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Homogenized Interface Model Describing Inhomogeneities Located on a Surface

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Abstract We study the influence of heterogeneities located near a planar surface on the elastic response of a three-dimensional elastic medium. These heterogeneities can be either reinforcements, like steel reinforcements in concrete, or defects, like micro-cracks periodically distributed. We prove that their influence is of the second order from an energetic viewpoint. Then, we propose an "up to second order effective model" in which the influence of the heterogeneities is given by a surface energy contribution involving both the jump of displacement across the surface and the tangential strain components on the surface. The effective coefficients entering in the definition of the surface energy are obtained by solving "elementary" elastic problems formulated on an infinite representative cell containing the defects. We analyze this model, in particular the properties of the effective surface coefficients, and establish its coherence with limit models previously described in the literature for stiff or soft interfaces. This approach is finally applied to several kinds of heterogeneities.

Keywords Homogenization · Separation of scales · Matched asymptotic expansions

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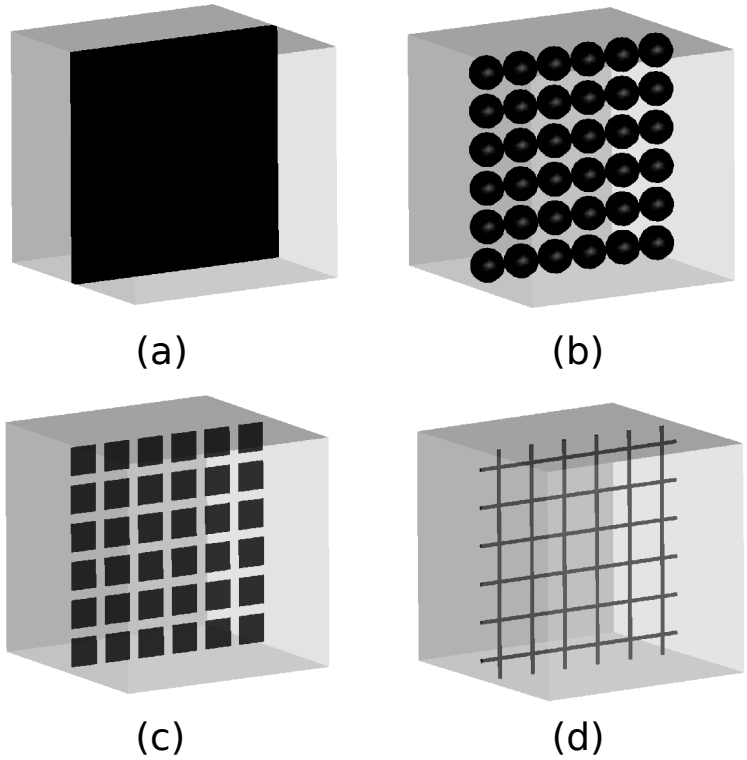

Fig. 1 Examples of solids with periodic inhomogeneities located on a surface. (a) Homogeneous layer (b) Spherical cavities (c) Periodic microcracks inside a material (d) Steel reinforcements in concrete

1 Introduction

Let us consider a three-dimensional elastic medium, which contains heterogeneities periodically distributed on a surface. The size of these heterogeneities is much smaller than the size of the overall structure. Some examples of such a problem are shown on Fig. 1: They range from steel reinforcements in concrete to microcracks or cavities located on a surface. In the context of finite element simulations, modeling these small heterogeneities results in huge computational costs. On the other hand, these heterogeneities do have an influence on the behavior of the overall structure. It is therefore interesting to construct a simplified model, which reproduces the effective behavior of the heterogeneities, while being much easier to compute.

Simplified approaches have been proposed in the simplest case where the heterogeneity is a thin homogeneous layer, whose constitutive behavior differs from the surrounding volume (see Fig. 1-a). This problem has been first analyzed by Huy and Sanchez-Palencia [21], then by Caillerie [13], Licht and Michaille [28], Abdelmoula *et al.* [1], Klarbring and Movchan [23], Geymonat *et al.* [19], Krasucki and Lenci [24, 25], Benveniste [5, 6], Bessoud *et al.* [7–9]. In such a case, the thickness of the layer is assumed to be much smaller than the global size of the structure and their ratio, say η, is considered as a small parameter. Other possible parameters are the stiffnesses of the different materials, and the asymptotic analysis is based on *a priori* as-

sumptions on the order of magnitude ratio between the stiffness of the layer and the stiffness of the material in the bulk. In particular it is shown that, if the stiffness ratio does not depend on η, then the layer does not have any influence on the behavior of the structure at first order, *i.e.* when η goes to 0. However, if one assumes that the layer is very stiff (of the order of η^{-1}), or very soft (of the order of η), the layer does have an influence on the structure at the order η^0. In the latter case, the layer behaves like an elastic interface where the transmission conditions are of Robin's type, while in the former case it is similar to a membrane and the transmission conditions are of Ventcel's type, see [26, 35]. These results are very satisfactory, but they only concern homogeneous layers.

Here we study a problem which is more general in the sense that the heterogeneous zone takes the form of a two dimensional periodic array of elastic inclusions (instead of a homogeneous layer of constant thickness), but also more particular in the sense that the stiffness ratio is constant and does not depend on the small parameter η characterizing the geometrical distribution of the inclusions. To identify a limit model, we need to separate the description of the zone which contains the inclusions from the description of the remaining part of the structure. The classical method used in such cases is the *separation of scales*, which was used by Sanchez-Palencia [32], Léné [27], Andrieux *et al.* [4], Suquet [34], Devries *et al.* [15], Briane [12] and Michel *et al.* [30] in the eighties to homogenize the behavior of three-dimensional periodic or quasi-periodic materials. By separating the so-called micro- and macro-scales, this method enables to identify an *effective behavior*, which characterizes the behavior of the material at large scales. This method will enable us to describe the heterogeneous zone as an interface with ad hoc transmission conditions.

Outside the zone where the inclusions are located, the body is homogeneous and does not contain heterogeneities. It can be described with a classical one-scale description. It is therefore necessary to specify matching conditions between the heterogeneous zone and the surrounding volume. Between these two regions, one expect to see boundary layer effects, like those exhibited by Dumontet [16] for composite materials or those which appear at the boundary of thin or slender structures [10]. These matching conditions will be established by using *matched asymptotic expansions*, which where first introduced in this context by Nguetseng and Sanchez-Palencia [31, 33].

Following previous results, we expect the heterogeneities to have no influence on the structure *at the order* η^0 when $\eta \to 0$. We will therefore need to take into account the next term in the expansion to find the influence at the order η^1. This approach has already been presented in previous papers for particular cases, see [2, 18, 29]. Here we extend these results in a more general case and propose an energetic model for the behavior of the homogenized interface. This energetic equivalence implicitly contains the desired transmission conditions by virtue of "classical" variational arguments.

The paper is organized as follows. In the second section, we detail the setting of the problem, explain the foundations of the method of separation of scales, and establish the main equations of our problem. In the third section, we solve iteratively the different equations governing the behavior of the structure, and show that the influence of the heterogeneities scales with their size. In the fourth section, we analyze the microscopic problems which characterize the effective behavior of the heterogeneities, and exhibit a so-called *interface energy*. The fifth section is devoted to constructing and analyzing a coupled energetic formulation describing the interaction between the homogeneous volume and the heterogeneities. Finally, the last section aims at validating the proposed model through several examples. We study the simplified case of an homogeneous layer, and therefore establish the coherence of our model with previous works. We also apply our method to more complex structures and analyze their effective behavior.

Throughout the paper we use the following notations: vectors and second order tensors are denoted by bold symbols (like $\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$) while third or fourth order tensors are denoted by sans serif letters (like A, B,

C, D). Their components are denoted by plain letters (like v_i, ε_{ij} and A_{ijkl} for the components of $\mathbf{v}$, $\boldsymbol{\varepsilon}$ and A). The single and double contraction of indices is indicated by dots and the convention of summation of repeated indices is implicitly used. For instance, $\mathbf{u}\cdot\mathbf{v}$ stands for $u_i v_i$, $\boldsymbol{\sigma}:\boldsymbol{\varepsilon}$ stands for $\sigma_{ij}\varepsilon_{ij}$, $\mathsf{A}:\boldsymbol{\varepsilon}$ denotes the second order tensor whose ij-component reads as $A_{ijkl}\varepsilon_{kl}$ and $\tilde{\boldsymbol{\varepsilon}}:\mathsf{A}:\boldsymbol{\varepsilon}$ stands for $A_{ijkl}\tilde{\varepsilon}_{ij}\varepsilon_{kl}$. For a field f which may be discontinuous across the plane surface Γ with unit normal vector $\mathbf{e}_1$, we denote by $f^+(\mathbf{x}')$ and $f^-(\mathbf{x}')$ (or by $f|^{\pm}(\mathbf{x}')$ to avoid a possible confusion) the limits, when they exist, of f at the point $\mathbf{x}'$ of Γ on each side of the plane Γ:

$$f^+(\mathbf{x}') = \lim_{\substack{\mathbf{x}\to\mathbf{x}' \\ (\mathbf{x}-\mathbf{x}')\cdot\mathbf{e}_1>0}} f(\mathbf{x}), \qquad f^-(\mathbf{x}') = \lim_{\substack{\mathbf{x}\to\mathbf{x}' \\ (\mathbf{x}-\mathbf{x}')\cdot\mathbf{e}_1<0}} f(\mathbf{x}).$$

Accordingly, we denote respectively by $[\![f]\!]$ and $\overline{f}$ the jump and the mean value of f on Γ:

$$[\![f]\!] = f^+ - f^-, \qquad \overline{f} = \frac{1}{2}\left(f^+ + f^-\right).$$

2 Separation of scales, asymptotic expansions of the real fields and matching conditions

2.1 The real problem

Let us consider a three-dimensional body, the natural reference configuration of which is the regular connected open subset Ω of $\mathbb{R}^3$. The behaviour of the material is linear elastic, characterized by the stiffness tensor $\mathsf{A}(\mathbf{x})$, and the density $\rho(\mathbf{x})$, $\mathbf{x}\in\Omega$ being the position vector. Except in the neighborhood of the surface Γ located in the plane $x_1 = 0$, the body is homogeneous, with a stiffness tensor $\mathsf{A} = \mathsf{A}^*$ and a density $\rho = \rho^*$. This material will be called the *matrix*. Near the plane surface Γ, the body contains heterogeneities which are periodically distributed along an array characterized by the two vectors $\mathbf{a}$ and $\mathbf{b}$. The material parameters A and ρ are therefore $\mathbf{a}$- and $\mathbf{b}$-periodic in the vicinity of Γ. The heterogeneity of the material may be due either to voids in the volume — like cavities or cracks — or to elastic inclusions. These two different cases are very similar, and give the same kind of homogenized behavior. But to simplify the presentation, we will explain the procedure in the case of elastic inclusions only. The case of cracks or voids is detailed in [29] and we will simply recall the main results in the last section. Note, however, that in the case of cracks or voids, the matrix must remain connected — *i.e.* the cracks or the voids must not separate the body into two parts — in order that our results remain valid. The size of the inclusions is characterized by their outer diameter e (that is the diameter of the smallest ball which contains one inclusion, or the thickness of the heterogeneous layer). The periodic vectors $\mathbf{a}$ and $\mathbf{b}$ and the outer diameter e are small compared to the size of the overall structure. The boundary $\partial\Omega$ of the solid is separated into two parts $\partial_F\Omega$ and $\partial_u\Omega$. A density $\mathbf{F}$ of surface forces are imposed on $\partial_F\Omega$, while a displacement $\mathbf{u}^d$ is prescribed on $\partial_u\Omega$ (see Fig. 2). The body is also subjected to a uniform gravity $\mathbf{g}$. In this context, the equilibrium of the body is governed by the following set of equations

$$\begin{cases} \boldsymbol{\sigma} = \mathsf{A}:\boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in} \quad \Omega, \\ \mathbf{div}(\boldsymbol{\sigma}) + \rho\mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{F} \quad \text{on} \quad \partial_F\Omega, \\ \mathbf{u} = \mathbf{u}^d \quad \text{on} \quad \partial_u\Omega, \end{cases} \tag{1}$$

which are respectively the constitutive equation, the equation of equilibrium and the boundary conditions. This problem is a classical linear elasto-static problem, and we know from classical theorems that, provided

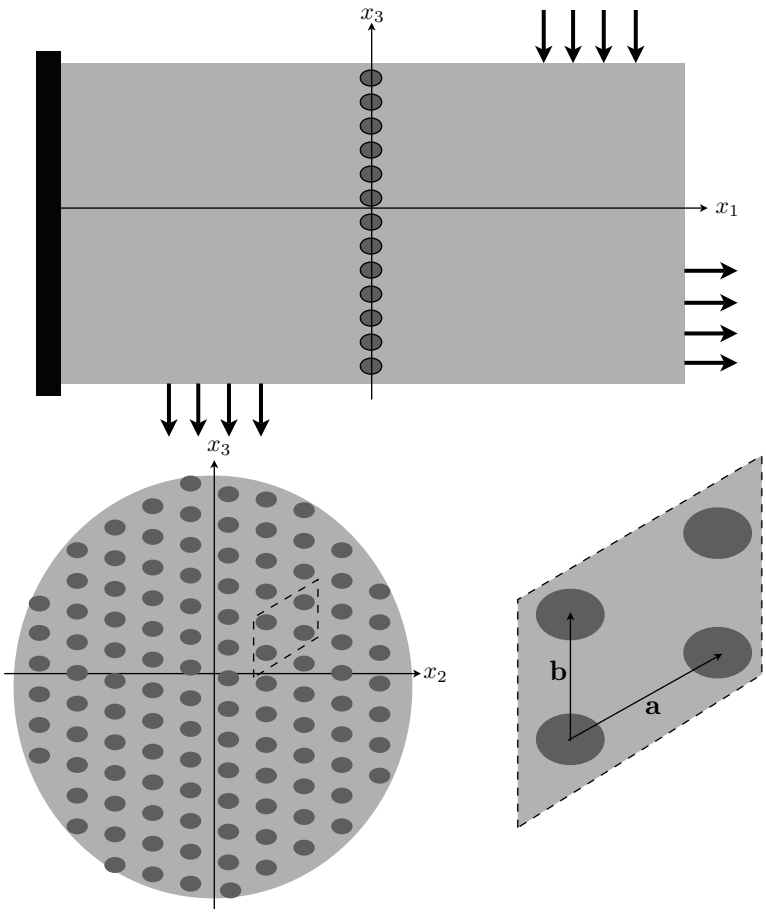

Fig. 2 The standard linear elastic problem, with prescribed forces and displacements. The heterogeneities are located on the plane Γ, and are periodically distributed along an array characterized by the vectors **a** and **b**.

the data $\mathbf{F}$ and $\mathbf{u}^d$ are sufficiently smooth, it admits a unique solution with finite energy. Moreover the displacement field solution is the minimizer of the potential energy functional

$$\mathcal{P}(\mathbf{u}) = \frac{1}{2}\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \mathsf{A} : \boldsymbol{\varepsilon}(\mathbf{u})\,\mathrm{d}\Omega - \int_\Omega \rho\mathbf{g}\cdot\mathbf{u}\,\mathrm{d}\Omega - \int_{\partial_F\Omega} \mathbf{F}\cdot\mathbf{u}\,\mathrm{dS}$$

over the set of kinematically admissible displacement fields.

2.2 Principles of scale separation and the two systems of coordinates

The purpose of scale separation is to identify an effective behavior for a given microscopic structure. This microscopic structure is assumed periodic, or quasi-periodic, and the idea is to compute the limit behavior of the medium when the size of the microstructure goes to zero. To this end, we need to give a sound definition of the micro- and macrostructure. We introduce the following length parameters: the size of the

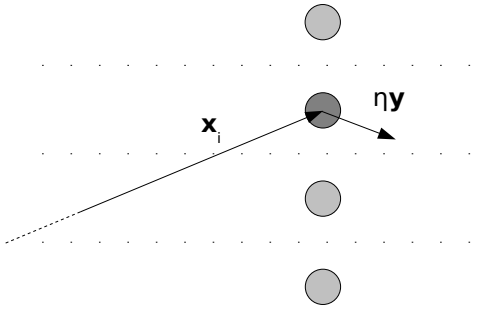

Fig. 3 Separation of the so-called macroscopic and microscopic coordinates. The macroscopic one $\mathbf{x}_i$ denotes the center of the periodic cell, while $\eta\mathbf{y}$ describes the relative position inside the periodic cell.

global structure is denoted by H, while the typical size of a periodic pattern is given by h. Since our model describes a wide variety of problems, it is difficult to give a general definition of these parameters, but in practice, H is the smallest dimension of the global structure, and h is the maximum of e, $\|\mathbf{a}\|$ and $\|\mathbf{b}\|$. In particular, in the case when the heterogeneities form a simple homogeneous layer, $\mathbf{a}$ and $\mathbf{b}$ are arbitrary, so we define $h = e$, with e the thickness of the layer. Throughout the paper, the ratio between these two scales is denoted by $\eta = h/H$, and η will be considered as a small dimensionless parameter.

The first key point in scale separation is to properly separate the description of the global structure from that of the microscopic details. This separation may be introduced in the following way. The volume Ω can be separated into two parts: the *outer domain* far enough from the surface Γ which contains no heterogeneity, and the *inner domain* near the plane Γ which contains the heterogeneities. The outer domain (which will be finally identified with $\Omega \setminus \Gamma$) can be described with a classical one-scale description. On the other hand, the inner domain must be described with a two-scale description, in order to distinguish the details of the heterogeneities. In this region, since the microstructure is periodic, the position of a given point may be defined by two independent data: the position of the periodic pattern where the point belongs, and its relative position inside the periodic pattern. The first data is denoted by $\mathbf{x}_i$, and describes the center of the pattern. It is a vector, which only takes discrete values, and verifies by definition $\mathbf{x}_i \in \Gamma$. It may be seen as the *macroscopic position of the point* (see Fig. 3).

