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Time dilation have opposite signs in hemispheres of recession and approach

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Abstract

In this paper we give a detailed analysis of the factual observation that time dilation or Doppler shift of frequency are oppositely signed relative to our line of sight in different regions of motion. We show that this depends on how these regions are connected to the actual path of motion and to the location of the observer/detector. For a static globe or circle/sphere of reference, we define two hemispheres; shifts will be red-shifted in the hemisphere of recession and blue-shifted, or violet shifted, in the hemisphere of approach, a fact which is not often mentioned. Indeed, it is a complete sphere where the motion can take place with respect to the line-of-sight, which we have studied in this paper, not just the direction of motion along a certain specific path. The transverse and longitudinal effects are special cases of this general effect. The transverse effect is always red-shifted consistent with the known effect of Relativity and it lies at the intersection of the hemispheres of approach and recession. By saying moving clocks run slower we conveniently forget this important effect. They also run faster, in the hemisphere of approach. We demonstrate this important result from the basic principles of the Special Theory of Relativity.

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I. INTRODUCTION

We will perform a detailed analysis of how relative to the line-of-sight a general motion of a source and equivalently an observer will affect observation of time dilation and Doppler shift. We use only two main references, so we mention this fact here; [1] and [2]. It is a straightforward but non-trivial analysis. It applies to any situation where one would like to have a binomial expansion of the Lorentz factors γ and β to any desired degree of accuracy by retaining terms up-to one's requirements. This will be useful in many studies.

Let us now define the context in which we have used terminology like "sphere of relative motion", "hemisphere of recession" and "sphere of reference"; see Figure 1 on page 8 and Figure 2 on page 9. Our definition is a natural extension of the concept of reference frames. This is helpful towards identification of the relations of different parts to each other in the problem/situation that we present. Simply put, one defines two spheres/circles, the first one is a sphere of reference whose center is where a detector or origin of reference is located. The detector can also be placed e.g. at the surface of another object whose center is at the center of this reference. Any concentric sphere that encloses or excludes a segment of the trajectory of another object is also a sphere of reference. Now a second sphere of reference can be defined. This second sphere will encompass a segment of the trajectory path so the path lies in either the lower or upper hemisphere with respect to the line of sight. Let us call it a sphere of relative motion as it defines whether an object is receding away from, or approaching the center of reference. The upper hemisphere of this second sphere, where an object is gradually moving away from the reference point, defined at the center of the first sphere, can be called a hemisphere of recession. On the other hand, if the object is in the lower hemisphere of the second sphere, we can call it a hemisphere of approach, as the object will gradually approach the center of reference. One must note that these spheres are not very arbitrary in nature, but rather, dependent on the specific nature of the trajectories, e.g. a satellite orbiting a planet in a circle will naturally always traverse the intersection of the two hemispheres, so it is a transverse motion to the line of sight. In hyperbolic motion, such as that of a flyby satellite moving away from a planet towards some other destination, part of the motion lies in the upper hemisphere or the hemisphere of recession. When dealing, as described above, with earthbound motion such as *perigee*, {the point of closest approach to earth}, the opposite is true. The segment of the trajectory is naturally proportional to the radial speed vector. The $radial-speed-vecor$ is very important for any study of celestial applications. Similarly a meteor falling towards Earth always lies in the hemisphere of approach. One sees that one can cautiously adjust the sphere of reference and the sphere of relative motion so that there is no ambiguity in the situation. Only then can one consistently apply the laws of Relativity, such as those that govern time dilation and Doppler shift of frequencies.

We also note as a reminder of convenience that the transverse effects are always $2nd$ order in speed of recession and the longitudinal effects are $1st$ and $3rd$ order in terms of the Lorentz factor, β. Higher order terms appear in the same pattern. Transverse effects contain β^2 and β^4 and longitudinal effects contain alternative odd terms given by factors from binomial expansion as presented in our analysis. The general case is accommodated by cosines of a suitable angle, which appear only in the orders of odd power terms. The analysis is valid for $0 \leq abs(\beta) \leq 1$, where abs stands for the modulus of a number or the regular absolute value function. The relation between fractional frequency shift f_ν and fractional time dilation or contraction f_t related to f_ν are computed here and these are equal in their modulus. This is the case when the f_ν's are small. When they are not small, additional review is necessary for the relation between the $f's$ in ν, t, λ and v. We note that this also follows a binomial analysis like the one presented in this paper, because the $f'_\nu s$ are fractions.

