New events in stick slip oscillators behaviour

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A two-degree-of-freedom oscillator excited by dry friction is considered. The system consists of two masses connected by a linear spring, one of which is connected to a fixed wall by another spring. The second mass is in contact with a driving belt moving at a constant velocity. Coulomb's friction force acts between the mass and the belt. Periodic orbits including stick phases and slip phases, during which the mass in contact with the belt moves faster than the belt, are found analytically. The stability of these "overshooting" orbits is also investigated.

Vibrating systems excited by dry friction are strongly nonlinear, and they are usually modeled as spring-mass oscillators. They have been the subject of a large number of publications, in the scope of several friction models and mainly by a numerical approach[1]. However, assuming Coulomb's law of dry friction, the corresponding dynamical model is a piecewise linear system and, even for multi- degree-of-freedom cases, some analytical results on the existence and the stability of periodic orbits including stick-slip phases have been obtained[2].

One of the most popular models of a stick-slip oscillator consists of several masses connected by linear springs, some of the masses are in contact with a driving belt moving at a constant velocity. Dry friction forces act between the mass and the belt. In the past, several authors investigated the behaviour of this system, with different friction laws and with or without external actions and damping. The simplest case includes only one mass: this one-degree-of-freedom system has been the subject of both analytical[3] and numerical[4] investigations. An interesting phenomenon is the existence, in the periodic orbits with stick and slip parts, of an "overshooting" [5] slip phase. In this part of the orbit, the mass in contact with the belt moves faster than the belt. Up to now, this phenomenon has only been observed for more complex friction models than Coulomb's one, and for systems with external actions. Moreover, it is easy to prove[6] that these overshooting orbits are impossible in the case of a one-degree-of-freedom system with Coulomb friction. In this paper, it is shown that, for a two-degree-of-freedom stick-slip oscillator, periodic orbits with an overshooting part is possible, assuming Coulomb's law of dry friction. A set of such periodic orbits is obtained by analytical and numerical methods.

1. Statement of the problem

The system (Fig. 1) consists of two masses m_1 and m_2 connected by a linear spring k_2. The mass m_1 is connected to a fixed wall by another spring k_1. The second mass is in contact with a driving belt moving at a constant velocity v_0. A friction force F acts between the mass m_2 and the belt.

This two-degree-of-freedom oscillator is governed by the following differential system.

$$x_1'' + x_1 - \chi x_2 = 0, \quad x_2'' + \chi \eta (x_2 - x_1) = \eta u$$

$$\chi = \frac{k_2}{k_1 + k_2}, \quad \eta = \frac{m_1}{m_2}, \quad u = \frac{F}{k_1 + k_2} \tag{1.1}$$

The prime denotes differentiation with respect to the dimensionless time

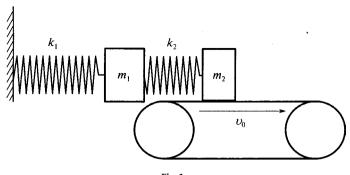$$t = \sqrt{(k_1 + k_2)/m_1}\, \tilde{t}$$

Fig. 1.

where $\tilde{t}$ is the dimensional time; x_1 and x_2 are the displacements of the masses m_1 and m_2. The friction force u is obtained from Coulomb's law

$$x_2' \neq V : \quad u = u_s \text{sign}(V - x_2') \quad \text{(slip motion)}$$

$$x_2' = V : \quad u = \begin{cases} \varepsilon u_s, & \varepsilon\chi(x_2 - x_1) > u_r, \quad \varepsilon = \pm 1 \quad \text{(slip motion)} \\ \chi(x_2 - x_1), & \chi|x_2 - x_1| < u_r \quad \text{(stick motion)} \end{cases}$$

$$V = \frac{\upsilon_0}{\sqrt{(k_1 + k_2)/m_1}}$$

(1.2)

where u_s is the dynamic friction force and u_r is the static friction force ($0 < u_s < u_r$). This model was investigated analytically in Ref.7 and a more complex model, where both masses m_1 and m_2 are in contact with the belt, is considered in Refs 2 and 8.

