Queueing approximation of suprema of spectrally positive Lévy process

M. Czystolowski, W. Szczotka

Queueing approximation of suprema of spectrally positive Lévy process

by

M.Czystołowski[1] and W. Szczotka[2]

University of Wrocław, Poland

November 29, 2009

Third Revision

Summary: Let $W = \sup_{0 \le t < \infty}(X(t) - \beta t)$, where X is a spectrally positive Lévy process with expectation zero and $0 < \beta < \infty$. One of the main results of the paper says that for such a process X there exists a sequence of M/GI/1 queues for which stationary waiting times converge in distribution to W. The second result shows that condition (III) of Proposition 2 in the paper is not implied by all other conditions.

AMS Subject Classification: 60K25, 60F17, 60E10, 60G99

Key words and phrases: Lévy process, Lévy measure, Laplace-Stielties transform, queueing systems, heavy traffic, stationary waiting time, Mittag-Leffler distribution.

1 Introduction

Let X be a Lévy process with expectation zero and let $W = \sup_{0 \le t < \infty}(X(t) - \beta t)$, where $0 < \beta < \infty$. The random variable W appears in many areas of applied probability, such as queueing theory, risk theory (see e.g. S. Asmussen [1] or P. Embrechts

[1]Supported by KBN grant N201 020 32/0816.

[2]Supported by grant 2787/W/IM.

et al. [2]). The Laplace-Stielties transform (LST) of W was considered by many authors (see among others G. Baxter and M.D. Donsker [3] and N.H. Bingham [5]). In the case when X is a centered Poisson process that distribution was given by R. Pyke [14]. In a more general case, i.e. for a spectrally positive Lévy process the LST of W was given for the first time by V.M. Zolotarev [22] and afterwards several different proofs of his theorem were formulated (see e.g. L. Takács, [19], N.H. Bingham [5], J.M. Harrison [9], O. Kella and W. Whitt [10] and others).

In the queueing theory the random variable W appears as a limit of appropriately normalized stationary waiting times in heavy traffic for some queues. This fact was shown several times with use of different techniques, approaches and assumptions (see e.g. Yu. V. Prokhorov [13], W. Whitt [20] and [21], S. Resnick and G. Samorodnitsky [15], W. Szczotka and W.A. Woyczyński [17] and [18], M. Czystołowski and W. Szczotka [7]). The form of the LST of W in the context of GI/GI/1 queues was given by O.J. Boxma and J.W. Cohen [4] for the cases with X being stable spectrally positive or spectrally negative Lévy processes.

A representation of stationary waiting time ω for G/G/1 queues given by $\omega = \sup_{0\le t<\infty}\big(Z(t)-\beta(t)\big)$, where Z is a process based on sums of differences of service times and inter-arrival times with sample paths in $D[0,\infty)$ and $\beta(t)$ is a function from $D[0,\infty)$, suggests a natural way of studying weak convergence $\omega_n = \sup_{0\le t<\infty}(X_n(t)-\beta_n(t)) \xrightarrow{D} W$. Namely, the method is based on interchanging the limit operation with supremum operation. This exchange is justified if: (I) $X_n \xrightarrow{D} X$ in $D[0,\infty)$ with Skorokhod J_1 topology and X is stochastically continuous; (II) $\beta_n(t) \to \beta t$ for each $t \ge 0$; and (IIIA) $\lim_{m\to\infty}\limsup_{n\to\infty} P(\sup_{t\ge m}(X_n(t)-\beta_n(t)) > \varepsilon) = 0$ for each $\varepsilon > 0$. This idea was first used by Yu. V. Prokhorov [13] to show convergence $\omega_n \xrightarrow{D} W$ for GI/GI/1 queues in heavy traffic with X being a Wiener process. It was also formulated by Asmussen [1] as an universal method of showing convergence $\sup_{0\le t<\infty}(X_n(t)-\beta_n(t)) \xrightarrow{D} \sup_{0\le t<\infty}(X(t)-\beta t)$ with general processes X_n and X and next used by S. Resnick and G. Samorodnitsky [15] to study convergence $\omega_n \xrightarrow{D} W$ with X being a Lévy process. W. Szczotka in (1990) and next in (1999) studying convergence $\omega_n \xrightarrow{D} W$ for queues with dependencies, replaced condition (IIIA) by condition (III): $\{\omega_n\}$ is tight. Notice that condition (III) is necessary for weak convergence of ω_n and in some queueing situations it may be easier to check (III) than (IIIA) (see proofs of Lemmas 2 and 3 in [7]). The method of proving $\omega_n \xrightarrow{D} W$ by verifying conditions (I), (II) and (III) was called by W. Szczotka and W. A. Woyczyński [17] the Heavy Traffic Invariance Principle (HTIP) (cf. Proposition 2).

The aim of the present paper is the following. Firstly, for a given Lévy measure ν concentrated on $(0,\infty)$ such that $\int_1^\infty x\nu(dx) < \infty$ construct a suitable sequence of M/GI/1 queueing systems in heavy traffic such that $\omega_n \xrightarrow{D} W$ with Lévy process X having measure ν. The construction of that sequence is given in Theorem 1. This theorem is necessary to formulate Theorem 2 and Corollary 2. However, it is also useful to give another derivation of W when X is a centered Poisson process, what is illustrated in Section: Application of Theorem 1.
As a consequence of Theorem 1 we get the fact that in the set of limiting distributions of stationary waiting times in heavy traffic are not only exponential distributions and Mittag-Leffler distributions, but also some convolutions of Mittag-Leffler distributions or distributions of suprema of: Poisson processes, Compound Poisson processes or Gamma processes with negative trends (cf. Remark 1).

The second aim of the paper is to answer the question whether condition (III) for GI/GI/1 queues is implied by conditions (I) and (II). The particular case of this question, i.e. when X is a Wiener process, was communicated to W. Szczotka by

W. Whitt. The negative answer is given in Theorem 2 and Corollary 2.

All theorems are based on a stronger version of Lemma 2 from [7], which we formulate here as Proposition 3. Essentially, the proposition extends the range of applications of Lemma 2 from [7], so may be treated as a new result.

The structure of the paper is as follows: Section 2 serves as a reminder of Lévy processes theory and queueing theory; Section 3 contains only novel results of the paper; finally, Appendix contains some technical facts needed in the paper and the proof of Proposition 3.

2 Preliminaries

2.1 Lévy Process

The terminology dealing with Lévy processes, which is used here, comes from [16] and we assume below that a Lévy process has sample paths in the space $D[0,\infty)$. Any Lévy process Y can be obtained as a limit in distribution of the following processes $Y_n(t) = \sum_{j=1}^{[nt]} \zeta_{n,j}, \quad t \geq 0, \ n \geq 1$, where for each $n \geq 1$, the random variables $\zeta_{n,k}, k \geq 1$, are mutually independent with distribution functions $F_{n,k}$, respectively, which satisfy conditions (3.35a)–(3.35d) given by Yu. V. Prokhorov in [12], p. 197. If for each $n \geq 1, \ F_{n,k} = F_n$, then condition (3.35b) is implied by all others. Further on we consider only the last case and only spectrally positive Lévy processes. Recall that Lévy process $Y = \{Y(t), t \geq 0\}$ is spectrally positive if its LST, $E \exp\Big(-sY(t)\Big) = \exp\Big(t\psi(s)\Big)$, $s \geq 0$, is such that

$$\psi(s) = -sb_r + s^2\sigma^2/2 + \int_0^\infty \Big(e^{-sx} - 1 + sx\mathbf{1}_{\{x \leq r\}}(x)\Big)\nu(dx) \tag{1}$$

where $b_r \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a *Lévy measure* concentrated on $(0, \infty)$; or equivalently, is a Lévy process with nonnegative jumps. The mentioned Prokhorov's result from [12], p. 197, adapted to spectrally positive Lévy processes has the following form.