If the real position of the point is denoted by $\mathbf{X}$, the difference $\mathbf{X} - \mathbf{x}_i$ is a small vector, describing the relative position of the point inside its periodic pattern. By introducing the ratio of scales η, we define the position of a point in the periodic pattern by

$$\mathbf{y} = \frac{1}{\eta}(\mathbf{X} - \mathbf{x}_i). \tag{2}$$

By definition, $\mathbf{y} \in \mathbb{R} \times \mathrm{Y}$, with Y the cross section of a periodic pattern. This volume is called *the periodic cell*, and will be denoted by $\mathbb{Y}$ in the following. Note that this periodic cell is not bounded in the transverse direction. The vector $\mathbf{y} \in \mathbb{Y}$ takes continuous values and may be seen as the *microscopic position of the point* at the scale of a pattern. Since the microstructure is periodic, the relative position inside the periodic cell is defined modulo the two vectors $\mathbf{a}/\eta$ and $\mathbf{b}/\eta$, and we have

$$\mathbf{y} + \frac{1}{\eta}(n\mathbf{a} + m\mathbf{b}) \equiv \mathbf{y}, \qquad \forall (n, m) \in \mathbb{Z}^2.$$

This relation implies that all physical fields must be periodic with respect to $\mathbf{a}/\eta$ and $\mathbf{b}/\eta$ at the micro-scale: the fields will be described as $\mathbf{y}'$*-periodic* in the following.

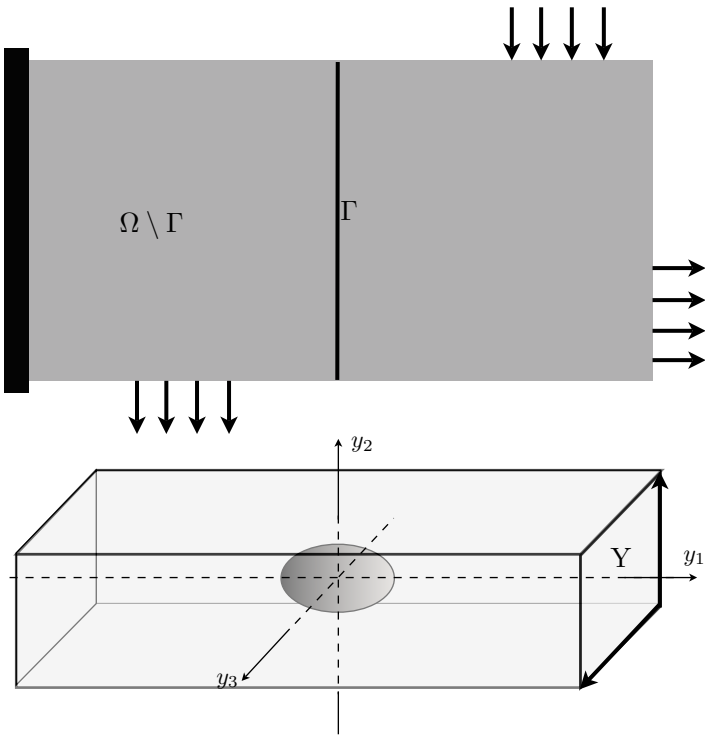

Fig. 4 The original problem is separated into a macroscopic problem and a microscopic one. At the macro scale, the heterogeneities reduce to a simple interface, while at the micro scale, the heterogeneities form a periodic infinite pattern. The periodic cell $\mathbb{Y}$ is bounded in the y_2 and y_3 directions, but is infinite in the y_1 direction.

The real position $\mathbf{X}$ of a given point near the interface Γ is given by the couple $(\mathbf{x}_i, \mathbf{y}) \in \Gamma \times \mathbb{Y}$, $\mathbf{x}_i$ taking discrete values and $\mathbf{y}$ taking periodic continuous values. Note that $\mathbf{x}_i$ and $\mathbf{y}$ are both of the order of H, which will simplify the following developments. As η decreases, the microstructure becomes smaller, and the discrete values taken by $\mathbf{x}_i$ become dense in the continuum set Γ. This justifies that we treat $\mathbf{x}_i$ as a continuous parameter, denoted by $\mathbf{x}'$ throughout the paper. With this approximation, the position of a given point is defined by two independent continuous parameters, which may be written

$$\mathbf{X}(\mathbf{x}', \mathbf{y}) = \mathbf{x}' + \eta \mathbf{y}. \tag{3}$$

Note that, if the microstructure is not periodic, it is much more difficult to define rigorously the micro- and macro-scales, while it should still be possible to identify an effective behavior. This is a difficult point in the theory of homogenization, which has been addressed by several authors, see for instance [12, 14].

Accordingly, we will use the two following coordinates systems depending whether we consider a material point at a macro- or at a micro-scale:

1. At a macro-scale, a point of the outer domain $\Omega\backslash\Gamma$ is defined by the macroscopic cartesian coordinates $\mathbf{x} = (x_1, x_2, x_3)$ with the associated orthonormal basis of vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. The coordinate x_1 gives the (signed) distance of a point to the plane Γ, the two other coordinates $\mathbf{x}' = (x_2, x_3)$ describe the tangent plane. $\mathbf{e}_2$ and $\mathbf{e}_3$ are not necessarily aligned with $\mathbf{a}$ or $\mathbf{b}$. A point $(0, x_2, x_3)$ of Γ will be identified with $\mathbf{x}' = (x_2, x_3)$.

2. At a micro-scale, a point of the inner domain $\Gamma \times \mathbb{Y}$ is characterized by the macroscopic coordinates $\mathbf{x}' = (x_2, x_3)$ and the microscopic coordinates $\mathbf{y} = (y_1, y_2, y_3)$, which are derived from the macroscopic ones near the interface by using the relation (2). The periodic coordinates $\mathbf{y}' = (y_2, y_3)$ describe the relative position in the tangent plane, while y_1 describes the distance of a point to the midplane Γ.

2.3 Decomposition of the fields

The displacement and stress fields solutions of (1) depend on the parameter η and hence will be denoted from now on by $\mathbf{u}^\eta$ and $\boldsymbol{\sigma}^\eta$. In the spirit of scale separation, the fields are decomposed in successive powers of η with two types of expansion, depending whether we consider the fields far from the surface Γ (outer expansion) or near the surface Γ (inner expansion). Specifically, we assume that $\mathbf{u}^\eta$ and $\boldsymbol{\sigma}^\eta$ can be expanded as follows.

Outer expansion

$$\mathbf{u}^\eta(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}) + \eta\mathbf{u}^1(\mathbf{x}) + \eta^2\mathbf{u}^2(\mathbf{x}) + \dots \tag{4}$$

$$\boldsymbol{\sigma}^\eta(\mathbf{x}) = \boldsymbol{\sigma}^0(\mathbf{x}) + \eta\boldsymbol{\sigma}^1(\mathbf{x}) + \eta^2\boldsymbol{\sigma}^2(\mathbf{x}) + \dots \tag{5}$$

Note that the outer fields $\mathbf{u}^i$ and $\boldsymbol{\sigma}^i$ ($i \in \mathbb{N}$) are defined on the whole outer domain $\Omega \setminus \Gamma$, even though the expansion is not a good approximation of the true solution in a small neighborhood of Γ.

Inner expansion

$$\mathbf{u}^\eta(\mathbf{X}) = \mathbf{v}^0(\mathbf{x}', \mathbf{y}) + \eta\mathbf{v}^1(\mathbf{x}', \mathbf{y}) + \eta^2\mathbf{v}^2(\mathbf{x}', \mathbf{y}) + \dots \tag{6}$$

$$\boldsymbol{\sigma}^\eta(\mathbf{X}) = \boldsymbol{\tau}^0(\mathbf{x}', \mathbf{y}) + \eta\boldsymbol{\tau}^1(\mathbf{x}', \mathbf{y}) + \eta^2\boldsymbol{\tau}^2(\mathbf{x}', \mathbf{y}) + \dots \tag{7}$$

The inner fields $\mathbf{v}^i$ and $\boldsymbol{\tau}^i$ are defined on the inner domain $\Gamma \times \mathbb{Y}$, and are periodic with respect to $\mathbf{y}'$.

It can be established[1] that the first non-zero order for all fields is the order 0. The terms indiced by 0 are the limit of the fields when η goes to zero, while the subsequent terms provide more accurate estimations of the fields when η is small. In the following, we will be most interested by the order 1, since it will appear that the main influence of the interface on the global structure scales with η. Note that, similarly to standard Taylor expansions, we do not know *a priori* if the series converge. To establish matching conditions between the outer and inner expansions of the fields, we will use the fact that both the inner and the outer expansions are valid in intermediate regions. These regions are both very close to the interface at the macro scale, and far away from the heterogeneities at the micro scale. Note that these regions are only well defined when the two scales are separated.

2.4 Set of equations at different orders

Let us introduce the two types of expansion into the elastic problem (1). For the inner expansions, it is necessary to reformulate the differential operators in terms of the macroscopic and microscopic coordinates.

[1] One can check that the negative order terms are zero. Conversely, if the series start at order 1, the boundary conditions cannot be satisfied.

Using relation (3), all derivative operators can be transformed in the following way:

$$\frac{\partial(\cdot)}{\partial \mathbf{X}} \quad \rightarrow \quad \frac{\partial(\cdot)}{\partial \mathbf{x}'} + \frac{1}{\eta}\frac{\partial(\cdot)}{\partial \mathbf{y}}.$$

This rule is very general, and can be applied to all first order differential operators. For example, if $\boldsymbol{\varepsilon}$ is the symmetric part of the gradient, one gets

$$\boldsymbol{\varepsilon}(\cdot) \quad \rightarrow \quad \boldsymbol{\varepsilon}_{x'}(\cdot) + \frac{1}{\eta}\boldsymbol{\varepsilon}_y(\cdot),$$

where $\boldsymbol{\varepsilon}_{x'}$ and $\boldsymbol{\varepsilon}_y$ denote respectively the symmetric part of the gradient with respect to the macroscopic and microscopic coordinates. The divergence operator can be transformed in the same way

$$\mathbf{div}(\cdot) \quad \rightarrow \quad \mathbf{div}_{x'}(\cdot) + \frac{1}{\eta}\mathbf{div}_y(\cdot).$$

In terms of components those relations read as

$$\boldsymbol{\varepsilon}_{x'}(\mathbf{v}) = \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)\mathbf{e}_\alpha \otimes_s \mathbf{e}_\beta + \frac{\partial v_1}{\partial x_\alpha}\mathbf{e}_\alpha \otimes_s \mathbf{e}_1, \qquad \boldsymbol{\varepsilon}_y(\mathbf{v}) = \left(\frac{\partial v_i}{\partial y_j} + \frac{\partial v_j}{\partial y_i}\right)\mathbf{e}_i \otimes_s \mathbf{e}_j, \tag{8}$$

$$\mathbf{div}_{x'}\boldsymbol{\tau} = \frac{\partial \tau_{i\alpha}}{\partial x_\alpha}\mathbf{e}_i, \qquad \mathbf{div}_y\boldsymbol{\tau} = \frac{\partial \tau_{ij}}{\partial x_j}\mathbf{e}_i, \tag{9}$$

where the Greek indices run from 2 to 3, the Latin indices run from 1 to 3 and $\mathbf{e}_i \otimes_s \mathbf{e}_j$ stands for the symmetrized dyadic product $\mathbf{e}_i \otimes_s \mathbf{e}_j = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i)$. Accordingly, for the inner expansions, the equilibrium and the constitutive equations read formally as

$$\begin{cases} \displaystyle\sum_{i=0}^{+\infty}\left(\eta^i \mathbf{div}_{x'}(\boldsymbol{\tau}^i) + \eta^{i-1}\mathbf{div}_y(\boldsymbol{\tau}^i)\right) + \rho\mathbf{g} = \mathbf{0}, \\ \displaystyle\sum_{i=0}^{+\infty}\eta^i\boldsymbol{\tau}^i = \sum_{i=0}^{+\infty}\mathsf{A}\!:\!\left(\eta^i\boldsymbol{\varepsilon}_{x'}(\mathbf{v}^i) + \eta^{i-1}\boldsymbol{\varepsilon}_y(\mathbf{v}^i)\right), \end{cases} \quad \text{in} \quad \Gamma \times \mathbb{Y}. \tag{10}$$

For the outer expansion, since the macroscopic coordinates $\mathbf{x}$ only are involved, the equilibrium and the constitutive equations also read as

$$\begin{cases} \displaystyle\sum_{i=0}^{+\infty}\eta^i\mathbf{div}(\boldsymbol{\sigma}^i) + \rho^*\mathbf{g} = \mathbf{0}, \\ \displaystyle\sum_{i=0}^{+\infty}\eta^i\boldsymbol{\sigma}^i = \sum_{i=0}^{+\infty}\eta^i\mathsf{A}^*\!:\!\boldsymbol{\varepsilon}(\mathbf{u}^i), \end{cases} \quad \text{in} \quad \Omega\backslash\Gamma. \tag{11}$$

After identifying the powers of η, we obtain the sequences of equations which are detailed in the following tables.

Equilibrium equations:

Order	Outer domain $\Omega\backslash\Gamma$	Inner domain $\Gamma \times \mathbb{Y}$
-1	*none*	$\mathbf{div}_y(\boldsymbol{\tau}^0) = \mathbf{0}$
0	$\mathbf{div}(\boldsymbol{\sigma}^0) + \rho^* \mathbf{g} = \mathbf{0}$	$\mathbf{div}_{x'}(\boldsymbol{\tau}^0) + \mathbf{div}_y(\boldsymbol{\tau}^1) + \rho \mathbf{g} = \mathbf{0}$
$i \geq 1$	$\mathbf{div}(\boldsymbol{\sigma}^i) = \mathbf{0}$	$\mathbf{div}_{x'}(\boldsymbol{\tau}^i) + \mathbf{div}_y(\boldsymbol{\tau}^{i+1}) = \mathbf{0}$

Constitutive equations:

Order	Outer domain $\Omega\backslash\Gamma$	Inner domain $\Gamma \times \mathbb{Y}$
-1	*none*	$\mathsf{A}\!:\!\boldsymbol{\varepsilon}_y(\mathbf{v}^0) = \mathbf{0}$
$i \geq 0$	$\boldsymbol{\sigma}^i = \mathsf{A}^*\!:\!\boldsymbol{\varepsilon}(\mathbf{u}^i)$	$\boldsymbol{\tau}^i = \mathsf{A}\!:\!\boldsymbol{\varepsilon}_{x'}(\mathbf{v}^i) + \mathsf{A}\!:\!\boldsymbol{\varepsilon}_y(\mathbf{v}^{i+1})$

Similarly, we can write the boundary conditions in the following way.

Boundary conditions:

Order	Dirichlet conditions	Neumann conditions
0	$\mathbf{u}^0 = \mathbf{u}^d$	$\boldsymbol{\sigma}^0 \cdot \mathbf{n} = \mathbf{F}$
$i \geq 1$	$\mathbf{u}^i = \mathbf{0}$	$\boldsymbol{\sigma}^i \cdot \mathbf{n} = \mathbf{0}$

Remark 1 We overlook willingly the boundary conditions which apply on the inner domain, *i.e.* when $\partial\Gamma \cap \partial_u\Omega$ or $\partial\Gamma \cap \partial_F\Omega$ are not empty. These boundary conditions do not have any influence up to order 1 in η, provided the external forces $\mathbf{F}$ or prescribed displacements $\mathbf{u}^d$ are sufficiently smooth in these regions. Identifying the influence of these boundary conditions would involve an additionnal separation of scales between the heterogeneities distributed on Γ, and those located on $\partial\Gamma$.

2.5 Matching conditions

We now need to establish matching conditions between the outer and inner expansions, using the fact that these expansions are both valid in some intermediate regions. These regions are far from the heterogeneities at the micro-scale, while being close to them at the macro-scale, which means $H \gg |x_1| \gg h$. We can therefore introduce the following expansions of the outer fields in the vicinity of Γ.

$$\mathbf{u}^i(\mathbf{x}) = \mathbf{u}^{i\pm}(\mathbf{x}') + x_1 \left.\frac{\partial \mathbf{u}^i}{\partial x_1}\right|^{\pm}(\mathbf{x}') + \frac{x_1^2}{2} \left.\frac{\partial^2 \mathbf{u}^i}{\partial x_1^2}\right|^{\pm}(\mathbf{x}') + \dots$$

$$\boldsymbol{\sigma}^i(\mathbf{x}) = \boldsymbol{\sigma}^{i\pm}(\mathbf{x}') + x_1 \left.\frac{\partial \boldsymbol{\sigma}^i}{\partial x_1}\right|^{\pm}(\mathbf{x}') + \frac{x_1^2}{2} \left.\frac{\partial^2 \boldsymbol{\sigma}^i}{\partial x_1^2}\right|^{\pm}(\mathbf{x}') + \dots$$

Since we have $x_1 = \eta y_1$, we obtain

$$\mathbf{u}^i(\eta y_1, \mathbf{x}') = \mathbf{u}^{i\pm}(\mathbf{x}') + \eta y_1 \left.\frac{\partial \mathbf{u}^i}{\partial x_1}\right|^{\pm}(\mathbf{x}') + \eta^2 \frac{y_1^2}{2} \left.\frac{\partial^2 \mathbf{u}^i}{\partial x_1^2}\right|^{\pm}(\mathbf{x}') + \dots, \tag{12}$$

$$\boldsymbol{\sigma}^i(\eta y_1, \mathbf{x}') = \boldsymbol{\sigma}^{i\pm}(\mathbf{x}') + \eta y_1 \left.\frac{\partial \boldsymbol{\sigma}^i}{\partial x_1}\right|^{\pm}(\mathbf{x}') + \eta^2 \frac{y_1^2}{2} \left.\frac{\partial^2 \boldsymbol{\sigma}^i}{\partial x_1^2}\right|^{\pm}(\mathbf{x}') + \dots, \tag{13}$$

where y_1 is such that $H/\eta \gg |y_1| \gg H$. In the intermediate regions, since the outer and inner expansions are both valid, we get

$$\mathbf{v}^0 + \eta \mathbf{v}^1 + \eta^2 \mathbf{v}^2 + \dots = \mathbf{u}^{0\pm} + \eta \left(\mathbf{u}^{1\pm} + y_1 \left.\frac{\partial \mathbf{u}^0}{\partial x_1}\right|^{\pm} \right) + \eta^2 \left(\mathbf{u}^{2\pm} + y_1 \left.\frac{\partial \mathbf{u}^1}{\partial x_1}\right|^{\pm} + \frac{y_1^2}{2} \left.\frac{\partial^2 \mathbf{u}^0}{\partial x_1^2}\right|^{\pm} \right) + \dots,$$

where the terms on the left depend upon $(\mathbf{x}', \mathbf{y})$, while the terms on the right only depend upon $(\mathbf{x}', y_1)$. This relation implies that the microscopic fields $\mathbf{v}^i$ should not depend upon (y_2, y_3) in the intermediate regions (i.e. when $|y_1| \gg H$). This property will be explained in section 3.3.