2. Slip and stick motions of the mass m_2

The dynamic behaviour of the dry friction oscillator under consideration is very complex and includes several phases of slip-stick motions of the mass m_2. For each kind of motion, the closed-form solution can be obtained.

The case $x_2' < V$ (slip motion). The solution is obtained from a modal analysis of system (1.1) where $u = u_s$:

$$Z(t) = H(t)Z_0; \quad Z = \begin{pmatrix} z \\ z' \end{pmatrix}, \quad Z_0 = Z(0), \quad z = X - d_0, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad d_0 = \begin{pmatrix} d_{01} \\ d_{02} \end{pmatrix}$$

$$H(t) = \begin{pmatrix} H_1(t) & H_2(t) \\ H_3(t) & H_1(t) \end{pmatrix}$$

(2.1)

The 2×2 matrices $H_i(t)$ (i=1,2,3) were obtained in analytical form[2], d_0 is the constant part of the solution.

The case $x_2' > V$ (slip overshooting motion). The solution is deduced from a modal analysis of system (1.1) where $u = -u_s$:

$$\bar{Z}(t) = H(t)\bar{Z}_0; \quad \bar{Z} = \begin{pmatrix} X + d_0 \\ X' \end{pmatrix}, \quad \bar{Z}_0 = \bar{Z}(0)$$

(2.2)

Taking into account the relation

$$\bar{Z} \equiv \begin{pmatrix} \bar{z} \\ \bar{z}' \end{pmatrix} = \begin{pmatrix} z \\ z' \end{pmatrix} + \begin{pmatrix} 2d_0 \\ 0 \end{pmatrix}$$

(2.3)

the solution for this overshooting slip motion is also written as

$$Z(t) = H(t)Z_0 + 2L(t)d_0; \quad L(t) = \begin{pmatrix} H_1(t) - I \\ H_3(t) \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(2.4)

The case $x_2' = V$ (stick motion). The motion is related to the following dynamical system

$$x_1'' + x_1 - \chi x_2 = 0, \quad x_2'' = 0$$

(2.5)

and was obtained in analytical form [2]

$$Z(t) = \Gamma(t)Z_0 \quad Z_0 = Z(0), \quad \Gamma(t) = \begin{pmatrix} \Gamma_1(t) & \Gamma_2(t) \\ \Gamma_3(t) & \Gamma_1(t) \end{pmatrix}$$

(2.6)

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The 2×2 matrices $\Gamma_i(t)\,(i=1, 2, 3)$ are derived from a modal analysis of system (2.5). Moreover, during all the oscillations of this kind, the following constraint holds

$$\chi|x_2 - x_1| < u_r \tag{2.7}$$

3. Periodic orbits with an overshooting phase

Several sets of periodic solutions including slip-stick motions of the system have been obtained in analytical form,[2,7,8] but for all these orbits, during the slip part, the velocity of the mass m_2 is less than the belt velocity. In what follows, a set of periodic orbits including a slip motion with an overshooting part, during which the mass m_2 moves faster than the belt, is obtained.

Let us assume that at $t=0$

$$x'_{20} = V, \quad \chi(x_{20} - x_{10}) = u_r \tag{3.1}$$

A motion with $u=u_s$ leads to $x''_2(0) < 0$, hence for $t>0$, x'_2 is a decreasing function of t and for some period of time $(0<t<\tau)$, the velocity of the mass m_2 is less than the belt velocity. The motion of the oscillator is given by relations (2.1). This motion terminates at $t=\tau$ if $x'_{2b} \equiv x'_2(\tau) = V$.

Let us assume that

$$\chi(x_{2b} - x_{1b}) < -u_r; \quad x_{ib} = x_i(\tau), \quad i = 1, 2 \tag{3.2}$$

For a new period of time $(\tau<t<\tau+\tau_1)$, the system undergoes an overshooting slip motion. During this phase, the mass m_2 moves faster than the belt. The solution is deduced from relation (2.2)

$$\bar{Z}(t) = H(t - \tau)\bar{Z}_b, \quad \bar{Z}_b = \bar{Z}(\tau) \tag{3.3}$$

A transition occurs at a time $t=\tau+\tau_1$ if $x'_{2c} \equiv x'_2(\tau+\tau_1) = V$.