Proposition 1. *Let Y be a spectrally positive Lévy process given by (b_r, σ^2, ν) and let processes Y_n be defined above for the case $F_{n,k} = F_n$. Then $Y_n \xrightarrow{\mathcal{D}} Y$ in $D[0, \infty)$ equipped with Skorokhod J_1 topology if and only if $\{F_n\}$ satisfies conditions*

P1 $\quad nF_n(y) \to \nu(-\infty, y) = 0 \quad$ *and* $\quad n\big(1 - F_n(x)\big) \to \nu(x, \infty)$, *as* $n \to \infty$, *for all continuity points $y < 0$ and $x > 0$ of the Lévy measure ν,*

P3 $\quad b_r \stackrel{df}{=} \lim_{n\to\infty} n \int_{|x|\leq r} x dF_n(x)$ *and* $|b_r| < \infty$, *for some* $0 < r < \infty$,

P4 $\quad$ *there exists σ^2 such that $0 < \sigma^2 < \infty$ and*

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} n \int_{|x|<\epsilon} x^2 dF_n(x) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} n \int_{|x|<\epsilon} x^2 dF_n(x) = \sigma^2.$$

If $E|Y(t)| < \infty$, then $EY(1) = b_r + \int_{|x|>r} x\nu(dx)$, so in case $EY(t) = 0$, we have

P5
$$b_r = -\int_{|x|>r} x\nu(dx).$$

In original formulation of Prokhorov's Theorem there was one extra condition, which in our context is not necessary. To be consistent with paper [7], where we use notation P1–P5 as well as P2 denoting that assumption, we do not change enumeration of conditions P3–P5.

2.2 Stationary waiting times in heavy traffic

Consider a sequence of GI/GI/1 queues with FIFO discipline of service. Let the n-th queue be generated by independent sequences $\{v_{n,k}, k \geq 1\}$, $\{u_{n,k}, k \geq 1\}$ of iid random variables with finite means. For generic random variables $v_{n,1}$ and $u_{n,1}$, denote by $\bar{v}_n$ and $\bar{u}_n$ their means, by B_n and A_n their distribution functions, and by F_n^B and F_n^A the distribution functions of $v_{n,1} - \bar{v}_n$ and $u_{n,1} - \bar{u}_n$, respectively. We interpret $v_{n,k}$ as the service time of the k-th unit in the n-th queue, and $u_{n,k}$ as the inter-arrival time between the k-th and $(k+1)$-st units in the n-th queue.

If $a_n \stackrel{df}{=} \bar{v}_n - \bar{u}_n < 0$, then $\omega_n \stackrel{df}{=} \sup_{k\geq 0} \sum_{j=1}^{k} (v_{n,j} - u_{n,j})$ is finite with probability one and is called a stationary waiting time. We will assume below that $a_n < 0$ for all n and $a_n \uparrow 0$, i.e. the systems we consider act in heavy traffic regime. Observe that

$$\omega_n = \sup_{0\leq t<\infty} (X_n(t) - |a_n|[nt]), \tag{2}$$

where $X_n(t) = \sum_{j=1}^{[nt]} (v_{n,j} - u_{n,j} - \bar{v}_n + \bar{u}_n)$, $t \geq 0$.

Now, we recall Heavy Traffic Invariance Principle from [17] for GI/GI/1 queues in a suitable form in which, for a convenience, the scaling constants c_n from [17] are included in random variables $v_{n,k}$ and $u_{n,k}$.

Proposition 2 (see [17], Theorem 1, Heavy Traffic Invariance Principle)**.** *Let the following conditions hold*

(I) $X_n \xrightarrow{\mathcal{D}} X$ *in* $D[0,\infty)$ *equipped with Skorokhod* J_1 *topology, where* X *is a Lévy process,*

(II) $\beta_n := n|a_n| \to \beta,\ 0 < \beta < \infty$,

(III) *the sequence* $\{\omega_n\}$ *is tight.*

Then $\omega_n \xrightarrow{\mathcal{D}} \sup_{0\leq t<\infty}(X(t) - \beta t)$.

The main results of the paper use Proposition 3 formulated below, which is a strengthened version of Lemma 2 from [7]. Roughly speaking, it states that for M/GI/1 queues tightness condition (III) is implied by conditions (I), (II), whenever condition P5 is true and $\{n\bar{v}_n^2\}$ is convergent to a finite limit. It also gives the LST of the limiting distribution of ω_n. For the sake of readability the proof is postponed to the Appendix.

Proposition 3. *Let the sequence $\{F_n^B\}$ defined for $\{B_n\}$ in M/GI/1 queues satisfy conditions P1 with ν, P3 with b_r and P4 with σ^2. Furthermore, let $\beta_n \to \beta, \quad 0 < \beta < \infty$ and $n\bar{v}_n^2 \to c^2, \quad 0 \le c^2 < \infty$. Then*

$$E\exp(-s\omega_n) \to \Big(1 + \frac{1}{s\beta}\psi(s)\Big)^{-1} \equiv \Psi(s), \tag{3}$$

where

$$\psi(s) = -sb_r + s^2(\sigma^2 + c^2)/2 + \int_0^\infty \Big(e^{-sx} - 1 + sx\mathbf{1}_{\{x \le r\}}(x)\Big)\nu(dx). \tag{4}$$

Moreover, if condition P5 holds, then Ψ is the LST of some nonnegative random variable.

Remark. Specifications of the exponential distributions of inter-arrival times are expressed by the assumption $\beta_n \to \beta$ and $0 < \beta < \infty$. Of course, it does not give precisely the parameters of these distributions, but it determines their asymptotic behavior, hence also behavior of partial sums processes built upon corresponding random variables. More precisely, if for a sequence of M/GI/1 queues the following convergences hold:

$$X_n^B \xrightarrow{D} X^B, \quad n\bar{v}_n^2 \to c^2, \ \ 0 \le c^2 < \infty, \quad \beta_n \to \beta, \ \ 0 < \beta < \infty,$$

where $X_n^B(t) = \sum_{k=0}^{[nt]}(v_{n,k} - \bar{v}_n)$, $t \ge 0$, and X^B is a Lévy process then $X_n \xrightarrow{D} X = X^B - c\mathcal{W}$, where $X_n(t) = \sum_{k=0}^{[nt]}(v_{n,k} - u_{n,k} - \bar{v}_n + \bar{u}_n)$, $t \ge 0$, and $\mathcal{W}$ is a Wiener process.

Remark. Under the assumptions of Proposition 3, we have $\omega_n \xrightarrow{D} \sup_{t \ge 0}\Big(X(t) - \beta t\Big)$, where X is a Lévy process characterized by parameters given in conditions P1, P3 and P4 (see e.g. [7]). One of the main drawbacks of Lemma 2 from [7] is that the most common process, i.e. Wiener process, is excluded. Now, we assume only that a limiting spectrally positive Lévy process has finite mean.

From Proposition 3 we get the following corollary.

Corollary 1. *Let the sequence* $\{F_n^B\}$ *defined for* $\{B_n\}$ *in M/GI/1 queues satisfy conditions* P1 *and* P3-P5. *Furthermore, let* $\beta_n := n|a_n| \to \beta$, $0 < \beta < \infty$, *and* $n\bar{v}_n^2 \to c^2$, $0 \leq c^2 < \infty$. *Then the sequence* $\{\omega_n\}$ *is tight.*

3 Main results

3.1 Relation between Lévy processes and M/G/1 queues.

Let X be a fixed spectrally positive Lévy process with mean zero, a Gaussian component σ^2 and a Lévy measure ν. Hence, by Remark 2 (cf. Appendix), the measure ν satisfies $\int_1^\infty \nu(x,\infty)dx < \infty$. Moreover, let us fix $\beta > 0$. Below, we define a sequence of M/GI/1 queues in heavy traffic, such that $\omega_n \xrightarrow{D} \sup_{0\leq t<\infty}(X(t) - \beta t) = W$. To do this we define distribution functions B_n and A_n. First, we specify B_n and then upon this specification we assume that A_n are exponential distribution functions with means $\bar{u}_n = \bar{v}_n + \beta/n$, respectively. The distribution functions $B_n, n \geq 1$, are defined separately in pure Poissonian case (i.e. $\sigma^2 = 0$), pure Gaussian case (i.e. $\nu = 0$) and in a general case (i.e. when both parameters are arbitrary).

