \sum_{m>k_n''} \Pi(k = m|X)\Pi_m(|d - d_o| \leq k_v w_n(\log n)^{-1}|X),\end{aligned}$$

where $\Pi_m(|d-d_o| \leq k_v w_n(\log n)^{-1}|X)$ is the posterior for d within the FEXP-model of dimension $m+1$, i.e. $\Pi_m(|d-d_o| \leq k_v w_n(\log n)^{-1}|X) := \Pi(|d-d_o| \leq k_v w_n(\log n)^{-1}|k = m, X)$.

To prove Theorem 2.2 it now suffices to show that

$$\sum_{k_n'' \leq m \leq k_n'} \Pi(k = m|X) = \Pi(k_n'' \leq k \leq k_n'|X) \xrightarrow{P_o} 1, \tag{5.1}$$

$$E_0^n \Pi_k(|d - d_o| \leq k_v w_n(\log n)^{-1}|X) \xrightarrow{P_o} 0, \qquad \forall k_n'' \leq k \leq k_n'. \tag{5.2}$$

The convergence in (5.1) is a by-product of Theorem 1 in Rousseau and Kruijer (2011). In the remainder we prove (5.2). For every $k \leq k_n'$ we can write, using the notation of (4.1),

$$\begin{aligned}\Pi_k(|d - d_o| < k_v w_n(\log n)^{-1}|X) &\leq \frac{N_{n,k}}{D_{n,k}}\\ &:= \frac{\int_{|d-d_o|<k_v w_n(\log n)^{-1}} \exp\{l_n(d,k) - l_n(d_o,k) + \log I_n(d,k)\}d\pi_d(d)}{\int_{|d-d_o|<w_n} \exp\{l_n(d,k) - l_n(d_o,k) + \log I_n(d,k)\}d\pi_d(d)}.\end{aligned} \tag{5.3}$$

Let $\delta_2 > 0$ and A_n be the set of $X \in \mathbb{R}^n$ such that

$$\left.\begin{aligned}|\log I_n(d,k) - \log I_n(d_o,k)| &\leq \epsilon_1,\\ \left|l_n^{(1)}(d_o,k) - \mathcal{D}(l_n^{(1)}(d_o,k))\right| &\leq n^{\frac{1}{2}}k^{-\frac{1}{2}}\sqrt{\log n},\\ \left|l_n^{(2)}(d,k) - \mathcal{D}(l_n^{(2)}(d_o,k))\right| &\leq \epsilon_1 n^{-(2+\delta_2)/(2\beta+1)}\end{aligned}\right\}$$

for all $|d - d_o| \leq w_n$ and $k''_n \leq k \leq k'_n$. Compared to the definition of A_n in the proof of Theorem 2.1, the constraints on $l_n^{(2)}(d,k)$ and I_n are different. For the latter, recall from Lemma 3.5 that $\log I_n(d,k) = \log I_n(d_o,k) + \mathbf{o_{P_o}}(1)$, uniformly over $d \in (d_o - w_n, d_o + w_n)$. As in the proof of Theorem 2.1, it now follows from Lemmas 3.4 and 3.5 that $P_o^n(A_n^c) \to 0$. We can write

$$E_0^n \left[\frac{N_{n,k}}{D_{n,k}}\right] \leq P_o^n(A_n^c) + E_0^n \left[\frac{N_{n,k}}{D_{n,k}} 1_{A_n}\right],$$

and bound $N_{n,k}/D_{n,k}$ pointwise for $X \in A_n$. Since when $k \in (k''_n, k'_n)$,

$$\frac{(l_n^{(1)}(d_o,k))^2}{2|l_n^{(2)}(d_o,k)|} n^{-(2+\delta_2)/(2\beta+1)} = o(1)$$

on A_n, for all $\delta_2 > 0$, analogous to (4.3) and (4.4), we find that for all $X \in A_n$, by definition of $b_n(d_o)$,

$$l_n(d,k) - l_n(d_o,k) + \log I_n(d,k) \leq 2\epsilon_1 - \frac{|l_n^{(2)}(d_o,k)|}{2} (d - d_o - b_n(d_o))^2 + \frac{(l_n^{(1)}(d_o,k))^2}{2|l_n^{(2)}(d_o,k)|}$$
$$l_n(d,k) - l_n(d_o,k) + \log I_n(d,k) \geq -2\epsilon_1 - \frac{|l_n^{(2)}(d_o,k)|}{2} (d - d_o - b_n(d_o))^2 + \frac{(l_n^{(1)}(d_o,k))^2}{2|l_n^{(2)}(d_o,k)|},$$

when n is large enough since $k > k''_n$. We now lower-bound $b_n(d_o)$ by bounding the terms on the right in (3.6) in Lemma 3.4. By construction of θ_o it follows that

$$r_k^{-1} \sum_{j>k} j^{-1}\theta_{o,j} = c_0 r_k^{-1} \sum_{j>k} j^{-\beta-\frac{3}{2}}/(\log j) \geq c k^{-\beta+\frac{1}{2}}(\log k)^{-1},$$

for some $c > 0$. Since $X \in A_n$, $2\mathcal{S}(l_n^{(1)}(d_o,k))/(nr_k) \leq 2\sqrt{k/n}\sqrt{\log n}$. Since $k \leq k'_n$, this bound is $o(k^{-\beta+\frac{1}{2}}(\log k)^{-1})$. The last term in (3.6) is $o(n^{\epsilon-1})$ when $\beta > 5/2$, and hence this term is also $o(k^{-\beta-\frac{1}{2}}(\log k)^{-1})$. Therefore, the last two terms in (3.6) are negligible with respect to $r_k^{-1}\sum_{j>k} j^{-1}\theta_{o,j}$. We deduce that $b_n(d_o) \geq ck^{-\beta+\frac{1}{2}}(\log k)^{-1} \geq cn^{-(2\beta-1)/(4\beta+2)}(\log n)^{-(2\beta+3)/(4\beta+2)}$ for n large enough.

Consequently, when the constant k_v is chosen sufficiently small, $\sqrt{nr_{k'_n}}(b_n(d_o) - k_v w_n(\log n)^{-1}) \geq (c-k_v)n^{1/(4\beta+2)}(\log n)^{-(\beta+1)/(2\beta+1)} := z_n \to \infty$. We now substitute the above bounds on $l_n(d,k) - l_n(d_o,k) + \log I_n(d,k)$ in the right hand side of (5.3), make the change of variables $u = d - d_o - b_n(d_o)$ and obtain

$$\frac{N_{n,k}}{D_{n,k}} \leq e^{5\epsilon_1} \frac{\int_{u \leq -k_v w_n(\log n)^{-1} - b_n(d_o)} \exp\{-\frac{nr_k u^2}{4}\} du}{\int_{|u|<w_n/2} \exp\{-\frac{nr_k u^2}{4}\} du}$$
$$\leq e^{5\epsilon_1} \frac{\int_{v>z_n} \exp\{-\frac{v^2}{2}\} dv}{\int_{|v|<w_n\sqrt{nr_k/8}} \exp\{-\frac{v^2}{2}\} dv} = \mathbf{o_{P_o}}(1).$$

This achieves the proof of Theorem 2.2.

6 Conclusion

In this paper we have derived conditions leading to a BVM type of result for the long memory parameter $d \in (-\frac{1}{2}, \frac{1}{2})$ of a stationary Gaussian process, for the class of FEXP-priors. To our knowledge such a result has not been obtained before. The result implies in particular that asymptotically credible intervals for d have good frequentist coverage.

A by-product of our results is that the *most natural prior* (Prior C) from a Bayesian perspective, which is also the prior leading to adaptive minimax rates under the loss function l on f, leads to sub-optimal estimators in terms of d. Prior A leads to optimal estimators for d however it is not adaptive. An interesting direction for future work would be to define an adaptive- minimax estimation procedure for d.

More broadly speaking, the approach considered here to derive the asymptotic posterior distribution of a finite dimensional parameter of interest in a semi-parametric problems could be used in other non - regular models, hence completing (not exhaustively) the recent works of Castillo (2010) and Bickel and Kleijn (2010).

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A Proof of Lemma 3.4

We decompose the first derivative of $l_n(d,k)$ as $l_n^{(1)}(d,k) = \mathcal{S}(l_n^{(1)}(d,k)) + \mathcal{D}(l_n^{(1)}(d,k))$, $\mathcal{S}(l_n^{(1)}(d,k))$ being a centered quadratic form and $\mathcal{D}(l_n^{(1)}(d,k))$ the remaining deterministic term. To simplify notations, in this proof we write $\mathcal{S} = \mathcal{S}(l_n^{(1)}(d_o,k))$ and $\mathcal{D} = \mathcal{D}(l_n^{(1)}(d_o,k))$. Using (1.6) (supplement) and defining $A = T_n^{-1}(f_{d_o,k})T_n(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k})$, we find that

$$\mathcal{D} = -\frac{1}{2}\mathrm{tr}\left[(T_n(f_{d_o,k}) - T_n(f_o))A\right], \quad \mathcal{S} = \frac{1}{2}\left(X^t AX - \mathrm{tr}\left[T_n(f_o)A\right]\right).$$

From (1.4) and (1.8) in the supplement it follows that

$$\begin{aligned} f_o - f_{d_o,k} &= f_{d_o,k}(e^{\Delta_{d_o,k}} - 1) = (\Delta_{d_o,k} + \frac{1}{2}e^{\xi}\Delta_{d_o,k}^2)f_{d_o,k} \\ &= f_{d_o,k}O(k^{-\beta+1/2}), \quad \xi \in (0, (\Delta_{d_o,k})_+)). \end{aligned} \tag{A.1}$$

Consequently, we have

$$\begin{aligned} \mathcal{D} &= \frac{1}{2}\mathrm{tr}\left[T_n(f_{d_o,k}(\Delta_{d_o,k} + O(\Delta_{d_o,k}^2)))T_n^{-1}(f_{d_o,k})T_n(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k})\right] \\ &= \frac{n}{4\pi}\int_{-\pi}^{\pi} H_k(x)(\Delta_{d_o,k}(x) + O(\Delta_{d_o,k}^2(x)))dx + \text{error} \\ &= \frac{n}{2}\sum_{j=k+1}^{\infty} \eta_j\theta_{o,j} + O(nk^{-2\beta-1}) + \text{error}. \end{aligned}$$

The last equality follows from (1.9) and (1.11) in the supplement. We bound the error term using Lemma 2.4 (supplement) applied to $H_k f_{d_o,k}$ and $f_{d_o,k}$, whose Lipschitz constants are bounded by $O(k)$ and $O(k^{(3/2-\beta)_+}$, respectively (see Lemma 3.1 in the supplement). Using that $\|\Delta_{d_o,k}\|_\infty = O(k^{-\beta+1/2})$ (see (1.8) in the the supplement) we then find that the error is $O(k^{3/2-\beta}n^{\epsilon})$.

The term $\mathcal{S}$ is a centered quadratic form with variance $\frac{1}{2}|T_n^{\frac{1}{2}}(f_o)AT_n^{\frac{1}{2}}(f_o)|^2$. Applying once more (A.1), we find that

$$\begin{aligned} \mathrm{tr}\left[(T_n(f_o)A)^2\right] &= \mathrm{tr}\left[\left(T_n^{-1}(f_{d_o,k})T_n(H_k f_{d_o,k})\right)^2\right](1 + O(\|\Delta_{d_o,k}\|_\infty)) \\ &= \frac{n}{2\pi}\int_{-\pi}^{\pi} H_k^2(x)dx + O(n^{\epsilon}k + \|\Delta_{d_o,k}\|_\infty) = nr_k(1 + o(n^{-\delta})), \end{aligned}$$

where the term $n^{\epsilon}k$ comes from Lemma 2.4 in the supplement, associated to $f_{d_o,k}$ and $f_{d_o,k}H_k$. This proves the first equality in Lemma 3.4.

Similar to the decomposition of $l_n^{(1)}(d_o,k)$, we decompose the second derivative as $l_n^{(2)}(d,k) = \mathcal{D}(l_n^{(2)}(d,k)) - 2\mathcal{S}_1(l_n^{(2)}(d,k)) + \mathcal{S}_2(l_n^{(2)}(d,k))$, where

$$\mathcal{S}_1(d) = X^t A_{1,d} X - \mathrm{tr}[T_n(f_o)A_{1,d}], \quad \mathcal{S}_2(d) = X^t A_{2,d} X - \mathrm{tr}[T_n(f_o)A_{2,d}],$$

$$\begin{aligned}\mathcal{D}_2(d) &:= \mathcal{D}(l_n^{(2)}(d,k)) \\ &= -\frac{1}{2}\mathrm{tr}\left[T_n(f_{d,k})A_{1,d}\right] + \mathrm{tr}\left[(T_n(f_{d,k}) - T_n(f_{d_o,k}))\left(A_{1,d} - \frac{1}{2}A_{2,d}\right)\right] \\ &\quad + \mathrm{tr}\left[(T_n(f_{d_o,k}) - T_n(f_o))\left(A_{1,d} - \frac{1}{2}A_{2,d}\right)\right],\end{aligned} \tag{A.2}$$

$A_{1,d} = T_n^{-1}(f_{d,k})(T_n(H_k f_{d,k})T_n^{-1}(f_{d,k}))^2$, $A_{2,d} = T_n^{-1}(f_{d,k})T_n(H_k^2 f_{d,k})T_n^{-1}(f_{d,k})$.

To control $\mathcal{D}_2(d)$ we use a first order Taylor expansion around d_o, implying that $\mathcal{D}_2(d) = \mathcal{D}_2(d_o) + O(|d-d_o|\sup_{|d'-d_o|\le \bar{v}_n}|\mathcal{D}_2'(d')|$. First we study $\mathcal{D}_2(d_o)$. At $d = d_o$, the right-hand side of (A.2) equals

$$\begin{aligned}-\frac{n}{4\pi}\int_{-\pi}^{\pi} H_k^2(x)\left(1 + (e^{\Delta_{d_o,k}} - 1)\right)dx + O(kn^{\epsilon}) \\ = -\frac{-nr_k}{2}\left(1 + O(k^{-\beta+1/2} + k^2/n^{1-\epsilon})\right).\end{aligned} \tag{A.3}$$

The $O(kn^{\epsilon})$ term is obtained from Lemma 2.4 (supplement), applied to $f_{2j} = H_k f_{d_o,k}$ and $f_{2j-1} = f_{d_o,k}$, with Lipschitz constants $O(k)$ for the former and $O(k^{(3/2-\beta)_+})$ for the latter, together with the bound $\|\Delta_{d_o,k}\|_\infty = O(k^{-\beta+1/2})$. Using

$$\begin{aligned}A_{1,d}' = &-3\left(T_n^{-1}(f_{d,k})T_n(H_k f_{d,k})\right)^3 T_n^{-1}(f_{d,k}) \\ &+ 2T_n^{-1}(f_{d,k})T_n(H_k^2 f_{d,k})T_n^{-1}(f_{d,k})T_n(H_k f_{d,k})T_n^{-1}(f_{d,k})\end{aligned} \tag{A.4}$$

and a similar expression for the derivative of $d \to A_{2,d}$, it follows that

$$\begin{aligned}|\mathcal{D}_2'(d')| &\lesssim \mathrm{tr}\left[(T_n(f_{d_o,k})T_n^{-1}(f_{d',k}) + I_n)(T_n(|H_k|f_{d',k})T_n^{-1}(f_{d',k}))^3\right] \\ &+ \mathrm{tr}\left[(T_n(f_{d_o,k})T_n^{-1}(f_{d',k}) + I_n)(T_n(|H_k|f_{d',k})T_n^{-1}(f_{d',k}))T_n(H_k^2 f_{d',k})T_n^{-1}(f_{d',k})\right] \\ &+ \mathrm{tr}\left[(T_n(f_{d_o,k})T_n^{-1}(f_{d',k}) + I_n)T_n(|H_k|^3 f_{d',k})T_n^{-1}(f_{d',k})\right]\end{aligned}$$

We control the first term of the right hand side of the above inequality, the second and third terms are controlled similarly. Note first that

$$\begin{aligned}&\mathrm{tr}\left[T_n(f_{d_o,k})T_n^{-1}(f_{d',k})(T_n(|H_k|f_{d',k})T_n^{-1}(f_{d',k}))^3\right] \\ &= |T_n^{\frac{1}{2}}(f_{d_o,k})T_n^{-1}(f_{d',k})T_n(|H_k|f_{d',k})T_n^{-1}(f_{d',k})T_n^{\frac{1}{2}}(|H_k|f_{d',k})|^2 \\ &\le \|T_n^{\frac{1}{2}}(f_{d_o,k})T_n^{-\frac{1}{2}}(f_{d',k})\|^2 \\ &\qquad \times \|T_n^{-\frac{1}{2}}(f_{d',k})T_n^{\frac{1}{2}}(|H_k|f_{d',k})\|^2 |T_n^{-\frac{1}{2}}(f_{d',k})T_n(|H_k|f_{d',k})T_n^{-\frac{1}{2}}(f_{d',k})|^2 \\ &\lesssim n^{\epsilon}|T_n^{-\frac{1}{2}}(f_{d',k})T_n(|H_k|f_{d',k})T_n^{-\frac{1}{2}}(f_{d',k})|^2,\end{aligned} \tag{A.5}$$

where the last inequality comes from Lemma 2.3 in the supplement. Note also that

$$|x|^{-2d'} \lesssim f_{d'}(x) \lesssim |x|^{-2d'} \quad \text{and } T_n(|H_k| f_{d',k}) \lesssim T_n(|H_k||x|^{-2d'}), \quad T_n(|f_{d',k}) \gtrsim T_n(|x|^{-2d'})$$

and replace $f_{d'}$ by $|x|^{-2d'}$ in (A.5), then

$$|T_n^{-1/2}(f_{d',k}) T_n(|H_k| f_{d',k}) T_n^{-1/2}(f_{d',k})|^2 \lesssim \left(\frac{n}{k} + O(k)\right)$$

using Lemma 2.4 in the supplement associated to $|H_k||x|^{-2d'}$ which has Lipschitz constant k. This leads to $\mathcal{D}_2(d') = O\left(n^{\epsilon}\frac{n}{k}\right)$, which implies that for all $\beta > 1$,

$$\mathcal{D}_2(d) = \mathcal{D}_2(d_o) + \mathbf{o}(|d - d_o| n^{\epsilon+1} k^{-1}) = -\frac{n r_k}{2}(1 + \mathbf{o}(n^{-\delta})).$$

For the stochastic terms in $l_n^{(2)}(d,k)$ we need a chaining argument to control the supremum over $d \in (d_o - \bar{v}_n, d_o + \bar{v}_n)$. We show that for all $\epsilon' > 0$ and $\gamma_n = n^{\frac{1}{2}+\epsilon'} k^{-\frac{1}{2}}$,

$$P_o^n\left(\sup_{|d-d_o|\le \bar{v}_n} |\mathcal{S}_1(d)| > \gamma_n\right) \quad = \quad o(1), \tag{A.6}$$

i.e. that $\mathcal{S}_1(d) = \mathbf{o_{P_o}}(\gamma_n)$. The same can be shown for $\mathcal{S}_2(d)$ using exactly the same arguments. Consider a covering of $(d_o - \bar{v}_n, d_o + \bar{v}_n)$ by balls of radius n^{-1} centered at d_j, $j = 1, \dots, J_n$ with $J_n \le 2\bar{v}_n n$. Then

$$\sup_{|d-d_o|<\bar{v}_n} |\mathcal{S}_1(d)| \le \max_j |\mathcal{S}_1(d_j)| + \sup_{|d-d'|\le n^{-1}} |\mathcal{S}_1(d) - \mathcal{S}_1(d)|,$$

and

$$\begin{aligned} P_o^n\left(\sup_{|d-d_o|\le \bar{v}_n} |\mathcal{S}_1(d)| > \gamma_n\right) \le P_o^n\left(\sup_{|d-d'|\le n^{-1}} |\mathcal{S}_1(d) - \mathcal{S}_1(d')| > \frac{1}{2}\gamma_n\right) \\ + J_n \max_{1\le j\le J_n} P_o^n\left(|\mathcal{S}_1(d_j)| > \frac{1}{2}\gamma_n\right). \end{aligned} \tag{A.7}$$

To control the first term on the right in (A.7), note that for a standard normal vector Z and some $d^* \in (d, d')$,

$$\mathcal{S}_1(d) - \mathcal{S}_1(d') = (d - d')\left(Z^t T_n^{\frac{1}{2}}(f_o) A'_{1,d^*} T_n^{\frac{1}{2}}(f_o) Z - \text{tr}\left[T_n(f_o) A'_{1,d^*}\right]\right),$$

with $A'_{1,d}$ as in (A.4). Using Lemma 2.3 (supplement) and the fact that $\|AB\| \le \|A\|\|B\|$ for all matrices A and B, it follows that $\|T_n^{\frac{1}{2}}(f_o) A'_{1,d^*} T_n^{\frac{1}{2}}(f_o)\| = O(n^{\epsilon})$, and hence $Z^t T_n^{\frac{1}{2}}(f_o) A'_{1,d^*} T_n^{\frac{1}{2}}(f_o) Z \le Z^t Z \|T_n^{\frac{1}{2}}(f_o) A'_{1,d^*} T_n^{\frac{1}{2}}(f_o)\| = O(n^{\epsilon}) Z^t Z$. Similarly, it follows that $\text{tr}\left[T_n(f_o) A'_{1,d^*}\right] \lesssim n$. Consequently, when $\epsilon = \epsilon'/2$ we have

$$|\mathcal{S}_1(d) - \mathcal{S}_1(d')| \lesssim n^{-1}\left(Z^t Z n^{\epsilon} + n\right),$$

uniformly over all d, d' such that $|d - d'| \leq n^{-1}$. Since $1 = o(\gamma_n)$,

$$P_o^n \left(\sup_{|d-d'|\leq n^{-1}} |\mathcal{S}_1(d) - \mathcal{S}_1(d')| \geq \frac{1}{2}\gamma_n \right) \leq P\left(Z^t Z > n^{1-\epsilon}\gamma_n/4\right) = o(1).$$

To bound the last term in (A.7), we apply Lemma 1.3 (supplement) to $(Z^tAZ - \text{tr}[A])|A|^{-1}$, with $A = T_n^{\frac{1}{2}}(f_o)A_{1,d}T_n^{\frac{1}{2}}(f_o)$ since as seen previously $|A|^2 = \mathbf{O}(n/k) = \mathbf{o}(\gamma_n^2 n^{-2\alpha})$ for α small enough, it follows that

$$P_o^n \left(\mathcal{S}_1(d) \geq \frac{1}{2}\gamma_n \right) \leq e^{-n^\alpha/8}.$$

Since J_n increases only polynomially with n, this finishes the proof of (A.6).