Let us assume that

$$\chi|x_{2c} - x_{1c}| < u_r; \quad x_{ic} = x_i(\tau + \tau_1), \quad i = 1, 2 \tag{3.4}$$

For $t>\tau+\tau_1$, the system undergoes a stick motion defined by

$$Z(t) = \Gamma(t - \tau - \tau_1)Z_c, \quad Z_c = Z(\tau + \tau_1) \tag{3.5}$$

A periodic motion of period $T=\tau+\tau_1+\tau_2$ is obtained if the following condition holds

$$Z(T) = \Gamma(\tau_2)Z_c = Z_0 \tag{3.6}$$

This periodic solution depends on the four initial data $(x_{i0}, x'_{i0}, i = 1, 2)$ and on the three duration times (τ, τ_1, τ_2). Relations (3.1) and the conditions of transition between the different parts of motion

$$x'_2(\tau) = V, \quad x'_2(\tau + \tau_1) = V \tag{3.7}$$

give four scalar equations for determining these parameters; the periodicity conditions (3.6), taking into account the last condition in (3.7), leads only to three additional scalar equations for solving the problem. It yields that this overshooting periodic orbit can be found for any set of data $(\chi, \eta, u_s, u_r, V)$.

4. Stability of the periodic orbit with an overshooting phase

For the stability analysis, we shall use an approach[9] that consists of employing Poincare map modeling. The phase space is partitioned into different configurations $(\Phi_1, ..., \Phi_n)$ limited by several boundaries $(\Sigma_1, ..., \Sigma_n)$. At the boundary Σ_k between two adjacent configurations, the system switches from one configuration to another. The time t_k of switching is obtained from the corresponding switching condition. Within each configuration Φ_k, the solution $\varphi_k(t)$ is known. For a periodic orbit, the final switching surface Σ_n has to be identical to Σ_0. A Poincare map from Σ_0 to Σ_0 is defined by $P=P_n \circ \ldots \circ P_k \circ \ldots \circ P_1$, where P_k is a map from Σ_{k-1} to Σ_k defined by

$$Z(t_k) = \varphi_k(t_k - t_{k-1})Z(t_{k-1})$$

The fixed points of the map define the periodic solutions. The stability of the fixed points of the map depends on the eigenvalues of the Jacobian matrix J of the map. If one (at least) of these eigenvalues lies (in the complex plane) outside the unit circle, the fixed point of the map is unstable, and the corresponding periodic orbit is also unstable. If all the eigenvalues of J are inside the unit circle, the fixed point of the map is stable, and the related periodic solution will be also stable, but in a special sense. This stability is not Lyapunov stability because the duration time of a full turn of the perturbed motion is not equal to the period of the unperturbed periodic orbit. This stability is not the orbital stability as well, because, any perturbation in a stick motion, with decreasing $\dot{x}_2$ will change the structure of the motion for a slip mode. This stability has the following sense: for any small perturbations in the vicinity of the unperturbed trajectory, except the part of stick mode, the related phase point will remain in the vicinity of the unperturbed trajectory for any time.

Let us introduce small perturbations into the initial data z_{00} and z''_{00} related to the unperturbed orbit:

$$z_0 = z_{00} + dz_0, \quad z'_0 = z'_{00} + dz'_0, \quad dz_0 = (x, y)^t, \quad dz'_0 = (v, w)^t \tag{4.1}$$

The new initial conditions are assumed to lie in the map Σ_0 of the phase space defined by

$$\chi(z_2 - z_1) \equiv \chi(x_2 - x_1) - u_s = u_r - u_s, \quad z'_2 \equiv x'_2 = V \tag{4.2}$$

leading to the relations

$$x = y, \quad w = 0 \tag{4.3}$$

The first part of the perturbed motion is a slip motion with a velocity of the mass m_2 slower than the belt velocity and is obtained from relations (2.1). This motion terminates at $t = \tau + d\tau$ when the velocity of the mass m_2 reaches the belt velocity:

$$Z(\tau + d\tau) = H(\tau + d\tau)(Z_{00} + dZ_0)$$

Taking only linear terms in $d\tau$, we obtain

$$dZ_b = Z(\tau + d\tau) - Z(\tau) \approx H(\tau)dZ_0 + H'(\tau)Z_{00}d\tau \tag{4.4}$$

where

$$dZ_b = \begin{pmatrix} dz_b \\ dz_b' \end{pmatrix}, \quad dz_b = \begin{pmatrix} x_b \\ y_b \end{pmatrix}, \quad dz_b' = \begin{pmatrix} \upsilon_b \\ w_b \end{pmatrix} \tag{4.5}$$