In the first case (pure Poissonian case) let B_n, $n \geq 1$, be equal to

$$B_n(x) = \begin{cases} 0, & \text{for } x < x_n, \\ 1 - \frac{1}{n}\nu(x,\infty), & \text{for } x \geq x_n, \end{cases} \tag{5}$$

where $\{x_n\}$ is a sequence of nonnegative numbers chosen for an infinite Lévy measure ν in such a way that

$$x_n \downarrow 0, \quad nx_n \to \infty \text{ and } \frac{1}{n}\nu(x_n, \infty) < 1; \tag{6}$$

while for a finite measure ν x_n is chosen as 0 for $n \geq \nu(0, \infty)$.

Generally, a choice of $\{x_n\}$ is not unique. For finite ν the sequence $\{x_n\}$ could be chosen in the same way as for the infinite one. However, as we will see at the end of this section, the option $x_n = 0$ simplifies computation leading to the distribution of W. Notice also that, simplicity of considerations is govern by a suitable definition of $\{x_n\}$, because the definition of B_n depends mainly on the sequence.

Observe that

$$\bar{v}_n = \int_0^\infty \Big(1 - B_n(x)\Big)dx = x_n + \frac{1}{n}\int_{x_n}^\infty \nu(x, \infty)dx.$$

Lemma 1. *If B_n, $n \geq 1$, are defined by (5), then $\bar{v}_n$ are finite and $n\bar{v}_n^2 \to 0$, as $n \to \infty$.*

Proof. In the case of finite ν we have

$$\bar{v}_n \leq \frac{1}{n}\nu(0, \infty) + \frac{1}{n}\int_1^\infty \nu(x, \infty)dx < \infty.$$

If $\nu(0, \infty) = \infty$, then (6) implies $x_n > 0$, for sufficiently large n, but without loss of generality we assume that it holds for all n. Hence we have

$$\bar{v}_n \leq x_n + \frac{1}{nx_n}\int_{x_n}^1 x\nu(x, \infty)dx + \frac{1}{n}\int_1^\infty \nu(x, \infty)dx < \infty.$$

By assertion (ii) of Remark 2 in Appendix we have $\limsup_n \int_{x_n}^1 x\nu(x, \infty)dx < \infty$, which implies $\bar{v}_n \to 0$.

Below, for simplicity, denote $v_n = v_{n,1}$ and $\mathbf{1}_A = \mathbf{1}(A)$, where A is an event. Notice that for every $\epsilon > 0$ we have

$$0 \le E(v_n^2 - 2v_n\overline{v}_n + \overline{v}_n^2)\mathbf{1}(v_n \le \epsilon) \le E(v_n^2)\mathbf{1}(v_n \le \epsilon) - \overline{v}_n^2 + 2\overline{v}_n E v_n \mathbf{1}(v_n > \epsilon).$$

Therefore

$$n\overline{v}_n^2 \le nE(v_n^2)\mathbf{1}(v_n \le \epsilon) + 2n\overline{v}_n E v_n \mathbf{1}(v_n > \epsilon).$$

But

$$\begin{aligned} nE(v_n^2)\mathbf{1}(v_n \le \epsilon) =& n\int_0^\epsilon x^2 dB_n(x) = n\int_{x_n}^\epsilon x^2 dB_n(x) \\ &= -\epsilon^2\nu(\epsilon,\infty) + x_n^2\nu(x_n,\infty) + 2\int_{x_n}^\epsilon x\nu(x,\infty)dx. \end{aligned}$$

Now applying Remark 2 from Appendix we get

$$\lim_{\epsilon\to 0}\lim_{n\to 0}\Big(-\epsilon^2\nu(\epsilon,\infty) + x_n^2\nu(x_n,\infty) + 2\int_{x_n}^\epsilon x\nu(x,\infty)dx\Big) = 0,$$

which gives $\lim_{\varepsilon\to 0}\limsup_{n\to\infty} nE(v_n^2)\mathbf{1}(v_n \le \epsilon) = 0$.

Furthermore, for sufficiently large n the following holds

$$nEv_n\mathbf{1}(v_n > \epsilon) = \int_\epsilon^\infty x\nu(dx),$$

so $2n\overline{v}_n E v_n \mathbf{1}(v_n > \epsilon) \to 0$ as $n \to \infty$. This, in view of the above, completes the proof of the Lemma. $\Box$

In the pure Gaussian case $B_n, n \ge 1$, are defined as the exponential distribution functions with means $\bar{v}_n = \sigma/\sqrt{2n}$, respectively, i.e.

$$B_n(x) = 1 - \exp(-x\sqrt{2n}/\sigma), \quad x \ge 0,\ n \ge 1. \tag{7}$$

Finally, for arbitrary processes B_n are defined as

$$B_n = B_{n,1} * B_{n,2},\ n \ge 1, \tag{8}$$

where $B_{1,n}$ are defined by (5) and $B_{2,n}$ by (7).

Theorem 1. *The following implications hold.*

(i) (Pure Poissonian case). If B_n are defined by (5), then $\{F_n^B\}$ satisfies conditions P1-P5 with Lévy measure ν and $\sigma^2 = 0$;

(ii) (Pure Gaussian case). If B_n are defined by (7), then $\{F_n^B\}$ satisfies conditions P1-P5, with Lévy measure $\nu \equiv 0$ and Gaussian component $\sigma^2/2$;

(iii) (General case). If B_n are defined by (8), then $\{F_n^B\}$ satisfies conditions P1-P5, with Lévy measure ν and Gaussian component $\sigma^2/2$;

(iv) If $\omega_n, n \geq 1$, are defined for M/GI/1 queues with B_n defined as in either (i) or (ii) or (iii) and A_n are exponential distribution functions with means $\bar{u}_n = \bar{v}_n + \beta/n$, respectively, then

$$\omega_n \xrightarrow{D} \sup_{0 \leq t < \infty} (X(t) - \beta t) = W, \tag{9}$$

where X is a spectrally positive Lévy process with mean zero, Gaussian component σ^2, Lévy measure ν and the LST of W is given by (3) and (4) with appropriate ν, σ^2 and $b_r = -\int_{|x|>r} x\nu(dx)$.

Proof. *(i)* To prove P1 notice that, by Lemma 1 for any $x > 0$ there exists n_0 such that $x - \bar{v}_n > x_n$ for all $n \geq n_0$. Then for any $x > 0$ being a continuity point of the measure ν we have

$$n\Big(1 - F_n^B(x)\Big) = n\Big(1 - B_n(x + \bar{v}_n)\Big) = \nu(x + \bar{v}_n, \infty) \to \nu(x, \infty).$$

Obviously $nF_n^B(x) \to 0$, for all $x < 0$.