II. BINOMIAL THEOREM

The form and properties of binomial expansion is given in [1] for:$-1 < x < 1$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + ... \tag{1}$$

By using the above, we derive the Lorentz factors γ and its products with β or its functions, and any powers in the following steps, $\gamma = (1-\beta^2)^{-1/2}$; expanding in the power of β^2,

$$\gamma = 1 + \alpha^{'}(-\beta^2) + \alpha^{''}(-\beta^2)^2 + \alpha^{'''}(-\beta^2)^3$$

where$\alpha^{'} = -\frac{1}{2}$, $\alpha^{''} = \frac{(-1/2)(-1/2-1)}{2}$ and

$\alpha''' = \frac{(-1/2)(-1/2-1)(-1/2-2)}{6}$,

therefore;

$\gamma = 1 + \frac{\beta^2}{2} + \frac{3\beta^4}{8} + \frac{5\beta^8}{48}$ and $1/\gamma = (1-\beta^2)^{1/2} = 1 - \frac{\beta^2}{2} - \frac{3\beta^4}{8} - \frac{5\beta^8}{48}$.

The following are expression for $(1 \pm \beta)^{\pm 1/2}$:

$\sqrt{1+\beta} = (1+\beta)^{1/2} = 1 + \frac{\beta}{2} - \frac{3\beta^2}{8} + \frac{5\beta^3}{48}$,

$\frac{1}{\sqrt{1+\beta}} = (1+\beta)^{-1/2} = 1 - \frac{\beta}{2} + \frac{3\beta^2}{8} - \frac{5\beta^3}{48}$,

$\sqrt{1-\beta} = (1-\beta)^{1/2} = 1 - \frac{\beta}{2} - \frac{3\beta^2}{8} - \frac{5\beta^3}{48}$,

to order β^2; $\gamma = 1 + \frac{\beta^2}{2}$, $1/\gamma = 1 - \frac{\beta^2}{2}$, $\sqrt{1+\beta} = 1 + \frac{\beta}{2} - \frac{3\beta^2}{8}$, $\frac{1}{\sqrt{1+\beta}} = 1 - \frac{\beta}{2} + \frac{3\beta^2}{8}$, $\sqrt{1-\beta} = 1 - \frac{\beta}{2} - \frac{3\beta^2}{8}$, $\frac{1}{\sqrt{1-\beta}} = 1 + \frac{\beta}{2} + \frac{3\beta^2}{8}$.

III. DOPPLER SHIFT AND ABERRATION IN SPECIAL RELATIVITY

A. Description of the general form of Doppler shift; recession, approach and transverse motion.

The time dilation and the Doppler shift are described here. You will notice that the usual form of time dilation given in textbooks and literature is nothing but a transverse Doppler-shift. Here we show how the Doppler transverse is always red-shifted and how the hemispheres of approach and recession are always oppositely shifted where the hemisphere of recession is entirely red-shifted. The special cases of longitudinal and transverse effects follow ipso-facto. A rigorous binomial and a basic logarithmic treatment is given.

The general form of special relativistic Lorentz transformation in terms of frequency is given in [2],

$$\nu = \nu' \frac{(1+\beta\cos\theta)}{\sqrt{1-\beta^2}} \quad and \quad \nu' = \nu \frac{(1-\beta\cos\theta)}{\sqrt{1-\beta^2}}. \tag{2}$$

The 2nd part of above equation is obtained by letting $\beta \to -\beta$, the reciprocity of the Lorentz Transformation of inertial frames. We can see that both ν' and ν are proper frequencies and that each with respect to the other is an observed frequency. Each of the alternative cases can induce a relative change in the signs of geometric and coordinate system connections, but not the Physics. Let us rewrite the above, still in its general form;

$$\frac{\nu'}{\nu} = 1 + \frac{\Delta\nu}{\nu} = \gamma(1-\beta\cos\theta) = \gamma - \gamma\beta\cos\theta.$$

Now, let us apply the result of all the care we took in deriving the binomial expansion of the Lorentz factors;

$\gamma - \gamma\beta\cos\theta = (1 + \frac{\beta^2}{2}) - (1 + \frac{\beta^2}{2})\beta\cos\theta$, or

$\gamma - \gamma\beta\cos\theta = 1 + \frac{\beta^2}{2} - \beta\cos\theta - \frac{\beta^3}{3}\cos\theta$;

$$\frac{\nu'}{\nu} = 1 + \frac{\Delta\nu}{\nu} = 1 + \frac{\beta^2}{2} - \beta\cos\theta - \frac{\beta^3}{3}\cos\theta.$$