B Control of the derivatives in θ on the log-likelihood

Before stating Lemma B.1 we first give a general expression for the derivatives of $l_n(d, k, \theta)$ with respect to θ. For all $j \geq 1$ and $l = (l_1, \ldots, l_j) \in \{0, 1, \ldots, k\}^j$, let $\sigma = (\sigma(1), \ldots, \sigma(|\sigma|))$ be a partition of $\{1, \ldots, j\}$. Let $|\sigma|$ be the number of subsets in this partition and $\sigma(i)$ the ith subset of $\{1, \ldots, j\}$ in the partition σ. Denoting $l_{\sigma(i)}$ the vector $(l_t, t \in \sigma(i))$, we can write

$$\nabla_{l_{\sigma(i)}} f_{d,k,\theta}(x) = \prod_{t\in\sigma(i)} \cos(l_t x) f_{d,k,\theta}(x).$$

For notational ease we write $\nabla_{\sigma(i)} f_{d,k,\theta} := \nabla_{l_{\sigma(i)}} f_{d,k,\theta}$. The derivative $\frac{\partial^j l_n(d,k,\theta)}{\partial\theta_{l_1}\ldots\partial\theta_{l_j}}$ can now be written in terms of the matrices

$$B_\sigma(d,\theta) = \prod_{i=1}^{|\sigma|} B_{\sigma(i)}(d,\theta), \quad B_{\sigma(i)}(d,\theta) = T_n(\nabla_{\sigma(i)} f_{d,k,\theta})T_n^{-1}(f_{d,k,\theta}). \tag{B.1}$$

There exist constants b_σ, c_σ and d_σ such that

$$\begin{aligned} &\frac{\partial^j l_n(d,k,\theta)}{\partial\theta_{l_1}\ldots\partial\theta_{l_j}} \\ &= \sum_{\sigma\in\mathcal{S}_j} b_\sigma \left(X^t T_n^{-1}(f_{d,k,\theta})B_\sigma(d,\theta)X - \text{tr}\left[T_n(f_o)T_n^{-1}(f_{d,k,\theta})B_\sigma(d,\theta)\right]\right) \\ &+ \sum_{\sigma\in\mathcal{S}_j} c_\sigma \text{tr}\left[B_\sigma(d,\theta)\right] + \sum_{\sigma\in\mathcal{S}_j} d_\sigma \text{tr}\left[(T_n(f_o)T_n^{-1}(f_{d,k,\theta}) - I_n)B_\sigma(d,\theta)\right], \end{aligned} \tag{B.2}$$

where $\mathcal{S}_j$ is the set of partitions of $\{1, \ldots, j\}$. For the first two derivatives $(j = 1, 2)$ the values of the constants b_σ, c_σ and d_σ are given below in Lemmas B.4 and B.5. For the higher order derivatives these values are not important for our purpose; we will only need that for any $j \geq 1$, the constant c_σ is zero if $|\sigma| = 1$.

The following lemma states that $l_n(d,k,\theta) - l_n(d,k)$ is the sum of a Taylor-approximation $\sum_{j=1}^{J} \frac{(\theta-\bar{\theta}_{d,k})^{(j)}\nabla^j l_n(d_o,k)}{j!}$ and terms whose dependence on d can be negligible. Since the proof is involved, some of the technical details are treated in Lemmas B.2 and B.3.

Lemma B.1. *Given $\beta > 1$, let $k \leq k_n$ and let d and θ be such that $l(f_o, f_{d,k,\theta}) \leq l_0^2\delta_n^2$. Then there exists an integer J and a constant $\epsilon > 0$ such that uniformly over $d \in (d_o - \bar{v}_n, d_o + \bar{v}_n)$ and $\theta \in B_k(\bar{\theta}_{d,k}, 2l_0\delta_n)$,*

$$\begin{aligned} l_n(d,k,\theta) - l_n(d,k) &= \sum_{j=1}^{J} \frac{(\theta-\bar{\theta}_{d,k})^{(j)}\nabla^j l_n(d_o,k)}{j!} \\ &\quad + (d-d_o)\sum_{j=2}^{J}\frac{1}{j!}g_{n,j}(\theta-\bar{\theta}_{d,k}) + S_n(d), \end{aligned} \tag{B.3}$$

where, for $u = \theta - \bar{\theta}_{d,k}$,

$$g_{n,j}(u) = \sum_{l_1,\ldots,l_j=0}^{k} u_{l_1}\cdots u_{l_j}\sum_{\sigma\in\mathcal{S}_j}\left(c_\sigma tr[T_{1,\sigma}(d_o,k)] + d_\sigma tr[T_{2,\sigma}(d_o,k)]\right), \tag{B.4}$$

$$\begin{aligned} T_{1,\sigma}(d_o,k) &= \sum_{i=1}^{|\sigma|}\left(\prod_{l<i} T_n(\nabla_{\sigma(l)}f_{d_o,k})T_n^{-1}(f_{d_o,k})\right)\times \\ &\quad \left[T_n(\nabla_{\sigma(i)}f_{d_o,k}H_k) - T_n(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k})T_n(\nabla_{\sigma(i)}f_{d_o,k})\right]\times \\ &\quad T_n^{-1}(f_{d_o,k})\left(\prod_{l>i} T_n(\nabla_{\sigma(l)}f_{d_o,k})T_n^{-1}(f_{d_o,k})\right), \end{aligned}$$

$$T_{2,\sigma}(d_o,k) = -T_n(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar{\theta}_{d_o,k}),$$

and $S_n(d)$ denotes any term of order

$$S_n(d) = \mathbf{o_{P_o}}(1) + \mathbf{o_{P_o}}\left(\frac{|d-d_o|n^{\frac{1}{2}-\delta}}{\sqrt{k}}\right) + \mathbf{o_{P_o}}\left((d-d_o)^2\frac{n^{1-\delta}}{k}\right). \tag{B.5}$$

When $\beta > 5/2$ and $k \leq k_n'$, we can choose $J = 2$, and (B.3) *simplifies to*

$$l_n(d,k,\theta) - l_n(d,k) = \sum_{j=1}^{2}\frac{(\theta-\bar{\theta}_{d,k})^{(j)}\nabla^j l_n(d_o,k)}{j!} + \mathbf{o_{P_o}}(1). \tag{B.6}$$

Proof. Recall that by (3.7),

$$\begin{aligned} l_n(d,k,\theta) - l_n(d,k) &= \sum_{j=1}^{J}\frac{(\theta-\bar{\theta}_{d,k})^{(j)}\nabla^j l_n(d_o,k)}{j!} \\ &+ \sum_{j=1}^{J}\frac{(\theta-\bar{\theta}_{d,k})^{(j)}\nabla^j(l_n(d,k)-l_n(d_o,k))}{j!} + R_{J+1,d}(\theta). \end{aligned} \tag{B.7}$$

To prove (B.3) we first show that, writing $u = \theta - \bar{\theta}_{d,k}$,

$$\sum_{j=1}^{J} \frac{u^{(j)} \nabla^j (l_n(d,k) - l_n(d_o,k))}{j!}$$

$$= \sum_{j=1}^{J} \frac{1}{j!} \sum_{l_1,\dots,l_j=0}^{k} u_{l_1} \dots u_{l_j} \left(\frac{\partial^j l_n(d,k,\bar{\theta}_{d,k})}{\partial\theta_{l_1}\dots\partial\theta_{l_j}} - \frac{\partial^j l_n(d_o,k,\bar{\theta}_{d_o,k})}{\partial\theta_{l_1}\dots\partial\theta_{l_j}} \right) \quad \text{(B.8)}$$

$$= (d - d_o) \sum_{j=1}^{J} \frac{1}{j!} g_{n,j}(u) + O(S_n(d)).$$

This result is combined with (B.7) and Lemma B.3 below, by which $g_{n,1}(u) = O(S_n(d))$. It then follows that $l_n(d,k,\theta) - l_n(d,k)$ equals

$$\sum_{j=1}^{J} \frac{(\theta - \bar{\theta}_{d,k})^{(j)} \nabla^j l_n(d_o,k)}{j!} + (d - d_o) \sum_{j=2}^{J} \frac{1}{j!} g_{n,j}(u) + R_{J+1,d}(\theta) + O(S_n(d)).$$

The final step is to prove that $R_{J+1,d}(\theta)$ is $\mathbf{o_{P_o}}(1)$ and hence $O(S_n(d))$; to this end J needs to be sufficiently large.

First we prove (B.8). For the factors $\frac{\partial^j}{\partial\theta} l_n(d,k,\bar{\theta}_{d,k}) - \frac{\partial^j}{\partial\theta} l_n(d,k,\bar{\theta}_{d_o,k})$ we substitute (B.2). In Lemma B.2 below we give expressions for each of the terms therein, which we substitute in (B.8). The main terms are $(d-d_o)\mathrm{tr}[T_{1,\sigma}(d_o,k)]$ and $(d-d_o)\mathrm{tr}[T_{2,\sigma}(d_o,k)]$ in (G.2) and (G.3), which after substitution in (B.8) give the term $(d-d_o)\sum_{j=1}^{J} \frac{1}{j!} g_{n,j}(u)$ on the right. The other terms in (G.1)-(G.3) that enter (B.8) through (B.2) are $O(S_n(d))$. This is due to the summation over $u_{l_1}, \dots, u_{l_j}$ in (B.8), and the Cauchy-Schwarz inequality by which

$$\left| \sum_{l_1,\dots,l_j=0}^{k} u_{l_1} \dots u_{l_j} \right| \leq (\sqrt{k}\|u\|)^j \leq (2l_0\sqrt{k}\delta_n)^j = o(n^{-\delta}), \quad \text{(B.9)}$$

for some $\delta > 0$, as $\|u\| \leq 2l_0\delta_n$ and (B.8) is proved. We now control $R_{J+1,d}(\theta)$.

Combining (3.8) and the first inequality in (B.9), we obtain

$$|R_{J+1,d}(\theta)| \leq \frac{1}{(J+1)!} (\sqrt{k}\delta_n)^{J+1} \max_{l_1,\dots,l_{J+1}} \sup_{\|\tilde{\theta}-\bar{\theta}_{d,k}\| \leq 2l_0\delta_n} \left| \frac{\partial^{J+1} l_n(d,k,\tilde{\theta})}{\partial\theta_{l_1}\dots\partial\theta_{l_{J+1}}}(x) \right|.$$

We give a direct bound on this derivative using (B.2). For all partitions σ of $\{l_1,\dots,l_{J+1}\}$ and all $(l_1,\dots,l_{J+1}) \in \{1,\dots,k\}^{J+1}$, we bound $\|B_{\sigma(i)}(d,\theta)\|$, using $\|B_{\sigma(i)}(d,\theta)\| \leq \|T_n^{\frac{1}{2}}(\nabla_{\sigma(i)} f_{d,k,\theta}) T_n^{-\frac{1}{2}}(f_{d,k,\theta})\|^2$ (see (1.6)). We bound $\|T_n^{\frac{1}{2}}(\nabla_{\sigma(i)} f_{d,k,\theta}) T_n^{-\frac{1}{2}}(f_{d,k,\theta})\|$ by application of Lemma 2.3 (supplement) with $f = f_{d,k,\theta}$ and $g = \nabla_{\sigma(i)} f_{d,k,\theta}$. The constant M in this lemma is bounded by

$$\sum_{j=0}^{k} |\theta_j| \leq \sum_{j=0}^{k} |(\bar{\theta}_{d,k})_j| + \sum_{j=0}^{k} |\theta_j - (\bar{\theta}_{d,k})_j| \leq 2\sqrt{L} + \sqrt{k}\|\theta - \bar{\theta}_{d,k}\| = O(1),$$

since $\sum_{i=0}^{k} |(\bar{\theta}_{d,k})_i| \leq 2\sqrt{L}$ (by Lemma 3.3) and $\|\theta - \bar{\theta}_{d,k}\| \leq \delta_n$. Consequently, Lemma 2.3 (supplement) implies that

$$\|B_{\sigma(i)}(d,\theta)\| \leq K, \tag{B.10}$$

where K depends only on L, L_o and not on n, d nor θ. From the relations in (1.6) and the definition of B_σ it follows that for any σ, d, θ,

$$|X^t B_\sigma(d,\theta) X| \leq X^t T_n^{-1}(f_o) X K^{|\sigma|} \|T_n^{\frac{1}{2}}(f_o) T_n^{-\frac{1}{2}}(f_{d,k})\|^2 \leq X^t T_n^{-1}(f_o) X K^{|\sigma|} n^{\epsilon},$$

$$|\mathrm{tr}\,[B_\sigma(d,\theta)]\,| \leq n K^{|\sigma|} \|T_n^{\frac{1}{2}}(f_o) T_n^{-\frac{1}{2}}(f_{d,k})\|^2 \leq n^{1+\epsilon} K^{|\sigma|}.$$

Therefore we have the bound

$$|R_{J+1}(d,\theta)| \leq C K^{J+1} n^{\epsilon} (\sqrt{k}\|\theta - \bar{\theta}_{d,k}\|)^{J+1} \left(X^t T_n^{-1}(f_o) X + n \right). \tag{B.11}$$

Since $k \leq k_n$, $\|\theta - \bar{\theta}_{d,k}\| \leq \delta_n$ and the term $X^t T_n^{-1}(f_o) X$ in (B.11) is the sum of n independent standard normal variables, there is a constant $c > 0$ such that

$$P_o \left(\sup_{|d-d_o| \leq \bar{v}_n} \sup_{\|\theta - \bar{\theta}_{d,k}\| \leq 2 l_0 \delta_n} |R_{J+1}(d,\theta)| > n^{-\epsilon} \right) \leq e^{-cn},$$

provided we choose J such that $(J+1)(1 - 1/\beta) > 2$. This concludes the proof of (B.3).

To prove (B.6) we first show that for $J = 2$, $|R_{J+1}(d,\theta)| = \mathbf{o_{P_o}}(1)$. Since $k \leq k_n'$, $\beta > 5/2$ and $\|\theta - \bar{\theta}_{d,k}\| \leq 2 l_0 \epsilon_n$, we can choose $J+1 = 3 > (2\beta+1)/(\beta - \frac{1}{2})$, and the preceding inequality becomes

$$P_o \left(\sup_{|d-d_o| \leq w_n} \sup_{\|\theta - \bar{\theta}_{d,k}\| \leq 2 l_0 \epsilon_n} |R_3(d,\theta)| > n^{-\epsilon} \right) \leq e^{-cn}.$$

Combining this result with (B.8), it only remains to be shown that $(d - d_o) g_{n,1}(u)$ and $(d - d_o) g_{n,2}(u)$ are $\mathbf{o_{P_o}}(1)$. Recall from Corollary 3.1 that $|d - d_o| = o(n^{\epsilon - (\beta - 1/2)/(2\beta+1)})$ for all $\epsilon > 0$. Consequently,

$$|d - d_o| n^{\frac{1}{2}} k^{-\frac{1}{2}} = o(n^{\epsilon + 1/(2\beta+1)} k^{-\frac{1}{2}}) = o((\sqrt{k}\epsilon_n)^{-1}), \quad (d - d_o)^2 \frac{n}{k} = o((\sqrt{k}\epsilon_n)^{-1})$$

for all $\beta > 2$. This implies that $S_n(d) = \mathbf{o_{P_o}}(1)$ and that, by Lemma B.3, $(d - d_o) g_{n,1}(u) = \mathbf{o}(1)$. Also, for all $(l_1, l_2) \in \{1, \ldots, k\}^j$ and all partitions σ of (l_1, l_2), the limiting integral of $\mathrm{tr}\,[T_{1,\sigma}]$ is equal to 0. Since $\beta > 5/2$ the Lipschitz constants of the functions $f_{d_o,k}$ or f_o are $O(1)$, so that Lemma 2.4 (supplement) implies $\mathrm{tr}\,[T_{1,\sigma}(d_o,k)] = O(n^{\epsilon} k)$. Similarly,

$$\mathrm{tr}\,[T_{2,\sigma}(d_o,k)] = \frac{n}{2\pi} \int_{-\pi}^{\pi} H_k(x) \cos(l_1 x) \cos(l_j x) dx + O(n^{\epsilon} k).$$

Thus we have

$$(d - d_o) g_{n,2}(u) = \frac{n(d - d_o)}{2\pi} \sum_{l_1, l_2 = 0}^{k} u_{l_1} u_{l_2} \int_{-\pi}^{\pi} H_k(x) \cos(l_1 x) \cos(l_j x) dx + \mathbf{o}(1),$$

which is $\mathbf{o}(1)$. This completes the proof of Lemma B.1. □

The proof of the following lemma is given in section 4 of the supplement.