To prove that $\{F_n^B\}$ satisfies P3 notice that for sufficiently large n such that $-r + \bar{v}_n < 0$ and $r + \bar{v}_n > x_n$ we have

$$\begin{aligned} n\int_{\{|x|\leq r\}} x dF_n^B(x) &= -\, n\int_{\{|x|>r\}} x dF_n^B(x) = -n\int_{\{|x-\bar{v}_n|>r\}} (x - \bar{v}_n) dB_n(x) \\ &= -\int_{\{x>r+\bar{v}_n\}} x\nu(dx) + n\bar{v}_n\Big(1 - B_n(r + \bar{v}_n)\Big). \end{aligned}$$

Hence

$$\lim_n n \int_{\{|x|\le r\}} x dF_n^B(x) = -\int_r^\infty x\nu(dx). \tag{10}$$

Furthermore, b_r in P3 equals to $b_r = -\int_r^\infty x\nu(dx)$, so condition P5 is satisfied.

To show P4 notice that for sufficiently large n we have

$$\begin{aligned} n\int_{-\varepsilon}^{\varepsilon} x^2 dF_n^B(x) &= n\int_{-\bar{v}_n}^{\varepsilon} x^2 dB_n(x+\bar{v}_n) \\ &= -n\varepsilon^2\Big(1-B_n(\varepsilon+\bar{v}_n)\Big) + n\bar{v}_n^2 + 2\int_{-\bar{v}_n}^{\varepsilon} xn\Big(1-B_n(x+\bar{v}_n)\Big)dx \\ &\to -\varepsilon^2\nu(\varepsilon,\infty) + 2\int_0^\varepsilon x\nu(x,\infty)dx = \int_0^\varepsilon x^2\nu(dx), \quad \text{as } n\to\infty. \end{aligned}$$

Hence

$$\lim_{\varepsilon\to 0}\limsup_n n\int_{-\varepsilon}^{\varepsilon} x^2 dF_n^B(x) = 0.$$

This completes the proof of part (i) of the Theorem.

(ii) By point (i) of Remark 3 in Appendix the sequence $\{F_n^B\}$ satisfies conditions P1-P4 with $\nu = 0$ and $\sigma^2/2$, so also P5. In the considered case $n\bar{v}_n^2 \to \sigma^2/2$. This completes the proof of point (ii) of the Theorem.

(iii) By point (i) of the Theorem the sequence $\{F_{n,1}^B\}$ satisfies conditions P1-P5 with Lévy measure ν and Gaussian component $\sigma^2 = 0$ and by point (ii) of the Theorem the sequence $\{F_{n,2}^B\}$ satisfies conditions P1-P5 with Lévy measure $\nu = 0$ and Gaussian component $\sigma^2/2$. Since B_n is the convolution of $B_{n,1}$ and $B_{n,2}$, the sequence $\{F_n^B\}$ satisfies conditions P1-P5 with Lévy measure ν and Gaussian component $\sigma^2/2$. This completes the proof of point (iii) of the Theorem.

(iv) The sequence $\{F_n^B\}$ satisfies conditions P1-P5 in all cases (i)-(iii), so by Proposition 1 we have convergence $\sum_{j=1}^{[n\cdot]}(v_{n,j} - \bar{v}_n) \xrightarrow{D} X^B$, where X^B is a spectrally positive Lévy process with mean zero, Lévy measure ν and Gaussian component equal

to zero (in point (i)) or $\sigma^2/2$ (in points (ii) and (iii)). It can be easily verified, that $\{F_n^A\}$ satisfies P1-P5 as well, so $\sum_{j=1}^{[n\cdot]}(u_{n,j}-\bar{u}_n) \xrightarrow{D} X^A$, where X^A is a Wiener process, degenerated in point (i) (i.e. equal to zero) and with variance $\sigma^2/2$ in the remaining points (cf. Remark 3 in Appendix). Since X^A and X^B are independent and $\beta_n \to \beta$, by Proposition 3 and HTIP we get convergence (9). Now using Proposition 3 once again we get that the LST of W is given by (3) and (4). This completes the proof of the Theorem. $\Box$

Application of Theorem 1. The first observation is that Proposition 3 jointly with Theorem 1 give another proof of the famous Zolotarev's theorem from [22]. The second observation is that immediately from Theorem 1 it follows that the class of limiting distributions of ω_n for M/GI/1 queues in heavy traffic contains distributions of W with X being a centered Poisson process, Compound Poisson process or Gamma process. Also the limiting distributions of ω_n may be the convolutions of some Mittag-Leffler distributions, what may be especially interesting from a queueing-theoretic point of view. The third observation is that Theorem 1 jointly with HTIP can help us to find distribution of W. Its usefulness is based on two points. The first one is the formula for the stationary waiting time in M/GI/1 queue, i.e.

$$(11) \qquad P(\omega_n \le x) = (1-\rho_n)\sum_{k=0}^{\infty}\rho_n^k\left(\frac{1}{\bar{v}_n}\int_0^x \Big(1-B_n(y)\Big)dy\right)^{*k}, \quad \text{where } \rho_n = \frac{\bar{v}_n}{\bar{u}_n},$$

which for the case $B(0+) = 0$ is given in [6], page 255 formula (4.82) and for the general case $B(0+) \ge 0$ it is given in the proof of Proposition 3. The second point is a proper choice of simple queues for which $\omega_n \xrightarrow{D} W$ and for which we are able to find the limiting distribution of ω_n. Below we demonstrate this way of getting distribution of W for $X(t) = Y(t) - \lambda t$, where $Y(t)$ is a Poisson process with intensity λ. In this case the LST of W is of the form

$$E\exp(-sW) = \frac{(b-\lambda)s}{bs+\lambda(e^{-s}-1)}, \quad s \geq 0, \text{ where } b = \beta+\lambda.$$

Instead of reversing this LST we find distribution of W using convergence $\omega_n \xrightarrow{D} W$. Namely, using formula (11) and convergence $\omega_n \xrightarrow{D} W$ for special M/GI/1 queues we show that

$$P(W \leq x) = P(\omega_n \leq x) = (1-\rho)\sum_{j=0}^{[x]} \frac{1}{j!}(j-x)^j \rho^j \exp\Big\{-\rho(j-x)\Big\}. \tag{12}$$

This distribution was obtained by R. Pyke [14] and here we give its queueing derivation. Notice that the Lévy measure corresponding to Poisson process Y and process X is $\nu(x,\infty) = \lambda$ for $0 \leq x \leq 1$ and $\nu(x,\infty) = 0$ for $x \geq 1$, so it is finite. Therefore taking $x_n = 0$ in the definition of B_n we have $B_n(x) = 1 - \lambda/n$ for $0 \leq x < 1$ and $B_n(x) = 1$ for $x \geq 1$ and $\rho_n = \lambda/(\lambda+\beta) \equiv \rho$. This implies that the residual distributions $\frac{1}{\bar{v}_n}\int_0^x \Big(1 - B_n(y)\Big)dy$ for each n are equal to the uniform distribution on $[0,1]$. According to formula (9.5) from Section I.9 in Feller [8], for the k-th convolution of uniform distributions on $[0,1]$, we have

$$\left(\frac{1}{\bar{v}_n}\int_0^x \Big(1-B_n(y)\Big)dy\right)^{*k} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(x-j)_+^k$$

where $x_+ = \max(0,x)$. This and the above give the following formula

$$P(\omega_n \leq x) = (1-\rho)\sum_{k=0}^{\infty}\rho^k \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(x-j)_+^k \equiv D(x),$$

where the right-hand side of the above does not depend on n, so $D(x) = P(W \leq x)$.

Notice that

$$\begin{aligned} D(x) &= (1-\rho)\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\rho^k\frac{1}{k!}(-1)^j\binom{k}{j}(x-j)_+^k \\ &= (1-\rho)\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j(x-j)_+^j\rho^j\sum_{k=j}^{\infty}\rho^{k-j}\frac{1}{(k-j)!}(x-j)_+^{k-j} \\ &= (1-\rho)\sum_{j=0}^{\infty}\frac{1}{j!}(-1)^j(x-j)_+^j\rho^j\exp\Big\{\rho(x-j)_+\Big\} \\ &= (1-\rho)\sum_{j=0}^{[x]}\frac{1}{j!}(j-x)^j\rho^j\exp\Big\{-\rho(j-x)\Big\}, \end{aligned}$$

which coincides with formula (12).