Identifying in the previous equality the first three powers of η, we obtain

$$\begin{aligned}
\mathbf{u}^{0\,\pm}(\mathbf{x}') &= \lim_{y_1 \to \pm\infty} \mathbf{v}^0(\mathbf{x}', \mathbf{y}), \\
\mathbf{u}^{1\,\pm}(\mathbf{x}') &= \lim_{y_1 \to \pm\infty} \left(\mathbf{v}^1(\mathbf{x}', \mathbf{y}) - y_1 \left.\frac{\partial \mathbf{u}^0}{\partial x_1}\right|^{\pm}(\mathbf{x}') \right), \\
\mathbf{u}^{2\,\pm}(\mathbf{x}') &= \lim_{y_1 \to \pm\infty} \left(\mathbf{v}^2(\mathbf{x}', \mathbf{y}) - y_1 \left.\frac{\partial \mathbf{u}^1}{\partial x_1}\right|^{\pm}(\mathbf{x}') - \frac{y_1^2}{2} \left.\frac{\partial^2 \mathbf{u}^0}{\partial x_1^2}\right|^{\pm}(\mathbf{x}') \right),
\end{aligned}$$

were we formally passed to the limit $y_1 \to \pm\infty$. By introducing the developments (5), (7) and (13), we obtain similarly

$$\begin{aligned}
\boldsymbol{\sigma}^{0\,\pm}(\mathbf{x}') &= \lim_{y_1 \to \pm\infty} \boldsymbol{\tau}^0(\mathbf{x}', \mathbf{y}), \\
\boldsymbol{\sigma}^{1\,\pm}(\mathbf{x}') &= \lim_{y_1 \to \pm\infty} \left(\boldsymbol{\tau}^1(\mathbf{x}', \mathbf{y}) - y_1 \left.\frac{\partial \boldsymbol{\sigma}^0}{\partial x_1}\right|^{\pm}(\mathbf{x}') \right), \\
\boldsymbol{\sigma}^{2\,\pm}(\mathbf{x}') &= \lim_{y_1 \to \pm\infty} \left(\boldsymbol{\tau}^2(\mathbf{x}', \mathbf{y}) - y_1 \left.\frac{\partial \boldsymbol{\sigma}^1}{\partial x_1}\right|^{\pm}(\mathbf{x}') - \frac{y_1^2}{2} \left.\frac{\partial^2 \boldsymbol{\sigma}^0}{\partial x_1^2}\right|^{\pm}(\mathbf{x}') \right).
\end{aligned}$$

Note that the high order fields are coupled to all previous orders, which makes the matching conditions quite complicated. However, in the following, we will only need the first two orders to characterize the effective behavior of the heterogeneities.

3 Iterative resolution of the inner and outer problems of order 0 and 1

We have established the equations of equilibrium, the constitutive equations and the matching conditions between the outer and the inner domains. Using these equations, we can solve the whole problem in an iterative way, starting by the order 0, and studying alternatively the inner problem near the interface, and the outer problem at the global scale.

3.1 Inner problem of order 0

Inside a pattern, the constitutive equation of order -1 can be written $\mathsf{A}:\boldsymbol{\varepsilon}_y(\mathbf{v}^0)=\mathbf{0}$. Since the stiffness of the material is positive definite, it is invertible and we have $\boldsymbol{\varepsilon}_y(\mathbf{v}^0)=\mathbf{0}$. This equation means that $\mathbf{v}^0$ is a rigid body motion at the scale of a periodic cell. Since $\mathbf{v}^0$ is also periodic in $\mathbf{y}'$, $\mathbf{v}^0$ can only be a translation, and does not depend upon $\mathbf{y}$. The value of the translation is left undefined at this step, it will be determined by the outer problem of order 0.

3.2 Outer problem of order 0

Using the previous result, we can now solve the outer problem of order 0. Inside the outer domain, the constitutive relation of order 0 and the equation of equilibrium of order 0 read respectively as $\boldsymbol{\sigma}^0=\mathsf{A}^*:\boldsymbol{\varepsilon}(\mathbf{u}^0)$ and $\mathbf{div}(\boldsymbol{\sigma}^0)+\rho^*\mathbf{g}=\mathbf{0}$, while on the boundary $\partial\Omega$ the boundary conditions of order 0 read as $\boldsymbol{\sigma}^0\cdot\mathbf{n}=\mathbf{F}$ on $\partial_F\Omega$ and $\mathbf{u}^0=\mathbf{u}^d$ on $\partial_u\Omega$.

We now need to identify the possible influence of the heterogeneities on the outer problem of order 0. This influence may result in jumps of stress and displacement across the interface Γ. To identify these possible jump conditions, we need to take into account the matching conditions between the inner and outer expansions. These matching conditions for the displacement of order 0 read

$$\mathbf{u}^{0\pm}(\mathbf{x}')=\lim_{y_1\to\pm\infty}\mathbf{v}^0(\mathbf{x}',\mathbf{y}),$$

and since $\mathbf{v}^0$ is a simple translation with respect to $\mathbf{y}$, we get

$$\mathbf{v}^0(\mathbf{x}')=\mathbf{u}^{0+}(\mathbf{x}')=\mathbf{u}^{0-}(\mathbf{x}')\quad\text{on}\quad\Gamma.$$

Consequently, the displacements of order 0 are continuous across the interface:

$$[\![\mathbf{u}^0]\!]=\mathbf{0}\quad\text{on}\quad\Gamma.$$

Let us now determine the jump of stress. The matching conditions of order 0 give

$$\boldsymbol{\sigma}^{0\pm}(\mathbf{x}')=\lim_{y_1\to\pm\infty}\boldsymbol{\tau}^0(\mathbf{x}',\mathbf{y}).$$

The inner equation of equilibrium of order -1 reads as $\mathbf{div}_y(\boldsymbol{\tau}^0)=\mathbf{0}$. Integrating over the surface Y gives

$$\frac{\partial}{\partial y_1}\int_{\mathrm{Y}}\boldsymbol{\tau}^0(\mathbf{x}',\mathbf{y})\cdot\mathbf{e}_1\,\mathrm{d}\mathbf{y}'=\mathbf{0},$$

where we used the periodicity conditions. If we introduce the notation

$$\langle f\rangle_{\mathrm{Y}}(\mathbf{x}',y_1)=\frac{1}{\mathrm{area(Y)}}\int_{\mathrm{Y}}f(\mathbf{x}',\mathbf{y})\,\mathrm{d}\mathbf{y}',\tag{14}$$

the former equation reads

$$\frac{\partial}{\partial y_1}\left\langle\boldsymbol{\tau}^0\cdot\mathbf{e}_1\right\rangle_{\mathrm{Y}}=\mathbf{0},\tag{15}$$

which means that the average stress $\left\langle\boldsymbol{\tau}^0\cdot\mathbf{e}_1\right\rangle_{\mathrm{Y}}$ is independent of y_1 (but depends on $\mathbf{x}'$). We emphasize that this result is only valid *on average*, because the normal stress $\boldsymbol{\tau}^0\cdot\mathbf{e}_1$ does depend upon $\mathbf{y}$ at the micro-scale,

as will be shown in the next section. In particular, this property implies that the average stresses in the limits $y_1 \to \pm\infty$ are the same. Accordingly, the matching conditions read

$$\boldsymbol{\sigma}^{0+}(\mathbf{x}')\cdot\mathbf{e}_1 = \boldsymbol{\sigma}^{0-}(\mathbf{x}')\cdot\mathbf{e}_1 = \left\langle \boldsymbol{\tau}^0\cdot\mathbf{e}_1 \right\rangle_{\mathrm{Y}}(\mathbf{x}'),$$

and one concludes that the stress of order 0 is continuous across the interface:

$$[\![\boldsymbol{\sigma}^0]\!]\cdot\mathbf{e}_1 = \mathbf{0} \quad \text{on} \quad \Gamma.$$

We have established that the heterogeneities do not introduce any jump of stress or displacement across the interface. Accordingly, the interface behaves like a perfect interface at order 0, with no influence on the behavior of the body. The entire body behaves as if it was completely homogeneous, and the set of equations of order 0 can be written

$$\begin{cases} \boldsymbol{\sigma}^0 = \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{u}^0) \quad \text{in} \quad \Omega, \\ \mathbf{div}(\boldsymbol{\sigma}^0) + \rho^*\mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega, \\ \boldsymbol{\sigma}^0\cdot\mathbf{n} = \mathbf{F} \quad \text{on} \quad \partial_F\Omega, \\ \mathbf{u}^0 = \mathbf{u}^d \quad \text{on} \quad \partial_u\Omega, \end{cases} \tag{16}$$

In conclusion, the defects do not have any influence on the outer structure at order 0. This result is well known from previous studies, and may be understood with simple arguments: when the thickness of the heterogeneous layer goes to zero, its own rigidity becomes negligible. Following this argument, we may guess that the influence of the layer scales with its thickness. We therefore have to push our analysis until order 1 to identify the effective behavior of the surface.

However, one must emphasize that this result is a direct consequence of the hypothesis that we made on the behavior of the material: we assumed that the stiffness of the heterogeneities does not depend on their size. Even though this assumption is rather natural, it could be questioned. In previous works, when studying the simplified case of a thin homogeneous layers, several authors made the hypothesis that the behavior of the heterogeneities depends on η. In particular, if the rigidity of the layer scales with either η or η^{-1} — that is if the layer is very soft, or very stiff, compared to the bulk material — the layer has an influence on the behavior of the structure at order 0. In the latter case, the layer behaves like a membrane, while in the former case, it behaves like an elastic interface. These models are very interesting, because they are very simple to implement, but they are not suitable for more complex heterogeneities. We will show in the last section that our general model covers much more applications, and is also consistent with these previous works.

3.3 Inner problem of order 1

We previously saw that the inner field of displacement $\mathbf{v}^0$ is a simple translation at the microscale (*i.e.* independent of $\mathbf{y}$). To characterize more accurately the influence of the heterogeneities, we need to determine $\mathbf{v}^1$. The inner equilibrium equation of order -1 reads $\mathbf{div}_y(\boldsymbol{\tau}^0) = \mathbf{0}$ and the constitutive equation of order 0 reads $\boldsymbol{\tau}^0 = \mathsf{A} : (\boldsymbol{\varepsilon}_{x'}(\mathbf{v}^0) + \boldsymbol{\varepsilon}_y(\mathbf{v}^1))$. Since $\mathbf{v}^0(\mathbf{x}') = \mathbf{u}^0(\mathbf{x}') \quad \text{on} \quad \Gamma$, the latter becomes

$$\boldsymbol{\tau}^0 = \mathsf{A} : (\boldsymbol{\varepsilon}_{x'}(\mathbf{u}^0) + \boldsymbol{\varepsilon}_y(\mathbf{v}^1)). \tag{17}$$

The inner fields $\mathbf{v}^1$ and $\boldsymbol{\tau}^0$ are related to the outer fields $\mathbf{u}^0$ and $\boldsymbol{\sigma}^0$ through the matching conditions

$$\mathbf{u}^{1\pm} = \lim_{y_1 \to \pm\infty} \left(\mathbf{v}^1 - y_1 \left. \frac{\partial \mathbf{u}^0}{\partial x_1} \right|^{\pm} \right), \qquad \boldsymbol{\sigma}^{0\pm} = \lim_{y_1 \to \pm\infty} \boldsymbol{\tau}^0.$$

Since the outer problem of order 0 is homogeneous, it can be established that $\boldsymbol{\sigma}^{0\pm}$ and $\left. \frac{\partial \mathbf{u}^0}{\partial x_1} \right|^{\pm}$ are continuous across Γ, provided the volume forces are sufficiently smooth in the volume. It is therefore possible to introduce the correctors $\hat{\mathbf{v}}^1$ and $\hat{\boldsymbol{\tau}}^0$, defined by

$$\mathbf{v}^1(\mathbf{x}', \mathbf{y}) = y_1 \frac{\partial \mathbf{u}^0}{\partial x_1}(0, \mathbf{x}') + \hat{\mathbf{v}}^1(\mathbf{x}', \mathbf{y}), \qquad \boldsymbol{\tau}^0(\mathbf{x}', \mathbf{y}) = \boldsymbol{\sigma}^0(0, \mathbf{x}') + \hat{\boldsymbol{\tau}}^0(\mathbf{x}', \mathbf{y}).$$

The constitutive equation (17) then reads

$$\begin{aligned} \boldsymbol{\sigma}^0 + \hat{\boldsymbol{\tau}}^0 &= \mathsf{A} : \boldsymbol{\varepsilon}_y(\hat{\mathbf{v}}^1) + \mathsf{A} : \boldsymbol{\varepsilon}_y \left(y_1 \frac{\partial \mathbf{u}^0}{\partial x_1} \right) + \mathsf{A} : \boldsymbol{\varepsilon}_{x'}(\mathbf{u}^0) \\ &= \mathsf{A} : \boldsymbol{\varepsilon}_y(\hat{\mathbf{v}}^1) + \mathsf{A} : \boldsymbol{\varepsilon}(\mathbf{u}^0), \end{aligned}$$

by definition of $\boldsymbol{\varepsilon}_{x'}$. Since $\boldsymbol{\sigma}^0 = \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{u}^0)$, we obtain

$$\hat{\boldsymbol{\tau}}^0(\mathbf{x}', \mathbf{y}) = \mathsf{A}(\mathbf{y}) : \boldsymbol{\varepsilon}_y(\hat{\mathbf{v}}^1)(\mathbf{x}', \mathbf{y}) + (\mathsf{A}(\mathbf{y}) - \mathsf{A}^*) : \boldsymbol{\varepsilon}(\mathbf{u}^0)(0, \mathbf{x}'),$$

where the arguments of the functions are explicitly stated to avoid any confusion. The behavior formulated in terms of the correctors is finally an elastic behavior with a prestress $(\mathsf{A} - \mathsf{A}^*) : \boldsymbol{\varepsilon}(\mathbf{u}^0)$. Since $\boldsymbol{\sigma}^0$ is homogeneous at the microscale (*i.e.* independent of $\mathbf{y}$), the inner equation of equilibrium of order -1 becomes $\mathbf{div}_y(\hat{\boldsymbol{\tau}}^0) = \mathbf{0}$ and the matching conditions for the stress reads $\lim_{y_1 \to \pm\infty} \hat{\boldsymbol{\tau}}^0 = \mathbf{0}$.

Accordingly, the inner problem of order 1 reads as

$$\begin{cases} \hat{\boldsymbol{\tau}}^0 = \mathsf{A} : \boldsymbol{\varepsilon}_y(\hat{\mathbf{v}}^1) + (\mathsf{A} - \mathsf{A}^*) : \boldsymbol{\varepsilon}(\mathbf{u}^0) \quad \text{in} \quad \Gamma \times \mathbb{Y}, \\ \mathbf{div}_y(\hat{\boldsymbol{\tau}}^0) = \mathbf{0} \quad \text{in} \quad \Gamma \times \mathbb{Y}, \\ \lim\limits_{y_1 \to \pm\infty} \hat{\boldsymbol{\tau}}^0 \cdot \mathbf{e}_1 = \mathbf{0}, \\ \hat{\mathbf{v}}^1 \quad \text{and} \quad \hat{\boldsymbol{\tau}}^0 \quad \text{are} \quad \mathbf{y}'\text{−periodic}, \end{cases}$$

where we only used the matching conditions for the normal components of the stress $\hat{\boldsymbol{\tau}}^0 \cdot \mathbf{e}_1$. This problem is an elastic problem, with $(\mathsf{A} - \mathsf{A}^*) : \boldsymbol{\varepsilon}(\mathbf{u}^0)$ as a prestress, posed in the infinite inner domain $\Gamma \times \mathbb{Y}$. This prestress is the only loading parameter of the problem, and is restricted to the inclusion, *i.e.* in the subdomain where $\mathsf{A} \neq \mathsf{A}^*$. This kind of problem has already been studied by Sanchez-Palencia and Dumontet [16,33]. It admits a solution unique for $\hat{\boldsymbol{\tau}}^0$ and unique up to a $\mathbf{x}'$–dependent translation for $\hat{\mathbf{v}}^1$. Because of the periodicity with respect to $\mathbf{y}'$, the stress field corrector $\hat{\boldsymbol{\tau}}^0$ decays exponentially to 0 when y_1 tends to $\pm\infty$, which ensures that the matching conditions for the stress are satisfied. Similarly, the displacement corrector $\hat{\mathbf{v}}^1$ converges exponentially to a translation when y_1 goes to $+\infty$ and to another translation when y_1 goes to $-\infty$. Both translations depend on $\mathbf{x}'$. Because of the non uniqueness of the solution, only the difference of these two translations is determined. Similarly to the inner problem of order 0, the two translations will be completely determined by the outer problem of order 1.

Remark 2 The previous system of equations has been established in the case where the heterogeneities are elastic inclusions; in the case of voids inside the matrix, the prestress is replaced by the following forces on the boundary of the void:

$$\hat{\boldsymbol{\tau}}^0 \cdot \mathbf{n} = -(\mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{u}^0)) \cdot \mathbf{n}.$$

Note also that, inside the void, the stress corrector is not equal to zero, but to $-\mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{u}^0)$, see [29] for more details.

In this inner problem, $\boldsymbol{\varepsilon}(\mathbf{u}^0)$ is given by the outer problem of order 0 and plays the role of the data. Since this problem is linear, the superposition principle is valid: there are six different modes of loading, corresponding to the six independent components of $\boldsymbol{\varepsilon}(\mathbf{u}^0)$; it is thus sufficient to solve six elementary problems posed in $\mathbb{Y}$. We denote by $\mathbf{V}^{ij}$ and $\mathbf{T}^{ij}$ the displacement and stress fields solution of the ij–elementary problem which reads as

$$\begin{cases} \mathbf{T}^{ij} = \mathsf{A} : \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}) + (\mathsf{A} - \mathsf{A}^*) : \mathbf{I}^{ij} \quad \text{in} \quad \mathbb{Y}, \\ \mathbf{div}_y(\mathbf{T}^{ij}) = \mathbf{0} \quad \text{in} \quad \mathbb{Y}, \\ \lim\limits_{y_1 \to \pm\infty} \mathbf{T}^{ij} \cdot \mathbf{e}_1 = \mathbf{0}, \\ \mathbf{V}^{ij} \quad \text{and} \quad \mathbf{T}^{ij} \quad \text{are} \quad \mathbf{y}'-\text{periodic}, \end{cases} \tag{18}$$

where $\mathbf{I}^{ij}$ denotes the second order tensor defined by $2I^{ij}_{kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$, $\boldsymbol{\delta}$ being the usual Kronecker symbol (*i.e.* the second order identity tensor). By symmetry, the problems ij and ji are the same and therefore there exist only 6 independent problems. The ij–elementary problem admits a unique solution for $\mathbf{V}^{ij}$ up to a translation. Since $\mathbf{V}^{ij}(\mathbf{y})$ tends to two constants, say $\mathbf{V}^{ij\,\pm}$, when y_1 goes to $\pm\infty$, the translation can be fixed by imposing the extra condition $\mathbf{V}^{ij\,+} + \mathbf{V}^{ij\,-} = \mathbf{0}$.