Lemma B.2. *Let $W_\sigma(d)$ denote any of the quadratic forms*

$$X^t T_n^{-1}(f_{d,k}) B_\sigma(d,\bar\theta_{d,k}) X - tr\left[T_n(f_o)T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})\right]$$

in (B.2). *For any $j \le J$, $(l_1,\dots,l_j) \in \{0,\dots,k\}^j$ and $\sigma \in \mathcal{S}_j$, we have*

$$|W_\sigma(d) - W_\sigma(d_o)| = \mathbf{o}_{\mathbf{P_o}}(|d-d_o| n^{\frac12+\epsilon} k^{-\frac12}), \tag{B.12}$$

$$\begin{aligned} &tr\left[B_\sigma(d,\bar\theta_{d,k})\right] - tr\left[B_\sigma(d_o,\bar\theta_{d_o})\right] \\ &\quad = (d-d_o)tr[T_{1,\sigma}(d_o,k)] + (d-d_o)^2 \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+}) \qquad \text{(B.13)}\\ &\quad = (d-d_o)tr[T_{1,\sigma}(d_o,k)] + (d-d_o)^2\mathbf{o}(n^{1-\delta}/k), \end{aligned}$$

$$\begin{aligned} &tr\left[(T_n(f_o)T_n^{-1}(f_{d,k}) - I_n)B_\sigma(d,\bar\theta_{d,k})\right] - tr\left[(T_n(f_o)T_n^{-1}(f_{d_o,k}) - I_n)B_\sigma(d_o,\bar\theta_{d_o,k})\right] \\ &\quad = (d-d_o)tr[T_{2,\sigma}(d_o,k)] + (d-d_o)^2\mathbf{o}(n/k) + (d-d_o)\mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12}). \end{aligned} \tag{B.14}$$

Lemma B.3. *For all $\beta > 1$ there exists a constant $\delta > 0$ such that uniformly over $\|\theta - \bar\theta_{d,k}\| \le \delta_n$,*

$$|g_{n,1}(\theta - \bar\theta_{d,k})| = \mathbf{o}(n^{1/2-\delta}k^{-1/2}).$$

Proof. For $u = \theta - \bar\theta_{d,k}$, we have

$$g_{n,1}(u) = -\frac{1}{2}\mathrm{tr}\left[T_n(H_k f_{d_o,\bar\theta_{d_o,k}})T_n^{-1}(f_{d_o,\bar\theta_{d_o,k}})T_n(u^t\nabla f_{d_o,\bar\theta_{d_o,k}})T_n^{-1}(f_{d_o,\bar\theta_{d_o,k}})\right].$$

This follows from (B.4) and Lemma below, by which $b_\sigma = d_\sigma = \frac12$ and $c_\sigma = 0$ (the only partition for $j=1$ being $\sigma = (\{l\})$). By Lemma 2.4 (supplement) $g_{n,1}(u)$ converges to zero, but at a rate slower than $n^{1/2-\delta}k^{-1/2}$. To obtain the $\mathbf{o}(n^{1/2-\delta}k^{-1/2})$ term, we write

$$g_{n,1}(u) = \Delta_1 + \Delta_2 + \Delta_3,$$

and bound the terms on the right using the other lemmas in section 2 of the supplement. We first prove that

$$\begin{aligned}\Delta_1 = tr&\left[T_n(H_k f_{d_o,k})T_n(f_{d_o,k}^{-1})T_n(u^t\nabla f_{d_o,k})T_n(f_{d_o,k}^{-1})\right] - \\ &(16\pi^4)tr\left[T_n(H_k)T_n(u^t\mathbf{cos})\right] = \mathbf{o}(1),\end{aligned}$$

where $\mathbf{cos}(x) = (1,\cos(x),\dots,\cos(kx))$. We then prove that

$$\Delta_2 \;=\; tr\left[T_n(H_k)T_n(u^t\mathbf{cos})\right] = 0,$$

and finally that

$$\begin{aligned}\Delta_3 \;=\;& tr\left[T_n(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k})T_n(u^t\nabla f_{d_o,k})T_n^{-1}(f_{d_o,k})\right] \\ & - tr\left[T_n(H_k f_{d_o,k})T_n\left(\frac{f_{d_o,k}^{-1}}{4\pi^2}\right)T_n(u^t\nabla f_{d_o,k})T_n\left(\frac{f_{d_o,k}^{-1}}{4\pi^2}\right)\right] \\ =\;& \mathbf{o}(n^{1/2-\delta}k^{-1/2}).\end{aligned}$$

To bound Δ_1 we use Lemma 2.5 (supplement) with $b_1(x) = H_k(x)$, $b_2(x) = u^t\mathbf{cos}$ and $L = k^{3/2-\beta}$. Equation (2.6) then implies that

$$|\Delta_1| \leq C\sqrt{k}\|u\|n^\epsilon\left(1 + k^{3/2-\beta}k^{-1/2}\right) = \mathbf{o}(1).$$

To bound $Delta_2$ note that for $l = 0, \dots, k$ and all $j_1, j_2 \leq n$,

$$(T_n(\cos(lx)))_{j_1,j_2} = \mathbb{I}_{|j_1-j_2|=l}, \quad (T_n(H_k))_{j_1,j_2} = \sum_{j=k+1}^{n} \eta_j \mathbb{I}_{j=|j_1-j_2|}.$$

Therefore

$$\text{tr}\left[T_n(H_k)T_n(\cos(l.)\right] = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j=k+1}^{n} \eta_j \mathbb{I}_{j=|j_1-j_2|}\mathbb{I}_{|j_1-j_2|=l} = 0,$$

since $l \leq k$ and $j > k$. We now turn to Δ_3. Following Lieberman et al. (2011), we consider separately the positive and negative parts of H_k and of $u^t\cos$. Hence we may treat these functions as if they were positive. We first define, for $\tilde{f}_{d_o,k} = (4\pi^2 f_{d_o,k})^{-1}$,

$$\begin{aligned}
&A_1 = T_n(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k}), \quad B_1 = T_n(H_k f_{d_o,k})T_n(\tilde{f}_{d_o,k}),\\
&A_2 = T_n(u^t\nabla f_{d_o,k})T_n^{-1}(f_{d_o,k}), \quad B_2 = T_n(u^t\nabla f_{d_o,k})T_n(\tilde{f}_{d_o,k}),\\
&\tilde{A} = T_n^{\frac{1}{2}}(H_k f_{d_o,k})T_n^{-1}(f_{d_o,k})T_n^{\frac{1}{2}}(u^t\nabla f_{d_o,k}),\\
&\tilde{B} = T_n^{\frac{1}{2}}(H_k f_{d_o,k})T_n(\tilde{f}_{d_o,k})T_n^{\frac{1}{2}}(u^t\nabla f_{d_o,k}),\\
&\Delta = I_n - T_n(f_{d_o,k})T_n(\tilde{f}_{d_o,k}).
\end{aligned}$$

Using the same computations as in Lieberman et al. (2011), we find that

$$\begin{aligned}
|\Delta_3| &\lesssim |\text{tr}\left[B_1B_2\Delta\right]| + |\tilde{A} - \tilde{B}||T_n^{\frac{1}{2}}(H_k f_{d_o,k})T_n(\tilde{f}_{d_o,k})\Delta T_n^{\frac{1}{2}}(u^t\nabla f_{d_o,k})|\\
&\quad + |\Delta|^2\sqrt{k}\|u\|n^\epsilon\\
&\lesssim \sqrt{k}\|u\|n^\epsilon k^{3/2-\beta} + |\text{tr}\left[B_1B_2\Delta\right]|.
\end{aligned}$$

The first term on the right is $\mathbf{o}(n^{1/2-\delta}k^{-1/2})$. We bound the last term using Lemma 2.5 (supplement) with $b_1 = H_k$, $b_2 = u^t\mathbf{cos}$ and $b_3 = 1$, which implies that $\text{tr}\left[B_1B_2\Delta\right] = 0 + O(\sqrt{k}\|u\|n^\epsilon k^{(3/2-\beta)_+}) = o(1)$. This achieves the proof of Lemma B.3. □

Lemma B.4. *Suppose that $k \leq k_n$ and that $l(f_o, f_{d_o,k}) \leq l_0^2\delta_n^2$. Then all elements of $\nabla_l l_n(d_o,k)$ ($l = 0, \dots, k$) are the sum of a centered quadratic form, $\mathcal{S}(\nabla_l l_n(d_o,k))$ with a variance equal to $\frac{n}{2}(1 + o(1))$ and a deterministic term, $\mathcal{D}(\nabla_l l_n(d_o,k))$ which is $o(k^{(3/2-\beta)_+}n^\epsilon)$.*

Proof. For all $l = 0, \dots, k$, we have

$$\nabla_l l_n(d_o,k) = \mathcal{S}(\nabla_l l_n(d_o,k)) + \mathcal{D}(\nabla_l l_n(d_o,k)),$$

where

$$\mathcal{S}(\nabla_l l_n(d_o,k)) = \frac{1}{2}X^t T_n^{-1}(f_{d_o,k})T_n(\nabla_l f_{d_o,k})T_n^{-1}(f_{d_o,k})X$$
$$- \frac{1}{2}\text{tr}\left[T_n(f_o)T_n^{-1}(f_{d_o,k})T_n(\nabla_l f_{d_o,k})T_n^{-1}(f_{d_o,k})\right],$$

$$\mathcal{D}(\nabla_l l_n(d_o,k)) = \frac{1}{2}\text{tr}\left[\left(T_n(f_o)T_n^{-1}(f_{d_o,k}) - I_n\right)T_n(\nabla_l f_{d_o,k})T_n^{-1}(f_{d_o,k})\right].$$

Note that this is a special case of (B.2), with $j = 1$, $b_\sigma = d_\sigma = \frac{1}{2}$ and $c_\sigma = 0$, the only partition being $\sigma = (\{l\})$. The variance of $\mathcal{S}(\nabla_l l_n(d_o,k))$ is equal to

$$\text{tr}[(T_n(f_o)T_n^{-1}(f_{d_o,k})T_n(\nabla_l f_{d_o,k})T_n^{-1}(f_{d_o,k}))^2]$$
$$= \frac{n}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f_o}{f_{d_o,k}}(x)\right)^2\cos^2(lx)dx + \mathbf{O}(n^\epsilon k),$$

since Lemma 2.4 (supplement) implies that the approximation error of the trace by its limiting integral is of order $O(n^\epsilon(k+k^{2(3/2-\beta)\vee 0}) = O(n^\epsilon k)$. Since $\frac{f_o}{f_{d_o,k}} = e^{\Delta_{d_o,k}}$ (see (A.1)), the integral in the preceding equation is

$$\frac{n}{2\pi}\int_{-\pi}^{\pi}\left(1 + 2\Delta_{d_o,k} + O(\Delta_{d_o,k}^2)\right)\cos^2(lx)dx$$
$$= \frac{n}{2} + 2na_{2l}(d_o) + O(n\delta_n^2) = \frac{n}{2}(1+\mathbf{o}(1)),$$

where a_l is defined at the beginning of the supplement. Lemma 1.3 (supplement) then implies that the centered quadratic form is of order $\mathbf{o}_{\mathbf{P_o}}(n^{\epsilon+1/2})$. Similarly, Lemma 2.4 (supplement) implies that

$$\mathcal{D}(\nabla_l l_n(d_o,k)) = \frac{n}{2\pi}\int_{-\pi}^{\pi}\frac{(f_o - f_{d_o,k})}{f_{d_o,k}}(x)\cos(lx)dx + \mathbf{O}(n^\epsilon\|\Delta_{d_o,k}\|_1 k)$$
$$= \frac{n}{2\pi}\int_{-\pi}^{\pi}\cos(lx)\Delta_{d_o,k}(x)dx + \mathbf{O}(nk^{-2\beta}) + \mathbf{O}(n^\epsilon k^{3/2-\beta})$$
$$= \mathbf{O}(n^\epsilon k^{(3/2-\beta)_+})$$

which completes the proof of Lemma B.4. □

Lemma B.5. *Let $A(d)$ be the $(k+1)\times(k+1)$ matrix with entries $A_{l_1,l_2}(d) = a_{l_1+l_2}(d)$, where $a_l(d) = 1_{l>k}(\theta_{o,l} - 2l^{-1}(d_o - d))$. Suppose that $k \le k_n$ and that $l(f_o, f_{d,k}) \le l_0^2\delta_n^2$. Then $J_n(d,k) = -\nabla^2 l_n(d,k,\theta)\Big|_{\theta=\bar\theta_{d,k}}$ satisfies*

$$\forall l_1, l_2 \le k, \quad |(J_n(d,k) - J_n(d_o,k))_{l_1,l_2}| = \mathbf{o}_{\mathbf{P_o}}(|d-d_o|n^\epsilon k + S_n(d)) = \mathbf{o}_{\mathbf{P_o}}(n/k) \tag{B.15}$$

uniformly over $d \in (d_o - \bar v_n, d_o + \bar v_n)$ and $k \le k_n$. We also have for all l_1, l_2

$$[J_n(d_o,k) - \frac{n}{2}I_{k+1} - \frac{n}{2}A(d_o)]_{l_1,l_2} := n(R_{2s})_{l_1,l_2} + n(R_{2d})_{l_1,l_2}, \tag{B.16}$$

where $(R_{2s})_{l_1,l_2}$ *is a centered quadratic form of order* $\mathbf{o_{P_o}}(n^{-1/2+\epsilon})$ *and* $(R_{2d})_{l_1,l_2}$ *is a deterministic term of order* $o(kn^{\epsilon-1})$*. For the matrix A, we have* $\|A(d_o)\| = o(1)$ *and* $|A(d_o)| = O(1)$*.*

In particular, (B.16) implies that $|J_n(d_o,k) - \frac{n}{2}I_{k+1} - \frac{n}{2}A(d_o)| = \mathbf{o_{P_o}}(kn^{1/2+\epsilon}) + o(k^2n^{\epsilon})$.

Proof. Let d and $k \le k_n$ be such that $l(f_o, f_{d,k}) \le l_0^2\delta_n^2$ so that $d \in (d_o - \bar{v}_n, d_o + \bar{v}_n)$ (see Corollary 3.1). Lemma B.1 implies that for all $l_1, l_2 \le k$,

$$\begin{aligned}&(J_n(d,k))_{l_1,l_2} - (J_n(d_o,k))_{l_1,l_2}\\ &:= -(d-d_o)\sum_{\sigma\in\mathcal{S}(l_1,l_2)}\left(c_\sigma \mathrm{tr}\left[T_{1,\sigma}(d_o,k)\right] + d_\sigma \mathrm{tr}\left[T_{2,\sigma}(d_o,k)\right]\right) + \mathbf{O_{P_o}}(S_n(d)).\end{aligned}$$

Lemma 2.4 (supplement) implies that

$$\mathrm{tr}\left[T_{i,\sigma}(d_o,k)\right] = O(kn^{\epsilon}), \quad i = 1,2$$

so that (B.15) is satisfied since this term is $\mathbf{o_{P_o}}(n^{1-\delta}/k)$. We then use expression (B.2), with $\sigma \in \{(\{1\},\{2\}),(\{1,2\})\}$ and we denote σ_1 and σ_2 the first and the second partition respectively. Note that $c_{\sigma_1} = d_{\sigma_2} = 1/2$, $c_{\sigma_2} = 0$ and $d_{\sigma_1} = 1$. From Lemma 2.4 (supplement), the quadratic form in $(J_n(d_o,k))_{l_1,l_2}$ is associated to a matrix whose Frobenius-norm is $O(\sqrt{n})$ and whose spectral norm is $O(n^{\epsilon})$. Hence, this quadratic form is $\mathbf{o_{P_o}}(n^{1/2+\epsilon})$. Also by Lemma 2.4 (supplement), the deterministic terms can be written as

$$\begin{aligned}&\frac{n}{4\pi}\int_{-\pi}^{\pi}\cos(l_1x)\cos(l_2x)\left(1+\Delta_{d_o,k}\right)(x)dx + o(kn^{\epsilon})\\ &\qquad = \frac{n}{2}\left(\mathbb{I}_{l_1=l_2} + a_{l_1+l_2}(d_o)\right) + o(kn^{\epsilon}),\end{aligned}$$

and Lemma B.5 is proved. □

C Proof of Lemma 3.5

Under the conditions of Theorem 2.1 we have $k = k_n$ and $\beta > 1$, and we may assume (by Lemma 3.2) that $l(f_o, f_{d,k,\theta}) \le l_0^2\delta_n^2$. Fixing d and k, we develop $\theta \to l_n(d,k,\theta)$ in $\bar{\theta}_{d,k}$. From Lemma B.1 in Appendix B it follows that

$$\begin{aligned}l_n(d,k,\theta) - l_n(d,k) &= \sum_{j=1}^{J}\frac{(\theta-\bar{\theta}_{d,k})^{(j)}\nabla^j l_n(d_o,k)}{j!}\\ &\quad + (d-d_o)\sum_{j=2}^{J}\frac{g_{n,j}(\theta-\bar{\theta}_{d,k})}{j!} + S_n(d),\end{aligned} \tag{C.1}$$

where $S_n(d)$ is as in (B.5). Substituting (C.1) in the definition of $I_n(d,k)$ in (3.1), we obtain

$$\begin{aligned}
I_n(d,k) &= \int_{\|\theta-\bar{\theta}_{d,k}\|\leq 2l_0\delta_n} e^{l_n(d,k,\theta)-l_n(d,k)} d\pi_{\theta|k}(\theta) \\
&= e^{S_n(d)}\pi_{\theta|k}(\bar{\theta}_{d,k}) \int_{\|u\|\leq 2l_0\delta_n} e^{\sum_{j=1}^J \frac{1}{j!}u^{(j)}\nabla^j l_n(d_o,k)+(d-d_o)\sum_{j=2}^J \frac{g_{n,j}(u)}{j!}+h_k u} du \\
&= e^{S_n(d)}\pi_{\theta|k}(\bar{\theta}_{d_o,k}) \int_{\|u\|\leq 2l_0\delta_n} e^{\sum_{j=1}^J \frac{1}{j!}u^{(j)}\nabla^j l_n(d_o,k)+(d-d_o)\sum_{j=2}^J \frac{g_{n,j}(u)}{j!}+h_k u} du.
\end{aligned} \tag{C.2}$$

The first equality follows from the definition of $I_n(d,k)$ and Lemma 3.3, by which we may replace the domain of integration by $\{\theta : \|\theta-\bar{\theta}_{d,k}\| \leq 2l_0\delta_n\}$. The second equality follows from the assumptions on $\pi_{\theta|k}$ in **prior A**, the transformation $u = \theta - \bar{\theta}_{d,k}$ and substitution of (C.1). Also the third equality follows from the assumptions on $\pi_{\theta|k}$: these imply that

$$\left|\log \pi_{\theta|k}(\bar{\theta}_{d,k}) - \log \pi_{\theta|k}(\bar{\theta}_{d_o,k})\right| = |d_o-d||h_k^t \eta_{[k]}|+o(1) = O(|d-d_o|(n/k)^{\frac{1}{2}-\epsilon})+o(1),$$

for some $\epsilon > 0$. Thus, the factor $e^{S_n(d)}\pi_{\theta|k}(\bar{\theta}_{d,k})$ on the second line of (C.2) may be replaced by $e^{S_n(d)}\pi_{\theta|k}(\bar{\theta}_{d_o,k})$. Because $S_n(d_o) = \mathbf{o_{P_o}}(1)$, (C.2) implies that

$$I_n(d_o,k) = (1+\mathbf{o_{P_o}}(1)) \int_{\|u\|\leq 2l_0\delta_n} \exp\left\{ h_k u + \sum_{j=1}^J \frac{u^{(j)}\nabla^j l_n(d_o,k)}{j!} \right\} du. \tag{C.3}$$

The most involved part of the proof is to establish the bounds

$$\begin{aligned}
&\pi_{\theta|k}(\bar{\theta}_{d_o,k}) \int_{\|u\|\leq l_0\delta_n} \exp\left\{ h_k u + \sum_{j=1}^J \frac{u^{(j)}\nabla^j l_n(d_o,k)}{j!} \right\} du \\
&\leq I_n(d,k) = e^{S_n(d)}\pi_{\theta|k}(\bar{\theta}_{d_o,k}) \int_{\|u\|\leq 2l_0\delta_n} e^{\sum_{j=1}^J \frac{1}{j!}u^{(j)}\nabla^j l_n(d_o,k)+(d-d_o)\sum_{j=2}^J \frac{g_{n,j}(u)}{j!}+h_k u} du \\
&\leq \pi_{\theta|k}(\bar{\theta}_{d_o,k}) \int_{\|u\|\leq 3l_0\delta_n} \exp\left\{ h_k u + \sum_{j=1}^J \frac{u^{(j)}\nabla^j l_n(d_o,k)}{j!} \right\} du.
\end{aligned} \tag{C.4}$$

Since the posterior distribution of θ conditional on $k = k_n$ and $d = d_o$ concentrates at $\bar{\theta}_{d_o,k}$ at a rate bounded by $l_0\delta_n$ (this follows from Lemma 3.2, with the restriction to $d = d_o$), the left- and right-hand side of (C.4) are asymptotically equal, up to a factor $(1 + \mathbf{o_{P_o}}(1))$. By (C.3), the left- and right-hand side are actually equal to $I_n(d_o,k)$. This implies that $I_n(d,k) = e^{S_n(d)}I_n(d_o,k)$, which is the required result.