The above equations were obtained by using the binomial approximations to the $2nd$ order in β and we have now an equation to the $3rd$ order in β. This might change slightly if we first accommodate up-to $3rd$ order and then obtain our result by chopping after $3rd$ order. This may give rise to new terms within the $3rd$ order. But for most purposes β itself is very small, usually 10^{-5}, e.g for satellites falling in cosmos at speeds ~ 50 times faster than jet airplanes in flight. That makes the β^2 terms 10^{-10} and β^3terms 10^{-15}. We would then make an error in that order by not accommodating enough terms before the expansion.

$$\frac{\nu'}{\nu} - 1 = \frac{\Delta\nu}{\nu} = f_\nu = \frac{\beta^2}{2} - \beta\cos\theta - \frac{\beta^3}{3}\cos\theta. \tag{3}$$

The above equation gives the most general Doppler shift terms in the chosen order and it accommodates the transverse and longitudinal effects with angular modifications giving the general shift. f_ν is called fractional shift in frequency. This is a very important variable and most calculations can be made very conveniently using such fractional shifts; this also relates to the fractional shifts in speed or wavelength and in time dilation/contraction. We have done another derivation, {see $NOTE-3$ in section IV}, that relates the fractional shift in frequency to that in time. The result is that f_ν is equal to f_t in magnitude but differs by a minus sign. It is valid only when f_ν is in itself a small fraction such that binomial expansion can be used for such a fraction. In this idealized limit $f_\nu = -f_t$, which means a redshift is a time dilation and a blue-shift is a time contraction, speed fractions will have the same signs as time fractions: $f_\nu \ll 1$, $f_\nu = -f_t$, or $\frac{\Delta\nu}{\nu} = -\frac{\Delta t}{t}$, $f_{\rm v} = f_t = -f_\nu$.

Thus $\cos\theta$ will vary between 0 to $90^0$and we'll accommodate the other variations of θ into sign ofβ. So we have 3 cases:

- $\star$ (i) Doppler hemisphere of recession, $\theta = 0$, $\beta > 0$, longitudinal Doppler effect (recession)
- $\star$ (ii) Doppler hemisphere of approach, $\theta = 0$, $\beta < 0$, longitudinal Doppler effect (approach)
- $\star$ (iii) Doppler plane of intersection of hemispheres of approach and recession, $\theta = 90^0$, $\beta > 0$, the transverse Doppler effect.

Note that for $2nd$ case $\beta < 0$ is just a convention, actually $0 < \beta < 1$, and there is a sign reversal for approach in our convention. So in all 3 cases the sign of f_ν and f_t, the numerical sign as opposed to our algebraic sign notation, will depend on the fractions as below. We need to work out which of the two inequalities is valid, and in the following we will show that only one of them is valid in all of the cases;

$$\frac{\beta^2}{2} > \beta + \frac{\beta^3}{2} \; or \; \frac{\beta^2}{2} < \beta + \frac{\beta^3}{2}. \tag{4}$$

Let us do some more mathematical exercise; let β be given as $\beta = ln(\alpha)$, {see $NOTE-2$ in section IV}. This means

$$\beta + \frac{2}{\beta} = ln\alpha + \frac{2}{ln(\alpha)} = \frac{(ln\alpha)^2 + 2}{ln(\alpha)}$$

under the conditions

$$0 < abs(\beta) < 1 \; and \; ln(1) = 0.$$

Let us now describe the approach, recession and transverse motion properties of the Doppler shift in terms of β; Approach, case (ii): $ln(\alpha) = \beta < 0$ and Recession/Transverse, case (i), (iii): $ln(\alpha) = \beta > 0$. So there are 2 conditions and the recognition that $(ln\alpha)^2 > 0$;

$$\beta + \frac{2}{\beta} = \frac{2 + \beta^2}{\beta} = \frac{(ln\alpha)^2 + 2}{ln(\alpha)} \lessgtr 0.$$

Hence the " $<$ " part holds for case (ii) and the " $>$ " part holds for case (i), (iii).

For case (ii); $\frac{2+\beta^2}{\beta} < 0$ or $2+\beta^2 > \beta$ (since $\beta < 0$)$\Rightarrow \beta < 2+\beta^2 \Rightarrow \frac{\beta}{2} < 1 + \frac{\beta^2}{2} \Rightarrow \frac{\beta^2}{2} > \beta + \frac{\beta^3}{2}$ as each time we multiply a $-ve$ sign we must reverse the inequality.