Remark 3 The unique solution of each elementary problem is $\mathbf{V}^{ij} = \mathbf{T}^{ij} = \mathbf{0}$ when the body is elastically homogeneous, *i.e.* when $\mathsf{A}(\mathbf{y}) = \mathsf{A}^*$ everywhere.

Using these elementary solutions, the general solution of the inner problem of order 1 can be written

$$\hat{\mathbf{v}}^1(\mathbf{x}', \mathbf{y}) = \mathbf{V}^{ij}(\mathbf{y}) \varepsilon_{ij}(\mathbf{u}^0)(0, \mathbf{x}') + \check{\mathbf{v}}^1(\mathbf{x}'), \tag{19}$$

$$\hat{\boldsymbol{\tau}}^0(\mathbf{x}', \mathbf{y}) = \mathbf{T}^{ij}(\mathbf{y}) \varepsilon_{ij}(\mathbf{u}^0)(0, \mathbf{x}'), \tag{20}$$

where the summation convention is used. In (19), $\check{\mathbf{v}}^1(\mathbf{x}')$ represents the translation which remains undetermined at this stage and which will be determined once the outer problem of order 1 is solved.

In the next section, it turns out that the important information to extract from the inner problem are the difference between the displacements at plus and minus infinity, and the average of the stress corrector in the cell. For this reason, we introduce the following notations

$$\mathbf{d}(\mathbf{x}') = \lim_{y_1 \to \infty} \hat{\mathbf{v}}^1(\mathbf{x}', y_1, \mathbf{y}') - \lim_{y_1 \to -\infty} \hat{\mathbf{v}}^1(\mathbf{x}', y_1, \mathbf{y}'), \tag{21}$$

$$\langle f \rangle_{\mathbb{Y}} = \frac{1}{H \, \text{area(Y)}} \int_{\mathbb{Y}} f(\mathbf{y}) \, \mathrm{d}\mathbf{y}. \tag{22}$$

In the definition (22), the characteristic length H of the body is introduced so that the $\langle f \rangle_{\mathbb{Y}}$ has the same physical dimension as f; $\langle f \rangle_{\mathbb{Y}}$ is finite as soon as f decays exponentially with $|y_1|$. By using (19) and (20), we obtain

$$\mathbf{d}(\mathbf{x}') = \left(\mathbf{V}^{ij\,+} - \mathbf{V}^{ij\,-} \right) \varepsilon_{ij}(\mathbf{u}^0)(0, \mathbf{x}'), \qquad \left\langle \hat{\boldsymbol{\tau}}^0 \right\rangle_{\mathbb{Y}} (\mathbf{x}') = \left\langle \mathbf{T}^{ij} \right\rangle_{\mathbb{Y}} \varepsilon_{ij}(\mathbf{u}^0)(0, \mathbf{x}'). \tag{23}$$

The average of the stress corrector $\left\langle \mathbf{T}^{ij} \right\rangle_{\mathbb{Y}}$ has *a priori* six independent components, but one can show that only the membrane components are non zero. Indeed, integrating $\mathbf{div}_y \mathbf{T}^{ij} = \mathbf{0}$ over Y and using the $\mathbf{y}'$ periodicity of $\mathbf{T}^{ij}$ yield

$$\frac{\partial}{\partial y_1} \left\langle \mathbf{T}^{ij} \cdot \mathbf{e}_1 \right\rangle_{\mathrm{Y}} = 0, \qquad \forall k \in \{1,2,3\}.$$

Since $\mathbf{T}^{ij}$ decreases to zero at infinity, we get $\left\langle \mathbf{T}^{ij} \cdot \mathbf{e}_1 \right\rangle_{\mathrm{Y}} (y_1) = 0$ for all k and all y_1. Integrating with respect to y_1 gives the desired result:

$$\left\langle T^{ij}_{k1} \right\rangle_{\mathbb{Y}} = \left\langle T^{ij}_{1k} \right\rangle_{\mathbb{Y}} = 0, \qquad \forall (i,j,k) \in \{1,2,3\}^3. \tag{24}$$

Therefore $\left\langle \hat{\boldsymbol{\tau}}^0 \right\rangle_{\mathbb{Y}}$ can be interpreted as the membrane stress living inside the homogenized interface. This will appear more clearly in the outer problem of order 1 considered below.

3.4 Outer problem of order 1

Whereas the heterogeneities are invisible in the outer problem of order 0, they appear in the outer problem of order 1. The goal of this section is to construct this problem and to study the influence of the heterogeneities on the solution.

At order 1, the outer constitutive equation reads $\boldsymbol{\sigma}^1 = \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{u}^1)$ and the equilibrium equation reads $\mathbf{div}(\boldsymbol{\sigma}^1) = \mathbf{0}$, both taking place in $\Omega \backslash \Gamma$, while the boundary conditions read $\boldsymbol{\sigma}^1 \cdot \mathbf{n} = \mathbf{0}$ on $\partial_F \Omega$ and $\mathbf{u}^1 = \mathbf{0}$ on $\partial_u \Omega$. To complete the set of equations, we need to identify the jump conditions across the surface Γ. As in the case of the outer problem of order 0, we use the matching conditions between the inner and outer problems. At the order 1, they read as

$$\mathbf{u}^{1\pm} = \lim_{y_1 \to \pm\infty} \hat{\mathbf{v}}^1, \qquad \boldsymbol{\sigma}^{1\pm} = \lim_{y_1 \to \pm\infty} \left(\boldsymbol{\tau}^1 - y_1 \frac{\partial \boldsymbol{\sigma}^0}{\partial x_1} \right).$$

Using the definition (21) of $\mathbf{d}$, the jump of displacement is therefore given by

$$[\![\mathbf{u}^1]\!](\mathbf{x}') = \mathbf{d}(\mathbf{x}'). \tag{25}$$

The jump of stress is given by

$$[\![\boldsymbol{\sigma}^1]\!](\mathbf{x}') = \lim_{y_1 \to \infty} \left(\boldsymbol{\tau}^1(\mathbf{x}', y_1, \mathbf{y}') - \boldsymbol{\tau}^1(\mathbf{x}', -y_1, \mathbf{y}') - 2y_1 \frac{\partial \boldsymbol{\sigma}^0}{\partial x_1}(0, \mathbf{x}') \right),$$

and, after averaging over the cross-section Y of the cell $\mathbb{Y}$, we get

$$[\![\boldsymbol{\sigma}^1]\!](\mathbf{x}') \cdot \mathbf{e}_1 = \lim_{y_1 \to \infty} \left(\left\langle \boldsymbol{\tau}^1 \right\rangle_{\mathrm{Y}} (\mathbf{x}', y_1) - \left\langle \boldsymbol{\tau}^1 \right\rangle_{\mathrm{Y}} (\mathbf{x}', -y_1) - 2y_1 \frac{\partial \boldsymbol{\sigma}^0}{\partial x_1}(0, \mathbf{x}') \right) \cdot \mathbf{e}_1. \tag{26}$$

Using the inner equilibrium of order 0

$$\mathbf{div}_{x'}(\boldsymbol{\tau}^0) + \mathbf{div}_y(\boldsymbol{\tau}^1) + \rho \mathbf{g} = \mathbf{0},$$

integrating over the part $[-y_1, y_1] \times \mathrm{Y}$ of the cell $\mathbb{Y}$ and using Ostrogradsky's theorem lead to

$$\left(\left\langle \boldsymbol{\tau}^1 \right\rangle_{\mathrm{Y}} (\mathbf{x}', y_1) - \left\langle \boldsymbol{\tau}^1 \right\rangle_{\mathrm{Y}} (\mathbf{x}', -y_1) \right) \cdot \mathbf{e}_1 + \int_{-y_1}^{y_1} \left(\left\langle \mathbf{div}_{x'}(\boldsymbol{\tau}^0) \right\rangle_{\mathrm{Y}} (\mathbf{x}', y_1) + \langle \rho \rangle_{\mathrm{Y}} (y_1) \mathbf{g} \right) \mathrm{d}y_1 = \mathbf{0}, \tag{27}$$

where we used the periodicity of $\boldsymbol{\tau}^1$. After combining (26) and (27), one obtains (the arguments of the functions are temporarily omitted)

$$[\![\boldsymbol{\sigma}^1]\!]\cdot\mathbf{e}_1 + \lim_{y_1\to\infty}\int_{-y_1}^{y_1}\left\langle \mathbf{div}_{x'}(\boldsymbol{\tau}^0) + \rho\mathbf{g} + \frac{\partial\boldsymbol{\sigma}^0}{\partial x_1}\cdot\mathbf{e}_1\right\rangle_{\mathrm{Y}} \mathrm{d}y_1 = \mathbf{0}.$$

Introducing the stress corrector $\hat{\boldsymbol{\tau}}^0$ yields

$$[\![\boldsymbol{\sigma}^1]\!]\cdot\mathbf{e}_1 + \lim_{y_1\to\infty}\int_{-y_1}^{y_1}\left\langle \mathbf{div}_{x'}(\hat{\boldsymbol{\tau}}^0) + \mathbf{div}_{x'}(\boldsymbol{\sigma}^0) + \frac{\partial\boldsymbol{\sigma}^0}{\partial x_1}\cdot\mathbf{e}_1 + \rho\mathbf{g}\right\rangle_{\mathrm{Y}} \mathrm{d}y_1 = \mathbf{0}. \tag{28}$$

By definition of $\mathbf{div}_{x'}$, we have

$$\mathbf{div}_{x'}(\boldsymbol{\sigma}^0) + \frac{\partial\boldsymbol{\sigma}^0}{\partial x_1}\cdot\mathbf{e}_1 = \mathbf{div}(\boldsymbol{\sigma}^0),$$

and, by the outer equilibrium equation of order 0, $\mathbf{div}(\boldsymbol{\sigma}^0) + \rho^*\mathbf{g} = \mathbf{0}$. Introducing these relations into (28) leads to

$$[\![\boldsymbol{\sigma}^1]\!](\mathbf{x}')\cdot\mathbf{e}_1 + H\,\mathbf{div}_{x'}\left\langle\hat{\boldsymbol{\tau}}^0\right\rangle_{\mathbb{Y}}(\mathbf{x}') + H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g} = \mathbf{0}. \tag{29}$$

Remark 4 This equation can be interpreted as an equilibrium equation of the surface Γ considered as a membrane: $H\left\langle\hat{\boldsymbol{\tau}}^0\right\rangle_{\mathbb{Y}}$ is the membrane stress tensor, $H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g}$ is the bulk force density of the membrane due to the gravity and $[\![\boldsymbol{\sigma}^1]\!]\cdot\mathbf{e}_1$ is the resultant of the forces applied by the surrounding volume on the membrane.

The complete system of equations for the outer problem of order 1 is finally

$$\begin{cases} \boldsymbol{\sigma}^1 = \mathsf{A}^*{:}\boldsymbol{\varepsilon}(\mathbf{u}^1) \quad \text{in} \quad \Omega\setminus\Gamma, \\ \mathbf{div}(\boldsymbol{\sigma}^1) = \mathbf{0} \quad \text{in} \quad \Omega\setminus\Gamma, \\ \boldsymbol{\sigma}^1\cdot\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial_F\Omega, \\ \mathbf{u}^1 = \mathbf{0} \quad \text{on} \quad \partial_u\Omega, \\ [\![\mathbf{u}^1]\!] = \mathbf{d} \quad \text{on} \quad \Gamma, \\ [\![\boldsymbol{\sigma}^1]\!]\cdot\mathbf{e}_1 + H\,\mathbf{div}_{x'}\left\langle\hat{\boldsymbol{\tau}}^0\right\rangle_{\mathbb{Y}} + H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g} = \mathbf{0} \quad \text{on} \quad \Gamma, \end{cases} \tag{30}$$

where $\mathbf{d}$ and $\left\langle\hat{\boldsymbol{\tau}}^0\right\rangle_{\mathbb{Y}}$ are given by (23). We see that the influence of the heterogeneities is taken into account by the jump of the displacement $\mathbf{d}$ and the membrane stress $\left\langle\hat{\boldsymbol{\tau}}^0\right\rangle_{\mathbb{Y}}$. By virtue of (23), both involve the outer strain field of order 0 $\boldsymbol{\varepsilon}(\mathbf{u}^0)$ and the solutions $\mathbf{V}^{ij}$ and $\mathbf{T}^{ij}$ of the 6 elementary problems posed on the inner cell $\mathbb{Y}$.

Remark 5 This outer problem of order 1 does not necessarily admit a solution with finite energy. Indeed, in general the jump conditions across Γ create singularities at the boundary of the surface Γ: to equilibrate the membrane stress $\left\langle\hat{\boldsymbol{\tau}}^0\right\rangle_{\mathbb{Y}}$, a linear density of forces is applied on the outer domain at $\partial\Gamma$. In such a case, such linear density of forces create singularities with infinite energy in a three-dimensional body. These singular border effects have been studied in [2] in a two-dimensional context. Their study is outside the scope of the present paper. We will focus on the elementary problems (18) and on the jump conditions (25) and (29).

Remark 6 When the body is homogeneous, *i.e.* when $\mathsf{A} = \mathsf{A}^*$ and $\rho = \rho^*$, the solution of the outer problem of order 1 is $\mathbf{u}^1 = \boldsymbol{\sigma}^1 = \mathbf{0}$ and it can be proved in the same way that all the subsequent terms of the outer expansion also vanish. That simply means that, in such a case, $\mathbf{u}^0$ and $\boldsymbol{\sigma}^0$ are the exact solutions of the real problem.

4 The effective behavior of the interface

In this section, we study more deeply the transmission conditions at the interface and propose an interpretation of the effective behavior of the heterogeneities in terms of a surface energy.

4.1 Properties of the transmission conditions

We have shown in the previous section that the jumps of the displacement $\mathbf{u}^1$ and of the normal stress $\boldsymbol{\sigma}^1\cdot\mathbf{e}_1$ of order 1 through the interface Γ are related to the strain $\boldsymbol{\varepsilon}^0=\boldsymbol{\varepsilon}(\mathbf{u}^0)$ of order 0 at the interface through

$$[\![u_k^1]\!](\mathbf{x}')=\left(V_k^{ij+}-V_k^{ij-}\right)\varepsilon_{ij}^0(0,\mathbf{x}'), \tag{31}$$

$$[\![\sigma_{11}^1]\!](\mathbf{x}')=-H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g}\cdot\mathbf{e}_1, \tag{32}$$

$$[\![\sigma_{\alpha 1}^1]\!](\mathbf{x}')=-H\left\langle T_{\alpha\beta}^{ij}\right\rangle_{\mathbb{Y}}\frac{\partial\varepsilon_{ij}^0}{\partial x_\beta}(0,\mathbf{x}')-H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g}\cdot\mathbf{e}_\alpha, \tag{33}$$

where the summation convention holds, the Latin indices running from 1 to 3 and the Greek indices from 2 to 3. These linear relations involve a third order tensor B and a fourth order tensor C whose components are obtained from the 6 elementary inner problems:

$$HB_{kij}=V_k^{ij+}-V_k^{ij-},\qquad C_{\alpha\beta ij}=\left\langle T_{\alpha\beta}^{ij}\right\rangle_{\mathbb{Y}}. \tag{34}$$

In (34) the characteristic length H is introduced so that B is dimensionless, and C has the dimension of a pressure. Because of the symmetries of the groups of indices ij and $\alpha\beta$, both tensors have 18 independent components. Some of them may vanish when the geometry of the inclusions and the elementary cell admits some symmetries, see the examples in the last section. (Note that both tensors vanish when the body is elastically homogeneous.) Accordingly, the transmission conditions can read as

$$[\![u_k^1]\!]=HB_{kij}\varepsilon_{ij}^0,\qquad [\![\sigma_{11}^1]\!]=-H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}g_1,\qquad [\![\sigma_{\alpha 1}^1]\!]=-HC_{\alpha\beta ij}\frac{\partial\varepsilon_{ij}^0}{\partial x_\beta}-H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}g_\alpha. \tag{35}$$

Let us introduce the fourth order tensor D whose components have the dimension of a pressure and correspond to the following elastic energy associated with the 6 elementary problems:

$$D_{ijkl}=\left\langle\boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}):\mathsf{A}:\boldsymbol{\varepsilon}_y(\mathbf{V}^{kl})\right\rangle_{\mathbb{Y}}-\left\langle\mathbf{I}^{ij}:(\mathsf{A}-\mathsf{A}^*):\mathbf{I}^{kl}\right\rangle_{\mathbb{Y}}. \tag{36}$$

These quantities are finite because $\boldsymbol{\varepsilon}_y(\mathbf{V}^{ij})$ decreases exponentially to $\mathbf{0}$ when y_1 goes to $\pm\infty$ and because $\mathsf{A}=\mathsf{A}^*$ outside the inclusion (whose volume is assumed finite). By construction D is symmetric and hence has at most 21 independent components, but it is not necessarily definite nor positive. Let us prove that the components of B and C are related to those of D by

$$B_{ikl}=S_{ij}^*D_{j1kl},\qquad C_{\alpha\beta kl}=A_{\alpha\beta i1}^*S_{ij}^*D_{j1kl}-D_{\alpha\beta kl}. \tag{37}$$

In (37), $\mathbf{S}^*$ is the second order tensor, inverse of the second order tensor $\mathbf{e}_1\cdot\mathsf{A}^*\cdot\mathbf{e}_1$ (whose components are A_{i1j1}^*). Both tensors $\mathbf{e}_1\cdot\mathsf{A}^*\cdot\mathbf{e}_1$ and $\mathbf{S}^*$ are symmetric definite positive by virtue of the symmetry and the positivity of A^*.