In the remainder we prove (C.4). To do so we construct below a change of variables $v = \psi(u)$, which satisfies

$$
\begin{aligned}
h_k v + \sum_{j=1}^{J} \frac{v^{(j)} \nabla^j l_n(d_o, k)}{j!} &= h_k u + \sum_{j=1}^{J} \frac{u^{(j)} \nabla^j l_n(d_o, k)}{j!} \\
&\quad + (d - d_o) \sum_{j=2}^{J} \frac{g_{n,j}(u)}{j!} + O(S_n(d)),
\end{aligned}
\tag{C.5}
$$

for all $\|u\| \leq 2l_0\delta_n$. We first define the notation required in the definition of ψ in (C.8) below. Recall from (B.4) in Lemma B.1 that $g_{n,j}(u)$ can be decomposed as

$$
g_{n,j}(u) = n \sum_{\sigma \in \mathcal{S}_j} (c_\sigma - d_\sigma) \sum_{l_1,\ldots,l_j=0}^{k} u_{l_1} \ldots u_{l_j} g^{(j)}_{l_1,\ldots,l_j}, \qquad j = 2, \ldots J,
$$

where $g^{(j)}_{l_1,\ldots,l_j}$ depends on σ. For ease of presentation however we omit this dependence in the notation. Using Lemma 2.4 (supplement) and (B.4) in Lemma B.1, it follows that for all $j \geq 2$ and $(l_1, \ldots, l_j) \in \{0, \ldots, k\}^j$,

$$
g^{(j)}_{l_1,\ldots,l_j} = \gamma^{(j)}_{l_1,\ldots,l_j} + r^{(j)}_{l_1,\ldots,l_j}, \tag{C.6}
$$

$$
\gamma^{(j)}_{l_1,\ldots,l_j} = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_k(x) \cos(l_1 x) \cdots \cos(l_j x) dx.
$$

Let $\bar{G}^{(2)}$ denote the matrix with elements $\gamma^{(2)}_{l_1,l_2}$, and $G^{(2)}$ the matrix with elements $g^{(2)}_{l_1,l_2}$. By direct calculation it follows that

$$
\gamma^{(2)}_{l_1,l_2} = \mathbb{I}_{l_1+l_2>k} \frac{1}{2(l_1 + l_2)}. \tag{C.7}
$$

Similarly, for all $j \geq 3$ and $l_1, \ldots, l_j \in \{0, 1, \ldots, k\}$ we define

$$
(G^{(j)}(u))_{l_1,l_2} = \sum_{l_3,\ldots,l_j=0}^{k} g^{(j)}_{l_1,\ldots,l_j} u_{l_3} \ldots u_{l_j}, \; (\bar{G}^{(j)}(u))_{l_1,l_2} = \sum_{l_3,\ldots,l_j=0}^{k} \gamma^{(j)}_{l_1,\ldots,l_j} u_{l_3} \ldots u_{l_j}.
$$

In contrast to $G^{(2)}$ and $\bar{G}^{(2)}$, $G^{(j)}(u)$ and $\bar{G}^{(j)}(u)$ depend on u. For notational convenience we will also write $G^{(2)}(u)$ and $\bar{G}^{(2)}(u)$. Finally, let $\tilde{I}_k = J_n(d_o, k)/n$ be the normalized Fisher information.

We now define the transformation ψ:

$$
\psi(u) = (I_{k+1} - (d - d_o) D(u)) u, \quad \text{with} \tag{C.8}
$$

$$
D(u) = (\tilde{I}_k + L(u))^{-1} G^t(u), \quad G(u) = \sum_{j=2}^{J} \frac{1}{j!} \sum_{\sigma \in \mathcal{S}_j} (c_\sigma - d_\sigma) G^{(j)}(u),
$$

$$(L(u))_{l_1,l_2} = -\sum_{j=3}^{J} \frac{1}{n(j-1)!} \sum_{l_3,\ldots,l_j=0}^{k} u_{l_3} \ldots u_{l_j} \nabla_{l_1,\ldots,l_j} l_n(d_o,k). \tag{C.9}$$

The construction of $G(u)$ is such that

$$nu^t G(u) u = \sum_{j=2}^{J} g_{n,j}(u). \tag{C.10}$$

Analogous to $G(u)$ and $D(u)$ we define $\bar{G}(u) = \sum_{j=2}^{J} \frac{1}{j!} \sum_{\sigma \in \mathcal{S}_j} (c_\sigma - d_\sigma) \bar{G}^{(j)}(u)$ and $\bar{D}(u) = (\tilde{I}_k + L(u))^{-1} \bar{G}^t(u)$. After substitution of $v = \psi(u)$, and using (C.25) in Lemma C.1 it follows that

$$\sum_{j=3}^{J} \frac{(v^{(j)} - u^{(j)}) \nabla^j l_n(d_o,k)}{j!} =$$
$$-(d-d_o) \sum_{j=3}^{J} \sum_{l_1,\ldots,l_j=0}^{k} (D(u)u)_{l_1} u_{l_2} \ldots u_{l_j} \frac{\nabla_{l_1,\ldots,l_j} l_n(d_o,k)}{(j-1)!} + \mathbf{O}(S_n(d)).$$

The definitions of $D(u)$ and $L(u)$ and (C.10) imply that

$$\begin{aligned}
&-n(v-u)^t \tilde{I}_k u = n(d-d_o) u^t D^t(u) \tilde{I}_k u = n(d-d_o) u^t G(u) (\tilde{I}_k + L(u))^{-1} \tilde{I}_k u \\
&= n(d-d_o) u^t G(u) \left(I_{k+1} - (\tilde{I}_k + L(u))^{-1} L(u) \right) u \\
&= (d-d_o) \sum_{j=2}^{J} \frac{1}{j!} g_{n,j}(u) - n(d-d_o)(D(u)u)^t L(u) u \\
&= (d-d_o) \sum_{j=2}^{J} \frac{1}{j!} g_{n,j}(u) - (d-d_o) \sum_{j=3}^{J-1} \sum_{l_1,\ldots,l_j=0}^{k} (D(u)u)_{l_1} u_{l_2} \ldots u_{l_j} \frac{\nabla_{l_1,\ldots,l_j} l_n(d_o,k)}{(j-1)!}.
\end{aligned}$$

At the same time, the definition of $\tilde{I}_k$ implies that

$$\frac{1}{2} \left(v^{(2)} - u^{(2)} \right) \nabla^2 l_n(d_o,k) = -n(v-u)^t \tilde{I}_k u - \frac{n(v-u)^t \tilde{I}_k (v-u)}{2}.$$

Combining the preceding results, we find that

$$\begin{aligned}
&h_k v + \sum_{j=1}^{J} \frac{v^{(j)} \nabla^j l_n(d_o,k)}{j!} - \left(h_k u + \sum_{j=1}^{J} \frac{u^{(j)} \nabla^j l_n(d_o,k)}{j!} + (d-d_o) \sum_{j=2}^{J} \frac{g_{n,j}(u)}{j!} \right) \\
&\quad = h_k(v-u) + (v-u)^t \nabla l_n(d_o,k) - \frac{n(v-u)^t \tilde{I}_k (v-u)}{2} + O(S_n(d)) \\
&\quad = (v-u)^t \nabla l_n(d_o,k) + O(S_n(d)),
\end{aligned}$$

where the last equality follows from (C.24) below in Lemma C.1, together with the assumption on h_k in **prior A** in (2.2).

Apart from the term $(v-u)^t \nabla l_n(d_o,k)$ on the last line, the preceding display implies (C.5). Hence, to complete the proof of (C.5) it suffices to show that

$$(v-u)^t \nabla l_n(d_o,k) = -(d-d_o)u^t D^t(u) \nabla l_n(d_o,k) = O(S_n(d)). \tag{C.11}$$

The proof of (C.11) consists of the following steps:

$$\begin{aligned} |u^t(D(u)-\bar{D}(u))^t \nabla l_n(d_o,k)| &= \mathbf{o_{P_o}}(n^{\frac{1}{2}-\delta}k^{-\frac{1}{2}}), & \text{(C.12)} \\ (d-d_o)u^t \bar{D}^t(u)\mathcal{D}(\nabla l_n(d_o,k)) &= O(S_n(d)), & \text{(C.13)} \\ (d-d_o)u^t \bar{D}^t(u)\mathcal{S}(\nabla l_n(d_o,k)) &= O(S_n(d)), & \text{(C.14)} \end{aligned}$$

where $\mathcal{S}(\nabla l_n(d_o,k))$ denotes the centered quadratic form in $\nabla l_n(d_o,k)$, and $\mathcal{D}(\nabla l_n(d_o,k))$ the remaining deterministic term. We will use the same notation below for $L(u)$.

Equation (C.12) follows from Lemma B.4 and (C.22) in Lemma C.1 below, which imply that the left-hand side equals $\mathbf{o_{P_o}}((\sqrt{k}\|u\|)^2 n^{-1+\epsilon}k\sqrt{n}) = \mathbf{o_{P_o}}(n^{\frac{1}{2}-\delta}k^{-\frac{1}{2}})$, for some $\delta > 0$. For the proof of (C.13), note that Lemma B.4 implies

$$|u^t \bar{D}^t(u)\mathcal{D}(\nabla l_n(d_o,k))| \lesssim \|\bar{D}(u)u\|\sqrt{k}k^{(3/2-\beta)_+ + \epsilon}.$$

Combined with Lemma C.1, this implies that the left-hand side is $O(\sqrt{k}k^{5/2-2\beta+\epsilon})$, which is $O(S_n(d))$. The proof of (C.14) is more involved. Recall that $\bar{D}(u)$ is defined as $\bar{D}(u) = (\tilde{I}_k + L(u))^{-1}\bar{G}^t(u)$. Using (B.16) in Lemma B.5, we obtain

$$(\tilde{I}_k + L(u))^{-1} = 2[I_{k+1} - (A(d_o) + R_{2s} + R_{2d} + L(u))(1 + \mathbf{o_{P_o}}(1))].$$

Substituting this in $\bar{D}(u)$, it follows that (C.14) can be proved by controlling $\bar{G}^{(j)}\mathcal{S}(\nabla l_n(d_o))$, $\bar{G}^{(j)}A(d_o)\mathcal{S}(\nabla l_n(d_o))$, $\bar{G}^{(j)}R_{2d}\mathcal{S}(\nabla l_n(d_o))$, $\bar{G}^{(j)}\mathcal{D}(L(u))\mathcal{S}(\nabla l_n(d_o))$, $\bar{G}^{(j)}R_{2s}\mathcal{S}(\nabla l_n(d_o))$ and $\bar{G}^{(j)}\mathcal{S}(L(u))\mathcal{S}(\nabla l_n(d_o))$ for all $j = 3,\dots,J$. To do so, first note that Lemma B.5 implies that $\|\bar{G}^{(j)}R_{2s}\mathcal{S}(\nabla l_n(d_o))\| = \mathbf{o_{P_o}}(n^\epsilon\sqrt{k})$. Hence,

$$|u^t\bar{G}^{(j)}R_{2s}\mathcal{S}(\nabla l_n(d_o))| = \mathbf{o_{P_o}}(k^{-\beta+1+\epsilon}) = \mathbf{o_{P_o}}(1),$$

which clearly is $O(S_n(d))$. The terms $\bar{G}^{(j)}\mathcal{S}(\nabla l_n(d_o))$, $\bar{G}^{(j)}A(d_o)\mathcal{S}(\nabla l_n(d_o))$, $\bar{G}^{(j)}R_{2d}\mathcal{S}(\nabla l_n(d_o))$ and $\bar{G}^{(j)}\mathcal{D}(L(u))\mathcal{S}(\nabla l_n(d_o))$ can be written as quadratic forms $Z^tMZ - tr[M]$, where, for a sequence $(b_l)_{l=0}^k$ and a function g with $\|g\|_\infty < \infty$, M is of the form

$$T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d_o,k})T_n\left(g(x)f_{d_o,k}(x)\sum_l b_l\cos(lx)\right)T_n^{-1}(f_{d_o,k})T_n^{\frac{1}{2}}(f_o),$$

Z being a vector of n independent standard Gaussian random variables. Using Lemma 2.4 (supplement) it can be seen that $|M|^2 \le n(\sum_l b_l^2 + k/n)$. Lemma 1.3 (supplement) with $\alpha = \epsilon + 1/2$ then implies that

$$P_o\left(|Z^tMZ - tr[M]| > n^{\epsilon+\frac{1}{2}}\left(\sum_l b_l^2 + \frac{k}{n}\right)^{\frac{1}{2}}\right) \le e^{-cn^\epsilon}. \tag{C.15}$$

For all $j \in \{3,\dots,J\}$, the four terms above can now be bounded for a particular choice of g and b_l.

- Bound on $\bar{G}^{(j)}\mathcal{S}(\nabla l_n(d_o))$. For all $l_2,\dots,l_j \in \{0,\dots,k\}$, set $b_l = \gamma^{(j)}_{l_2,l,l_4,\dots,l_j}$ and $g(x)=1$. Then we have

$$\sum_{l=0}^{k} b_l \mathcal{S}(\nabla l_n(d_o)))_{l_1} = \mathbf{o_{P_o}}\left(n^{\frac{1}{2}+\epsilon}\left(\sum_l b_l^2\right)^{\frac{1}{2}}\right).$$

By induction it can be shown that

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}&\cos(l_0x)\cos(l_1x)\cos(l_2x)\cdots\cos(l_jx)dx\\ &= 2^{-j}\sum_{\epsilon_1,\dots,\epsilon_j\in\{-1,1\}} \mathbb{I}_{l_0+\sum_{i=1}^{j}\epsilon_i l_i=0}\end{aligned} \tag{C.16}$$

Consequently, $\sum_l b_l^2 = O(k^{-1})$ for all $l_2, l_4,\dots,l_j \in \{0,\dots,k\}$. Using the fact that $\sum_l |u_l| = o(1)$, we obtain that $(\bar{G}^{(j)}\mathcal{S}(\nabla l_n(d_o)))_{l_2} = \mathbf{o_{P_o}}(n^{1/2+\epsilon}k^{-1/2})$, for all $l_2 \in \{0,\dots,k\}$. This implies that

$$\|\bar{G}^{(j)}\mathcal{S}(\nabla l_n(d_o))\| = \mathbf{o_{P_o}}(n^{1/2+\epsilon}) = O(S_n(d)). \tag{C.17}$$

- Bound on $A(d_o)\mathcal{S}(\nabla l_n(d_o))$. Set $b_l = \mathbb{I}_{l+l_1\geq k}\theta_{o,l+l_1}$ and $g(x)=1$, then

$$(A(d_o)\mathcal{S}(\nabla l_n(d_o)))_{l_1} = \mathbf{o_{P_o}}(n^{1/2+\epsilon}k^{-\beta}), \quad \forall l_1 \in \{0,\dots,k\}.$$

Combined with Lemma C.1 this implies that

$$\|\bar{G}^{(j)}A(d_o)\mathcal{S}(\nabla l_n(d_o))\| = \mathbf{o_{P_o}}(n^{1/2+\epsilon}k^{-\beta+1/2}).$$

- Bound on $\bar{G}^{(j)}R_{2d}\mathcal{S}(\nabla l_n(d_o)$. Set $b_l = (R_{2d})_{l_1,l}$ and $g(x)=1$, for all $l_1 = 0,\dots,k$, then Lemmas B.5 and C.1 lead to

$$\|\bar{G}^{(j)}R_{2d}\mathcal{S}(\nabla l_n(d_o))\| = \mathbf{o_{P_o}}(n^{1/2+\epsilon}k^2n^{-1}) = \mathbf{o_{P_o}}(n^{-1/2+\epsilon}k^2).$$

- Bound on $\bar{G}^{(j)}\mathcal{D}(L(u))\mathcal{S}(\nabla l_n(d_o))$. For all $l_1, l_3,\dots,l_j, l_3',\dots,l_{j'}' \in \{0,\dots,k\}$, set

$$\begin{aligned}b_l = \frac{1}{n}\mathrm{tr}\Bigg[&T_n^{-1}(f_{d_o})T_n\left(\sum_{l_2=0}^{k}\gamma^{(j)}_{l_1,l_2,\dots,l_j}\cos(l_2x)g_1(x)f_{d_o}(x)\right)\times\\ &T_n^{-1}(f_{d_o})T_n(\cos(lx)g_2(x)f_{d_o}(x))\cdots T_n^{-1}(f_{d_o})T_n(g_r(x)f_{d_o}(x))\Bigg]\end{aligned}$$

where $g_1(x),\dots,g_r(x)$ are products of functions of the form $\cos(l_i'x)$ and $g_1(x)\dots g_r(x) = \cos(l_3'x)\dots\cos(l_{j'}'x)$. Lemmas 2.1 and 2.6 in the supplement, together with (C.16), imply that

$$\sum_l b_l^2 = O(k^{-1}), \quad \text{and} \quad \|\bar{G}^{(j)}\mathcal{D}(L(u))\mathcal{S}(\nabla l_n(d_o))\| = \mathbf{o_{P_o}}(n^{\epsilon+1/2}). \tag{C.18}$$

Consequently, the contribution to all these terms in $(v-u)^t \nabla l_n(d_o,k)$ is of order $\mathbf{O}(S_n(d))$.

We control $u^t \bar{G}^{(j)} \mathcal{S}(L(u))\mathcal{S}(\nabla l_n(d_o,k))$, by bounding $\|\bar{G}^{(j)}\mathcal{S}(L(u))\|$ using a similar idea. Indeed, for all $l_1, l_2 \le k$, $(\bar{G}^{(j)}\mathcal{S}(L(u)))_{l_1,l_2}$ can be written as a sum of terms of the form $(Z^t M_{l_1,l_2} Z - tr(M_{l_1,l_2}))/n$, where Z is a vector of n independent standard Gaussian random variables, and M_{l_1,l_2} has the form

$$T_n^{\frac{1}{2}}(f_o) T_n^{-1}(f_{d_o,k}) \left(\prod_{i<i_0} T_n(\nabla_\sigma(i) f_{d_o,k}) T_n^{-1}(f_{d_o,k}) \right) \times$$
$$T_n\left(\sum_{l=0}^{k} \gamma^{(j)}_{l,l_1,l_2,\ldots,l_{j-1}} \cos(lx) \nabla_{\sigma(i_0)-\{l\}} f_{d_o,k} \right) T_n^{-1}(f_{d_o,k}) \prod_{i<i_0} T_n(\nabla_\sigma(i) f_{d_o,k}) T_n^{\frac{1}{2}}(f_o).$$

We can use the same argument as in (C.15) since for all $l_1, l_2, \ldots, l_{j-1}$

$$\begin{aligned} |M_{l_1,l_2}| &\lesssim |T_n^{-\frac{1}{2}}(f_{d_o,k} T_n(\sum_{l=0}^{k} \gamma^{(j)}_{l,l_1,l_2,\ldots,l_{j-1}} \cos(lx) \nabla_{\sigma(i_0)-\{l\}} f_{d_o,k}) T_n^{-\frac{1}{2}}(f_{d_o,k})| \\ &= O(n^{1/2+\epsilon} k^{-1/2}). \end{aligned}$$

Hence, it follows that $n^{-1}[Z^t M_{l_1,l_2} Z - tr(M_{l_1,l_2})] = \mathbf{o_{P_o}}(n^{-1/2+\epsilon} k^{-1/2})$ and

$$u^t \bar{G} \mathcal{S}(L(u)) \mathcal{S}(\nabla l_n(d_o,k)) = \mathbf{o_{P_o}}(\|u\| n^\epsilon k) = \mathbf{o_{P_o}}(n^{1/2-\delta} k^{-1/2}). \tag{C.19}$$

Combining (C.19) and (C.17)-(C.18), we obtain (C.14). This in turn finishes the proof of (C.11), since

$$(v-u)^t \nabla l_n(d_o,k) = \mathbf{o_{P_o}}(|d-d_o| n^{\frac{1}{2}-\delta} k^{-\frac{1}{2}}) = O(S_n(d)).$$

We now prove that $\psi(u)$ is a one-to-one transformation. First note that $\psi(u)$ is continuously differentiable for all $\|u\| \le 2 l_0 \delta_n$. This follows from the definition $\psi(u) = (I_{k+1} - (d-d_o)(\tilde{I}_k + L(u))^{-1} G^t(u))u$, the fact that $G(u)$ and $L(u)$ are polynomial in u and Lemma C.1, by which $\|L(u)\| = \mathbf{o_{P_o}}(1)$. To prove that $\psi(u)$ is also one-to-one, we bound the spectral norm of the Jacobian

$$\psi'(u) = I_{k+1} - (d-d_o) D(u) - (d-d_o)(D'(u)u),$$

where $(D'(u)u)$ is the $(k+1)\times(k+1)$ matrix with elements

$$\sum_{l=0}^{k} u_l \frac{\partial (D(u))_{l_1,l}}{\partial u_{l_2}}, \qquad l_1, l_2 = 0, \ldots, k.$$

For $\psi(u)$ to be one-to-one, it suffices to have $\psi'(u) = I_{k+1}(1 + \mathbf{o_{P_o}}(1))$.