For case (i), (iii) therefore, since in our convention, $\beta > 0$, $\frac{\beta^2}{2} > \beta + \frac{\beta^3}{2}$, then;

$$\begin{aligned} \beta \gtrless 0,\ \frac{\beta^2}{2} &> \beta + \frac{\beta^3}{2} \\ \Rightarrow \beta \gtrless 0,\ \frac{\beta^2}{2} &> \beta + \frac{\beta^3}{3}. \end{aligned} \tag{5}$$

Thus, THE SAME INEQUALITY IS VALID IN ALL CASES *even though* the mathematical and physical conditions are different. We note here that although we derived $\beta \gtrless 0,\ \frac{\beta^2}{2} > \beta + \frac{\beta^3}{2}$ it is also true that $\beta \gtrless 0,\ \frac{\beta^2}{2} > \beta + \frac{\beta^3}{3}$, which is what we need for our purpose. We show this in our notes, $\{NOTE - 4$ in section IV$\}$.

B. Conclusions on nature of frequency shifts or time dilation/contraction in different regions of relative motion

Now let us return to f_ν and f_t, or equivalently, $\frac{\Delta\nu}{\nu}$ and $\frac{\Delta t}{t}$;

$$\frac{\Delta t}{t} = f_t = -\{-\beta\cos\theta + \beta^2/2 - (\beta^3/3)\cos\theta\}.$$

Case (i), (iii) can be described as: $(i) \rightarrow \{\theta = 0^0,\ \beta > 0\}$, and $\{\theta = 90^0,\ \beta > 0\} \leftarrow (iii)$. We deal with (iii), (i) before moving onto (ii).

- ⋆ case (iii): $\{\theta = 90^0,\ \beta > 0\}$ $f_t = -f_\nu = -[\frac{\beta^2}{2}]$, thus: $f_\nu = \frac{\beta^2}{2}$, $f_t = -\beta^2/2\frac{\beta^2}{2}$. This is redshift, frequency drop and time dilation. Since observed time is dilated we need to subtract $abs(f_t) = \beta^2/2$ from where this time is observed and dilated, e.g. laboratory clocks, hence the $-ve$ sign. This is also called the Transverse Doppler effect as well as time dilation. By recognizing β in terms of γ, as done in our binomial analysis, and keeping the requisite order of terms we can recognize the "footprint" of time dilation. In general, time can be either contracted or dilated. It hinges on the question "can we formalize a way of measuring time without involving processes in nature that are described by their frequencies" and the answer is NO. Hence Doppler shift is equivalent to time dilation and contraction depending on the exact situation of the motion, in other words, in reality there is no difference between time dilation/contraction and Doppler shift.

- ⋆ case (i): $\theta = 0,\ \beta > 0,\ f_t = -f_\nu = -[-\beta + \frac{\beta^2}{2} - \frac{\beta^3}{3}]$, $f_t = -f_\nu = -\beta + \frac{\beta^2}{2} - \frac{\beta^3}{3}$, in terms up-to $3rd$ order in β, this is deliberate. We apply Eq.(5): $\frac{\beta^2}{2} > \beta + \frac{\beta^3}{3}$, as obtained above. Then we can see that $f_t = -f_\nu < 0$. The sign of $\Delta\nu$, Δt and "primeness" of t and ν are important. This will be described in a note at the end. This is redshift, frequency drop and time dilation. It is a longitudinal Doppler shift when the source and observer are receding from each other, relatively.

- ⋆ case (ii): $\theta = 0^0,\ \beta < 0$, $f_t = -f_\nu = \frac{\beta^2}{2} - (\beta + \frac{\beta^3}{3})$. We apply the inequality of Eq.(5), $\frac{\beta^2}{2} > \beta + \frac{\beta^3}{3}$, again. But we have to be a little careful and this applies to case (i) above, as well. We have $\beta \rightarrow -\beta$ which is not a Lorentz reciprocity. It represents the fact that we have chosen it to be in the lower quadrants of $\{\pi \leftrightsquigarrow 2\pi\}$ whilst we require that a "line of sight" and "line of motion" extend null angle between them; see Figure 1 on page 8. It can be seen that the only terms that are affected are the terms that are odd in power of β as they have the cosine factors. In general $f_\nu = \frac{\beta^2}{2} - \beta\cos\theta - \frac{\beta^3}{3}\cos\theta$, $0 < \theta < 2\pi$, but we accommodate the $3rd$ and $4th$ quadrant to $\beta < 0$ and $0 < abs(\beta) < 1$. This leads to $f_t = -f_\nu > 0$, which means a violet/blue-shift, frequency increase or time contraction. Take notice of the fact that Eq.(5) is still valid, as $0 < \cos\theta < 1$ in this quadrant with the above convention.