The following remark shows that convolutions of some distributions of type W are also of type W. Its proof is omitted.

Remark 1. *Let nonnegative numbers p_j and σ_j^2, $j \geq 0$, be such that $\sum_{j=0}^{\infty} p_j = 1$, $\sum_{j=0}^{\infty}\sigma_j^2 p_j \stackrel{df}{=} \sigma^2 < \infty$ and let ν_j, $j \geq 0$, be Lévy measures concentrated on $(0,\infty)$ such that $\int_1^{\infty} x\nu_j(dx) < \infty$, $\sup_{j\geq 0}\int_0^{\infty}\min(1,x^2)\nu_j(dx) < \infty$ and let $\nu \stackrel{df}{=} \sum_{j=0}^{\infty} p_j\nu_j$. Furthermore, let Ψ_j and ψ_j be defined by (3) and (4), for ν_j and σ_j^2 respectively. Then we have the following:*

(i) If X is a spectrally positive Lévy process with Lévy measure ν and Gaussian component σ^2 then the LST of the random variable W is

$$E\exp(-sW) = \Big(1+\frac{1}{\beta s}\sum_{j=0}^{\infty}p_j\psi_j(s)\Big)^{-1}. \qquad (13)$$

(ii) If $\sigma^2 = 0$ and $s^{-1}\psi_1(s)\psi_2(s) = \psi_3(s)$, $s \geq 0$, then $\Psi_1(s)\Psi_2(s)$ is LST of W with X being a spectrally positive Lévy process with finite mean.

From Remark 1 it follows that if G_i $i = 1, 2$, are the Mittag-Leffler distributions with LSTs equal to $(1+\lambda_i^{-\kappa_i} s^{\kappa_i})^{-1}$, $s \geq 0$, where $0 < \kappa_i < 1$ are such that $\kappa_1 + \kappa_2 < 1$ and $\lambda_i^{-\kappa_i} = \beta^{-1}\alpha_i\Gamma(-\alpha_i)$, with $1 < \alpha_i < 2$ and the function Γ is the analytical extension to $\mathbb{R} \setminus \{0, -1, -2, \ldots\}$ of the gamma function $\widetilde{\Gamma}(y) = \int_0^\infty x^{y-1}e^{-x}dx,\ y > 0$, (cf. N. N. Lebedev [11]) then convolution $G_1 * G_2$ is the distribution of W with LST of the form $\Psi(s) = \Big(1 + \lambda_1^{-\kappa_1}s^{\kappa_1} + \lambda_2^{-\kappa_2}s^{\kappa_2} + \lambda_1^{-\kappa_1}\lambda_2^{-\kappa_2}s^{\kappa_1+\kappa_2}\Big)^{-1}, \quad s \geq 0.$

3.2 Non sufficiency of conditions I and II in HTIP.

One can raise a question whether conditions (I) and (II) in HTIP imply condition (III), i.e. the tightness of $\{\omega_n\}$. A similar question was communicated privately to W. Szczotka by W. Whitt. Namely, W. Whitt asked if the convergence in distribution to a Wiener process with a negative trend of the processes $X_n(t) = \sum_{j=1}^{[nt]}(v_{n,j} - u_{n,j}), t \geq 0, n \geq 1$, for GI/GI/1 queues in the heavy traffic, implies tightness of $\{\omega_n\}$. An answer to this question give Theorem 2 and Corollary 2 formulated below. They state that for any spectrally positive Lévy process X with the Lévy measure ν satisfying $\int_1^\infty x\nu(dx) < \infty$, there exists a sequence of M/GI/1 queues for which conditions (I) and (II) hold, but (III) does not.

Let us consider a sequence of M/GI/1 queues with B_n defined as

$$B_n(x) = \begin{cases} 1 - q_n, & \text{for } 0 \leq x < b_n, \\ 1, & \text{for } x \geq b_n, \end{cases} \tag{14}$$

where $0 < b_n \uparrow \infty,\ q_n b_n \downarrow 0$ (monotonically) and $nb_nq_n \to d,\ 0 < d < \infty$; $A_n, n \geq 1$, being exponential distribution functions with means $\bar{v}_n + \beta/n, n \geq 1$, respectively.

Theorem 2. *The sequence $\{F_n^B\}$ induced by B_n defined in (14) satisfies conditions P1-P4 and $\beta_n \to \beta$, but $\{\omega_n\}$ is not tight.*

Proof. Let $x > 0$, then for n such that $b_n > x + \bar{v}_n$ we have $n\Big(1 - F_n^B(x)\Big) = nq_n$. From the assumption $nb_nq_n \to d$, $0 < d < \infty$ and $b_n \uparrow \infty$ we get that $\{F_n^B\}$ satisfies P1 with $\nu \equiv 0$, i.e. $\nu(x, \infty) = \nu(-\infty, -x) = 0$ for all $x > 0$.

Notice that

$$\lim_{n\to\infty} n \int_{|x|<r} xF_n^B(x) = \lim_{n\to\infty} -n\bar{v}_n(1 - q_n) = -d,$$

so P3 is satisfied with $b_r = -d$. Similarly

$$\lim_{n\to\infty} n \int_{|x|<\epsilon} x^2F_n^B(x) = \lim_{n\to\infty} n\bar{v}_n^2(1 - q_n) = 0,$$

which shows that P4 holds with $\sigma^2 = 0$.

Hence by Prokhorov's result (Proposition 1) we obtain $X_n^B \xrightarrow{D} -de$, where $e(t) = t, t \geq 0$. Moreover, P5 does not hold. Because of $\beta_n = \beta$ and $n\bar{v}^2 \to 0$, as $n \to \infty$, and next by Proposition 3 we get

$$E(e^{-s\omega_n}) \to \frac{1}{1 + d/\beta}, \quad \text{as } n \to \infty.$$

But $1/(1 + d/\beta)$ is not LST of any probability measure, so $\{\omega_n\}$ is not tight. This completes the proof. $\square$

The most crucial point of the previous construction and assertion of the theorem is that the first moments of the approximating sequence do not converge to the corresponding moment of the limiting process. The fact that the weak limit in Theorem 2 is degenerated can be easily removed, which shows the following corollary:

Corollary 2. *For any spectrally positive Lévy process X with finite mean, Lévy measure ν and Gaussian component σ^2 there exists a sequence of M/GI/1 queues, such that $X_n \xrightarrow{D} X$, $\beta_n \to \beta$, but $\{\omega_n\}$ is not tight.*

Proof. Let us define a sequence of M/GI/1 queues by defining B_n and A_n in the following way: $B_n = B_n^{(1)} * B_n^{(2)}$, where $B_n^{(1)}$ are defined by (8) for the Lévy measure ν and $B_n^{(2)}$ by (14) while A_n, $n \geq 1$ are exponential distribution functions with means $\bar{u}_n = \bar{v}_n + \beta/n$, respectively. Since $F_n^B(x) = F_n^{B^{(1)}} * F_n^{B^{(2)}}(x)$, by Proposition 3 and Theorem 2, we get $X_n \xrightarrow{D} X - de$, with $e(t) = t$. Because of $n\bar{v}^2 \to \sigma^2/2$, as $n \to \infty$, and $\beta_n = \beta$, and next by Proposition 3, we have

$$E(e^{-s\omega_n}) \to \left(1 + d/\beta + s^2\sigma^2/2 + \int_0^\infty \left(e^{-sx} - 1 + sx\right)\nu(dx)\right)/(s\beta)\Big)^{-1},$$

as $n \to \infty$. However, the right-hand side of the above is not LST of any probability measure, so $\{\omega_n\}$ is not tight. This completes the proof. $\square$

4 Appendix

4.1 Auxiliary results

Here we give auxiliary technical facts which we used in Section 3.