Proof of (37). Let $\boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\varepsilon}}$ be two symmetric macroscopic strain tensors and let $\boldsymbol{\tau}$ and $\tilde{\boldsymbol{\tau}}$ be the associated microscopic stress tensor fields, *i.e.* $\boldsymbol{\tau}(\mathbf{y}) = \boldsymbol{\sigma} + \varepsilon_{ij}\mathbf{T}^{ij}(\mathbf{y})$ and $\tilde{\boldsymbol{\tau}}(\mathbf{y}) = \tilde{\boldsymbol{\sigma}} + \tilde{\varepsilon}_{ij}\mathbf{T}^{ij}(\mathbf{y})$, with $\boldsymbol{\sigma} = \mathsf{A}^* : \boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\sigma}} = \mathsf{A}^* : \tilde{\boldsymbol{\varepsilon}}$. Let $\mathbf{v}$ and $\tilde{\mathbf{v}}$ be the associated microscopic displacement fields, *i.e.* $\mathbf{v}(\mathbf{y}) = \varepsilon_{ij}\mathbf{V}^{ij}(\mathbf{y})$ and $\tilde{\mathbf{v}}(\mathbf{y}) = \tilde{\varepsilon}_{ij}\mathbf{V}^{ij}(\mathbf{y})$. Let us first remark that

$$\langle \tilde{\boldsymbol{\tau}} : \boldsymbol{\varepsilon}(\mathbf{v}) \rangle_{\mathbb{Y}} = B_{pij} A^*_{p1kl} \tilde{\varepsilon}_{kl} \varepsilon_{ij}. \tag{38}$$

Indeed, using (18), we have

$$\int_{\mathbb{Y}} \tilde{\boldsymbol{\tau}} : \boldsymbol{\varepsilon}_y(\mathbf{v}) \, \mathrm{d}\mathbf{y} = -\int_{\mathbb{Y}} \mathbf{div}_y(\tilde{\boldsymbol{\tau}}) \cdot \mathbf{v} \, \mathrm{d}\mathbf{y} + \int_{\partial\mathbb{Y}} \mathbf{n} \cdot \tilde{\boldsymbol{\tau}} \cdot \mathbf{v} \, \mathrm{dS}.$$

Since $\mathbf{div}_y \tilde{\boldsymbol{\tau}} = \mathbf{0}$, $\lim_{y_1 \to \pm\infty} \tilde{\boldsymbol{\tau}}(\mathbf{y}) = \tilde{\boldsymbol{\sigma}}$, $\lim_{y_1 \to \pm\infty} \mathbf{v}(\mathbf{y}) = \mathbf{V}^{ij\pm}\varepsilon_{ij}$ and by virtue of the $\mathbf{y}'$-periodicity of $\tilde{\boldsymbol{\tau}}$ and $\mathbf{v}$, we get

$$\int_{\mathbb{Y}} \tilde{\boldsymbol{\tau}} : \boldsymbol{\varepsilon}_y(\mathbf{v}) \, \mathrm{d}\mathbf{y} = \mathrm{area(Y)} \mathbf{e}_1 \cdot \tilde{\boldsymbol{\sigma}} \cdot (\mathbf{V}^{ij+} - \mathbf{V}^{ij-}) \varepsilon_{ij},$$

and (38) follows from (34). Let us now express D in terms of B and C. Starting from its definition, we get

$$\begin{aligned} D_{ijkl}\tilde{\varepsilon}_{ij}\varepsilon_{kl} &= \left\langle \boldsymbol{\varepsilon}_y(\tilde{\mathbf{v}}) : \mathsf{A} : \boldsymbol{\varepsilon}_y(\mathbf{v}) - \tilde{\boldsymbol{\varepsilon}} : (\mathsf{A} - \mathsf{A}^*) : \boldsymbol{\varepsilon} \right\rangle_{\mathbb{Y}} \\ &= \langle \tilde{\boldsymbol{\tau}} : \boldsymbol{\varepsilon}_y(\mathbf{v}) \rangle_{\mathbb{Y}} - \tilde{\boldsymbol{\varepsilon}} : \left\langle \mathsf{A} : \boldsymbol{\varepsilon}_y(\mathbf{v}) + (\mathsf{A} - \mathsf{A}^*) : \boldsymbol{\varepsilon} \right\rangle_{\mathbb{Y}} \\ &= A^*_{p1ij} B_{pkl} \tilde{\varepsilon}_{ij} \varepsilon_{kl} - C_{\alpha\beta kl} \tilde{\varepsilon}_{\alpha\beta} \varepsilon_{kl}, \end{aligned}$$

where we used (38), (24) and (34). Since this equality holds for all $\boldsymbol{\varepsilon}$ and $\tilde{\boldsymbol{\varepsilon}}$, one gets

$$0 = (D_{ijkl} - A^*_{p1ij} B_{pkl}) \tilde{\varepsilon}_{ij} + C_{\alpha\beta kl} \tilde{\varepsilon}_{\alpha\beta}, \qquad \forall \tilde{\boldsymbol{\varepsilon}}.$$

Taking first $\tilde{\boldsymbol{\varepsilon}} = \mathbf{e}_1 \otimes_s \mathbf{e}_i$ yields $D_{i1kl} = A^*_{i1j1} B_{jkl}$. Inverting this relation gives the first part of (37). Then taking $\tilde{\boldsymbol{\varepsilon}} = \mathbf{e}_\alpha \otimes_s \mathbf{e}_\beta$ yields $D_{\alpha\beta kl} - A^*_{p1\alpha\beta} B_{pkl} + C_{\alpha\beta kl} = 0$. Using the previously obtained expression of B_{pkl} gives the second part of (37). □

To give an energetic interpretation of these transmission conditions, it is more convenient to express the transmission conditions (35) not in terms of the whole strain tensor $\boldsymbol{\varepsilon}^0$ but in terms of its in-plane part $\boldsymbol{\varepsilon}^0_\Gamma = \varepsilon^0_{\alpha\beta} \mathbf{e}_\alpha \otimes_s \mathbf{e}_\beta$ and the normal stress vector $\boldsymbol{\sigma}^0 \cdot \mathbf{e}_1$. To this end, introducing the stress-strain relation

$$\begin{pmatrix} \sigma^0_{11} \\ \\ \sigma^0_{\delta 1} \end{pmatrix} = \begin{pmatrix} A^*_{1111} & A^*_{11\gamma 1} \\ \\ A^*_{\delta 111} & A^*_{\delta 1\gamma 1} \end{pmatrix} \begin{pmatrix} \varepsilon^0_{11} \\ \\ 2\varepsilon^0_{\gamma 1} \end{pmatrix} + \begin{pmatrix} A^*_{11\alpha\beta}\varepsilon^0_{\alpha\beta} \\ \\ A^*_{\delta 1\alpha\beta}\varepsilon^0_{\alpha\beta} \end{pmatrix},$$

and inverting it, $\boldsymbol{\varepsilon}^0 \cdot \mathbf{e}_1$ can be written as follows:

$$\begin{pmatrix} \varepsilon^0_{11} \\ \\ 2\varepsilon^0_{\delta 1} \end{pmatrix} = \begin{pmatrix} S^*_{11} & S^*_{1\gamma} \\ \\ S^*_{\delta 1} & S^*_{\delta\gamma} \end{pmatrix} \begin{pmatrix} \sigma^0_{11} - A^*_{11\alpha\beta}\varepsilon^0_{\alpha\beta} \\ \\ \sigma^0_{\gamma 1} - A^*_{\gamma 1\alpha\beta}\varepsilon^0_{\alpha\beta} \end{pmatrix}. \tag{39}$$

Inserting (39) into (35) leads to the following transmission conditions

$$[\![u^1_k]\!] = H B_{ki1} S^*_{ij} \sigma^0_{j1} + H \Big(B_{k\alpha\beta} - B_{ki1} S^*_{ij} A^*_{j1\alpha\beta} \Big) \varepsilon^0_{\alpha\beta},$$

$$[\![\sigma^1_{\alpha 1}]\!] = -H C_{\alpha\beta i1} S^*_{ij} \frac{\partial \sigma^0_{j1}}{\partial x_\beta} - H \Big(C_{\alpha\beta\gamma\delta} - C_{\alpha\beta i1} S^*_{ij} A^*_{j1\gamma\delta} \Big) \frac{\partial \varepsilon^0_{\gamma\delta}}{\partial x_\beta} - H \left\langle \rho - \rho^* \right\rangle_{\mathbb{Y}} g_\alpha.$$

Using the relations (37) one obtains the final form of the transmission conditions.

Proposition 1 *The jump conditions on* Γ *for the outer problem of order 1 can read as*

$$[\![\mathbf{u}^1]\!] = H\mathsf{L}\cdot(\boldsymbol{\sigma}^0\cdot\mathbf{e}_1)+H\mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\mathbf{u}^0), \qquad [\![\boldsymbol{\sigma}^1]\!]\cdot\mathbf{e}_1 = -H\mathbf{div}_{x'}\Big(-(\boldsymbol{\sigma}^0\cdot\mathbf{e}_1)\cdot\mathsf{M}+\mathsf{N}:\boldsymbol{\varepsilon}_\Gamma(\mathbf{u}^0)\Big)-H\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g}, \quad (40)$$

where L*,* M *and* N *are respectively the symmetric second order tensor, the third order tensor and the symmetric fourth order tensor defined in terms of the tensors* A^* *and* D *by*

$$L_{ij} = S^*_{ik}D_{k1l1}S^*_{lj}, \qquad M_{i\alpha\beta} = S^*_{ij}D_{j1\alpha\beta} - L_{ij}A^*_{j1\alpha\beta}, \quad (41)$$

$$N_{\alpha\beta\gamma\delta} = -D_{\alpha\beta\gamma\delta} + S^*_{ij}\Big(D_{i1\alpha\beta}A^*_{j1\gamma\delta} + D_{i1\gamma\delta}A^*_{j1\alpha\beta}\Big) - A^*_{i1\alpha\beta}L_{ij}A^*_{j1\gamma\delta}, \quad (42)$$

the other components vanishing.

Remark 7 The tensors L, M and N admit the following symmetries, consequences of the symmetries of A^* and D:

$$L_{ij} = L_{ji}, \qquad M_{i\alpha\beta} = M_{i\beta\alpha}, \qquad N_{\alpha\beta\gamma\delta} = N_{\beta\alpha\gamma\delta} = N_{\alpha\beta\delta\gamma} = N_{\gamma\delta\alpha\beta}.$$

Accordingly, in the case where there exists no additional symmetry, the tensors L, M and N have respectively 6, 9 and 6 independent components. Therefore we have *a priori* 21 independent coefficients characterizing the homogenized behavior of the interface. If the periodic cell $\mathbb{Y}$ has symmetry properties, the number of independent coefficients is reduced. For example, if the three planes normal to the basis vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are symmetry planes, the behavior of the interface is characterized by only 9 independent coefficients. Furthermore, if the periodic pattern is left invariant by a rotation of 90° around $\mathbf{e}_1$, the number of independent coefficients is reduced to 6.

Note that the tensors L and N have *a priori* no property of definiteness or positivity. We will show in the examples of the last sections that L can have positive, negative or zero eigenvalues and that N can be negative or positive, depending whether the inclusion is stiffer or softer than the matrix.

4.2 The up to order 1 effective problem

In the third section, we presented the iterative resolution of the complete problem, where we solve alternatively the inner and outer mechanics equations. In practice, this method could be implemented in the following way:

1. We solve the six elementary problems (18) at the microscale. These problems are solved once and for all and they give the two tensors B and C or equivalently the tensor D.
2. We solve the order 0 outer problem (16) which does not involve the heterogeneities. One obtains the macroscopic fields $\mathbf{u}^0$ and $\boldsymbol{\sigma}^0$.
3. We extract from the previous limit problem the strain field $\boldsymbol{\varepsilon}(\mathbf{u}_0)$ on the surface Γ. Using the relation (35), we identify the jump conditions across the interface for the order 1 outer problem.
4. We finally solve the order 1 outer problem (30), which is a correction of the outer mechanical fields due to the presence of heterogeneities in the material.

This approach is not very satisfactory in practice for several reasons. First, that needs to solve sequentially two outer problems, the solution of the first giving the data for the second. Second, these data involve the strain field and its tangential derivatives on the interface, but these quantities (particularly the derivatives of the strains) are not, in general, well approximated by classical numerical methods. Finally, the order 1 outer problem can exhibit singular boundary effects, which makes the numerical resolution tricky. Here we present a method which can correct all these drawbacks. It consists in constructing a unique outer problem which is a combination of the first two outer problems and whose solution $\mathbf{U}^\eta$ admits the same expansion up to order 1 in η as the real solution $\mathbf{u}^\eta$. Accordingly, this new outer problem can be seen as the *up to order 1 effective problem* . Moreover, it turns out that this problem admits a variational formulation in which the solution $\mathbf{U}^\eta$ is the stationary point of a potential energy. This energy is the sum of the usual potential energy and a surface energy over Γ which contains the information on the heterogeneities.

Let us consider the following problem which consists in finding the displacement field $\mathbf{U}$ and the stress field $\boldsymbol{\Sigma}$ such that

$$\begin{cases} \boldsymbol{\Sigma} = \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{U}) \quad \text{in} \quad \Omega \setminus \Gamma, \\ \mathbf{div}(\boldsymbol{\Sigma}) + \rho^* \mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega \setminus \Gamma, \\ \boldsymbol{\Sigma} \cdot \mathbf{n} = \mathbf{F} \quad \text{on} \quad \partial_F \Omega, \\ \mathbf{U} = \mathbf{u}^d \quad \text{on} \quad \partial_u \Omega, \\ [\![\mathbf{U}]\!] = h \left(\mathsf{L} \cdot (\overline{\boldsymbol{\Sigma}} \cdot \mathbf{e}_1) + \mathsf{M} : \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}}) \right) \quad \text{on} \quad \Gamma, \\ [\![\boldsymbol{\Sigma}]\!] \cdot \mathbf{e}_1 + h \, \mathbf{div}_{x'} \left(-(\overline{\boldsymbol{\Sigma}} \cdot \mathbf{e}_1) \cdot \mathsf{M} + \mathsf{N} : \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}}) \right) + h \left\langle \rho - \rho^* \right\rangle_{\mathbb{Y}} \mathbf{g} = \mathbf{0} \quad \text{on} \quad \Gamma. \end{cases} \tag{43}$$

Note that the jumps of $\mathbf{U}$ and $\boldsymbol{\Sigma} \cdot \mathbf{e}_1$ depend on the *mean values* of the normal stress $\overline{\boldsymbol{\Sigma}} \cdot \mathbf{e}_1$ and of the tangential strain at the interface $\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}})$, the latter being defined by

$$\boldsymbol{\varepsilon}_\Gamma(\overline{U}) = \left(\frac{\partial \overline{U}_\alpha}{\partial x_\beta} + \frac{\partial \overline{U}_\beta}{\partial x_\alpha} \right) \mathbf{e}_\alpha \otimes_s \mathbf{e}_\beta .$$

Remark 8 Some extra conditions could appear at the boundary $\partial\Gamma$ of Γ, like a linear density of forces, see Remark 5. Their study is outside the scope of the present paper, see [2] for their analysis in a particular case. However, there exists no extra conditions when $\partial\Gamma \subset \partial_u\Omega$ and $\mathbf{u}^d = \mathbf{0}$ in a neighborhood of $\partial\Gamma$.

Since $h = \eta H$, if we consider again that η is a small parameter, the solution of (43) depends on this parameter. After reintroducing explicitly this dependence and denoting the solution by $(\mathbf{U}^\eta, \boldsymbol{\Sigma}^\eta)$, let us assume that the solution can be expanded in terms of the powers of η as follows:

$$\mathbf{U}^\eta = \sum_{i=0}^{\infty} \eta^i \mathbf{U}^i, \qquad \boldsymbol{\Sigma}^\eta = \sum_{i=0}^{\infty} \eta^i \boldsymbol{\Sigma}^i .$$

Inserting these expansions into the set of equations above and identifying the different powers of η immediately show that the set of equations at order 0 are those of the outer problem of order 0, see (16), and in particular that $[\![\mathbf{U}^0]\!] = 0, [\![\boldsymbol{\Sigma}^0]\!] \cdot \mathbf{e}_1 = \mathbf{0}$ on Γ. That means that $\mathbf{U}^0 = \mathbf{u}^0$ and $\boldsymbol{\Sigma}^0 = \boldsymbol{\sigma}^0$. In the same way, since $\mathbf{u}^0$ and $\boldsymbol{\sigma}^0 \cdot \mathbf{e}_1$ are continuous across Γ and hence $\overline{\mathbf{u}^0} = \mathbf{u}^0$, $\overline{\boldsymbol{\sigma}^0} \cdot \mathbf{e}_1 = \boldsymbol{\sigma}^0 \cdot \mathbf{e}_1$, the set of equations at order 1 are those of the outer problem of order 1, see (30) and (40). Therefore, $\mathbf{U}^1 = \mathbf{u}^1$ and $\boldsymbol{\Sigma}^1 = \boldsymbol{\sigma}^1$. Thus, the pair $(\mathbf{U}^\eta, \boldsymbol{\Sigma}^\eta)$, solution of (43), admits the same expansion as the solution $(\mathbf{u}^\eta, \boldsymbol{\sigma}^\eta)$ of the real problem (1) up to order 1 in η. (In general, the subsequent terms of the expansions are different.) Accordingly, (43) can be seen as an *up to the order 1 effective problem.*

Remark 9 Note that the introduction of $\overline{\mathbf{U}}$ and $\overline{\boldsymbol{\Sigma}}$ in the system of equations (43) is somehow arbitrary: Replacing $(\overline{\mathbf{U}}, \overline{\boldsymbol{\Sigma}})$ by $(\mathbf{U}^+, \boldsymbol{\Sigma}^+)$ or $(\mathbf{U}^-, \boldsymbol{\Sigma}^-)$ in (43) still leads to an effective problem accurate up to order 1. In other words, there is no uniqueness of the *up to the order 1 effective problem*. The virtue of introducing $\overline{\mathbf{U}}$ and $\overline{\boldsymbol{\Sigma}}$ in the formulation is that it preserves a kind of symmetry between the two sides of Γ.