IV. NOTES

1. In section−III we used a convention where a primed frame is moving with respect to the unprimed at speed $v = \beta$ in units of speed of light, $0 \leq \beta \leq 1$. This is the convention used by [2]. In this reference ν' is proper frequency and ν is observed frequency due to the motion/speed β. Here one has to be careful in preventing oneself from using Lorentz reciprocity inconsistently which results in accruing $-ve$ signs. The Physics in the end has to be consistent. One rule of caution is therefore to check the signs of $\Delta\nu$ and ν and the relative primeness of ν and ν' which introduce additional $-ve$ signs at unspecified points in the calculation if used incorrectly. All this could be a source of confusion under Lorentz reciprocation.

2. One can prove the above results without using $ln()$ by just employing the properties of numbers, viz the square of any real number is always positive. But having $ln()$ may be useful if one intends to study a functional form of β. It is widely customary in Relativity to employ logarithmic functions and we think it is very useful for generalization purposes.

3. By elementary definition $\frac{\nu'}{\nu} - 1 = \frac{t}{t'} - 1$ and $\frac{t-t'}{t'} = -\frac{\triangle t}{t'}$ we deduce:
$\frac{\triangle\nu}{\nu} = \frac{\nu'-\nu}{\nu} = \left(\frac{1}{t'} - \frac{1}{t}\right)/\frac{1}{t} = \frac{t-t'}{t'} = -\frac{\triangle t}{t'}$. Then, $\frac{t'}{\nu} = -\frac{\triangle t}{\triangle\nu}$. Because $\frac{t'}{t} - 1 = \frac{\triangle t}{t}$ we have $1 + \frac{\Delta t}{t} = \frac{t'}{t}$, it follows then from $\frac{t'}{t} \times \frac{t}{\nu} = -\frac{\triangle t}{\triangle\nu}$ that $\frac{t'}{t} = -\frac{\nu}{t} \times \frac{\triangle t}{\triangle\nu}$, so

$$1 + \frac{\triangle t}{t} = -\frac{\nu}{t} \times \frac{\triangle t}{\triangle\nu}.$$

$$\left(1 + \frac{\triangle t}{t}\right)\frac{t}{\nu} = -\frac{\triangle t}{\triangle\nu}.$$

$$\frac{t}{\nu} + \frac{\triangle t}{\nu} = -\frac{\triangle t}{\triangle\nu}.$$

$$\frac{1}{\nu}\left(t + \triangle t\right) = -\frac{\triangle t}{\triangle\nu}.$$

$$\frac{\triangle\nu}{\nu} = -\frac{\triangle t}{t + \triangle t}$$
$$= -\frac{1}{1 + t/\triangle t}.$$

So

$$-\frac{1}{\triangle\nu/\nu} = 1 + \frac{t}{\triangle t},$$

this leads to

$$\frac{t}{\triangle t} = -\left[\frac{1 + (\triangle\nu/\nu)}{\triangle\nu/\nu}\right].$$

Thus we have

$$\frac{\triangle t}{t} = -\left[\frac{\triangle\nu/\nu}{1 + (\triangle\nu/\nu)}\right].$$

So this shows why fractional shifts in frequency and time are equal in magnitude but opposite in sign if $f_\nu \ll 1$. Hence we also see how one can actually expand f_ν and f_t in the binomial series if $f_\nu \ll 1$. One can extend this to other limits of f_ν.

4. Here we show how $\beta \gtrless 0$, $\frac{\beta^2}{2} > \beta + \frac{\beta^3}{3}$ follows from $\beta \gtrless 0$, $\frac{\beta^2}{2} > \beta + \frac{\beta^3}{2}$. We write $\beta + \frac{\beta^3}{2} = \beta(1 + \frac{\beta^2}{2}) = \beta(1 + \frac{\beta^2}{3} + \frac{\beta^2}{2} - \frac{\beta^2}{3}) = \beta(1 + \frac{\beta^2}{3} + \frac{\beta^2}{6}) = \beta(1 + \frac{\beta^2}{3}) + \frac{\beta^3}{6}$ as $\frac{\beta^2}{2} > \beta + \frac{\beta^3}{2}$ this implies $\frac{\beta^2}{2} > \beta(1 + \frac{\beta^2}{3}) + \frac{\beta^3}{6}$, if $\beta > 0$, $\frac{\beta^3}{6} > 0$ which implies $\frac{\beta^2}{2} > \beta(1 + \frac{\beta^2}{3})$ if $\beta < 0$, LHS is $+ve$ and RHS is $-ve$ which again implies $\frac{\beta^2}{2} > \beta(1 + \frac{\beta^2}{3})$ since both terms of RHS are $-ve$.