By (C.24) in Lemma C.1 below, we have $|d-d_o|\|D(u)\| = \mathbf{O}_{P_o}(|d-d_o|)$. Therefore we only need to control the spectral norm of $D'(u)u$. For all l_1, l_2, we have

$$(D'(u)u)_{l_1,l_2} = \left[-(\tilde{I}_k + L(u))^{-1} \frac{\partial L(u)}{\partial u_{l_2}} (\tilde{I}_k + L(u))^{-1} G^t(u) u + (\tilde{I}_k + L(u))^{-1} \frac{\partial G^t(u)}{\partial u_{l_2}} u \right]_{l_1}. \tag{C.20}$$

Both $(G(u))_{l_1,l_2}$ and $(L(u))_{l_1,l_2}$ can be written as

$$F_{l_1,l_2}(u;\tau,b) := \sum_{j=2}^{J} \tau_j \sum_{l_3,\ldots,l_j=0}^{k} u_{l_3}\cdots u_{l_j} b_{l_1,l_2,\ldots,l_j},$$

where the constants $\tau_j, b_{l_1,\ldots,l_j}$ are different for G and L, and b is symmetric in its indices. In particular, $\tau_2 = 0$ in the case of L. Using this generic notation for $G(u)$ and $L(u)$, we find that for all $v \in \mathbb{R}^{k+1}$ and all $l_1, l_2 \le k$,

$$\left(\frac{\partial F(u;\tau,b)v}{\partial u_{l_2}}\right)_{l_1} = \sum_{j=3}^{J} \tau_j(j-3+1) \sum_{l_3,\ldots,l_j=0}^{k} v_{l_3} u_{l_4}\cdots u_{l_j} b_{l_1,l_2,\ldots,l_j} := F(v,u;\tau',b),$$

where $\tau'_j = \tau_j(j-3+1)$, $j = 3,\ldots,J$. It therefore has the same form as $F(u;\tau',b)$, with v replacing one of the u's. Applying this to the first term of (C.20), with $v = (\tilde{I}_k + L(u))^{-1}G^t(u)u$, we find that

$$|(\tilde{I}_k + L(u))^{-1}F(v,u;\tau',b)| \lesssim |F(v,u;\tau',b)| = \mathbf{O}(1),$$

where we used (C.21) and (C.24) from Lemma C.1. The second term of (C.20) is treated similarly with $v = u$ so that we finally obtain

$$|D'(u)u| = \mathbf{O}(1),$$

and ψ is one-to-one on $\{u : \|u\| \le 2l_0\delta_n\}$. Using the above bounds we also deduce that the Jacobian is equal to $\exp(O(S_n(d)))$, since

$$\begin{aligned}
&\log\det[\mathrm{Jac}] = \log\det\left[I_{k+1} - (d-d_o)D(u) - (d-d_o)D'(u)u\right] \\
&= O[(d-d_o)(|\mathrm{tr}[D(u)]| + \mathrm{tr}[D'(u)u]) + (d-d_o)^2(|D(u)|^2 + |D'(u)u|^2))] \\
&= O(\sqrt{k}(d-d_o) + k(d-d_o)^2) = O(S_n(d)).
\end{aligned}$$

This finishes the proof of (C.4), and hence the proof of Lemma 3.5.

Lemma C.1. *Let $v = \psi(u)$, with ψ as in* (C.8)*. Under the conditions of Lemma 3.5, we have*

$$\begin{aligned}
|L(u)| &= \mathbf{o}_{\mathbf{P_o}}(n^{-1/2+\epsilon}k) = \mathbf{o}_{\mathbf{P_o}}(1), && \text{(C.21)}\\
|G - \bar{G}| &= \mathbf{o}_{\mathbf{P_o}}(n^{-1/2+\epsilon}k) = \mathbf{o}_{\mathbf{P_o}}(1), && \text{(C.22)}\\
|D(u)| &= \mathbf{O}_{P_o}(1), && \text{(C.23)}\\
\|u - \psi(u)\| &\lesssim |d-d_o|\mathbf{O}_{P_o}(\|u\|), && \text{(C.24)}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=3}^{J} \frac{(v^{(j)} - u^{(j)})\nabla^j l_n(d_o,k)}{j!} \\
&= -(d-d_o)\sum_{j=3}^{J}\sum_{l_1,\ldots,l_j=0}^{k} (D(u)u)_{l_1} u_{l_2}\ldots u_{l_j} \frac{\nabla_{l_1,\ldots,l_j} l_n(d_o,k)}{(j-1)!} + O(S_n(d)),
\end{aligned} \qquad \text{(C.25)}$$

uniformly over $\|u\| \le 2l_0\delta_n$.

Proof. We first prove (C.21). From (B.2), we recall that $\nabla_{l_1,\dots,l_j} l_n(d_o,k)$ is the sum of a centered quadratic form $\mathcal{S}(\nabla_{l_1,\dots,l_j} l_n(d_o,k))$ and a deterministic term $\mathcal{D}(\nabla_{l_1,\dots,l_j} l_n(d_o,k))$. For all $l_1,\dots,l_j$, $\mathcal{S}(\nabla_{l_1,\dots,l_j} l_n(d_o,k))$ equals

$$X^t\left(T_n^{-1}(f_{d_o,k})\sum_{\sigma\in\mathcal{S}_j} b_\sigma B_\sigma(d_o)\right)X - \mathrm{tr}\left[T_n(f_o)T_n^{-1}(f_{d_o,k})\sum_{\sigma\in\mathcal{S}_j} b_\sigma B_\sigma(d_o)\right],$$

with $B_\sigma(d_o) := B_\sigma(d_o,\bar\theta_{d_o,k})$ as defined in (B.1). Using Lemma 1.3 (supplement) together with (B.10) we obtain that for all $l_1,\dots,l_j$, $\mathcal{S}(\nabla_{l_1,\dots,l_j} l_n(d_o,k)) = \mathbf{o_{P_o}}(n^{\frac{1}{2}+\epsilon})$, and its contribution to $|L(u)|$ is $\mathbf{o_{P_o}}(k(\sqrt{k}\|u\|)^{j-2}n^{-\frac{1}{2}+\epsilon}) = \mathbf{o_{P_o}}(n^{-1/2-\delta}k)$. The deterministic term in (B.2) is

$$\mathcal{D}(\nabla_{l_1,\dots,l_j} l_n(d_o,k)) = \sum_\sigma c_\sigma \mathrm{tr}\left[B_\sigma(d_o)\right] + \sum_\sigma d_\sigma \mathrm{tr}\left[(T_n(f_o)T_n^{-1}(f_{d_o,k}) - I_n)B_\sigma(d_o)\right].$$

We bound the contribution of the first term to $|L(u)|$; the second term can be treated similarly. Let $\tilde L(u)$ be the matrix when in (C.9) we replace $\nabla_{l_1,\dots,l_j} l_n(d_o,k)$ by $\sum_\sigma c_\sigma \mathrm{tr}[B_\sigma(d_o)]$. Hence,

$$(\tilde L(u))_{l_1,l_2} = -\sum_{j=3}^{J}\frac{1}{(j-1)!}\sum_{\sigma\in\mathcal{S}_j} c_\sigma \sum_{l_3,\dots,l_j=0}^{k} u_{l_3}\dots u_{l_j}\frac{\mathrm{tr}\left[B_\sigma(d_o)\right]}{n}, \tag{C.26}$$

$$\text{where}\quad \frac{1}{n}\mathrm{tr}\left[B_\sigma(d_o)\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(l_1x)\dots\cos(l_jx)dx + E_\sigma, \tag{C.27}$$

E_σ being the approximation error. For each σ and $j\ge 4$, the contribution of the integral in (C.27) to $(\tilde L(u))_{l_1,l_2}$ is $O(\int_{-\pi}^{\pi}|u^t\mathbf{cos}|^{j-2}(x)dx) = O(\|u\|^2)$; hence its contribution to $|\tilde L(u)|$ is $k\|u\|^2 = o(n^{-1/2-\delta}k)$. For $j=3$, we have

$$\frac{1}{4\pi}\sum_{l_3=1}^{k} u_{l_3}\int_{-\pi}^{\pi}\cos(l_1x)\cos(l_2x)\cos(l_3x)dx = \frac{1}{2}\left(u_{l_1+l_2}\mathbb{I}_{l_3=l_1+l_2} + u_{|l_1-l_2|}\mathbb{I}_{l_3=|l_1-l_2|}\right),$$

and the contribution of this term to $|\tilde L(u)|$ is of order $\sqrt{k}\|u\| = o(n^{-1/2+\epsilon}k)$. Next we bound the contribution to $|\tilde L(u)|$ of the error term E_σ in (C.27). Note that we can write the last sum in (C.26) as

$$\sum_{l_3,\dots,l_j=0}^{k} u_{l_3}\dots u_{l_j}\frac{\mathrm{tr}\left[B_\sigma(d_o)\right]}{n} = \frac{1}{n}\mathrm{tr}\left[\prod_{i=1}^{p} T_n\left(b_i(x)f_{d_o,k}\right)T_n^{-1}(f_{d_o,k})\right], \tag{C.28}$$

where

$$b_i(x) = (u^t\mathbf{cos}(x))^{|\sigma(i)|-\delta_1(i)-\delta_2(i)}\cos(l_1\cdot)^{\delta_1(i)}\cos(l_2\cdot)^{\delta_2(i)}, \tag{C.29}$$

$\delta_1(i) = \mathbb{I}_{1\in\sigma(i)}$, $\delta_2(i) = \mathbb{I}_{2\in\sigma(i)}$ and $p = |\sigma|$. If $p\le 3$, then Lemma 2.4 (supplement) implies that

$$E_\sigma = O((\sqrt{k}\|u\|)^{j-2}n^{\epsilon-1}[k^{2(3/2-\beta)_+} + k]) = o(n^{-1-\delta}k). \tag{C.30}$$

If $p \geq 4$, then Lemma 2.6 (supplement) together with (C.28), with

$$f = f_{d_o,k}, \quad f_{2i} = b_i f_{d_o,k}, i \leq |\sigma|,$$

$L = k^{(3/2-\beta)_+}$, $M, m^{-1} = O(1)$, $M^{(i)} = O((\sqrt{k}\|u\|)^{|\sigma(i)|-\delta_1(i)-\delta_2(i)}$ and $L^{(i)} = O(k(\sqrt{k}\|u\|)^{|\sigma(i)|-\delta_1(i)-\delta_2(i)})$, leads to the bound

$$\sum_{l_3,\ldots,l_j=0}^{k} u_{l_3}\ldots u_{l_j}\frac{\mathrm{tr}\,[B_\sigma(d_o)]}{n} - \frac{1}{n}\mathrm{tr}\left[\prod_{i=1}^{|\sigma|} T_n(b_i f_{d_o,k}) T_n\left(\frac{1}{4\pi^2 f_{d_o,k}}\right)\right]$$
$$= \mathbf{o}(k^{(3/4-\beta/2)_+} n^{-1/2+\epsilon}\|u\|(\sqrt{k}\|u\|)^{j-3}) = \mathbf{o}(n^{-1/2-\delta}).$$

Using Lemma 2.1 (supplement) we finally obtain that

$$\frac{1}{n}\mathrm{tr}\left[\prod_{i=1}^{|\sigma|} T_n(b_i f_{d_o,k})\right] - \frac{1}{2\pi}\int_{-\pi}^{\pi}(u^t\mathbf{cos})^{j-2}(x)\cos(l_1x)\cos(l_2x)dx = \mathbf{o}(n^{-1-\delta}k).$$

Therefore the contribution of the approximation error E_σ in $|\tilde{L}(u)|$ is of order $\mathbf{o}(n^{-1/2-\delta}k)$. Using a similar argument we control the terms in the form $\mathrm{tr}\left[T_n(f_o)(T_n^{-1}(d_o,k) - I_n)B_\sigma(d_o)\right]$ and (C.21) is proved.

We now prove (C.22) and bound

$$(G-\bar{G})_{l_1,l_2} = \sum_{j=2}^{J}\frac{1}{j!}\sum_{\sigma\in\mathcal{S}_j}(c_\sigma - d_\sigma)\sum_{l_3,\ldots,l_j=0}^{k} r^{(j)}_{l_1,\ldots,l_j}u_{l_3}\ldots u_{l_j},$$

with $r^{(j)}_{l_1,\ldots,l_j}$ as in (C.6). These are the approximation errors which occur when replacing $\frac{1}{n}\mathrm{tr}[T_{1,\sigma}(d_o,k)]$ and $\frac{1}{n}\mathrm{tr}[T_{2,\sigma}(d_o,k)]$ by their limiting integrals (see also (B.4)). Therefore, for each $\sigma \in \mathcal{S}_j$, $\sum_{l_3,\ldots,l_j=0}^{k} r^{(j)}_{l_1,\ldots,l_j}u_{l_3}\ldots u_{l_j}$ is a combination of terms of the form

$$\frac{1}{n}\mathrm{tr}\left[\prod_{i=1}^{p} T_n\left(b_i(x)f_{d_o,k}\right)T_n^{-1}(f_{d_o,k})\right] - \frac{1}{2\pi}\int_{-\pi}^{\pi}(u^t\mathbf{cos}(x))^{j-2}H_k(x)\cos(l_1x)\cos(l_2x)dx,$$

with $p \in \{|\sigma|, |\sigma|+1\}$ and the functions b_i defined as in (C.29) apart from $b_1(x) = H_k(x)(\sum_{l=0}^{k} u_l\cos(lx))^{|\sigma(1)|-\delta_1(1)-\delta_2(1)}\cos(l_1.)^{\delta_1(1)}\cos(l_2.)^{\delta_2(1)}$. Therefore, using the same construction as in (C.28)-(C.30), we obtain that

$$|(G-\bar{G})_{l_1,l_2}| = O(n^{-1/2+\epsilon}), \quad |G-\bar{G}| = O(n^{-1/2+\epsilon}k) = o(1).$$

To prove (C.23), we use the just obtained bound on $|G-\bar{G}|$, and in addition establish a bound $|\bar{G}|$. We treat each term $G^{(j)}$ in $G(u) = \sum_{j=2}^{J}\frac{1}{j!}\sum_{\sigma\in\mathcal{S}_j}(c_\sigma - d_\sigma)G^{(j)}(u)$ separately. First we show that $|\bar{G}^{(2)}| = O(1)$, which follows from definition (C.7), by which

$$\mathrm{tr}\left[(\bar{G}^{(2)})^2\right] = \sum_{l_1,l_2,l_1+l_2\geq k}^{k}\frac{1}{(l_1+l_2)^2} \leq 1.$$

Consequently, $|\bar{G}^{(2)}| \leq 1$. For $j \geq 3$, note that for all $0 \leq l_1, l_2 \leq k$,

$$\begin{aligned}
\left|\bar{G}^{(j)}_{l_1,l_2}(u)\right| &= \left|\sum_{l_3,\ldots,l_j=0}^{k} \gamma^{(j)}_{l_1,l_2,\ldots,l_j} u_{l_3}\ldots u_{l_j}\right| \\
&\leq \sum_{l_4,\ldots,l_j=0}^{k} |u_{l_4}\ldots u_{l_j}| \int_{-\pi}^{\pi} |H_k(x)| \left|\sum_{l_3=0}^{k} \cos(l_3 x) u_{l_3}\right| dx \\
&\leq (\sqrt{k}\|u\|)^{j-3}\frac{\|u\|}{\sqrt{k}} = (\sqrt{k})^{j-4}(\|u\|)^{j-2}.
\end{aligned}$$

Therefore, $|\bar{G}^{(j)}(u)| \leq k(\sqrt{k})^{j-4}(\|u\|)^{j-2} = (\sqrt{k}\|u\|)^{j-2} = o(1)$, for all $j \geq 3$. Hence $|\bar{G}(u)| = \mathbf{O}(1)$, which combined with $\|(\tilde{I}_k^{+} L(u))-1\| = \mathbf{O}_{P_o}(1)$ (see Lemma B.5) and (C.21)), imply that

$$|\bar{D}(u)| = |(\tilde{I}_k + L(u))^{-1}\bar{G}^t(u)| = O(1),$$

uniformly over $\|u\| \leq 2l_0\delta_n$. It follows that

$$|D(u)| \leq \|(\tilde{I}_k + L(u))^{-1}\| \left(|\bar{G}| + |G - \bar{G}|\right) = \mathbf{O}_{P_o}(1).$$

This concludes the proof of (C.23); (C.24) directly follows from this result since $\|u - \psi(u)\| \leq |d - d_o||D(u)|\|u\|$. Finally, we prove (C.25). We have

$$\begin{aligned}
\sum_{j=3}^{J} \frac{(v^{(j)} - u^{(j)})\nabla^j l_n(d_o,k)}{j!} &= -(d-d_o)\sum_{j=3}^{J}\sum_{l_1,\ldots,l_j=0}^{k} (D(u)u)_{l_1} u_{l_2}\ldots u_{l_j} \frac{\nabla_{l_1,\ldots,l_j} l_n(d_o,k)}{(j-1)!} \\
&+ (d-d_o)^2 \sum_{j=3}^{J} \binom{j}{2} \sum_{l_1,\ldots,l_j=0}^{k} (D(u)u)_{l_1}(D(u)u)_{l_2}\ldots u_{l_j} \frac{\nabla_{l_1,\ldots,l_j} l_n(d_o,k)}{(j-1)!} \\
&+ \ldots + (-1)^J \sum_{l_1,\ldots,l_J=0}^{k} (D(u)u)_{l_1}(D(u)u)_{l_2}\ldots(D(u)u)_{l_J} \frac{\nabla_{l_1,\ldots,l_J} l_n(d_o,k)}{(J-1)!}
\end{aligned}$$

Using the same argument as in the proof of (C.21), we find that for all for all $j \geq 3$

$$\begin{aligned}
&\sum_{l_1,\ldots,l_j=0}^{k} (D(u)u)_{l_1}(D(u)u)_{l_2} u_{l_3}\ldots u_{l_j} \frac{\nabla_{l_1,\ldots,l_j} l_n(d_o,k)}{j!} \\
&= n\int_{-\pi}^{\pi} (D(u)u)^t \cos(x))^2 (u^t \cos(x))^{j-2} dx \\
&\quad + (\sqrt{k}\|u\|)^{j-1} O(\sqrt{n} n^{\epsilon}(\sqrt{k}\|u\|) + k + \sqrt{n}\|u\| k^{1/2(3/2-\beta)_+}) \\
&= \mathbf{o}(n^{1-\delta}k^{-1}), \quad \text{for some } \delta > 0.
\end{aligned}$$

Similarly, the higher-order terms in the above expression for $\sum_{j=3}^{J} \frac{(v^{(j)}-u^{(j)})\nabla^j l_n(d_o,k)}{j!}$ can be shown to be $O(S_n(d))$, which terminates the proof of Lemma C.1. □

The rest of the paper corresponds to the suppelmentary material

D Technical results

Let $\eta_j = -1_{j>0} 2/j$ and recall that $\bar\theta_{d,k} = \theta_{o[k]} + (d_o - d)\eta_{[k]}$. Let the sequence $\{a_j\}$ be defined as $a_j = \theta_{o,j} + (d_o - d)\eta_j$ when $j > k$ and $a_j = 0$ when $j \le k$. In addition, define

$$H_k(x) = \sum_{j=k+1}^{\infty} \eta_j \cos(jx), \qquad G_k(x) = \sum_{j=1}^{k} \eta_j \cos(jx), \tag{D.1}$$

$$\Delta_{d,k}(x) = \sum_{j=k+1}^{\infty} (\theta_{o,j} + (d_o - d)\eta_j)\cos(jx) = \sum_{j=k+1}^{\infty} a_j \cos(jx). \tag{D.2}$$