Remark 10 The advantages of this formulation are: (i) the problem is set in a homogeneous body (the matrix) cut by a surface where the fields can be discontinuous (like a cohesive crack), all the information relative to the heterogeneities are included in the transmission conditions on that interface; (ii) the problem can be solved in one step; (iii) the solution is not singular and is even more regular than the solution of the true problem as seen in the next section; (iv) the practical implementation of such a problem in a finite element code only requires to introduce the *ad hoc* interfacial elements; (v) the last advantage of this formulation is that it leads to finding a stationary point of an energetic functional, *i.e.* the problem (43) has a variational character; *but* the important drawback is that this energy functional is, in general, *non convex* because the positivity of the tensors L and N is not guaranteed, as we explain in the next section.

Remark 11 This formulation is similar to other effective problems formulated in very different contexts: In the field of viscous flow simulations, several authors were interested in formulating effective boundary conditions for rough boundaries, which gives rise to similar problems [3, 22]. In the domain of wave simulations, similar equations also appear in the formulation of ideal absorbing boundary conditions [17, 20]. These artificial boundary conditions are used to simulate infinite media with a low computational cost.

4.3 The variational formulation of the up to the order 1 effective problem

We show in this section that the problem (43) has a variational character, *i.e.* its solution is the stationary point of an energetic functional. An important step in this proof consists in the identification of this energy functional which contains a surface energy on Γ. The form of the variational formulation depends on whether the tensor L is invertible. Let us begin by the simplest case where L is invertible, the general case will be considered afterwards.

Let us assume that L is invertible and let us define the set $\mathcal{C}$ of kinematically admissible displacement fields by

$$\mathcal{C} = \left\{ \mathbf{w} = (w_1, w_2, w_3) \;\middle|\; \mathbf{w} \in H^1(\Omega \setminus \Gamma)^3,\ (\overline{w_2}, \overline{w_3}) \in H^1(\Gamma)^2,\ \mathbf{w} = \mathbf{u}^d \quad \text{on} \quad \partial_u \Omega \right\}, \tag{44}$$

where $H^1(D)$ denotes the Sobolev space of functions which belong to $L^2(D)$ and whose distributional first derivatives also belong to $L^2(D)$. The set $\mathcal{C}$ is an affine space whose associated linear space is denoted by $\mathcal{C}_0$ and consists in the fields with the same regularity than those of $\mathcal{C}$ and which satisfy $\mathbf{w} = \mathbf{0}$ on $\partial_u \Omega$. Let us introduce the potential energy $\mathcal{P}_{\text{eff}}$ as the functional defined on $\mathcal{C}$ by

$$\begin{aligned} \mathcal{P}_{\text{eff}}(\mathbf{w}) = & \frac{1}{2} \int_{\Omega \setminus \Gamma} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{w}) \, \mathrm{d}\mathbf{x} - \int_{\partial_F \Omega} \mathbf{F} \cdot \mathbf{w} \, \mathrm{dS} - \int_{\Omega} \rho^* \mathbf{g} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} - h \int_{\Gamma} \left\langle \rho - \rho^* \right\rangle_{\mathbb{Y}} \mathbf{g} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}' \\ & + \frac{h}{2} \int_{\Gamma} \left(\left(\frac{[\![\mathbf{w}]\!]}{h} - \mathsf{M} : \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}}) \right) \cdot \mathsf{L}^{-1} \cdot \left(\frac{[\![\mathbf{w}]\!]}{h} - \mathsf{M} : \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}}) \right) + \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}}) : \mathsf{N} : \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}}) \right) \mathrm{d}\mathbf{x}'. \end{aligned} \tag{45}$$

The regularity assumed for $\mathbf{w}$ is exactly what is necessary and sufficient for $\mathcal{P}_{\text{eff}}(\mathbf{w})$ to be finite. Specifically, $\mathbf{w}$ must belong to $H^1(\Omega \setminus \Gamma)^3$ so that the bulk elastic energy is finite. Accordingly, the traces $\mathbf{w}^+$ and

$\mathbf{w}^-$ associated with $\mathbf{w}$ on each side of the plane surface Γ belong to $H^{1/2}(\Gamma)$. Hence their mean value $\overline{\mathbf{w}}$ belongs also to $H^{1/2}(\Gamma)$. But this does not ensure that the surface energy remains finite, because of the term $\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}}):\mathsf{N}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})$. For that, it is necessary (in the case where N is definite) that $\boldsymbol{\varepsilon}_{\alpha\beta}(\overline{\mathbf{w}})$ belongs to $L^2(\Gamma)$ and hence that $\overline{v_\alpha}$ belongs to $H^1(\Gamma)$. The first three terms in the definition of $\mathcal{P}_{\text{eff}}$ are the usual ones in the definition of the potential energy for a (homogeneous) elastic body, while the last two are the "effective" contribution of the heterogeneities. In particular the last one seems to correspond to a surface elastic energy due to two contributions: one associated with a density of springs whose stiffness would be L^{-1} and the other associated with an elastic membrane whose stiffness tensor would be N; the main difference however is that these quantities are not necessarily positive since the signs of L and N are not known in advance and will depend on whether the inclusion is stiffer or softer than the matrix.

Let us prove the following fundamental result:

Proposition 2 *Let us assume that the boundary $\partial\Gamma$ of Γ is included in the part of the boundary which is fixed, i.e. $\partial\Gamma \subset \partial_u\Omega$ and $\mathbf{u}^d = \mathbf{0}$ in a neighborhood of $\partial\Gamma$, and let us assume that the tensor L is invertible. Let $\mathbf{U}$ be a kinematically admissible displacement field and $\boldsymbol{\Sigma}$ its associated stress field, i.e. $\mathbf{U} \in \mathcal{C}$ and $\boldsymbol{\Sigma} = \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{U})$. Then, the pair $(\mathbf{U}, \boldsymbol{\Sigma})$ is solution of the up to order 1 effective problem (43) if and only if $\mathbf{U}$ renders stationary the potential energy $\mathcal{P}_{\text{eff}}$ over $\mathcal{C}$.*

Proof The proof will be formal, the questions of regularity lying outside the scope of our paper. Let $(\mathbf{U}, \boldsymbol{\Sigma})$ be a solution of (43) with $\mathbf{U} \in \mathcal{C}$. We have to prove that $\mathcal{P}'_{\text{eff}}(\mathbf{U})(\mathbf{w}) = 0$ for all $\mathbf{w} \in \mathcal{C}_0$, $\mathcal{P}'_{\text{eff}}(\mathbf{U})$ denoting the Gâteaux derivative of $\mathcal{P}_{\text{eff}}$ at $\mathbf{U}$, *i.e.* the linear form defined on $\mathcal{C}_0$ by $\mathcal{P}'_{\text{eff}}(\mathbf{U})(\mathbf{w}) = \frac{d}{dh}\mathcal{P}_{\text{eff}}(\mathbf{U}+h\mathbf{w})|_{h=0}$. Accordingly, we have to prove the following variational equality

$$\begin{aligned} 0 = &\int_{\Omega\setminus\Gamma} \boldsymbol{\Sigma}:\boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}\mathbf{x} + h\int_\Gamma \left(\left(\frac{[\![\mathbf{U}]\!]}{h} - \mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}})\right)\cdot\mathsf{L}^{-1}\cdot\left(\frac{[\![\mathbf{w}]\!]}{h} - \mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})\right) + \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}}):\mathsf{N}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})\right)\mathrm{d}\mathbf{x}' \\ &- \int_{\partial_F\Omega} \mathbf{F}\cdot\mathbf{w}\,\mathrm{dS} - \int_\Omega \rho^*\mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} - h\int_\Gamma \langle\rho-\rho^*\rangle_{\mathbb{Y}}\,\mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}', \qquad \forall\mathbf{w}\in\mathcal{C}_0. \end{aligned} \tag{46}$$

Let $\mathbf{w} \in \mathcal{C}_0$, multiplying the equilibrium equation in (43) by $\mathbf{w}$ and integrating over $\Omega\setminus\Gamma$ lead to

$$0 = -\int_{\Omega\setminus\Gamma} (\mathbf{div}\boldsymbol{\Sigma} + \rho^*\mathbf{g})\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}.$$

After an integration by parts, taking into account that $\mathbf{w} = \mathbf{0}$ on $\partial_u\Omega$ and that $\boldsymbol{\Sigma} = \mathbf{F}$ on $\partial_F\Omega$ gives

$$0 = \int_{\Omega\setminus\Gamma} \boldsymbol{\Sigma}:\boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}\mathbf{x} - \int_{\partial_F\Omega} \mathbf{F}\cdot\mathbf{w}\,\mathrm{dS} - \int_\Omega \rho^*\mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} + \int_\Gamma [\![\mathbf{w}\cdot\boldsymbol{\Sigma}\cdot\mathbf{e}_1]\!]\,\mathrm{d}\mathbf{x}'. \tag{47}$$

Since the jump of a product can read as $[\![fg]\!] = [\![f]\!]\overline{g} + \overline{f}[\![g]\!]$, the integral over Γ in (47) becomes

$$\int_\Gamma [\![\mathbf{w}\cdot\boldsymbol{\Sigma}\cdot\mathbf{e}_1]\!]\,\mathrm{d}\mathbf{x}' = \int_\Gamma \left([\![\mathbf{w}]\!]\cdot\overline{\boldsymbol{\Sigma}}\cdot\mathbf{e}_1 + \overline{\mathbf{w}}\cdot[\![\boldsymbol{\Sigma}]\!]\cdot\mathbf{e}_1\right)\mathrm{d}\mathbf{x}'.$$

By virtue of the jump condition of $\boldsymbol{\Sigma}$ in (43), the second part of the integral becomes

$$\int_\Gamma \overline{\mathbf{w}}\cdot[\![\boldsymbol{\Sigma}]\!]\cdot\mathbf{e}_1\,\mathrm{d}\mathbf{x}' = -h\int_\Gamma \mathbf{div}_{x'}\left(-(\overline{\boldsymbol{\Sigma}}\cdot\mathbf{e}_1)\cdot\mathsf{M} + \mathsf{N}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}})\right)\cdot\overline{\mathbf{w}}\,\mathrm{d}\mathbf{x}' - h\int_\Gamma \langle\rho-\rho^*\rangle_{\mathbb{Y}}\,\mathbf{g}\cdot\overline{\mathbf{w}}\,\mathrm{d}\mathbf{x}'.$$

After an integration by parts of the divergence term, using the hypothesis that $\mathbf{w} = \mathbf{0}$ on $\partial_u\Omega$ and the symmetries of N gives

$$\int_\Gamma \overline{\mathbf{w}}\cdot[\![\boldsymbol{\Sigma}]\!]\cdot\mathbf{e}_1\,\mathrm{d}\mathbf{x}' = +h\int_\Gamma \left(-(\overline{\boldsymbol{\Sigma}}\cdot\mathbf{e}_1)\cdot\mathsf{M} + \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}}):\mathsf{N}\right):\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})\,\mathrm{d}\mathbf{x}' - h\int_\Gamma \left\langle \rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g}\cdot\overline{\mathbf{w}}\,\mathrm{d}\mathbf{x}'.$$

Then, (47) becomes

$$\begin{aligned} 0 = &\int_{\Omega\setminus\Gamma}\boldsymbol{\Sigma}:\boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}\mathbf{x} + h\int_\Gamma\left((\overline{\boldsymbol{\Sigma}}\cdot\mathbf{e}_1)\cdot\left(\frac{[\![\mathbf{w}]\!]}{h} - \mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})\right) + \boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}}):\mathsf{N}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})\right)\mathrm{d}\mathbf{x}' \\ &- \int_{\partial_F\Omega}\mathbf{F}\cdot\mathbf{w}\,\mathrm{d}\mathrm{S} - \int_\Omega \rho^*\mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} - h\int_\Gamma\left\langle\rho-\rho^*\right\rangle_{\mathbb{Y}}\mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}'. \end{aligned} \tag{48}$$

Since L is assumed to be invertible, the jump condition of $\mathbf{U}$ in (43) can read as

$$\overline{\boldsymbol{\Sigma}}\cdot\mathbf{e}_1 = \mathsf{L}^{-1}\cdot\left(\frac{[\![\mathbf{U}]\!]}{h} - \mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}})\right). \tag{49}$$

Inserting (49) into (48) gives (46).

The proof of the converse, *i.e.* if $(\mathbf{U},\boldsymbol{\Sigma})$ satisfies (46) then $(\mathbf{U},\boldsymbol{\Sigma})$ is solution of (43), does not present any difficulty and is left to the reader. □

Remark 12 The assumption that $\partial\Gamma \subset \partial_u\Omega$ plays an important role in the proof of the proposition above since it allows us to eliminate the boundary terms in the integration by parts of the divergence term over Γ. Otherwise a linear density of forces would appear on $\partial\Gamma$ and should be combined with the linear density of forces which exists in general in the formulation of the problem (43), see Remark 5.

Let us consider now the case when L is not invertible. Let $\operatorname{Ker}\mathsf{L}$ and $\operatorname{Im}\mathsf{L}$ be the kernel and the image of L. Since L is symmetric, the jump condition of $\mathbf{U}$ in (43) implies that $[\![\mathbf{U}]\!] - h\,\mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}})$ must belong to $\operatorname{Im}\mathsf{L}$ and hence be orthogonal to $\operatorname{Ker}\mathsf{L}$. This condition constitutes a kinematic condition which must be introduced in the definition of $\mathcal{C}$. Accordingly, $\mathcal{C}$ reads now

$$\mathcal{C} = \left\{\mathbf{w}\,\middle|\,\mathbf{w}\in H^1(\Omega\setminus\Gamma),\ (\overline{w_2},\overline{w_3})\in H^1(\Gamma),\ \mathbf{w}=\mathbf{u}^d \text{ on } \partial_u\Omega,\ [\![\mathbf{w}]\!] - h\,\mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{w}})\in\operatorname{Im}\mathsf{L} \text{ on } \Gamma\right\}.$$

The restriction $\tilde{\mathsf{L}}$ of L to the orthogonal of $\operatorname{Ker}\mathsf{L}$ is invertible and the jump condition of $\mathbf{U}$ in (43) can read

$$\overline{\boldsymbol{\Sigma}}\cdot\mathbf{e}_1 = \tilde{\mathsf{L}}^{-1}\cdot\left(\frac{[\![\mathbf{U}]\!]}{h} - \mathsf{M}:\boldsymbol{\varepsilon}_\Gamma(\overline{\mathbf{U}})\right).$$

Since all the proof of Proposition 2 is unchanged until (49), Proposition 2 remains valid, provided we replace L^{-1} by $\tilde{\mathsf{L}}^{-1}$ in the definition of $\mathcal{P}_{\text{eff}}$.

Remark 13 From the mathematical viewpoint, the fact that the tensors L and N can be negative (and in general one of the two tensors is really negative as it is shown in the examples presented in the next section) has several drastic consequences: (i) neither the existence nor the uniqueness of a stationary point for $\mathcal{P}_{\text{eff}}$ on $\mathcal{C}$ is ensured; (ii) a stationary point is not necessarily a minimizer for $\mathcal{P}_{\text{eff}}$ on $\mathcal{C}$, and a global minimum may not even exist. However, the mathematical analysis made in [11] in a simplified context suggests that, even though (43) is a saddle point problem, it generally admits one unique solution, except for a countable set of parameters of the model. The verification that this result remains true in our general context will be the subject of future works.

5 Examples

In all the examples considered below, the matrix and the inclusion are assumed isotropic with respective Lamé coefficients (λ^*, μ^*), $(\lambda^{\mathsf{i}}, \mu^{\mathsf{i}})$ and respective Young modulus and Poisson ratio (E^*, ν^*), $(E^{\mathsf{i}}, \nu^{\mathsf{i}})$. These coefficients satisfy the usual inequalities : $3\lambda^* + 2\mu^* > 0$, $\mu^* > 0$, $E^* > 0$, $-1 < \nu^* < 1/2$ for the matrix, $3\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}} \geq 0$, $\mu^{\mathsf{i}} \geq 0$, $E^{\mathsf{i}} \geq 0$, $-1 \leq \nu^{\mathsf{i}} \leq 1/2$ for the inclusion. The limit case $E^{\mathsf{i}} = 0$, corresponding to a void or a crack, can be studied in the same framework, provided the matrix of the body remains connected. Throughout this section, the difference between the value of an elastic coefficient in the inclusion and that in the matrix is denoted by a double bracket, for example $[\![\mu]\!] = \mu^{\mathsf{i}} - \mu^*$. By virtue of the isotropy hypothesis, the tensors $\mathbf{e}_1 \cdot \mathsf{A}^* \cdot \mathbf{e}_1$ and $\mathbf{S}^*$ read

$$\mathbf{e}_1 \cdot \mathsf{A}^* \cdot \mathbf{e}_1 = (\lambda^* + 2\mu^*)\mathbf{e}_1 \otimes \mathbf{e}_1 + \mu^*(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \quad \mathbf{S}^* = \frac{1}{\lambda^* + 2\mu^*}\mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{1}{\mu^*}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3).$$

5.1 Layer of constant thickness

Here the heterogeneity is due to a thin layer with constant thickness $h = \eta H$. Accordingly, the vectors of periodicity $(\mathbf{a}, \mathbf{b})$ are arbitrary and the elementary cell $\mathbb{Y}$ can be reduced to the real line $\mathbb{R}$, the heterogeneity lying in the interval $(-H/2, +H/2)$. In other words, the inner fields only depend on the y_1 coordinate and

$$(\lambda(y_1), \mu(y_1)) = \begin{cases} (\lambda^*, \mu^*) & \text{if} \quad |y_1| > H/2, \\ (\lambda^{\mathsf{i}}, \mu^{\mathsf{i}}) & \text{if} \quad |y_1| < H/2. \end{cases}$$

Our asymptotic analysis is valid if and only if $3\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}} > 0$ and $\mu^{\mathsf{i}} > 0$. The solutions of the 6 elementary problems (18) can be obtained in a closed form. Indeed, since $\mathbf{T}^{ij}$ depends only on y_1, we deduce from the equilibrium equation and the conditions at infinity that $T^{ij}_{k1} = 0$ for all (i, j, k) and everywhere in $\mathbb{Y}$. In the same way, one has $\varepsilon_{\alpha\beta}(\mathbf{V}^{ij}) = 0$ for all (i, j, α, β) and everywhere in $\mathbb{Y}$. Accordingly, the stress-strain relation gives $\boldsymbol{\varepsilon}(\mathbf{V}^{ij}) = \mathbf{0}$ in the matrix while in the layer one gets

$$\varepsilon_{11}(\mathbf{V}^{ij}) = -\frac{[\![\lambda + 2\mu]\!]}{\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}}} I^{ij}_{11} - \frac{[\![\lambda]\!]}{\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}}} I^{ij}_{\alpha\alpha}, \qquad \varepsilon_{\alpha 1}(\mathbf{V}^{ij}) = -\frac{[\![\mu]\!]}{\mu^{\mathsf{i}}} I^{ij}_{\alpha 1}, \qquad \text{in} \quad (-H/2, +H/2).$$