5. We already mentioned this but summarize again: There is a $\cos\theta$ multiplied to odd powers of β which comes in $-ve$ sign relative to the even powers in the general frequency shift. For our considerations $0 < \theta < 90^0$, $0 < \cos\theta < 1$,

$$\frac{\beta^2}{2} > \beta + (\frac{\beta^3}{3}) \Rightarrow \frac{\beta^2}{2} > \beta\cos\theta + (\frac{\beta^3}{3})\cos\theta$$

always. Hence the hemisphere of approach is always oppositely shifted with respect to the hemisphere of recession; recession being red-shifted, approach must be blue in the entire hemisphere. At the intersection the $\cos\theta = 0$ and odd terms vanishes to order of β^3. We intuit that this is valid to all higher powers as well. Therefore a transverse shift is always $+ve$ or red-shifted by an amount $\frac{\beta^2}{2}$, and with the correct binomial-fractions it applies to all higher even powers of β.

6. The aberration such as that of star-light is the angular deflection caused by the relative motion of the source and the observer. If our detectors are mounted on moving frames, {e.g. earth itself}, even very small speed differences cause the frequency to differ due to the angle extended by such relative velocity. The aberration of an order β is given by $\beta = \tan\theta$. For small values of β therefore, the fractional speed increment can be measured by the fractional angular deflection, and these are equal. We have not studied this in detail, hence we are not giving any results here. But one can see that when there is a β the Lorentz transformations of the θ or $\tan\theta$ can be treated as it has been in this paper.

7. The fractional shifts f_ν can themselves be binomially expanded when they are small, which is so for small values of β. It is then equal to the fractional shifts in time f_t and f_λ etc, except for a relative $-ve$ sign in case of f_t. So there are two scenarios. In the first f_ν has to be expanded to higher orders binomially, in-case of small β-values. This will perhaps be rigorous, but unnecessary, since we will otherwise incur errors of only inconsequential proportion. In the $2nd$ scenario, in-case $\beta \twoheadrightarrow 1$, we have to review by some other method to find what the consequences are. In this latter case the binomial expansion is still valid, since $\beta \leq 1$, always. But, the relation between fractional shifts in the variables may need to be reviewed.

8. The fractional increments or reduction in speed, $f_{\rm v} = \frac{\Delta {\rm v}}{\rm v}$ is related, in the case of small values of f_ν, by a factor of 2 to the same. In other words, $f_{\rm v} = 0.5 \times f_\nu$. That is so, assuming that the final absolute shifts in frequency are shifts by the square of the speed. e.g. in telemetry of celestial objects the frequency used will give different results by this factor for the same frequency, if this relation between f_ν and $f_{\rm v}$ is not readily recognized.

9. A note on terminology and nature of the factors β and γ. We ask ourselves a fundamental notional question, which is more fundamental β or γ? One defines β from the basic speed-of-light perspective and then defines γ based on this definition because the latter makes calculations easier as the $\beta's$ appear in denominator, in numerator, inside square-roots and inside particle detectors and where-not. It is a serious factual observation that inside a particle detector, like in the case of the recent OPERA experiment [3], one must proceed from basic relativistic equations and gradually factor in the contributions from various sources if one wants to see the total effect in a consistent way, Quantum Mechanics included. So defining these two parameters instead of just one would make life easier for concise studies. If $\beta > 1$ it poses problems to the mathematical nature of Relativity because what comes out is a complex number γ, and only Quantum Mechanics can deal with such a problem, which is why it is interesting to note that the answer to the OPERA anomaly is likely inherent here; because $\Delta\beta = 7.5\ km/s = 25\ ppm$ in terms of speed-of-light, and without the application of Quantum Mechanics, this is a pointer to a "non-physical" character of Relativity as stated above.