Using this notation we can write

$$-2\log|1 - e^{ix}| = -\log(2 - 2\cos(x)) = G_k(x) + H_k(x), \tag{D.3}$$

$$\begin{aligned} f_{d,k}(x) = f_{d,k,\bar\theta_{d,k}}(x) &= f_o(x)\exp\left\{ - \sum_{j=k+1}^{\infty} a_j \cos(jx) \right\} \\ &= f_o(x) e^{-\Delta_{d,k}(x)} = f_o(x) e^{(d-d_o)H_k(x) - \Delta_{d_o,k}(x)}. \end{aligned} \tag{D.4}$$

Given d, k and θ_o, the sequence $\{a_j\}$ represents the closest possible distance between f_o and $f_{d,k,\theta}$, since

$$l(f_o, f_{d,k}) = l(f_o, f_{d,k,\bar\theta_{d,k}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Delta_{d,k}^2(x)dx = \sum_{j>k} a_j^2. \tag{D.5}$$

From (D.4) it also follows that for all d,

$$\frac{\partial}{\partial d} f_{d,k} = H_k f_{d,k}. \tag{D.6}$$

Lemma D.1. *When $\theta_o \in \Theta(\beta, L_o)$, there exist constants such that for any positive integer k,*

$$k^{-1} \lesssim \int_{-\pi}^{\pi} H_k^2(x)dx \lesssim k^{-1}, \tag{D.7}$$

$$\sum_{l>k} |\theta_{o,l}| = O(k^{-\beta+\frac{1}{2}}), \quad \sum_{l\ge 0} |\theta_{o,l}| = O(1), \tag{D.8}$$

$$\int_{-\pi}^{\pi} \Delta_{d_o,k}(x) H_k(x)dx = \sum_{j>k} \eta_j \theta_{o,j} = O\left(k^{-\frac{1+2\beta}{2}}\right), \tag{D.9}$$

$$\int_{-\pi}^{\pi} \Delta_{d_o,k}^2(x)dx = \sum_{l>k} \theta_{o,l}^2 = O\left(k^{-2\beta}\right), \tag{D.10}$$

$$\int_{-\pi}^{\pi} \Delta_{d_o,k}^2(x) H_k(x)dx = O\left(k^{-2\beta-1}\right), \tag{D.11}$$

$$\int_{-\pi}^{\pi} H_k^4(x)dx \lesssim \frac{\log k}{k}. \tag{D.12}$$

When $k \to \infty$, the big-O in (D.8)-(D.11) *may be replaced by a small-o, since* $\sum_{l>k} \theta_{o,l}^2 l^{2\beta}$ *then tends to zero.*

Proof. The result for $\int H_k^2(x)dx$ follows directly from the definition of H_k. The assumption that $\theta_o \in \Theta(\beta, L_o)$ and the Cauchy-Schwarz inequality imply that

$$\sum_{l>k} |\theta_{o,l}| \leq \sqrt{\sum_{l>k} \theta_{o,l}^2 l^{2\beta}} \sqrt{\sum_{l>k} l^{-2\beta}} = O(k^{-\beta+\frac{1}{2}}),$$

proving the first result in (D.8). Similarly, one can prove (D.9). For (D.10), note that $\sum_{l>k} \theta_{o,l}^2 \leq k^{-2\beta} \sum_{l>k} \theta_{o,l}^2 l^{2\beta}$. For the other bounds we omit the details of the proof. They follow from the fact that for all sequences a, b and c,

$$\begin{aligned} 2 &\sum_{l,m,n>k} a_l b_m c_n \int_{-\pi}^{\pi} \cos(lx)\cos(mx)\cos(nx)dx \\ &= \sum_{m,n>k} b_m c_n \sum_{l>k} a_l \int_{-\pi}^{\pi} \cos(lx)\left(\cos((m+n)x) + \cos((m-n)x)\right)dx \\ &= \sum_{m,n>k} a_{m+n} b_m c_n + \sum_{m,n>k;m-n>k} a_{m-n} b_m c_n. \end{aligned}$$

□

Before stating the next lemma we give bounds for the functions H_k and G_k. Since $-2\log|1-e^{ix}| = -\log(x^2 + O(x^4))$, there exist positive constants c, B_0, B_1 and B_2 such that

$$|H_k(x)| \geq B_0 |\log x|, \quad |x| \leq ck^{-1}, \tag{D.13}$$

$$|H_k(x)| \leq B_1 |\log x| + B_2 \log k, \quad x \in [-\pi, \pi]. \tag{D.14}$$

Lemma D.2. *Let $a_j = (\theta_{o,j} - (d - d_o)\eta_j)1_{j>k}$, as in* (D.2). *Then for $p \geq 1$ and $q = 2, 3, 4$ there exist constants $c(p,q)$ such that for all $d \in (-\frac{1}{2}, \frac{1}{2})$ and $k \leq \exp(|d - d_o|^{-1})$,*

$$\begin{aligned} \int_{-\pi}^{\pi} \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx &= O\left(\frac{(\log k)^{c(p,q)}}{k}\right) \\ &+ O((\log k)^{q+pB_2|d-d_o|}|d-d_o|^{-\frac{q}{2}} e^{-|d-d_o|^{-1}}), \end{aligned} \tag{D.15}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{f_o}{f_{d,k}}(x) - 1\right)\cos(ix)\cos(jx)dx = \frac{1}{2}a_{i+j}1_{i+j>k} + O\left(\sum_{j>k} a_j^2\right), \tag{D.16}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{f_o}{f_{d,k}}(x) - 1\right) H_k^2(x)dx = O(|d-d_o|k^{-1}\log k), \tag{D.17}$$

where the constant B_2 in (D.15) *is as in* (D.14), *and the constants in* (D.16) *and* (D.17) *are uniform in d. The constant $c(p,q)$ in* (D.15) *equals $0, \frac{1}{2}, 1$ when respectively $q = 2, 3, 4$.*

Proof. When $d = d_o$, (D.15) directly follows from (D.7) and (D.12), because of the boundedness of $(f_o/f_{d_o,k})^p = \exp\{p\Delta_{d_o,k}\}$. Now suppose $d \neq d_o$. Let $C_k = \max_{x\in[-\pi,\pi]} \exp\{|\Delta_{d_o,k}(x)|\}$ and $b_m = \max_{x\in[m,\pi]} |(d-d_o)H_k(x)|$, for $m = e^{-\frac{1}{|d-d_o|}} < e^{-1}$. Since $\sum_{j=0}^{\infty} |\theta_{o,j}| < \infty$, the sequence C_k is bounded by some constant C. To prove (D.15) we write

$$\begin{aligned}
&\frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx \\
&\quad= \int_0^m \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx + \int_m^{\pi}\left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx.
\end{aligned} \tag{D.18}$$

We first bound the last integral in the preceding display, by substitution of $(f_o/f_{d,k})^p = \exp\{p\Delta_{d,k}\} = \exp\{-p(d-d_o)H_k + p\Delta_{d_o,k}\}$. From (D.14) it follows that

$$b_m \leq |d-d_o|(B_1|d-d_o|^{-1} + B_2\log k) \leq B_1 + B_2,$$

as $k \leq \exp(|d-d_o|^{-1})$. Hence we obtain $(f_o/f_{d,k})^p \leq Ce^{b_m}$ on (m,π). For $q = 2$ and $q = 4$ the bound on the last integral in (D.18) therefore follows from (D.7) and (D.12); for $q = 3$ the bound follows from the Cauchy-Schwarz inequality.

Next we bound the first integral in (D.18). Because the function $x^{|d-d_o|}(\log x)^2$ has a local maximum of $4|d-d_o|^{-2}e^{-2}$ at $x = e^{-2/|d-d_o|}$, $(\log x)^2 \leq 4x^{-|d-d_o|}|d-d_o|^{-2}e^{-2}$ for all $x \in [0,m]$. Again using (D.14) we find that

$$\begin{aligned}
&\int_0^m \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx \lesssim \sum_{j=0}^{q}\binom{q}{j}\int_0^m (B_1|\log x|)^j\,(B_2\log k)^{q-j}\,e^{-p(d-d_o)H_k(x)}dx \\
&\lesssim \sum_{j=0}^{q}\binom{q}{j}(\log k)^{q-j+pB_2|d-d_o|}\int_0^m (B_1|\log x|)^j\,x^{-pB_1|d-d_o|}dx \\
&\leq \sum_{j=0}^{q}\binom{q}{j}(\log k)^{q-j+pB_2|d-d_o|}\left(\frac{2B_1^2}{e|d-d_o|}\right)^{\frac{j}{2}}\int_0^m x^{-(j/2+pB_1)|d-d_o|}dx \\
&\lesssim (\log k)^{q+pB_2|d-d_o|}|d-d_o|^{-\frac{q}{2}}e^{-1/|d-d_o|}.
\end{aligned}$$

We now prove (D.16).

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f_o}{f_{d,k}}(x) - 1\right)\cos(ix)\cos(jx)dx \\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{\Delta_{d,k}(x)} - 1\right)\cos(ix)\cos(jx)dx \\
&\quad\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\Delta_{d,k}(x) + \frac{1}{2}\Delta_{d,k}^2(x)e^{(\Delta_{d,k}(x))_+}\right)\cos(ix)\cos(jx)dx.
\end{aligned}$$

The linear term equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}(x)\cos(ix)\cos(jx)dx$$
$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\sum_{l>k}a_l\cos(lx)\right)(\cos((i+j)x)+\cos((i-j)x))\,dx=\frac{1}{2}a_{i+j}1_{i+j>k}.$$

For the quadratic term we have

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}^2(x)e^{(\Delta_{d,k}(x))_+}\cos(ix)\cos(jx)dx\right|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}^2(x)e^{(\Delta_{d,k}(x))_+}dx$$
$$\leq\frac{1}{2\pi}\int_0^m\Delta_{d,k}^2(x)e^{-\Delta_{d,k}(x)}dx+\frac{(1+Ce^{b_m})}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}^2(x)dx. \tag{D.19}$$

This is $O(\sum_{j>k}a_j^2)$, which follows from (D.5) and integration over $(0,e^{-\frac{1}{\bar{v}_n}})$ and $(e^{-\frac{1}{\bar{v}_n}},\pi)$ as above.

To prove (D.17), write $\exp(\Delta_{d,k})-1=\Delta_{d,k}+\Delta_{d,k}^2e^{\xi}$ with $\Delta_{d,k}=-(d-d_o)H_k(x)+\Delta_{d_o,k}(x)$ and $|d-d_o|\leq\bar{v}_n$, substitute (D.14) and proceed as in the proof of (D.15) above. The biggest term is a multiple of $|d-d_o|\int_{-\pi}^{\pi}|H_k(x)|^3dx$, which is $O(\bar{v}_nk^{-1})$. This is larger than the approximation error when $\beta>\frac{1}{2}(1+\sqrt{2})$. □

Lemma D.3. *Let A be a symmetric matrix matrix such that $|A|=1$ and let $Y=(Y_1,\dots,Y_n)$ be a vector of independent standard normal random variables. Then for any $\alpha>0$,*

$$P\left(Y^tAY-tr(A)>n^{\alpha}\right)\leq\exp\{-n^{\alpha}/8\}.$$

Proof. Note that $\|A\|\leq|A|=1$ so that for all $s\leq1/4$, $sy^tAy\leq s_0y^ty\|A\|\leq y^ty/4$ and $\exp\{sY^tAY\}$ has finite expectation. Choose $s=1/4$, then by Markov's inequality,

$$\begin{aligned}P\left(Y^tAY-\mathrm{tr}(A)>n^{\alpha}\right)&\leq e^{-n^{\alpha}/4}Ee^{(Y^tAY-\mathrm{tr}(A))/4}\\&=\exp\left\{-n^{\alpha}/4-\frac{1}{2}\log\det[I_n-A/2]-\mathrm{tr}(A)/4\right\}\\&\leq\exp\left\{-n^{\alpha}/4+\mathrm{tr}(A^2)/4\right\}.\end{aligned}$$

The last inequality follows from the fact that $A(I_n-\tau A/2)^{-1}$ has eigenvalues $\lambda_j(1-\tau\lambda_j/2)^{-1}$, where λ_j are the eigenvalues of A for all $\tau\in(0,1)$. Hence, $\mathrm{tr}(A^2(I_n-\tau A/2)^{-2})$ is bounded by $4\mathrm{tr}(A^2)$. The result follows from the fact that when n is large enough $n^{\alpha}>2\mathrm{tr}(A^2)=2$. □

E Convergence of the trace of a product of Toeplitz matrices

Suppose $T_n(f_j)$ $(j = 1, \dots, p)$ are covariance matrices associated with spectral densities f_j. According to a classical result by Grenander and Szégö (Grenander and Szegö (1958)),

$$\frac{1}{n}\mathrm{tr}\left[\prod_{j=1}^{p} T_n(f_j)\right] \to (2\pi)^{2p-1}\int_{-\pi}^{\pi}\prod_{j=1}^{p} f_j(x)dx.$$

In this section we give a series of related results. We first recall a result from Rousseau et al. (2010).

Lemma E.1. *Let* $1/2 > t > 0$ *and* $L^{(i)}, M^{(i)} > 0$, $\rho_i \in (0,1]$, $d_i \in [-1/2 + t, 1/2-t]$ *for all* $i = 1, ..., 2p$ *and let* f_i, *(*$i \le 2p$*) be functions on* $[-\pi,\pi]$ *satisfying*

$$|f_i(x)| = |x|^{-2d_i} g_i(x),\ |g_i(x)| \le M^{(i)},\ |g_i(x)-g_i(y)| \le \frac{M^{(i)}|x-y|}{|x| \wedge |y|} + L^{(i)}|x-y|^{\rho_i} \tag{E.1}$$

and assume that $\sum_{i=1}^{p}(d_{2i-1} + d_{2i}) < \frac{1}{2}$. *Then for all* $\epsilon > 0$ *there exists a constant* K *depending only on* ϵ, t *and* $q = \sum_{j=1}^{p}(d_{2j-1} + d_{2j})_+$ *such that*

$$\left|\frac{1}{n}tr\left[\prod_{j=1}^{p} T_n(f_{2j-1})T_n(f_{2j})\right] - (2\pi)^{2p-1}\int_{-\pi}^{\pi}\prod_{j=1}^{2p} f_j(x)dx\right|$$
$$\le K\sum_{j=2}^{2p}\left(\prod_{i\neq j} M^{(i)}\right) L^{(j)} n^{-\rho_j+\epsilon+2q} + K\prod_{i=1}^{2p} M^{(i)} n^{-1+q+\epsilon}.$$

To prove a similar result involving also inverses of matrices, we need the following two lemmas. They can be found elsewhere, but as we make frequent use of them they are included for easy reference and are formulated in a way better suited to our purpose. The first lemma can be found on p.19 of Rousseau et al. (2010), and is an extension of Lemma 5.2 in Dahlhaus (1989).

Lemma E.2. *Suppose that for* $0 < t < 1/2$ *and* $d \in [-1/2 + t, 1/2 - t]$

$$|f(x)| = |x|^{-2d} g(x), \quad m \le |g(x)| \le M, \quad |g(x) - g(y)| \le L|x - y|^{\rho} \tag{E.2}$$

and assume that $0 < m \le 1 \le M < +\infty$ *and* $L \ge 1$. *Then, for all* $\epsilon > 0$, *there exists a constant* K *depending on* t *and* ϵ *only such that*

$$|I_n - T_n^{\frac{1}{2}}(f)T_n\left(\frac{1}{4\pi^2 f}\right)T_n^{\frac{1}{2}}(f)|^2 \le KL\frac{M^2}{m^2}n^{1-\rho+\epsilon}.$$

Proof. By Lemma E.1,

$$|I_n - T_n^{\frac{1}{2}}(f)T_n\left(\frac{1}{4\pi^2 f}\right)T_n^{\frac{1}{2}}(f)|^2 = \text{tr}\left\{I_n - 2T_n^{\frac{1}{2}}(f)T_n\left(\frac{1}{4\pi^2 f}\right)T_n^{\frac{1}{2}}(f) + T_n^{\frac{1}{2}}(f)T_n\left(\frac{1}{4\pi^2 f}\right)T_n(f)T_n\left(\frac{1}{4\pi^2 f}\right)T_n^{\frac{1}{2}}(f)\right\}$$

converges to zero, the approximation error being bounded by $K[L(1+M^2/m^2)+M^2/m^2]$. □

The next result can be found as Lemma 3 in Lieberman et al. (2011), and is an extension of Lemma 5.3 in Dahlhaus (1989).

Lemma E.3. *Suppose that f_1 and f_2 are such that $|f_1(x)| \geq m|x|^{-2d_1}$ and $|f_2(x)| \leq M|x|^{-2d_2}$ for constants $d_1, d_2 \in (-\frac{1}{2}, \frac{1}{2})$ and $m, M > 0$. Then*

$$\|T_n^{-\frac{1}{2}}(f_1)T_n^{\frac{1}{2}}(f_2)\| \leq C\frac{M}{m}n^{(d_2-d_1)_+ + \epsilon}.$$

Proof. In the proof of Lemma 5.3 on p. 1761 in Dahlhaus (1989), the first inequality only depends on the upper and lower bounds m and M. □

Using the preceding lemmas, the approximation result given in Lemma E.1 for traces of matrix products can be extended to include matrix inverses.

Lemma E.4. *Suppose that f satisfies* (E.2) *with constants d, ρ, L, m and M. For f_{2j}, $j = 1, \dots, p$, assume that* (E.1) *holds with constants d_{2j}, ρ_{2j}, $L^{(2j)}$ and $M^{(2j)}$ ($j = 1, \dots, p$). For convenience, we denote $M^{(2j-1)} = m^{-1}$, $\rho_{2j-1} = \rho$ and $L^{(2j-1)} = L$ ($j = 1, \dots, p$). Suppose in addition that $d, d_{2j} \in [-\frac{1}{2}+t, \frac{1}{2}-t]$ satisfy $\sum_{j=1}^{p}(d_{2j} - d)_+ < \frac{1}{2}(\rho - \frac{1}{2})$, and let $q = \sum_{j=1}^{p}(d_{2j} - d)_+$. Then for all $\epsilon > 0$ there exists a constant K such that*

$$\begin{aligned} &\left|\frac{1}{n}tr\left\{\prod_{j=1}^{p} T_n^{-1}(f)T_n(f_{2j})\right\} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\prod_{j=1}^{p}\frac{f_{2j}(x)}{f(x)}dx\right| \\ &\leq K\left[\sum_{j=2}^{2p}\left(\prod_{i\neq j}^{2p} M^{(i)}\right)L^{(j)}n^{-\rho_j} + n^{-1}\prod_{i\leq 2p}M^{(i)}\right]n^{\epsilon+2q} \\ &+ \left(\prod_{j=1}^{p}M^{(2j)}\right)\left(L\frac{M}{m}\right)^{\frac{(p+1)}{2}} n^{(1-\rho)\frac{(p+1)}{2}-1+\epsilon+2q}, \end{aligned} \tag{E.3}$$

and setting $\tilde{f} = 1/(4\pi^2 f)$,

$$\begin{aligned} &\frac{1}{n}\left|tr\left\{\prod_{j=1}^{p} T_n^{-1}(f)T_n(f_{2j})\right\} - tr\left\{\prod_{j=1}^{p} T_n(\tilde{f})T_n(f_j)\right\}\right| \\ &\leq \left(\prod_{j=1}^{p}M^{(2j)}\right)\left(L\frac{M}{m}\right)^{\frac{(p+1)}{2}} n^{(1-\rho)\frac{(p+1)}{2}-1+\epsilon+2q}. \end{aligned} \tag{E.4}$$

Proof. Without loss of generality, we consider the f_{2j}'s to be nonnegative When this is not the case, we write $f_{2j} = f_{2j}^+ - f_{2j}^-$ and treat the positive and negative part separately; see also Dahlhaus (1989) , p. 1755-56. To prove (E.4), we use the construction of Lemma 5 from Lieberman et al. (2011), who treat the case $\rho = 1$ and $d_{2j} = d'$. Inspection of their proof shows that this extends to $\rho \neq 1$ and d_{2j} that differ with j. To prove (E.3), we use the construction of Dahlhaus' Theorem 5.1 (see also the remark on p. 744 of Lieberman and Phillips (2004), after (28)), and apply Lemma E.1 with $f_{2j-1} = \tilde{f} = \frac{1}{4\pi^2 f}$, $j = 1, \dots, p$. This gives the first term on the right in (E.3). The last term in (E.3) follows from (E.4). □

Although the bound provided by Lemma E.4 is sufficiently tight for most purposes, certain applications require sharper bounds. These can only be obtained if we exploit specific properties of f and f_{2j}. In Lemma E.5 below we improve on the first term on the right in (E.3). This is useful when for example $b_i(x) = cos(jx)$; the Lipschitz constant L is then of order $O(k)$, but the boundedness of b_i actually allows a better result. In Lemma E.6 we improve on the last term of (E.3).