Inserting these values of $\boldsymbol{\varepsilon}(\mathbf{V}^{ij})$ into the definition (36) of D yields

$$D_{1111} = -\frac{\lambda^* + 2\mu^*}{\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}}}[\![\lambda + 2\mu]\!], \qquad D_{\alpha 1 \beta 1} = -\frac{\mu^*}{\mu^{\mathsf{i}}}[\![\mu]\!]\delta_{\alpha\beta},$$

$$D_{11\alpha\beta} = -[\![\lambda]\!]\frac{\lambda^* + 2\mu^*}{\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}}}\delta_{\alpha\beta}, \qquad D_{\alpha\beta\gamma\delta} = \left(\frac{[\![\lambda]\!]^2}{\lambda^{\mathsf{i}} + 2\mu^{\mathsf{i}}} - [\![\lambda]\!]\right)\delta_{\alpha\beta}\delta_{\gamma\delta} - [\![2\mu]\!] I_{\alpha\beta\gamma\delta},$$

where $I_{\alpha\beta\gamma\delta} = \frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ is the projection operator on the vector space of second order symmetric tensors. The other components of D are obtained by symmetry or vanish. By virtue of (41)-(42), the tensors L, M and N read as

$$\mathsf{L} = \left[\!\!\left[\frac{1}{\lambda + 2\mu} \right]\!\!\right] \mathbf{e}_1 \otimes \mathbf{e}_1 + \left[\!\!\left[\frac{1}{\mu} \right]\!\!\right] (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \tag{50}$$

$$\mathsf{M} = -\left[\!\!\left[\frac{\nu}{1 - \nu} \right]\!\!\right] \mathbf{e}_1 \otimes (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \tag{51}$$

$$N_{\alpha\beta\gamma\delta} = \left[\!\!\left[\frac{2\mu\lambda}{\lambda + 2\mu} \right]\!\!\right] \delta_{\alpha\beta}\delta_{\gamma\delta} + [\![2\mu]\!] I_{\alpha\beta\gamma\delta}, \tag{52}$$

The jumps of stress and displacement can be readily obtained from these relations by using equation (40). The tensor L is generally invertible, except in the particular cases where $[\![\lambda + 2\mu]\!] = 0$ or $[\![\mu]\!] = 0$. In the general case, the sign of L depends on the sign of these coefficients. The coupling tensor M only depends on the Poisson ratio of the materials, and vanishes when the matrix and the inclusion have the same Poisson ratio: the coupling between the membrane deformations and the jump of displacement is typically a Poisson effect. The sign of N is not obvious at this step. In the trivial case where the matrix and the inclusion have the same elastic moduli, the three tensors L, M and N vanish, which means that the jump of displacement and membrane stress are zero.

Let us consider the particular case when the matrix and the layer have the same Poisson ratio, *i.e.* when $\nu^* = \nu^{\mathsf{i}} = \nu \in (-1, \frac{1}{2})$. In this case, $\mathsf{M} = \mathsf{0}$ while the tensors L and N can read as

$$\mathsf{L} = \frac{(1+\nu)(1-2\nu)}{1-\nu} \left[\!\!\left[\frac{1}{E}\right]\!\!\right] \mathbf{e}_1 \otimes \mathbf{e}_1 + 2(1+\nu) \left[\!\!\left[\frac{1}{E}\right]\!\!\right] (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3),$$
$$N_{\alpha\beta\gamma\delta} = \frac{[\![E]\!]}{1+\nu} \left(I_{\alpha\beta\gamma\delta} + \frac{\nu}{1-\nu} \delta_{\alpha\beta}\delta_{\gamma\delta} \right).$$

In the latter expressions, L and N have opposite signs: if $[\![E]\!] > 0$, *i.e.* if the layer is stiffer than the matrix, N is positive while L is negative. Conversely, if the layer is softer than the matrix, L becomes positive while N is negative. Since the coercitivity of the interface energy depends on the positiveness of these two tensors, this result suggests that our interface model is generally not stable, as emphasized previously.

It is also interesting to show that our model is consistent with other models previously described in the literature for stiff or soft interfaces. This proof can only be formal, since the derivation of our model was based on the hypothesis that the material parameters were independent of η. Let us first consider the case when the inclusion is much stiffer than the matrix, *i.e.* when $E^{\mathsf{i}} \gg E^*$. We can therefore introduce the ratio $\theta = E^*/E^{\mathsf{i}}$ as a small parameter. If we introduce $E^{\mathsf{i}} = \theta^{-1} E^*$ in (50–52), we see that L and M remain bounded when $\theta \to 0$, while N is not bounded. The unbounded part of N is

$$N_{\alpha\beta\gamma\delta} = \theta^{-1} \frac{E^*}{1+\nu^{\mathsf{i}}} I_{\alpha\beta\gamma\delta} + \theta^{-1} \frac{E^* \nu^{\mathsf{i}}}{(1+\nu^{\mathsf{i}})(1-\nu^{\mathsf{i}})} \delta_{\alpha\beta}\delta_{\gamma\delta} + O_{\theta=0}(\theta^0).$$

In other words, the membrane stress becomes predominant over the displacement jump. Formally, if θ is of the order of η, the membrane stress appears at order 0 instead of order 1, and depends only on the membrane components of the deformation $\boldsymbol{\varepsilon}_\Gamma(\mathbf{u}^0)$. The jump of stress at order 0 is therefore given by:

$$[\![\boldsymbol{\sigma}^0]\!] \cdot \mathbf{e}_1 + h \, \mathbf{div}_{x'} \left(\frac{E^{\mathsf{i}}}{1+\nu^{\mathsf{i}}} \boldsymbol{\varepsilon}_\Gamma(\mathbf{u}^0) + \frac{E^{\mathsf{i}} \nu^{\mathsf{i}}}{(1+\nu^{\mathsf{i}})(1-\nu^{\mathsf{i}})} \mathbf{div}_{x'}(\mathbf{u}^0) \, (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \right) = \mathbf{0}.$$

In this case, we can construct a model which is *effective up to order 0 in* η, where the interface behaves like a membrane. Specifically, the effective energy (45) can read as

$$\mathcal{P}_{\text{eff}}(\mathbf{w}) = \frac{1}{2} \int_{\Omega\setminus\Gamma} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{w}) \, \mathrm{d}\mathbf{x} - \int_{\partial_F \Omega} \mathbf{F} \cdot \mathbf{w} \, \mathrm{dS} - \int_\Omega \rho^* \mathbf{g} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} - h \int_\Gamma \left\langle \rho - \rho^* \right\rangle_{\mathbb{Y}} \mathbf{g} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}'$$
$$+ \frac{h E^{\mathsf{i}}}{2(1+\nu^{\mathsf{i}})} \int_\Gamma \left(\boldsymbol{\varepsilon}_\Gamma(\mathbf{w}) : \boldsymbol{\varepsilon}_\Gamma(\mathbf{w}) + \frac{\nu^{\mathsf{i}}}{1-\nu^{\mathsf{i}}} (\mathbf{div}_{x'} \mathbf{w})^2 \right) \mathrm{d}\mathbf{x}'. \tag{53}$$

The displacements are continuous across Γ and the set of admissible displacements becomes

$$\mathcal{C} = \left\{ \mathbf{w} = (w_1, w_2, w_3) \;\middle|\; \mathbf{w} \in H^1(\Omega)^3, \; (w_2, w_3) \in H^1(\Gamma)^2, \; \mathbf{w} = \mathbf{u}^d \quad \text{on} \quad \partial_u \Omega \right\}. \tag{54}$$

This model is consistent with the so-called *stiff model* described in the literature [6, 7, 21], where the transmission conditions across the interface are of Ventcel's type.

Conversely, we can consider the case when the layer is much softer than the matrix. This corresponds to the case when $\theta \to \infty$. Following the previous method, when $\theta \to \infty$, M and N remain bounded while L is not bounded:

$$\mathsf{L} = \frac{\theta(1+\nu^{\mathsf{i}})(1-2\nu^{\mathsf{i}})}{E^*(1-\nu^{\mathsf{i}})}\ \mathbf{e}_1 \otimes \mathbf{e}_1 + \frac{2\theta(1+\nu^{\mathsf{i}})}{E^*}(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) + O_{\theta\to\infty}(\theta^0).$$

In other words, the jump of displacement become predominant over the membrane stress. Formally, if θ is of the order of η^{-1}, the jump of displacement appears at order 0 instead of order 1, and depends only on the force applied on the interface $\boldsymbol{\sigma}^0 \cdot \mathbf{e}_1$. The jump of displacement at order 0 is therefore given by:

$$[\![\mathbf{u}^0]\!] = h\left(\frac{\sigma^0_{11}}{\lambda^{\mathsf{i}}+2\mu^{\mathsf{i}}}\mathbf{e}_1 + \frac{\sigma^0_{12}}{\mu^{\mathsf{i}}}\mathbf{e}_2 + \frac{\sigma^0_{13}}{\mu^{\mathsf{i}}}\mathbf{e}_3\right).$$

In this case, we can construct a model which is *effective up to order 0 in* η, where the interface behaves like an elastic interface. Specifically, the effective energy (45) can read as

$$\begin{aligned}\mathcal{P}_{\text{eff}}(\mathbf{w}) =& \frac{1}{2}\int_{\Omega\setminus\Gamma} \boldsymbol{\varepsilon}(\mathbf{w}) : \mathsf{A}^* : \boldsymbol{\varepsilon}(\mathbf{w})\,\mathrm{d}\mathbf{x} - \int_{\partial_F\Omega} \mathbf{F}\cdot\mathbf{w}\,\mathrm{dS} - \int_\Omega \rho^*\mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x} - h\int_\Gamma \left\langle \rho - \rho^* \right\rangle_{\mathbb{Y}} \mathbf{g}\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}' \\ &+ \frac{1}{2h}\int_\Gamma \left((\lambda^{\mathsf{i}}+2\mu^{\mathsf{i}})[\![w_1]\!]^2 + \mu^{\mathsf{i}}[\![w_2]\!]^2 + \mu^{\mathsf{i}}[\![w_3]\!]^2\right)\mathrm{d}\mathbf{x}', \end{aligned} \tag{55}$$

while the space of admissible displacements becomes

$$\mathcal{C} = \left\{ \mathbf{w} = (w_1, w_2, w_3) \ \middle|\ \mathbf{w} \in H^1(\Omega\setminus\Gamma)^3,\ \mathbf{w} = \mathbf{u}^d \quad \text{on} \quad \partial_u\Omega \right\}. \tag{56}$$

This model is consistent with the so-called *soft model* described in the literature [6, 19], where the interface is replaced by a uniform density of springs. Let us note that in both limit cases the energy functional retrieves its positive definiteness, which was lost in the general case.

5.2 Square array of transverse penny-shaped cracks

The main interest of this formulation does not lie in the study of homogeneous layers: it is able to describe a much wider variety of localized defects. We can study for example the influence of penny-shaped cracks. These cracks lie in the plane $(\mathbf{e}_2, \mathbf{e}_3)$, their radius is rh, $0 \le r < 1/\sqrt{2}$, while the distance between two neighboring cracks is h. Thus, the periodicity vectors at the macroscale are $\mathbf{a} = h\mathbf{e}_2$ and $\mathbf{b} = h\mathbf{e}_3$. At the microscale, the elementary cell is $\mathbb{Y} = \mathbb{R}\times(-H/2, H/2)^2$ and contains a crack which is the disk $\mathcal{D} = \left\{(0, \mathbf{y}') \ \middle|\ \|\mathbf{y}'\| < rH\right\}$ with unit normal vector $\mathbf{e}_1$, cf. Figure 5. This case is treated in [29] and we simply recall here the main results. The 6 elementary problems (18) must be slightly modified because of the presence of the crack. In particular, the displacement fields $\mathbf{V}^{ij}$ are discontinuous across $\mathcal{D}$ and the stress vector $\mathbf{T}^{ij}\cdot\mathbf{e}_1$ is prescribed on the lips of the crack (see Remark 2). Accordingly the elementary problems read now

$$\begin{cases} \mathbf{T}^{ij} = \mathsf{A}^* : \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}) \quad \text{in} \quad \mathbb{Y}\setminus\mathcal{D}, \\ \mathbf{div}_y(\mathbf{T}^{ij}) = \mathbf{0} \quad \text{in} \quad \mathbb{Y}\setminus\mathcal{D}, \\ \mathbf{T}^{ij}\cdot\mathbf{e}_1 = -(\mathsf{A}^* : \mathbf{I}^{ij})\cdot\mathbf{e}_1 \quad \text{on} \quad \mathcal{D}, \\ \lim\limits_{y_1\to\pm\infty} \mathbf{T}^{ij}\cdot\mathbf{e}_1 = \mathbf{0}, \\ \mathbf{V}^{ij} \quad \text{and} \quad \mathbf{T}^{ij} \quad \text{are} \quad \mathbf{y}'-\text{periodic}. \end{cases} \tag{57}$$

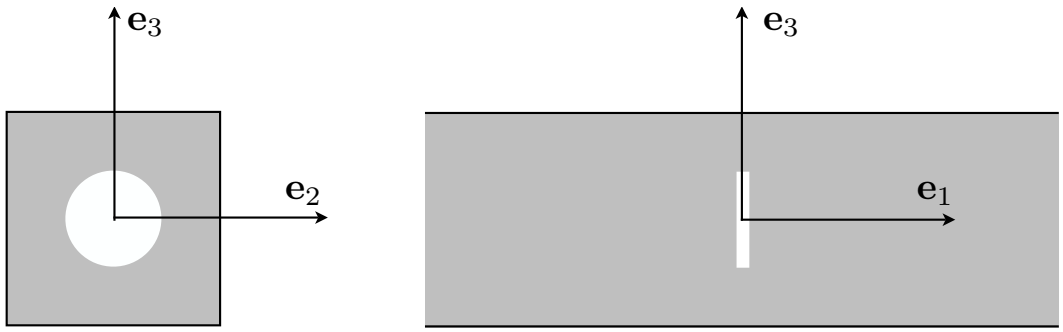

Fig. 5 Case when the defect is a penny-shaped crack in the plane $y_1 = 0$

The force applied on the lips of the crack is $-(\mathsf{A}^*\!:\!\mathbf{I}^{ij})\cdot\mathbf{e}_1 = -\lambda^*\delta_{ij}\mathbf{e}_1 - 2\mu^*\mathbf{I}^{ij}\cdot\mathbf{e}_1$. It therefore appears that $\mathbf{V}^{23} = \mathbf{V}^{32} = \mathbf{0}$, while $\mathbf{V}^{22} = \mathbf{V}^{33} = \lambda^*/(\lambda^* + 2\mu^*)\mathbf{V}^{11}$. The definition of the tensor D reads

$$\begin{aligned} D_{ijkl} &= \Big\langle \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij})\!:\!\mathsf{A}\!:\!\boldsymbol{\varepsilon}_y(\mathbf{V}^{kl})\Big\rangle_{\mathbb{Y}} - \Big\langle \mathbf{I}^{ij}\!:\!(\mathsf{A}-\mathsf{A}^*)\!:\!\mathbf{I}^{kl}\Big\rangle_{\mathbb{Y}} \\ &= \Big\langle \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij})\!:\!\mathsf{A}^*\!:\!\boldsymbol{\varepsilon}_y(\mathbf{V}^{kl})\Big\rangle_{\mathbb{Y}}. \end{aligned}$$

The tensor D is related to the effective damage compliance tensor $\tilde{\mathsf{D}}$ defined in [29] (it was denoted there by D) by $\mathsf{A}^*\!:\!\tilde{\mathsf{D}}\!:\!\mathsf{A}^* = \mathsf{D}$. By virtue of the relations above between the fields $\mathbf{V}^{ij}$ and owing to the symmetries of the elementary cell, the tensor D can be written as follows

$$\boldsymbol{\varepsilon}\!:\!\mathsf{D}\!:\!\boldsymbol{\varepsilon} = \frac{d_n}{E^*}\Big((\lambda^*+2\mu^*)\varepsilon_{11} + \lambda^*\varepsilon_{\alpha\alpha}\Big)^2 + \frac{d_m}{E^*}4{\mu^*}^2\varepsilon_{\alpha 1}\varepsilon_{\alpha 1}, \qquad \forall\boldsymbol{\varepsilon} = \big(\varepsilon_{ij}\big). \tag{58}$$

In (58), d_n and d_m are *positive* dimensionless coefficients which only depend on the parameter r characterizing the ratio between the radius of the cracks and their mutual distance. It can also be proved that d_n and d_m are increasing functions of r, growing from 0 to $+\infty$ when r grows from 0 to $1/\sqrt{2}$. (When $r = 1/\sqrt{2}$, the crack separates the elementary cell into two parts, the matrix is no more connected and our analysis is no more valid.)

Using Proposition 1 and inserting (58) into (41)-(42) yield

$$\mathsf{L} = \frac{d_n}{E^*}\mathbf{e}_1\otimes\mathbf{e}_1 + \frac{d_m}{E^*}(\mathbf{e}_2\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3), \qquad \mathsf{M} = 0, \qquad \mathsf{N} = 0.$$

Therefore, since $\langle\rho - \rho^*\rangle_{\mathbb{Y}} = 0$ (the volume of the void is zero), the jump conditions in (43) become

$$[\![\mathbf{U}]\!] = \frac{d_n h}{E^*}\Sigma_{11}\mathbf{e}_1 + \frac{d_m h}{E^*}\Sigma_{\alpha 1}\mathbf{e}_\alpha, \qquad [\![\boldsymbol{\Sigma}]\!]\cdot\mathbf{e}_1 = \mathbf{0}, \tag{59}$$

where h is the distance between two neighboring cracks. The jump conditions (59) correspond to Robin's transmission conditions, see [29]. In other words, this array of cracks behaves like a surface density of normal and tangential springs whose stiffness is equal to $E^*/d_n h$ and $E^*/d_m h$, respectively. We have obtained here the simplest effective behavior. That example could suggest that the effective behavior of planar cracks is always that of a surface density of springs. Other examples treated in [29], where one simply changes the orientation of the cracks, proves the converse.