   On terminology we would like to further note that the "Lorentz factor" is the most general terminology as it defines both β and γ, when we say β: speed factor is enough, although it is somewhat casual. When we say γ: Lorentz factor, Lorentz boost, speed factor, boost factor, or simply boost, all work to define suitably. Actually β and γ are equivalents as implicitly pointed out above in their definitions, hence all of the terminology is correct in any of the cases. So a β can be said to be a boost-factor as well; once one is defined or measured, the other is immediately known in practice. In theory it is β that is more fundamental, —we can see this just from intuitive understanding and the correspondence principle, as there never was a γ explicitly prior to

relativistic formulations, but there was a β, just by the fact that people compared ratios of various velocities: "How fast is object A compared to object B", a ratio of their speeds deals with this; "how fast is object A or B compared to light"; that ratio is known as β and the relativistic formulation $0 \leq \beta \leq 1$ reminds us that nothing travels faster than light, hence this ratio is always slated to be a fraction where as γ is not.

10. In Figure 2 on the next page we have defined a sphere of relative motion, (- drawn in a blue-dotted pattern -), whose center is the position where an observer or detector is located. The LOS, or line of sight, extends from the detector by a blue line. We show 3 trajectories of actual importance.

⋆ The 1*st* one is a meteor or crashing satellite falling e.g. towards Earth's surface, lies in the region of approach as gradually its distance is reduced from the earth-centered reference.

⋆ The 2*nd* one is a circle or sphere which serves as a "sphere of reference" concentric with many such spheres of reference. But, we can also chose one of these circle/sphere as one which defines the circular orbits of some satellites, such as the orbit of most GPS satellites. The speed vector will then be transverse/tangential to the circle or sphere of reference. Hence this defines the intersection of two hemispheres, which add to give us the sphere of relative motion. We may call it the "sphere of relative motion" as it is this sphere that contains a speed-vector or trajectory segment of the actual motion.

⋆ In the 3*rd* scenario the speed vector or radial velocity vector of a hyperbolic motion is also shown; this one belongs to the hemisphere of recession, as the trajectory implies a configuration of gradually larger radial separation. Note that this only describes an instantaneous situation as depicted in this diagram. Also for a hyperbolic trajectory where an object is approaching us from below or above, the region represented by a segment of the hyperbola will lie in the hemisphere of approach instead of recession.

By the same reasoning, when we have accumulated enough change from the instantaneous situation, we need to define larger or smaller circles/spheres of reference and spheres of relative motion. We call it the sphere of relative motion as the LOS defines the "relativeness" of the speed-vectors which then define which region is a region of approach and which is a region of recession relative to the references

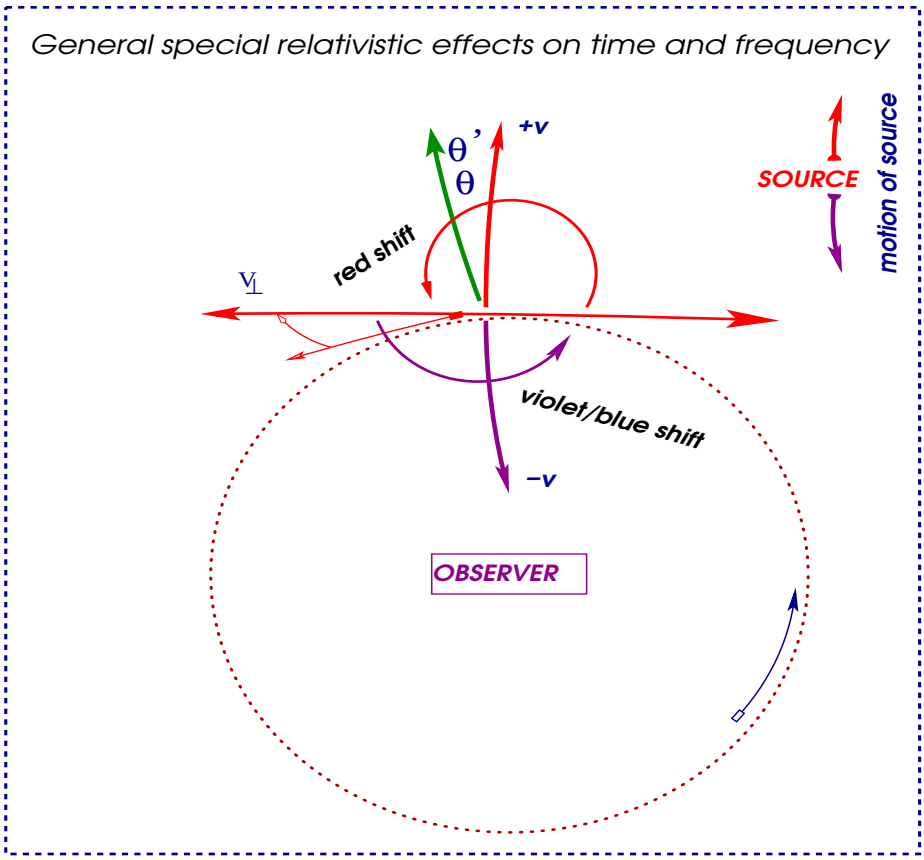

Figure 1: The time dilation/contraction effects as frequency red-shift/blue-shift, shown for above configuration.