Remark 2. *(i) If ν is a Lévy measure such that $\int_1^a x^\delta \nu(dx) < \infty$ for some $1 \leq a \leq \infty$ and $\delta \geq 1$, then for any r, $0 < r \leq a \leq \infty$, we have*

$$\int_r^a x^\delta \nu(dx) = r^\delta \nu(r,a] + \delta \int_r^a x^{\delta-1}\nu(x,a]dx. \tag{15}$$

(ii) If ν is a Lévy measure, then

$$\lim_{\epsilon\to 0} \epsilon^2 \nu(\epsilon, a] < \infty \quad \textit{and} \quad \lim_{\epsilon\to 0} \int_\epsilon^1 x\nu(x,a]dx < \infty, \quad \textit{where } 1 \leq a \leq \infty. \tag{16}$$

Proof. The assertion (i) in case $\nu(r,a] = 0$ is trivial. To prove it for $\nu(r,a] > 0$ first recall that every Lévy measure is finite on any interval (x,∞), $x > 0$ and then define the distribution function F by $F(x) = 0$ for $x \leq r$; $F(x) = 1$ for $x > a$ and

$$F(x) = 1 - \frac{\nu(x,a]}{\nu(r,a]}, \quad \text{for } r \leq x \leq a,$$

where $\nu(x,\infty] = \nu(x,\infty)$.

Notice that for $0 < r \leq a$ we have

$$\begin{aligned}\int_r^a x^\delta \nu(dx) &= \nu(r,a] \int_r^a x^\delta F(dx) = -\nu(r,a] \int_r^a x^\delta d\big(1 - F(x)\big) \\ &= \nu(r,a]\Big(-x^\delta\big(1 - F(x)\big)\Big|_r^a + \delta \int_r^a x^{\delta-1}\big(1 - F(x)\big)dx\Big) \\ &= \nu(r,a]\Big(r^\delta + \delta \int_r^a x^{\delta-1} \frac{\nu(x,a]}{\nu(r,a]} dx\Big) = r^\delta \nu(r,a] + \delta \int_r^a x^{\delta-1} \nu(x,a] dx.\end{aligned}$$

This completes the proof of point (i).

To prove assertion (ii) we use assertion (i), i.e. (15) with $r = \epsilon$, $a = 1$ and $\delta = 2$. Then (15) jointly with $\lim_{\epsilon \to 0} \int_\epsilon^1 x^2 \nu(dx) = \int_0^1 x^2 \nu(dx) < \infty$ imply (16) and this completes the proof of assertion (ii) and the Remark. $\square$

The following remark gives conditions under which the distribution functions of exponentially distributed random variables centered by their expectations satisfy conditions P1-P4.

Remark 3. *Let* $G_n(x) = 1 - \exp(-\lambda_n x)$, *for* $x \geq 0$ *and* $F_n^G(x) = G_n(x + 1/\lambda_n)$.
(i) If $\lambda_n/\sqrt{n} \to \lambda, 0 < \lambda < \infty$, *then* $\{F_n^G\}$ *satisfies conditions P1-P4 with* $\nu = 0, b_r = 0, \sigma^2 = 1/\lambda^2$.
(ii) If $\lambda_n/\sqrt{n} \to \infty$, *then* $\{F_n^G\}$ *satisfies conditions P1-P4 with* $\nu = 0, b_r = 0, \sigma^2 = 0$.
(iii) If $\lambda_n/n \to \lambda, 0 < \lambda < \infty$, *then* $\{F_n^G\}$ *satisfies conditions P1-P4 with* $\nu = 0$, $b_r = 0, \sigma^2 = 0$.

Proof. Let $\{\eta_k, k \geq 1\}$ be a sequence of iid nonnegative random variables, exponentially distributed with parameter 1 and for each $n \geq 1$ let $\{\eta_{n,k}, k \geq 1\}$

be a sequence of iid nonnegative random variables with distribution function G_n, being exponential with parameter λ_n. Then $\eta_{n,k} - E\eta_{n,k} \stackrel{D}{=} (\eta_k - 1)/\lambda_n$. Let $X_n^G(t) = \sum_{j=1}^{[nt]}(\eta_{n,j} - E\eta_{n,j})$ and $Z_n(t) = \sum_{j=1}^{[nt]}(\eta_j - 1)$. Then we have relation $X_n^G \stackrel{D}{=} Z_n/\lambda_n$ and convergence $\frac{1}{\sqrt{n}} Z_n \stackrel{D}{\to} \mathcal{W}$, where $\mathcal{W}$ is a standard Wiener process. Hence and by relations

$$\frac{1}{\lambda_n} Z_n(t) = \frac{\sqrt{n}}{\lambda_n} \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (\eta_j - 1) \quad \text{and} \quad X_n^G \stackrel{D}{=} Z_n/\lambda_n$$

and next by Proposition 1 we get assertions (i) and (ii) of the Remark.

In case (iii) we have $\frac{n}{\lambda_n} \frac{1}{n} \sum_{j=1}^{[nt]}(\eta_j - 1) \stackrel{D}{\to} 0$. Hence by the same argumentation as before we get assertion (iii). This completes the proof. □

4.2 Proof of Proposition 3

Formula (4.82) in [6], p. 252, gives the distribution function of the stationary waiting time ω for M/GI/1 queues in the case when the distribution function B of the service times is such that $B(0+) = 0$. Namely, it has the form

$$P(\omega \le x) = (1-\rho) \sum_{j=0}^{\infty} \rho^j \left(\frac{1}{\bar{v}} \int_0^x \Big(1 - B(s)\Big) ds \right)^{*j}, \quad x \ge 0, \tag{17}$$

where $\rho = \bar{v}/\bar{u}$ and H^{*j} is the j-th fold convolution of a distribution function H. However, this formula is also true for the case $0 < B(0+) < 1$ and $0 < \rho < 1$. To see it, let us take a sequence of M/GI/1 queues with distribution function $B^{(k)}$ of service times in the k-th queue, such that $B^{(k)}(x) = p\delta_{x_k}(x) + (1-p)B(x)$, $B(0+) = 0$ and $x_k \downarrow 0$, where $\delta_{x_k}(x) = 0$ for $x < x_k$ and $\delta_{x_k}(x) = 1$ for $x \ge x_k$. Whereas, $A^{(k)}$'s (distribution functions of the inter-arrival times) do not depend on k. Then $B^{(k)}(0+) = 0,\ 0 < \rho^{(k)} < 1$. Therefore, for any $x > 0$ the distribution function of the stationary waiting time $\omega^{(k)}$ for the k-th queue has the form

$$(18)\qquad P(\omega^{(k)} \le x) = \Big(1-\rho^{(k)}\Big)\sum_{j=0}^{\infty}\Big(\rho^{(k)}\Big)^j\left(\frac{1}{\bar{v}^{(k)}}\int_0^x\Big(1-B^{(k)}(s)\Big)ds\right)^{*j}.$$

But $B^{(k)} \Rightarrow B^{(0)} = p\delta_0 + (1-p)B$ and $\omega^{(k)} \xrightarrow{D} \omega^{(0)}$, where $\omega^{(0)}$ is the stationary waiting time for the M/GI/1 queue with $B^{(0)}$ being the distribution function of service times. Since the right-hand side and the left-hand side of (18) converge, for each $x > 0$ we have

$$(19)\qquad P(\omega^{(0)} \le x) = \Big(1-\rho^{(0)}\Big)\sum_{j=0}^{\infty}\Big(\rho^{(0)}\Big)^j\left(\frac{1}{\bar{v}^{(0)}}\int_0^x\Big(1-B^{(0)}(s)\Big)ds\right)^{*}.$$

This completes the proof that (17) is also valid for B such that $0 < B(0+) < 1$.