Lemma E.5. *Let $f(x) = |x|^{-2d} g(x)$ with $-1/2 < d < 1/2$ and g a bounded Lipschitz function satisfying $m < g < M$, with Lipschitz constant L.*

- *Let $b_1, \dots, b_p$ be bounded functions and let $\|b\|_\infty$ denote a common upper bound for these functions. Then for all $\epsilon > 0$,*

$$\begin{aligned} &\left| tr\left[\prod_{i=1}^p T_n(b_i f) T_n(f^{-1})\right] - (2\pi)^p tr\left[\prod_{i=1}^p T_n(b_i)\right]\right| \\ &\leq C n^\epsilon \left(\frac{M}{m}\right)^p \|b\|_\infty^{p-1} \left(\|b\|_\infty + L \sum_{j=1}^p \|b_j\|_2\right). \end{aligned} \tag{E.5}$$

- *Let b_j $(j \geq 2)$ be bounded functions. Let b_1 be such that $\|b_1\|_2 < +\infty$, and assume that for all $a > 0$ there exists $M'(a) > 0$ such that*

$$\int_{-\pi}^{\pi} |b_1(x)||x|^{-1+a} dx \leq M'(a).$$

Then for all $a > 0$

$$\begin{aligned} &\left| tr\left[\prod_{i=1}^p T_n(b_i f) T_n(f^{-1})\right] - (2\pi)^p tr\left[\prod_{i=1}^p T_n(b_i)\right]\right| \\ &\leq C \left(\frac{M}{m}\right)^p \prod_{i \geq 2} \|b_i\|_\infty \left(n^{3pa} M'(a) + L(\log n)^{2p-1} \|b_1\|_2\right). \end{aligned} \tag{E.6}$$

Proof. We prove (E.5); the proof of (E.6) follows exactly the same lines. We define $\Delta_n(x) = e^{ix}$ and $L_n(x) = n \wedge |x|^{-1}$ where the latter is an upper bound of the former. Using the decomposition as on p. 1761 in Dahlhaus (1989) or as in the proof of we find that

$$
\begin{aligned}
&\left|\mathrm{tr}\left[\prod_{i=1}^{p} T_n(b_i f) T_n(f^{-1})\right] - (2\pi)^p \mathrm{tr}\left[\prod_{i=1}^{p} T_n(b_i)\right]\right| \\
&\leq C\left|\int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{p} b_i(x_{2i-1}) \left(\prod_{i=1}^{p} \frac{f(x_{2i-1})}{f(x_{2i})} - 1\right) \Delta_n(x_1 - x_2)\ldots\Delta_n(x_{2p} - x_1)dx\right| \\
&\leq C\left|\int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{p} b_i(x_{2i-1}) \frac{g(x_{2i-1})}{g(x_{2i})} \left(\prod_{i=1}^{p} \frac{|x_{2i-1}|^{-2d}}{|x_{2i}|^{-2d}} - 1\right) \Delta_n(x_1 - x_2)\ldots\Delta_n(x_{2p} - x_1)dx\right| \\
&+ C\left|\int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{p} b_i(x_{2i-1}) \left(\prod_{i=1}^{p} \frac{g(x_{2i-1})}{g(x_{2i})} - 1\right) \Delta_n(x_1 - x_2)\ldots\Delta_n(x_{2p} - x_1)dx\right| \\
&\leq C\left(\frac{M\|b\|_\infty}{m}\right)^p \sum_{j=1}^{p} \int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{j} \frac{|x_{2i-1} - x_{2i}|^{1-3a}}{(|x_{2i}| \wedge |x_{2i-1}|)^{1-a}} L_n(x_1 - x_2)\ldots L_n(x_{2p} - x_1)dx \\
&+ CL\left(\frac{M\|b\|_\infty}{m}\right)^{p-1} \sum_{j=1}^{p} \int_{[-\pi,\pi]^{2p}} |b_j(x_{2j-1})||x_{2j-1} - x_{2j}| L_n(x_1 - x_2)\ldots L_n(x_{2p} - x_1)dx \\
&\leq C\left(\frac{M\|b\|_\infty}{m}\right)^p n^{3pa} \left(\int_{[-\pi,\pi]} |x|^{-1+a} dx\right)^p + CL\left(\frac{M\|b\|_\infty}{m} \log n\right)^{p-1} (\log n)^{2p-1} \sum_{j=1}^{p} \|b_j\|_2.
\end{aligned}
$$

□

Lemma E.6. *Let $\tilde{f} = 1/(4\pi^2 f)$, and let $\rho > 1/2$ and $L > 1$, then under the conditions of Lemma E.4 we have the following alternative bound for* (E.4)*:*

$$
\begin{aligned}
&\left| tr\left\{\prod_{j=1}^{p} T_n^{-1}(f) T_n(f_{2j})\right\} - tr\left\{\prod_{j=1}^{p} T_n(\tilde{f}) T_n(f_{2j})\right\}\right| \\
&\lesssim \sqrt{L} n^{(1-\rho/2)+2q+\epsilon} \left\{\sum_{j=1}^{p-1} \left(\sqrt{M_{2p}} \prod_{l=j+1}^{p-1} M^{(2l)}\right) \times \left(\int_{-\pi}^{\pi} \frac{|f_{2p}(x)|}{f(x)} \prod_{l=1}^{j} \frac{f_{2l}^2}{f^2}(x) dx\right)^{\frac{1}{2}}\right. \\
&\left. + \prod_{l=2}^{p} M^{(2l)} \left(\int_{-\pi}^{\pi} \frac{f_2^2}{f^2}(x) dx\right)^{\frac{1}{2}} + error\right\}
\end{aligned}
\tag{E.7}
$$

where

$$
error \leq L^{3/4} n^{(1-3\rho)/4} \prod_{l=1}^{p} M^{(2l)} + \sum_{j=1}^{p} \sqrt{L^{(2j)} n^{-\rho_{2j}} M^{(2j)}} \prod_{l \neq j} M^{(2l)}
$$

Remark E.1. *The constant appearing on the right hand side of* (E.7) *depends on M and m, but in all our applications of Lemma E.6, the constants M and m will be bounded and of no consequence.*

Proof. Following the construction of Dahlhaus (1989), equation (13), we write $|\mathrm{tr}\{\prod_{j=1}^{p} T_n^{-1}(f)T_n(f_{2j})\} - \mathrm{tr}\{\prod_{j=1}^{p} T_n(\tilde{f})T_n(f_{2j})\}|$ as

$$\begin{aligned}
&\left|\mathrm{tr}\left\{\prod_{j=1}^{p} A_j - \prod_{j=1}^{p} B_j\right\}\right| \\
&= \left|\mathrm{tr}\left\{(A_1 - B_1)\prod_{l=2}^{p} A_l + \sum_{j=2}^{p}\left(\prod_{l=1}^{j-1} B_l\right)(A_j - B_j)\prod_{l=j+1}^{p} A_l\right\}\right|,
\end{aligned} \tag{E.8}$$

where $A_j = T_n^{\frac{1}{2}}(f_{2j-2})T_n^{-1}(f)T_n^{\frac{1}{2}}(f_{2j})$, $B_j = T_n^{\frac{1}{2}}(f_{2j-2})T_n(\tilde{f})T_n^{\frac{1}{2}}(f_{2j})$ and $f_0 := f_{2p}$ (similarly for ρ_0, $L^{(0)}$ and $M^{(0)}$). When $j = p$, the factor $\prod_{l=j+1}^{p} A_l$ is understood to be the identity. Without loss of generality, the functions f_{2j} are assumed to be positive (it suffices to write $f_{2j} = f_{2j+} - f_{2j-}$). Lemma E.3 implies that for each j,

$$\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_{2j})\| \lesssim \frac{M^{(2j)}}{m} n^{(d_{2j}-d)_+ + \epsilon}. \tag{E.9}$$

Using the relations in (1.6) (main paper) it then follows that

$$\begin{aligned}
\left\|\prod_{l=j+1}^{p} A_l\right\| &\leq \prod_{l=j+1}^{p} \|T_n^{\frac{1}{2}}(f_{2l})T_n^{-\frac{1}{2}}(f)\|^2 \\
&\lesssim \left(\prod_{l=j+1}^{p-1} M^{(2l)}\right) n^{2\sum_{l=j+1}^{p-1}(d_{2l}-d)_+ + (d_{2j}-d)_+ + (d_{2p}-d)_+} \sqrt{M^{(2p)}M^{(2j)}}.
\end{aligned} \tag{E.10}$$

First we treat the term $(A_1 - B_1)\prod_{l=2}^{p} A_l$ on the right in (E.8). Writing $R =$

$I_n - T_n^{\frac{1}{2}}(f)T_n(\tilde{f})T_n^{\frac{1}{2}}(f)$, it follows that

$$\begin{aligned}
&\left|\text{tr}\left[(A_1 - B_1)\prod_{l=2}^{p} A_l\right]\right| \\
&\quad = \left|\text{tr}\left[T_n^{\frac{1}{2}}(f_{2p})T_n^{-\frac{1}{2}}(f)RT_n^{-\frac{1}{2}}(f)T_n(f_2)T_n^{-\frac{1}{2}}(f)T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\prod_{l=3}^{p} A_l\right]\right| \\
&\quad \le |R||T_n^{-\frac{1}{2}}(f)T_n(f_2)T_n^{-\frac{1}{2}}(f)|\|T_n^{\frac{1}{2}}(f_{2p})T_n^{-\frac{1}{2}}(f)\|\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\|\left\|\prod_{l=3}^{p} A_l\right\| \\
&\quad \lesssim L^{\frac{1}{2}} n^{(1-\rho)/2+\epsilon+\frac{1}{2}+2q}\prod_{l=2}^{p} M^{(2l)}\left(\int_{-\pi}^{\pi}\frac{f_2^2(x)}{f^2(x)}dx + \text{error}\right)^{\frac{1}{2}} \\
&\quad \lesssim L^{\frac{1}{2}} n^{(1-\rho)/2+\epsilon+\frac{1}{2}+2q}\prod_{l=2}^{p} M^{(2l)} \times \\
&\qquad \left(\int_{-\pi}^{\pi}\frac{f_2^2(x)}{f^2(x)}dx + n^{\epsilon+2q}\left(L^{3/2}\left(M^{(2)}\right)^2 n^{(1-3\rho)/2} + M^{(2)}L^{(2)}n^{-\rho_2}\right)\right)^{\frac{1}{2}}.
\end{aligned} \tag{E.11}$$

The first inequality follows from the relations in (1.6) (main paper). The second inequality follows after writing $|T_n^{-\frac{1}{2}}(f)T_n(f_2)T_n^{-\frac{1}{2}}(f)|$ as the sum of a limiting integral and an approximation error; in addition we use (E.9) and Lemma E.2, by which

$$|R|^2 \le KL(M/m)^2 n^{1-\rho+\epsilon} \lesssim Ln^{1-\rho+\epsilon}. \tag{E.12}$$

This follows from Lemma E.4, which we use to bound the approximation error. The second term within the brackets in (E.11) constitutes part of the term *error*.

Next we bound the term $(\prod_{l=1}^{j-1} B_l)(A_j - B_j)\prod_{l=j+1}^{p} A_l$ in (E.8) for $j = 2$. Similar to the preceding decomposition, we have

$$\begin{aligned}
\left|\text{tr}\left[B_1(A_2 - B_2)\prod_{l=3}^{p} A_l\right]\right| &= \left|\text{tr}\left[B_1 T_n^{\frac{1}{2}}(f_2)T_n^{-\frac{1}{2}}(f)RT_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\prod_{l=3}^{p} A_l\right]\right| \\
&\le |B_1 T_n^{\frac{1}{2}}(f_2)T_n^{-\frac{1}{2}}(f)||R|\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\|\left\|\prod_{l=3}^{p} A_l\right\|.
\end{aligned}$$

The terms $|R|$, $\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\|$ and $\|\prod_{l=3}^{p} A_l\|$ are bounded as in (E.9), (E.10) and (E.12). For the term $|B_1 T_n^{\frac{1}{2}}(f_2) T_n^{-\frac{1}{2}}(f)|$ we have the decomposition

$$\begin{aligned}
&|B_1 T_n^{\frac{1}{2}}(f_2) T_n^{-\frac{1}{2}}(f)|^2 = \mathrm{tr}\left[T_n^{-\frac{1}{2}}(f) T_n^{\frac{1}{2}}(f_2) B_1^t B_1 T_n^{\frac{1}{2}}(f_2) T_n^{-\frac{1}{2}}(f)\right] \\
&= \mathrm{tr}\left[B_1^t B_1 T_n^{\frac{1}{2}}(f_2) T_n^{-1}(f) T_n^{\frac{1}{2}}(f_2)\right] = \mathrm{tr}\left[B_1^t B_1 T_n^{\frac{1}{2}}(f_2) T_n(\tilde{f}) T_n^{\frac{1}{2}}(f_2)\right] \\
&\quad + \mathrm{tr}\left[B_1^t B_1 T_n^{\frac{1}{2}}(f_2) T_n^{-\frac{1}{2}}(f) R T_n^{-\frac{1}{2}}(f) T_n^{\frac{1}{2}}(f_2)\right] \\
&\leq |B_1 T_n^{\frac{1}{2}}(f_2) T_n^{\frac{1}{2}}(\tilde{f})|^2 + |B_1^t B_1| |R| \| T_n^{\frac{1}{2}}(f_2) T_n^{-\frac{1}{2}}(f)\|^2.
\end{aligned}$$

Using again Lemmas E.1, E.2 and E.3, we find that the first term on the right is bounded by

$$n\left\{\int_{-\pi}^{\pi} \frac{|f_{2p}(x)| f_2^2(x)}{f^3(x)} dx + M^{(2p)} M^{(2)} n^{4(d_2-d)_+ + 2(d_{2p}-d)_+ + \epsilon}\left(M^{(2)} L n^{-\rho} + L^{(2)} n^{-\rho_2}\right)\right\}$$

and the second term by

$$\begin{aligned}
&n^{\frac{1}{2}-\frac{\rho}{2}+(d_2-d)_+ + \epsilon}\sqrt{L} M^{(2)}\left[n\int_{-\pi}^{\pi} \frac{f_{2p}^2(x) f_2^2(x)}{f^4(x)} dx + M^{(2p)} M^{(2)} n^{4(d_2-d)_+ + 2(d_{2p}-d)_+ + \epsilon} \times \right. \\
&\quad \left. \times\left((M^{(2)})^2 (M^{(2p)})^2 L n^{1-\rho} + (M^{(2p)})^2 M^{(2)} L^{(2)} n^{1-\rho_2} + (M^{(2)})^2 M^{(2p)} L^{(2p)} n^{1-\rho_{2p}}\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left|\mathrm{tr}\left[B_1 (A_2 - B_2) \prod_{l=3}^{p} A_l\right]\right| \\
&\lesssim L^{\frac{1}{2}} n^{\frac{1}{2}+(1-\rho)/2+\epsilon+2q} \sqrt{M^{(2p)}} \prod_{l=2}^{p-1} M^{(2l)} \left[\int_{-\pi}^{\pi} \frac{f_2^2(x)|f_{2p}(x)|}{f^3(x)} dx + M^{(2p)} (M^{(2)})^2 L n^{-\rho}\right. \\
&+ n^{-\rho/2} (L)^{1/2} \sqrt{M^{(2)}} \left(\int_{-\pi}^{\pi} \frac{f_2^2(x) f_{2p}^2(x)}{f^4(x)} dx\right)^{\frac{1}{2}} + L^{(2)} M^{(2)} M^{(2p)} n^{-\rho_2} \\
&\left. + (LL^{(2)})^{1/2} M^{(2p)} (M^{(2)})^{3/2} n^{-(\rho+\rho_2)/2} + (LL^{(2p)} M^{(2p)})^{1/2} (M^{(2)})^2 n^{-(\rho+\rho_{2p})/2}\right]^{\frac{1}{2}}.
\end{aligned}$$

Note that

$$\left(\int_{-\pi}^{\pi} \frac{f_2^2(x) f_{2p}(x)^2}{f^4(x)} dx\right)^{\frac{1}{2}} \lesssim M^{(2)} M^{(2p)},$$

$$(LL^{(2)} M^{(2p)})^{1/2} (M^{(2)})^{3/2} n^{-(\rho+\rho_2)/2} \leq L (M^{(2)})^2 n^{-\rho} + L^{(2)} M^{(2)} M^{(2p)} n^{-\rho_2}$$

and $Ln^{-\rho} \lesssim L^{3/2} n^{(1-3\rho)/2}$. Therefore the terms on the right are of the same order as the right hand side of (E.11). A similar argument applies to the term $(LL^{(2p)} M^{(2p)})^{1/2} (M^{(2)})^2 n^{-(\rho+\rho_{2p})/2}$.