The effective potential energy reads now

$$\mathcal{P}_{\text{eff}}(\mathbf{v}) = \frac{1}{2}\int_{\Omega\setminus\Gamma}\boldsymbol{\varepsilon}(\mathbf{v})\!:\!\mathsf{A}^*\!:\!\boldsymbol{\varepsilon}(\mathbf{v})\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Gamma\left(\frac{E^*}{d_n h}[\![v_1]\!]^2 + \frac{E^*}{d_m h}[\![v_\alpha]\!][\![v_\alpha]\!]\right)\mathrm{d}\mathbf{x}' - \int_{\partial_F\Omega}\mathbf{F}\cdot\mathbf{v}\,\mathrm{dS} - \int_\Omega\rho^*\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.$$

It is a (strictly) convex functional by virtue of the positivity of the coefficients d_n and d_m. Moreover, since M and N vanish, $\mathcal{P}_{\text{eff}}$ does not involve the membrane strain $\boldsymbol{\varepsilon}'$ on Γ, and $\mathcal{P}_{\text{eff}}(\mathbf{v})$ is finite if and only if $\mathbf{v}$ belongs to $H^1(\Omega \setminus \Gamma)^3$. Accordingly, the set of admissible displacement fields reads now

$$\mathcal{C} = \left\{ \mathbf{v} \;\middle|\; \mathbf{v} \in H^1(\Omega \setminus \Gamma)^3,\ \mathbf{v} = \mathbf{u}^d \quad \text{on} \quad \partial_u \Omega \right\}.$$

Since $\mathcal{P}_{\text{eff}}$ is coercive and (strictly) convex on $\mathcal{C}$, there exists a unique displacement field $\mathbf{U}$ which renders stationary $\mathcal{P}_{\text{eff}}$ on $\mathcal{C}$ (up to admissible rigid motions). Moreover, $\mathbf{U}$ is the global minimizer of $\mathcal{P}_{\text{eff}}$ on $\mathcal{C}$.

The effective behavior of the cracks is characterized by the dimensionless coefficients (d_n, d_m). To our knowledge, there is no analytical solution to the elementary problems (57). These coefficients must therefore be computed numerically, using for example the finite element method. Some precautions must however be taken. First, the solution of the elementary problems is only unique up to an arbitrary translation. To obtain a numerical solution, it is necessary to prescribe this uniform translation. This can be easily achieved by prescribing the displacement of one single node of the finite element mesh. Second, the elementary cell is theoretically unbounded in the $\mathbf{e}_1$ direction. However, since the stress correctors decay exponentially when y_1 goes to $\pm\infty$, the elementary cell can be artificially bounded in this direction without much influence on the numerical result. Some numerical investigations show that reducing the cell in the y_1 direction to $6H$, *i.e.* six times the size of the inclusion, is sufficient to obtain an accurate approximation of the solution.

We computed the values of (d_n, d_m) for several values of the radius of the cracks, ranging from 0 to $1/\sqrt{2}$. The Poisson ratio of the matrix was set to 0.3. The results are presented in the following tabular.

r	0.0	0.1	0.2	0.3	0.4	0.5	0.6
d_n	0.0	0.0042	0.037	0.134	0.355	0.96	2.25
d_m	0.0	0.0047	0.043	0.156	0.413	1.11	2.52

One can see that d_n and d_m are not very large when the crack is small, but that they grow very fast in the vicinity of $1/\sqrt{2} \approx 0.707$. This was expected since, when $r \to 1/\sqrt{2}$, the periodic array of microscopic cracks transforms into one macroscopic crack.

5.3 An hexagonal array of spherical holes

Here we analyse a second non-trivial example, where the heterogeneity is due to a periodic distribution of spherical voids in the plane $(\mathbf{e}_2, \mathbf{e}_3)$. The radius of a void is rh, $0 \le r < 1/\sqrt{3}$, $h = \eta H$ being the distance between two neighboring voids. The periodicity vectors at the macroscale are $\mathbf{a} = \ell \mathbf{e}_2$ and $\mathbf{b} = \ell(\mathbf{e}_2 + \sqrt{3}\mathbf{e}_3)/2$. At the microscale, we can choose for Y a perfect hexagon with center $\mathbf{0}$, see Figure 6. Accordingly, the elementary cell is $\mathbb{Y} = \mathbb{R} \times \mathrm{Y}$ and contains a void corresponding to the ball $\mathcal{D} = \{\mathbf{y} \,|\, \|\mathbf{y}\| < rH\}$. This case is also treated in [29] and we simply recall here the main results. The 6 elementary problems (18) must be modified because of the presence of the void. In particular, the displacement fields $\mathbf{V}^{ij}$ cannot be defined in $\mathcal{D}$, whereas the stress tensor $\mathbf{T}^{ij}$ can be extended in $\mathcal{D}$ so that the equilibrium equations hold in the whole

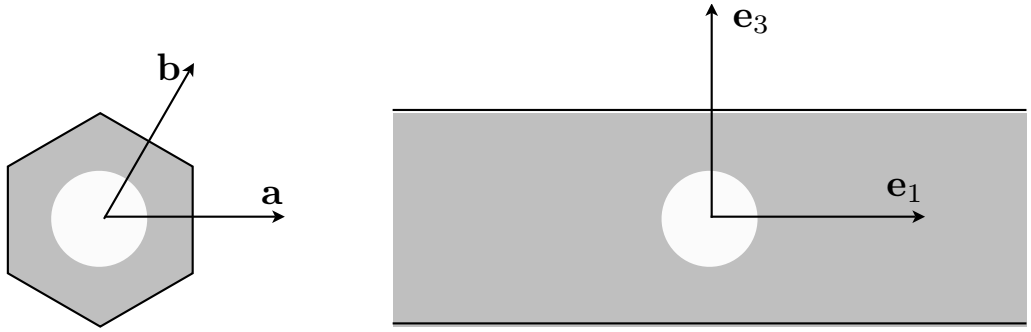

Fig. 6 Case when the defect is a spherical hole

cell $\mathbb{Y}$. Accordingly the elementary problems read now

$$\begin{cases} \mathbf{T}^{ij} = \mathsf{A}^* : \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}) \quad \text{in} \quad \mathbb{Y} \setminus \mathcal{D}, \\ \mathbf{T}^{ij} = -\mathsf{A}^* : \mathbf{I}^{ij} \quad \text{in} \quad \mathcal{D}, \\ \mathbf{div}_y(\mathbf{T}^{ij}) = \mathbf{0} \quad \text{in} \quad \mathbb{Y}, \\ \lim\limits_{y_1 \to \pm\infty} \mathbf{T}^{ij} \cdot \mathbf{e}_1 = \mathbf{0}, \\ \mathbf{V}^{ij} \quad \text{and} \quad \mathbf{T}^{ij} \quad \text{are} \quad \mathbf{y}'-\text{periodic}. \end{cases} \tag{60}$$

Unlike the previous example of planar cracks, none of the elementary problems are trivial. The definition of the tensor D reads

$$\begin{aligned} D_{ijkl} &= \left\langle \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}) : \mathsf{A} : \boldsymbol{\varepsilon}_y(\mathbf{V}^{kl}) \right\rangle_{\mathbb{Y}} - \left\langle \mathbf{I}^{ij} : (\mathsf{A} - \mathsf{A}^*) : \mathbf{I}^{kl} \right\rangle_{\mathbb{Y}} \\ &= \left\langle \mathbf{T}^{ij} : \mathsf{A}^{*-1} : \mathbf{T}^{kl} \right\rangle_{\mathbb{Y}}, \end{aligned}$$

where $\mathbf{T}^{ij}$ is extended inside the void by $-\mathsf{A}^* : \mathbf{I}^{ij}$ (see Remark 2). This tensor is still related to the effective damage compliance tensor $\tilde{\mathsf{D}}$ defined in [29] by $\mathsf{A}^* : \tilde{\mathsf{D}} : \mathsf{A}^* = \mathsf{D}$. Since the material is isotropic, the void is spherical and the periodic array is hexagonal, we can use the results of [27] to obtain that $\tilde{\mathsf{D}}$ and D are *positive transversely isotropic* fourth order tensors with axis $\mathbf{e}_1$. Therefore, D can be written

$$\boldsymbol{\varepsilon} : \mathsf{D} : \boldsymbol{\varepsilon} = \Delta_n E^* \varepsilon_{11}^2 + 2\delta_n E^* \varepsilon_{11}\varepsilon_{\alpha\alpha} + \delta_m E^* (\varepsilon_{22} + \varepsilon_{33})^2 + 2d_m E^* \varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta} + 4d_n E^* \varepsilon_{1\alpha}\varepsilon_{1\alpha}, \tag{61}$$

where the five dimensionless coefficients Δ_n, d_m, d_n, δ_m and δ_n are functions of r and ν^* which satisfy the following inequalities:

$$\Delta_n > 0, \quad d_m > 0, \quad d_n > 0, \quad \delta_m + d_m > 0, \quad (\delta_m + d_m)\Delta_n > \delta_n^2. \tag{62}$$

Inserting (61) into (41)–(42) gives the tensors L, M and N. The details of the calculations are left to the reader and we finally obtain

$$\begin{cases} \mathsf{L} = \dfrac{\Delta_n E^*}{(\lambda^* + 2\mu^*)^2} \mathbf{e}_1 \otimes \mathbf{e}_1 + \dfrac{d_n E^*}{\mu^{*2}} (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \\ \mathsf{M} = \left(\delta_n - \dfrac{\lambda^*}{\lambda^* + 2\mu^*} \Delta_n \right) \dfrac{E^*}{\lambda^* + 2\mu^*} \mathbf{e}_1 \otimes (\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3), \\ N_{\alpha\beta\kappa\zeta} = -2d_m E^* I_{\alpha\beta\kappa\zeta} - \left(\delta_m - \dfrac{2\lambda^* \delta_n}{\lambda^* + 2\mu^*} + \dfrac{\lambda^{*2} \Delta_n}{(\lambda^* + 2\mu^*)^2} \right) E^* \delta_{\alpha\beta}\delta_{\kappa\zeta}. \end{cases} \tag{63}$$

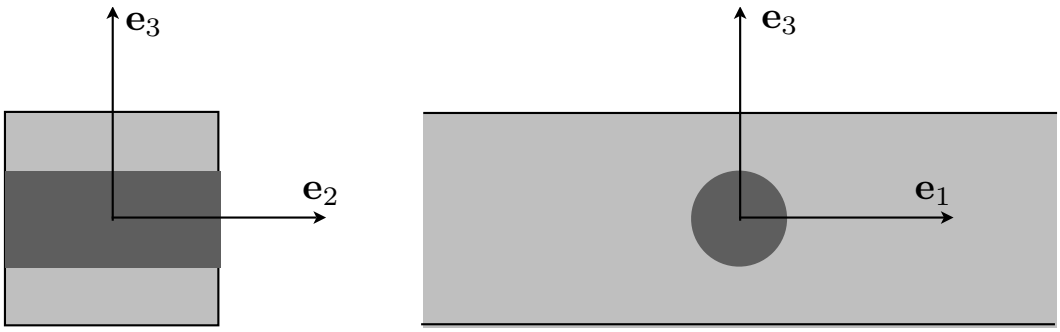

Fig. 7 Unidirectional reinforcements

By virtue of the positivity of the tensor D and hence of the inequalities (62), we immediately see that L is *positive* while N is *negative*. That means that in terms of the effective surface energy, the spring-like energy is positive while the membrane-like energy is negative as in the case of a soft layer. Note that the effective behavior of the holes cannot be assimilated to a surface density of springs and is hence strongly different from the effective behavior of transverse penny-shaped cracks.

These coefficients were numerically computed, in the case where the Poisson ratio of the matrix is 0.3, and the radius of the sphere is 3/8. The results are presented in the following tabular.

Parameter	Δ_n	d_m	d_n	δ_m	δ_n
Value	1.05	0.15	0.30	0.41	0.54

5.4 Unidirectional reinforcements

Finally, we can study the effective behavior of unidirectional reinforcements. These reinforcements are cylinders, with radius rh and axis $\mathbf{e}_2$. They are periodically distributed in the $\mathbf{e}_3$ direction, with a spacing h. The periodic vectors at the macro-scale are $\mathbf{a} = \alpha\mathbf{e}_2$ and $\mathbf{b} = h\mathbf{e}_3$, with α an arbitrary constant. At the microscopic level, the fields are therefore independent of y_2, and the elementary cell can be reduced to $\mathbb{Y} = \mathbb{R}\times(-H/2, H/2)$ in the plane $(\mathbf{e}_1, \mathbf{e}_3)$. The reinforcement cylinder is located in the domain $\mathcal{D} = \{(y_1, y_3) \mid \|\mathbf{y}\| < rH\}$, so we have

$$(\lambda(y_1,y_3),\mu(y_1,y_3)) = \begin{cases} (\lambda^*,\mu^*) & \text{if} \quad \mathbf{y}\notin\mathcal{D}, \\ (\lambda^{\mathsf{i}},\mu^{\mathsf{i}}) & \text{if} \quad \mathbf{y}\in\mathcal{D}. \end{cases}$$

Since there is no void in the volume, the elementary problems are the ones derived previously

$$\begin{cases} \mathbf{T}^{ij} = \mathsf{A}{:}\boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}) + (\mathsf{A}-\mathsf{A}^*){:}\mathbf{I}^{ij} \quad \text{in} \quad \mathbb{Y}, \\ \mathbf{div}_y(\mathbf{T}^{ij}) = \mathbf{0} \quad \text{in} \quad \mathbb{Y}, \\ \lim\limits_{y_1\to\pm\infty} \mathbf{T}^{ij}\cdot\mathbf{e}_1 = \mathbf{0}, \\ \mathbf{V}^{ij} \quad \text{and} \quad \mathbf{T}^{ij} \quad \text{are} \quad \mathbf{y}'-\text{periodic}, \end{cases}$$

and the tensor D is defined by

$$D_{ijkl} = \left\langle \boldsymbol{\varepsilon}_y(\mathbf{V}^{ij}) : \mathsf{A} : \boldsymbol{\varepsilon}_y(\mathbf{V}^{kl}) \right\rangle_{\mathbb{Y}} - \left\langle \mathbf{I}^{ij} : (\mathsf{A} - \mathsf{A}^*) : \mathbf{I}^{kl} \right\rangle_{\mathbb{Y}}.$$

Since the planes with unit normal vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are symmetry planes, the tensor D has only 9 independent components:

$$\mathsf{D} = c_{ij} E^* (\mathbf{e}_i \otimes \mathbf{e}_i) \otimes (\mathbf{e}_j \otimes \mathbf{e}_j) + 4\tau_\alpha E^* (\mathbf{e}_1 \otimes_s \mathbf{e}_\alpha) \otimes (\mathbf{e}_1 \otimes_s \mathbf{e}_\alpha) + 4\tau_m E^* (\mathbf{e}_2 \otimes_s \mathbf{e}_3) \otimes (\mathbf{e}_2 \otimes_s \mathbf{e}_3),$$

where c_{ij}, τ_α and τ_m are dimensionless coefficients which depend upon $(r, \theta, \nu^{\mathsf{i}}, \nu^*)$, and c_{ij} obeys a symmetry condition $c_{ij} = c_{ji}$. The tensors L, M and N can be computed in terms of these coefficients by using the formulas (41)–(42).

These coefficients were numerically computed in the case where $\theta = E^*/E^{\mathsf{i}} = 0.1$ and $\nu^{\mathsf{i}} = \nu^* = 0.3$, for two values of the relative radius r, and one obtains:

r	c_{11}	c_{22}	c_{33}	c_{12}	c_{13}	c_{23}	τ_2	τ_3	τ_m
1/20	-0.014	-0.07	-0.014	-0.0059	-0.0055	-0.0059	-0.0052	-0.0043	-0.0053
1/3	-0.54	-3.30	-0.87	-0.21	-0.18	-0.31	-0.17	-0.14	-0.32

Note that the tensor D is negative definite. Unsurprisingly, the bigger the fibers are, the more influence they have on the behavior of the structure. Moreover, one may notice that, when the relative radius of the fibers r is small, the interaction between the fibers is small. In this case, several coefficients are almost equal : $c_{11} \approx c_{33}$, $c_{12} \approx c_{23}$ and $\tau_2 \approx \tau_m$.

6 Conclusion and perspectives

We presented a comprehensive method to identify the effective behavior of periodic heterogeneities located on a plane surface. A wide variety of problems may be described with this approach, ranging from reinforcements and rigid inclusions to microcracks or cavities in solids. Preliminary works showed that the influence of these heterogeneities results in a correction of order 1 of the mechanical fields. The effective behavior of the interface is obtained by solving six elementary problems at the scale of a pattern. It is then necessary to solve twice the outer problem: the first time with an entirely homogeneous body, and the second by taking into account the jumps of displacement and stress due to the heterogeneities.

To simplify the implementation of this model, the order 0 and 1 problems are reformulated in an energetic framework by introducing an interface energy. This formulation simplifies the analysis of the effective behavior of the interface, along with its implementation in a finite element code. The general model couples elastic interface and membrane behaviors, and is thus able to exhibit most kinds of interfacial behaviors. This general model may also be reduced to simpler models when it is relevant, depending on the energies involved in the different phenomena. The main drawback of this model is that the interface energy is generally not positive definite, which fosters unphysical instabilities in the vicinity of the interface. This is a consequence from the fact that we identified the effective behavior of a zero-thickness interface. We believe that this problem could be solved by considering a finite thickness interface, for which the stiffness of the homogeneous volume would control the instabilities of the interface. This problem will be the subject of future works.

This approach can be easily extended to all kinds of linear differential equations of second order. This includes thermal diffusion, electrostatics, magnetism or elastodynamics. Note that, in the latter case, the wave length of the vibrations must be much larger than the size of the heterogeneities to avoid any diffraction effects. We also believe that this model may be generalized to take into account a wider variety of problems. We assumed in particular that the defects are distributed on a surface which is planar. This hypothesis could be removed by introducing differential geometry in the derivation of the model. However, the final model should be very similar. We also assumed that the materials on each side of the heterogeneities are the same, which may be untrue for interfacial cracks for example. Such problems could be treated in a very similar manner.

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