Later on we assume that the distribution function B in (17) may be such that $0 \le B(0+) < 1$ and the sequence of distribution functions B_n can be written as follows:

$$B_n(x) = p_n + (1-p_n)D_n(x), \quad x \ge 0,$$

where $0 \le p_n < 1$ and distribution functions D_n are such that $D_n(0+) = 0$. Then the expectations of B_n and D_n are denoted by $\bar{v}_n$ and $\bar{v}_n^D$, respectively, and $\bar{v}_n = (1-p_n)\bar{v}_n^D$. Furthermore, notice that

$$B_{n,0}(x) \stackrel{\text{df}}{=} \frac{1}{\bar{v}_n}\int_0^x\Big(1-B_n(x)\Big)dx = \frac{1}{\bar{v}_n^D}\int_0^x\Big(1-D_n(x)\Big)dx.$$

From formula (17), for this general case, the LST of ω_n has the following form:

$$\begin{aligned} E\exp(-s\omega_n) &= \frac{1-\rho_n}{1-\rho_n\hat{B}_{n,0}(s)} = \left(1+\frac{\rho_n}{1-\rho_n}\Big(1-\hat{B}_{n,0}(s)\Big)\right)^{-1} \\ &= \left(1+\frac{\rho_n}{1-\rho_n}\frac{1}{s\bar{v}_n^D}\int_0^\infty(e^{-sx}-1+sx)dD_n(x)\right)^{-1}, \end{aligned}$$

where $\hat{B}_{n,0}$ is the LST of the distribution function $B_{n,0}$ and $\rho_n = \bar{v}_n/\bar{u}_n$. By the relation

$$\frac{\rho_n}{1-\rho_n}\frac{1}{\bar{v}_n^D} = \frac{1-p_n}{|a_n|},$$

we get

$$E\exp(-s\omega_n) = \left(1 + \frac{1-p_n}{n|a_n|}\ \frac{1}{s}\ n\int_0^\infty (e^{-sx} - 1 + sx)dD_n(x)\right)^{-1}.$$

By the definition of B_n we finally obtain

$$E\exp(-s\omega_n) = \left(1 + \frac{1}{s\beta_n}n\int_0^\infty (e^{-sx} - 1 + sx)dB_n(x)\right)^{-1}.$$

Putting $I_n = \int_0^\infty (e^{-sx} - 1 + sx)dB_n(x)$, and

$$I_{n,1} = \int_{-\bar{v}_n}^\infty (e^{-sx} - 1 + sx)dF_n^B(x), \qquad I_{n,2} = \int_0^\infty \left(e^{-sx}(1 - e^{s\bar{v}_n}) + s\bar{v}_n\right)dB_n(x)$$

we get $I_n = I_{n,1} + I_{n,2}$. But for $\epsilon > 0$ we have

$$I_{n,1} = C_{n,1}(\epsilon) + C_{n,2}(\epsilon) + C_{n,3} + C_{n,4}, \tag{20}$$

where

$$\begin{aligned}
C_{n,1}(\epsilon) &= \int_{-\bar{v}_n}^{\varepsilon} (e^{-sx} - 1 + sx)dF_n^B(x),\\
C_{n,2}(\epsilon) &= \int_{\varepsilon}^{r} (e^{-sx} - 1 + sx)dF_n^B(x),\\
C_{n,3} &= \int_r^\infty (e^{-sx} - 1)dF_n^B(x),\\
C_{n,4} &= s\int_r^\infty x dF_n^B(x).
\end{aligned}$$

By the assumption $n\bar{v}_n^2 \to c^2$, $0 \le c^2 < \infty$, we have $\bar{v}_n \to 0$. Therefore, for any $\varepsilon > 0$ there exists sufficiently large n such that $\bar{v}_n \le \varepsilon$. Hence for such n we have

$$\begin{aligned}
nC_{n,1}(\epsilon) &= n\int_{|x|\le\varepsilon} (e^{-sx} - 1 + sx)dF_n^B(x)\\
&= \frac{s^2}{2}n\int_{|x|\le\epsilon} x^2 dF_n^B(x) + n\int_{|x|\le\epsilon}\sum_{k=3}^\infty \frac{s^k x^k}{k!}dF_n^B(x).
\end{aligned}$$

But

$$|n\int_{|x|\le\epsilon}\sum_{k=3}^{\infty}\frac{s^k x^k}{k!}dF_n^B(x)| \;\le\; s^3 n\int_{|x|\le\epsilon}|x|^3 e^{s|x|}dF_n^B(x) \;\le\; s^3\epsilon n e^{s\epsilon}\int_{|x|\le\epsilon}|x|^2 dF_n^B(x).$$

Hence by P4 we get

$$\lim_{\epsilon\to 0}\limsup_n nC_{n,1}(\epsilon)=\lim_{\epsilon\to 0}\liminf_n nC_{n,1}(\epsilon)=s^2\sigma^2/2. \tag{21}$$

Now notice that

$$nC_{n,2}(\epsilon)=-\int_{\epsilon}^{r}(e^{-sx}-1+sx)dn\Big(1-F_n^B(x)\Big).$$

Therefore, by P1 and then by the continuity and boundedness of the function $e^{-sx}-1+sx$ on $(0,r)$ we get

$$\lim_{\epsilon\to 0}\lim_n nC_{n,2}(\epsilon)=\int_0^r(e^{-sx}-1+sx)\nu^B(dx). \tag{22}$$

Now notice that by P1 and continuity and boundedness of the function $e^{-sx}-1$ on (r,∞) we get

$$\lim_{\epsilon\to 0}\lim_n nC_{n,3}=\int_r^{\infty}(e^{-sx}-1)\nu^B(dx). \tag{23}$$

Finally notice that for sufficiently large n we have $\int_r^\infty x dF_n^B(x)=-\int_{-r}^{r}x dF_n^B(x)$, which jointly with P3 imply

$$\lim_{\epsilon\to 0}\lim_n nC_{n,4}=-s\lim_n n\int_{|x|\le r}x dF_n^B(x)=-sb_r. \tag{24}$$

Hence by (20) we get

$$\lim_{\epsilon\to 0}\lim_n nI_{n,1}=-sb_r+s^2\sigma^2/2+\int_0^{\infty}\Big(e^{-sx}-1+sxI_{\{|x|\le r\}}(x)\Big)\nu^B(dx). \tag{25}$$

Now we show that $nI_{n,2} \to s^2c^2/2$. To do this notice that

$$\begin{aligned}
I_{n,2} &= \int_0^\infty \Big(1 - sx + (sx)^2/2 - 1 + s(x-\bar{v}_n) - s^2(x-\bar{v}_n)^2/2 + s\bar{v}_n\Big) dB_n(x) \\
&\quad + \int_0^\infty \sum_{k=3}^\infty \frac{s^k(-1)^k}{k!}\Big(x^k - (x-\bar{v}_n)^k\Big) dB_n(x) \\
&= s^2 \int_0^\infty (x\bar{v}_n - \bar{v}_n^2/2) dB_n(x) + \int_0^\infty R_n(x,s) dB_n(x) \\
&= s^2\bar{v}_n^2/2 + \int_0^\infty R_n(x,s) dB_n(x),
\end{aligned}$$

where

$$R_n(x,s) \stackrel{\text{df}}{=} \sum_{k=3}^\infty \frac{s^k(-1)^k}{k!}\Big(x^k - (x-\bar{v}_n)^k\Big).$$