Finally, we bound the term $(\prod_{l=1}^{j-1} B_l)(A_j - B_j) \prod_{l=j+1}^{p} A_l$ in (E.8) for $j \geq 3$. For $j \geq 3$, Lemma E.1 implies that

$$|\prod_{l=1}^{j-1} B_l|^2 = n\left(\int_{-\pi}^{\pi} \frac{f_{2p}(x) f_{2j-2}(x)}{f^2(x)} \prod_{l=1}^{j-2} \frac{f_{2l}^2(x)}{f^2(x)} dx + error_j\right),$$

where

$$error_j \lesssim n^{\epsilon+2\sum_{l=1}^{j-1}(d_{2l}-d)_+}\prod_{l=1}^{j-1}M^{(2l)}M^{(2l-2)}\left(Ln^{-\rho}+\sum_{l=1}^{j-1}\frac{L^{(2l)}}{M^{(2l)}}n^{-\rho_{2l}})\right).$$

Consequently, we have for all $j \geq 2$

$$\left|\mathrm{tr}\left[\left(\prod_{l=1}^{j-1}B_l\right)(A_j-B_j)\prod_{l=j+1}^{p}A_l\right]\right|$$
$$\leq |\prod_{l=1}^{j-1}B_l||R|\prod_{l=j+1}^{p}\|A_l\|\|T_n^{\frac{1}{2}}(f_{2j})T_n^{-\frac{1}{2}}(f)\|\|T_n^{\frac{1}{2}}(f_{(2j-2)})T_n^{-\frac{1}{2}}(f)\|$$
$$\lesssim L^{\frac{1}{2}}n^{\frac{1}{2}+(1-\rho)/2+2q+\epsilon}\sqrt{M^{(2p)}}\prod_{l=j+1}^{p-1}M^{(2l)}\left(\int_{-\pi}^{\pi}\frac{f_{2p}(x)f_{2j}(x)}{f^2(x)}\prod_{l=1}^{j-1}\frac{f_{2l}^2(x)}{f^2(x)}dx+error_j\right)^{\frac{1}{2}}.$$

for all $j \geq 3$, which finishes the proof of Lemma E.6. □

F Hölder constants of various functions

Lemma F.1. *Let $\theta_o \in \Theta(\beta, L_o)$. Then f_o satisfies condition* (E.2) *with $\rho = 1$ when $\beta > \frac{3}{2}$, and with any $\rho < \beta - \frac{1}{2}$ when $\beta \leq \frac{3}{2}$. The Hölder-constant only depends on L_o. When $\theta \in \Theta_k(\beta, L)$, $f_{d,k,\theta}$ satisfies* (E.2) *with $\rho = 1$, regardless of β. The Hölder-constant is of order $k^{\frac{3}{2}-\beta}$. The function $-\log(2-2\cos(x))f_{d,k,\theta}$ satisfies condition* (E.1) *with $\rho = 1$ and Hölder-constant of order $k^{\frac{3}{2}-\beta}$. The functions $G_k f_{d,k,\theta}$ and $H_k f_{d,k,\theta}$, with G_k and H_k as in* (D.1)*, satisfy* (E.1) *with $\rho = 1$ and Hölder-constant of order k.*

Proof. The function $\sum_{j=0}^{\infty}\theta_{o,j}\cos(jx)$ (i.e. the logarithm of the short-memory part of f_o), has smoothness $\rho < \beta - \frac{1}{2}$, since

$$\sum_{j=0}^{\infty}|\theta_{o,j}||\cos(jx)-\cos(jy)| \leq \left(\sum_{j=0}^{\infty}|\theta_{o,j}|j^{\rho}\right)|x-y|^{\rho}$$
$$\leq \left(\sum_{j=0}^{\infty}\theta_{o,j}^2 j^{2\beta}\right)^{\frac{1}{2}}\left(\sum_{j=0}^{\infty}j^{-2(\beta-\rho)}\right)^{\frac{1}{2}}|x-y|^{\rho},$$

which is finite only when $\rho < \beta - \frac{1}{2}$. Since $\sum_{j=0}^{\infty}|\theta_j| \lesssim \sqrt{L}$ when $\theta \in \Theta(\beta - 1/2, L)$ and $\beta > 1$, the functions $\sum_{j=0}^{\infty}\theta_j\cos(jx)$ and $\exp\{\sum_{j=0}^{\infty}\theta_j\cos(jx)\}$ have the same smoothness; only the values of L and M differ. The same calculation can be made when the FEXP-expansion is finite: when $\theta \in \Theta_k(\beta, L)$, then for all $x, y \in [-\pi, \pi]$,

$$\left|\sum_{j=0}^{k}\theta_j(\cos(jx)-\cos(jy))\right| \leq |x-y|\sum_{j=0}^{k}j|\theta_j| \lesssim \sqrt{L}k^{\frac{3}{2}-\beta}|x-y|. \tag{F.1}$$

Since

$$|G_k(x) - G_k(y)| \leq 2\sum_{j=1}^{k} \eta_j |\cos(jx) - \cos(jy)| = O(k)|x-y|, \tag{F.2}$$

$G_k f_{d,k,\theta}$ has Hölder-smoothness $\rho = 1$, its Hölder-constant being $O(k)$. The same result holds for $H_k f_{d,k,\theta}$, since $H_k(x) = -\log(2 - 2\cos(x)) - G_k(x)$ (see (D.3)) and $k^{\frac{3}{2}-\beta} = o(k)$ for all $\beta > 1$. □

G Proof of Lemma B.2

For easy reference we first restate the result. Let $W_\sigma(d)$ denote any of the quadratic forms

$$X^t T_n^{-1}(f_{d,k}) B_\sigma(d, \bar{\theta}_{d,k}) X - \text{tr}\left[T_n(f_o) T_n^{-1}(f_{d,k}) B_\sigma(d, \bar{\theta}_{d,k})\right]$$

in (B.2) (in the main paper). Then for any $j \leq J$, $(l_1, \ldots, l_j) \in \{0, \ldots, k\}^j$ and $\sigma \in \mathcal{S}(l_1, \ldots, l_j)$, we have

$$|W_\sigma(d) - W_\sigma(d_o)| = \mathbf{o_{P_o}}(|d - d_o| n^{\frac{1}{2}+\epsilon} k^{-\frac{1}{2}}), \tag{G.1}$$

$$\begin{aligned}
&\text{tr}\left[B_\sigma(d, \bar{\theta}_{d,k})\right] - \text{tr}\left[B_\sigma(d_o, \bar{\theta}_{d_o})\right] \\
&\quad = (d - d_o)\text{tr}[T_{1,\sigma}(d_o, k)] + (d - d_o)^2 \mathbf{o}(n^{\epsilon+\frac{1}{2}} k^{-\frac{1}{2}+(1-\beta/2)_+}) \qquad \text{(G.2)}\\
&\quad = (d - d_o)\text{tr}[T_{1,\sigma}(d_o, k)] + (d - d_o)^2 \mathbf{o}(n^{1-\delta}/k),
\end{aligned}$$

$$\begin{aligned}
&\text{tr}\left[(T_n(f_o) T_n^{-1}(f_{d,k}) - I_n) B_\sigma(d, \bar{\theta}_{d,k})\right] - \text{tr}\left[(T_n(f_o) T_n^{-1}(f_{d_o,k}) - I_n) B_\sigma(d_o, \bar{\theta}_{d_o,k})\right] \\
&\quad = (d - d_o)\text{tr}[T_{2,\sigma}(d_o, k)] + (d - d_o)^2 \mathbf{o}(n/k) + (d - d_o)\mathbf{o}(n^{\epsilon+\frac{1}{2}} k^{-\frac{1}{2}}).
\end{aligned} \tag{G.3}$$

Proof of Lemma B.2. We first prove (G.2). Developing the left-hand side in d we obtain, for all j, $(l_1, \ldots, l_j) \in \{0, \ldots, k\}^j$ and $\sigma \in \mathcal{S}_j$,

$$\begin{aligned}
&\text{tr}\left[B_\sigma(d, \bar{\theta}_{d,k})\right] - \text{tr}\left[B_\sigma(d_o, \bar{\theta}_{d_o})\right] \\
&\quad = (d - d_o)\text{tr}\left[B'_\sigma(d_o, \bar{\theta}_{d_o,k})\right] + \frac{(d - d_o)^2}{2}\text{tr}\left[B''_\sigma(\bar{d}, \bar{\theta}_{\bar{d}})\right],
\end{aligned} \tag{G.4}$$

where $\bar{d} \in (d, d_o)$, and B' and B'' denote the first and second derivative with respect to d, respectively. Writing

$$\begin{aligned}
\tilde{B}_{\sigma(i)}(d, k) &= T_n(H_k \nabla_{\sigma(i)} f_{d,k}) T_n^{-1}(f_{d,k}) \\
&\quad - T_n(\nabla_{\sigma(i)} f_{d,k}) T_n^{-1}(f_{d,k}) T_n(H_k f_{d,k}) T_n^{-1}(f_{d,k}),
\end{aligned}$$

it follows that $B'_\sigma(d, \bar{\theta}_{d,k})$ equals

$$B'_\sigma(d, \bar{\theta}_{d,k}) = \sum_{i=1}^{|\sigma|} \prod_{j<i} T_n(\nabla_{\sigma(j)} f_{d,k}) T_n^{-1}(f_{d,k}) \tilde{B}_{\sigma(i)}(d, k) \prod_{j>i} T_n(\nabla_{\sigma(j)} f_{d,k}) T_n^{-1}(f_{d,k}).$$

We recall the definition of $T_{1,\sigma}$ in Lemma B.1 (main paper), and conclude that $B'_\sigma(d,\bar\theta_{d,k}) = T_{1,\sigma}(d,k)$. Consequently, the first term on the right in (G.4) equals $(d-d_o)\mathrm{tr}[T_{1,\sigma}(d_o,k)]$.

The second derivative $B''_\sigma(d,\bar\theta_{d,k})$ equals

$$
\begin{aligned}
&2\sum_{i_1<i_2}^{|\sigma|}\prod_{j<i_1} T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\tilde B_{\sigma(i_1)}(d,k)\prod_{i_1<j<i_2} T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\\
&\qquad\times \tilde B_{\sigma(i_2)}(d,k)\prod_{i_2<j} T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\\
&\qquad+\sum_{i=1}^{|\sigma|}\prod_{j<i} T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\tilde B'_{\sigma(i)}(d,k)\prod_{i<j} T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k}).
\end{aligned}
$$

We now show that $\mathrm{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right] = \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+})$. From Lemma E.4 and the above expression for $B''_\sigma(d,\bar\theta_{d,k})$, it can be seen that $\mathrm{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]$ converges to zero. To bound the approximation error, we cannot use directly Lemma E.4 because the bound in (E.4) becomes too large when $\beta<2$ and $|\sigma|$ is larger than 1. We therefore use Lemmas E.1 and E.6. Let $A''_\sigma(d,\bar\theta_{d,k})$ be the matrix obtained after replacing every factor $T_n^{-1}(f_{d,k})$ in $B''_\sigma(d,\bar\theta_{d,k})$ by $T_n(\tilde f_{d,k})$, for $\tilde f_{d,k} = f_{d,k}^{-1}/(4\pi^2)$. We recall from Lemma F.1 that the Lipschitz constant of $f_{d,k}$ is $O(k^{(2-\beta)_+})$, and for $H_k^j f_{d,k}$ and $H_k^j\nabla_{\sigma(m)}f_{d,k}$ ($m\le|\sigma|$, $j=1,2$) it is $O(k\log k)$. Consequently, Lemma E.1 implies that

$$
\left|\mathrm{tr}\left[A''_\sigma(d,\bar\theta_{d,k})\right]\right| \quad=\quad O(kn^\epsilon) = \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+})
$$

when $k\le k_n$ and $\beta>1$. It follows from Lemma E.6 that

$$
\begin{aligned}
\left|\mathrm{tr}\left[A''_\sigma(d,\bar\theta_{d,k})\right]-\mathrm{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]\right| &= O(n^{1/2+\epsilon}k^{(1-\beta/2)_+})\left(\int_{-\pi}^{\pi}H_k^2(x)dx\right)^{\frac12}\\
&= \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+}).
\end{aligned}
$$

Note that in the case where $B''_\sigma(d,\bar\theta_{d,k})$ contains a Toeplitz matrix of the form $T_n(H_k^2 f_{d,k})$ or $T_n(H_k^2\nabla_{\sigma(m)}f_{d,k})$ then it contains no other Toeplitz matrix involving H_k and we can set $f_2 = H_k^2 f_{d,k}$ or $f_2 = H_k^2\nabla_{\sigma(m)}f_{d,k}$ and use Remark 2.1; this leads to the above error rate. Combining the preceding results for $|\mathrm{tr}[A''_\sigma(d,\bar\theta_{d,k})]|$ and $|\mathrm{tr}[A''_\sigma(d,\bar\theta_{d,k})]-\mathrm{tr}[B''_\sigma(d,\bar\theta_{d,k})]|$ we obtain that

$$
\left|\mathrm{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]\right| \quad=\quad \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+}) = \mathbf{o}(n^{1-\delta}/k),
$$

which completes the proof of (G.2).

Next, we prove (G.3). Writing $f_o - f_{d,k} = f_o - f_{d_o,k} + f_{d_o,k} - f_{d,k}$, it follows that the left-hand side of (G.3) equals

$$\begin{aligned}
&\mathrm{tr}\left[T_n(f_o - f_{d,k})T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})\right] - \mathrm{tr}\left[T_n(f_o - f_{d_o,k})T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right]\\
&= \mathrm{tr}\left[T_n(f_o - f_{d_o,k})\left\{T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k}) - T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right\}\right]\\
&+ \mathrm{tr}\left[T_n(f_{d_o,k} - f_{d,k})T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})\right]\\
&:= C_1 + C_2.
\end{aligned}$$

Using (D.4) we write $f_{d,k} = f_{d_o,k}e^{(d-d_o)H_k}$ and $f_o = f_{d_o,k}e^{\Delta_{d_o,k}}$, and we develop $C_\sigma(d,\bar\theta_{d,k}) = T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})$ around $d = d_o$. It follows that

$$\begin{aligned}
C_1 &= \mathrm{tr}\left[T_n(f_o - f_{d_o,k})\left\{T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k}) - T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right\}\right]\\
&= (d-d_o)\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C_\sigma^{'}(d_o,\bar\theta_{d_o,k})\right]\\
&\quad + (d-d_o)^2\int_0^1(1-u)\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C_\sigma^{''}(d_u,\bar\theta_{d_u,k})\right]du,
\end{aligned}$$

with $d_u = ud + (1-u)d_o$. For the first term on the right, we write, using Lemmas E.1 and E.6,

$$\begin{aligned}
&\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C_\sigma^{'}(d_o,\bar\theta_{d_o,k})\right]\\
&\quad = \frac{n}{2\pi}\int_{-\pi}^{\pi}\frac{f_o - d_{d_o,k}}{f_{d_o,k}}H_k(x)\cos(l_1x)\ldots\cos(l_{|\sigma|}x)dx + \text{error},
\end{aligned}$$

where σ is a partition of $\{1,...,j\}$ and the error term is

$$O\left(\|\Delta_{d_o,k}\|_\infty n^\epsilon(k + k^{0.5(3/2-\beta)_+}\frac{\sqrt{n}}{\sqrt{k}} + k^{0.5(3/2-\beta)_+}\left(\frac{n}{k^{2\beta}} + k\|\Delta_{d_o,k}\|_\infty\right)^{\frac12}\right),$$

which is $o(k^{-1/2}n^{1/2-\delta})$. Similarly, Lemmas E.1 and E.6 imply that there exists $c\in\mathbb{R}$ such that for al $d\in(d_o - \bar v_n, d_o + \bar v_n)$

$$\begin{aligned}
&\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C_\sigma^{''}(d,\bar\theta_{d,k})\right]\\
&\quad = \frac{cn}{2\pi}\int_{-\pi}^{\pi}\frac{f_o - d_{d_o,k}}{f_{d_o,k}}H_k^2(x)\cos(l_1x)\ldots\cos(l_{|\sigma|}x)dx + \text{error},
\end{aligned}$$

where the error term is of order

$$O\left(\|\Delta_{d_o,k}\|_\infty n^\epsilon(k + k^{\frac12(2-\beta)_+}\frac{\sqrt{n}}{\sqrt{k}} + k^{\frac12(2-\beta)_+}(\frac{n}{k^{2\beta}} + k\|\Delta_{d_o,k}\|_\infty)^{\frac12}\right) = o\left(\frac{n^{1-\delta}}{k}\right).$$

This implies that $C_1 = O(S_n(d))$.

Using a Taylor expansion of $C_\sigma(d,\bar\theta_{d,k})$ and of $e^{-(d-d_o)H_k}$ around d_o, it follows that

$$\begin{aligned}
C_2 &= -(d-d_o)\mathrm{tr}\left[T_n(f_{d_o,k}H_k)T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right]\\
&\quad - \frac12(d-d_o)^2\mathrm{tr}\left[T_n(f_{d_o,k}H_k^2e^{-t(d'-d_o)H_k})C_\sigma(d',\bar\theta_{d',k}) + 2T_n(f_{d_o,k}H_k)C_\sigma^{'}(d',\bar\theta_{d',k})\right],
\end{aligned}$$

for some $d^{'}$ between d and d_o. The first term equals $\mathrm{tr}[T_{2,\sigma}]$. The second equals

$$-\frac{1}{2}(d-d_o)^2 \mathrm{tr}\left[T_n(f_{d',k}H_k^2)C_\sigma(d',\bar\theta_{d',k}) + 2T_n(f_{d_o,k}H_k)C_\sigma^{'}(d^{'},\bar\theta_{d',k})\right]$$
$$= -\frac{n(d-d_o)^2}{2\pi}\int_{-\pi}^{\pi} H_k^2(x)\cos(l_1x)\ldots\cos(l_{|\sigma|}x)dx + \mathrm{error},$$

where the error term is $O\left(n^\epsilon\left(k + k^{0.5(2-\beta)_+}(nk^{-1}+kn^\epsilon)^{1/2}\right)\right) = o(k^{-1}n^{1-\delta})$. Therefore

$$C_2 = (d-d_o)\mathrm{tr}[T_{2,\sigma}] + O(n/k).$$

Finally, to prove (G.1), let $Z = T_n^{-\frac{1}{2}}(f_o)X$ and let $A_d = T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})T_n^{\frac{1}{2}}(f_o)$. Then for any $|d-d_o| \le \bar v_n$, we have

$$W_\sigma(d) - W_\sigma(d_o) = Z^t(A_d - A_{d_o})Z - \mathrm{tr}\left(A_d - A_{d_o}\right).$$

Writing A'_d for the derivative of A_d with respect to d, it follows that

$$A_d - A_{d_o} = (d-d_o)A'_{\bar d}, \tag{G.5}$$

for some $\bar d$ between d and d_o. Using (D.6), we find that

$$A'_d = T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d,k})T_n(H_kf_{d,k})T_n^{-1}(f_{d,k})T_n(B_\sigma(d,\bar\theta_{d,k}))T_n^{\frac{1}{2}}(f_o)$$
$$+ T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d,k})B'_\sigma(d,\bar\theta_{d,k})T_n^{\frac{1}{2}}(f_o).$$

Therefore, Lemma 2 of Lieberman et al. (2011) and the inequalities in (1.6) (main paper) imply that

$$|A_d - A_{d_o}| \le |d-d_o||A_{\bar d}^{'}|$$
$$\le C|d-d_o|\|T_n^{\frac{1}{2}}(f_o)T_n^{-\frac{1}{2}}(f_{\bar d,k})\|^2 \prod_{i=1}^{|\sigma|}\|T_n^{-\frac{1}{2}}(f_{\bar d,k})B_{\sigma(i)}(\bar d,\bar\theta_{\bar d,k})T_n^{\frac{1}{2}}(f_{\bar d,k})\|$$
$$+ |T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k\nabla_{\sigma(i)}f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})|$$
$$= |d-d_o|n^\epsilon O\left(|T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_kf_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})| + |T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k\nabla_{\sigma(i)}f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})|\right), \tag{G.6}$$

where $\sigma(i)$ can also be the empty set, in which case $\nabla_{\sigma(i)}f_{\bar d,k} = f_{\bar d,k}$. We bound the terms between brackets using Lemma E.4, with $p=2$, $f = f_{\bar d,k}$ and $g_1 = g_2$ equalling either $H_kf_{\bar d,k}$ or $H_k\nabla_{\sigma(i)}f_{\bar d,k}$. The Hölder constants of these functions are given by Lemma F.1. Hence we find that

$$|T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_kf_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})|^2 = \mathrm{tr}\left[\left(T_n^{-1}(f_{\bar d,k})T_n(H_kf_{\bar d,k})\right)^2\right]$$
$$= \frac{n}{2\pi}\int_{-\pi}^{\pi}H_k^2(x)dx + O(n^\epsilon(k+k^{2-\beta})) = O(n^{1-1/(2\beta)}(\log n)^{1/(2\beta)}). \tag{G.7}$$

The last inequality follows from equation (D.7) in Lemma D.1 and the fact that $k = k_n$ and $\beta > 1$. Similarly, it follows that

$$|T_n^{-\frac{1}{2}}(f_{\bar{d},k})T_n(H_k \nabla_{\sigma(i)} f_{\bar{d},k})T_n^{-\frac{1}{2}}(f_{\bar{d},k})|^2 = O(n^{1-1/(2\beta)}(\log n)^{1/(2\beta)}). \quad \text{(G.8)}$$

Inserting (G.6), (G.7) and (G.8) in (G.5), we find that $|A_d - A_{d_o}| \leq |d - d_o| n^{1/2-1/(4\beta)+\epsilon}$, for all $|d - d_o| \leq \bar{v}_n$ and all $\epsilon > 0$, when n is large enough. Consequently, we can apply Lemma 1.3 with $A = (A_d - A_{d_o})/|A_d - A_{d_o}|$, so that when n is large enough

$$\sup_{|d-d_o|\leq \bar{v}_n} P_o\left(|W_\sigma(d) - W_\sigma(d_o)| > |d - d_o| n^{2\epsilon + \frac{1}{2} - \frac{1}{4\beta}}\right) \leq e^{-n^\epsilon/8}. \quad \text{(G.9)}$$

Using the above computations with $|d - d'| \leq n^{-2}$, we obtain

$$|W_\sigma(d) - W_\sigma(d')| \leq n^{-2+\epsilon}\left(n + Z^t Z\right).$$

Hence, for all $\epsilon < \frac{1}{2}$ and $c > 0$,

$$P_o\left(\sup_{|d'-d|\leq n^{-2}} |W_\sigma(d) - W_\sigma(d')| > n^{-\epsilon}\right) \leq P_o\left(Z^t Z > n^{2-2\epsilon}\right) \leq e^{-cn}, \quad \text{(G.10)}$$

provided n is large enough. Hence, we obtain (G.1) by combining (G.9) and (G.10) in a simple chaining argument over the interval $(d_o - \bar{v}_n, d_o + \bar{v}_n)$.

□