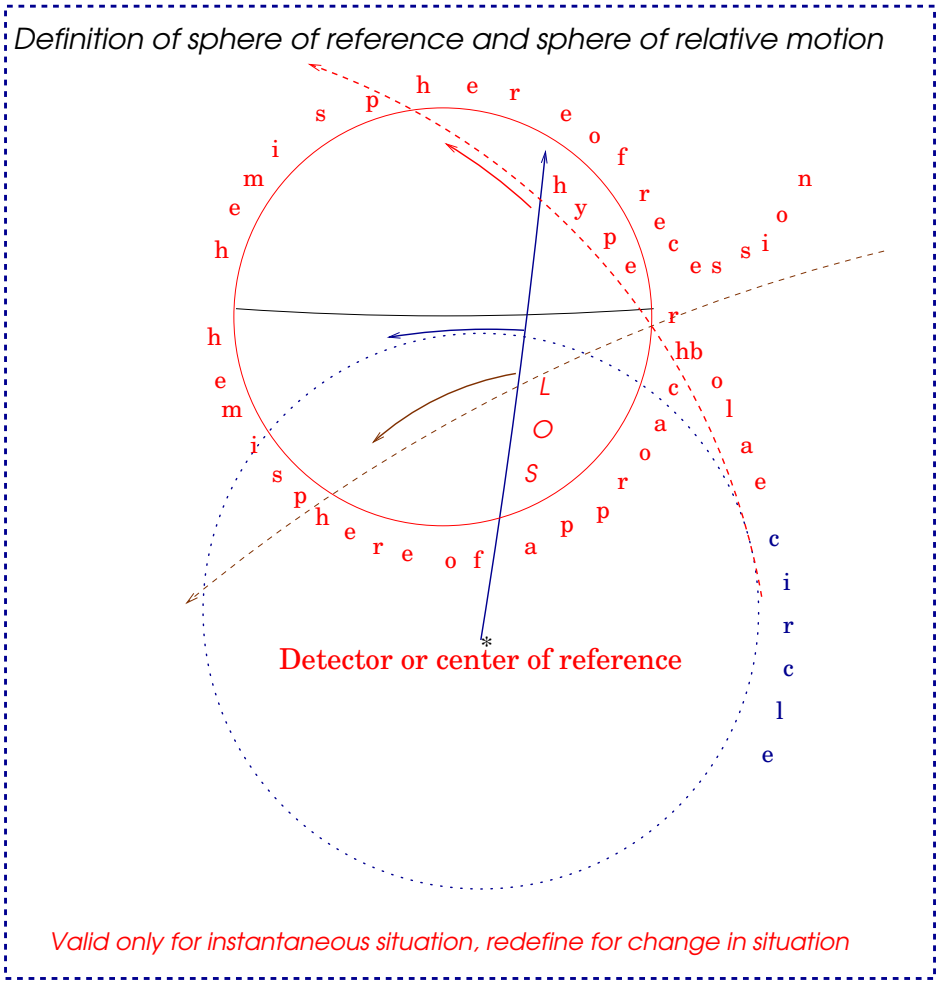

Figure 2: This diagram explains our terminology, sphere of reference and sphere of relative motion. LOS = Line Of Sight. See $NOTE -$ in section IV.

V. SUMMARY/CONCLUSION

Our definition of special notions of reference of observation, and reference of the actual motion path, enables an investigation of important relativistic effects. We have deduced from an application of the binomial theorem and properties of both logarithmic and real numbers that Doppler shift and time dilation of Special Relativity are oppositely signed in two hemispheres, which naturally define where the motion trajectory lies with respect to the observer and the line of sight. An application on aberration of special relativistic recession and approach is also inherent here. The analysis draws on very basic, yet non-trivial, and not-so-often cited facts of Relativity. This paper serves as an explicit reminder of the mathematical apparatus of Relativity. Thus, the concept and methodology presented in this paper should be applicable to a wide variety of investigations, especially where one seeks accuracy to very high powers of β through binomial expansion. We purposely kept this paper limited to a generalized treatment suitable for future reference.

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