Now notice that

$$\begin{aligned}
R_n(x,s) &= \Big(e^{-sx} - 1 + sx - (sx)^2/2\Big) \\
&\quad - \Big(e^{-s(x-\bar{v}_n)} - 1 + s(x-\bar{v}_n) - s^2(x-\bar{v}_n)^2/2\Big) \\
&= -\Big(e^{s\bar{v}_n} - 1 - s\bar{v}_n - s^2\bar{v}_n^2/2\Big) \\
&\quad - \Big((e^{s\bar{v}_n} - 1)(e^{-sx} - 1 + sx) - sx(e^{s\bar{v}_n} - 1 - s\bar{v}_n)\Big) \\
&= -\sum_{k=3}^\infty s^k\bar{v}_n^k/k! - (e^{s\bar{v}_n} - 1)(e^{-sx} - 1 + sx) + sx\sum_{k=2}^\infty s^k\bar{v}_n^k/k!.
\end{aligned}$$

Hence

$$\begin{aligned}
I_{n,2} &= s^2\bar{v}_n^2/2 - \sum_{k=3}^\infty s^k\bar{v}_n^k/k! - (e^{s\bar{v}_n} - 1)\int_0^\infty (e^{-sx} - 1 + sx) dB_n(x) \\
&\quad + s\bar{v}_n \sum_{k=2}^\infty s^k\bar{v}_n^k/k! \\
&= s^2\bar{v}_n^2/2 - \sum_{k=3}^\infty s^k\bar{v}_n^k/k! + s\bar{v}_n\sum_{k=2}^\infty s^k\bar{v}_n^k/k! - (e^{s\bar{v}_n} - 1)I_n.
\end{aligned}$$

This and the relation $I_n = I_{n,1} + I_{n,2}$ give

$$e^{s\bar{v}_n} nI_{n,2} = n(1 - e^{s\bar{v}_n})I_{n,1} + s^2n\bar{v}_n^2/2 - n\sum_{k=3}^\infty s^k\bar{v}_n^k/k! + sn\bar{v}_n\sum_{k=2}^\infty s^k\bar{v}_n^k/k!. \tag{26}$$

But, by the convergence $n\bar{v}_n^2 \to c^2,\ 0 \le c < \infty$, we get the following:

$$(27)\quad 0 \le n \sum_{k=3}^{\infty} (s\bar{v}_n)^k/k! \le s^3 n\bar{v}_n^3 \sum_{k=3}^{\infty} (s\bar{v}_n)^{k-3}/(k-3)! = s^3 n\bar{v}_n^3 e^{s\bar{v}_n} \to 0, \quad \text{as } n \to \infty$$

$$(28)\quad 0 \le s\bar{v}_n n \sum_{k=2}^{\infty} (s\bar{v}_n)^k/k! \le s^3 n\bar{v}_n^3 \sum_{k=2}^{\infty} (s\bar{v}_n)^{k-2}/(k-2)! = s^3 n\bar{v}_n^3 e^{s\bar{v}_n} \to 0, \quad \text{as } n \to \infty,$$

which, in view of (26), gives the convergence $nI_{n,2} \to s^2c^2/2$. Hence and from (25) and (20) we get the first assertion of Proposition 3.

To prove the second assertion of the proposition we need to show that $\Psi^B(s)$ is continuous at $s = 0$, i.e. $\lim_{s\to 0} \Psi^B(s) = 1$, which is equivalent to $\psi^B(s)/s \to 0$ as $s \to 0$. But the last holds because of condition P5 and equality $\lim_{s\to 0} \psi^B(s)/s = -b_r^B - \int_r^\infty x\nu^B(dx)$. This gives the second assertion of the proposition and completes its proof.

Acknowledgements. The authors are very grateful to the Referee for his/her valuable comments and suggestions.

References

[1] Asmussen, S. (1987). Applied Probability and Queues. Wiley.

[2] Embrechts P., Klüppelberg C., Mikosch T. (1997). Modeling Extremal Events for Insurance and Finance. Springer.

[3] Baxter, G. and Donsker, M.D. (1957). On the distribution and the supremum functional for processes wih stationary independent increments. *Trans. Amer. Math. Soc.* 85, 73-87.

[4] Boxma, O.J. and Cohen, J.W. (1999). Heavy-traffic analysis for GI/G/1 queue with heavy-tailed distributions. *Queueing Systems,* 33, 177-204.

[5] Bingham, N.H. (1975). Fluctuation theory in continuous time. *Adv. in Appl. Probab.* 7, 705-766.

[6] Cohen, J.W. (1969). The single server queue, Wiley.

[7] Czystołowski, M. and Szczotka, W. (2007). Tightness of stationary waiting times in Heavy Traffic for GI/GI/1 queues with thick tails *Probab. Math. Stat*, 27.1 pp. 109–123.

[8] Feller, W. (1971). An Introduction to Probability Theory and its Applications, Vol. II, second edition, Wiley, New York.

[9] Harrison, J.M. (1977). The supremum distribution of the Lévy process with no negative jumps, *Adv. in Appl. Probbab.* 9, pp. 417-422.

[10] Kella, O. and Whitt, W. (1991). Queues with server vacations and Lévy processes with secondary jump input, *Ann. Appl. Prob.* 1, pp. 104-117.

[11] Lebedev, N.N., (1972). Special Functions and Their Applications, Dover Publications, Inc., New York.

[12] Prokhorov, Yu., V. (1956). Convergence of Random Processes and Limit Theorems in Probability Theory. *Theory Probability and its Applications,* Vol 1, pp.157-214.

[13] Prokhorov, Yu., V. (1963) The Transitional Phenomena in the Queueing Processes I, *Lietouvos Mathematikos Rinkinys* III, Nr 1, pp 199-205.

[14] Pyke, R., (1959). The supremum and infimum of the Poisson process, *Ann. Math. Stat.*, Vol. 30, No. 2., pp. 568-576.

[15] Resnick, S. and Samorodnitsky, G. (2000). A Heavy Traffic Approximation for Workload Processes with Heavy Tailed Service Requirements, *Management Science* Vol. 46, No. 9 pp. 1236-1248.

[16] Sato, R. (2001). Basic Results on Lévy Processes, *Lévy Processes, Theory and Applications*, Ed. O.E.Barndorff-Nielsen, T.Mikosch, S.I.Resnik, Birkhauser.

[17] Szczotka, W. and Woyczyński, W.A. (2003). Distributions of Suprema of Lévy Processes via Heavy Traffic Invariance Principle, *Probab. Math. Stat.* 23, pp 251-172.

[18] Szczotka, W. and Woyczyński, W.A. (2004). Heavy tailed Dependent Queues in Heavy Traffic, *Probab. Math. Stat.* 24.1, pp. 67-96.

[19] Takács, L. (1965). On the distribution of the supremum of stochastic processes with exchangeable increments. *Trans. Amer. Math. Soc.* 119, 367-379.

[20] Whitt, W. (1974). Heavy traffic limit theorems for queues: A survey Mathematical methods in queueing theory (proc. Conf., Western Michigan Univ., Kalamazoo, Mich 1973) *Lecture notes in Econom. and Math. Systems,* Vol. 98, Springer, Berlin, Germany, 3007-350.

[21] Whitt, W. (2002). Stochastic-Process Limits. An Introduction to Stochastic-Process Limits and Their Application to Queues. Springer.

[22] Zolotarev, V. M. (1964). The first passage time to a level and the behaviour at infinity of processes with independent increments, Theory Probab. Appl. 9, pp. 653-661.

Marek Czystołowski
Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50-384 Wrocław
Poland
E-mail:
Marek.Czystolowski@math.uni.wroc.pl

Władysław Szczotka
Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50-384 Wrocław
Poland
E-mail:
Wladyslaw.Szczotka@math.uni.wroc.pl