The value of $d\tau$ is obtained from the condition

$$x_2'(\tau + d\tau) = V \Rightarrow w_b = 0 \tag{4.6}$$

From (4.4) and (4.6) we deduce

$$\begin{pmatrix} x_b \\ y_b \\ \upsilon_b \end{pmatrix} = J_b \begin{pmatrix} x \\ \upsilon \end{pmatrix}, \quad J_b = (J_{bij}), \quad i = 1, 2, 3, \quad j = 1, 2,$$

$$J_{bi1} = H_{i1} + H_{i2} - \kappa_i(H_{41} + H_{42}), \quad J_{bi2} = H_{i3} - \kappa_i H_{21}, \quad i = 1, 2, 3$$

$$H(\tau) = (H_{ij}), \quad i, j = 1, 2, 3, 4, \quad H_{33} = H_{11},$$

$$\kappa_1 = \frac{z_{1b}'}{\chi \eta (z_{1b} - z_{2b})}, \quad \kappa_2 = \frac{V}{z_{1b}'}\kappa_1, \quad \kappa_3 = \frac{\chi z_{2b} - z_{1b}}{z_{1b}'}\kappa_1, \quad z_{ib} = z_i(\tau), \quad i = 1, 2, \quad z_{ib}' = z_i'(\tau) \tag{4.7}$$

For small perturbations, condition (3.2) also holds for the perturbed motion, and for $t > \tau + d\tau$ the system undergoes an overshooting slip motion ($x_2' > V$) given by

$$Z(t) = H(t - \tau - d\tau)Z(\tau + d\tau) + 2L(t - \tau - d\tau)d_0 \tag{4.8}$$

This motion terminates at $t = \tau + d\tau + \tau_1 + d\tau_1$, when the velocity of the mass m_2 reaches the belt velocity:

$$x_2'(\tau + d\tau + \tau_1 + d\tau_1) = V \tag{4.9}$$

Taking only linear terms in (4.8), we obtain

$$dZ_c = Z(\tau + d\tau + \tau_1 + d\tau_1) - Z(\tau + \tau_1) \approx H(\tau_1)dZ_c + (H'(\tau_1)Z_b + 2L'(\tau_1)d_0)d\tau_1 \tag{4.10}$$

Components of the vector dZ_c are given by formulae (4.5) with the subscript b replaced by c.
The value of $d\tau_1$ is deduced from condition (4.9), i.e., from the equality $w_c = 0$, and from (4.10) we obtain the relation

$$\begin{pmatrix} x_c \\ y_c \\ \upsilon_c \end{pmatrix} = J_c \begin{pmatrix} x_b \\ y_b \\ \upsilon_b \end{pmatrix}, \quad J_c = (J_{cij}), \quad J_{cij} = h_{ij} - \kappa_{i+3}h_{4j}, \quad i, j = 1, 2, 3$$

$$H(\tau_2) = (h_{ij}) \quad i, j = 1, 2, 3, 4, \quad h_{33} = h_{11}, \quad \kappa_4 = \frac{z_{1c}'}{\chi \eta (z_{1c} - z_{2c}) - 2\eta u_s}$$

$$\kappa_5 = \frac{V}{z_{1c}'}\kappa_4, \quad \kappa_6 = \frac{\chi z_{2c} - z_{1c}}{z_{1c}'}\kappa_4, \quad z_{ic} = z_i(\tau + \tau_1), \quad i = 1, 2, \quad z_{1c}' = z_i'(\tau + \tau_1) \tag{4.11}$$

For small perturbations, condition (3.4) also holds for the perturbed motion, and for $t > \tau + d\tau + \tau_1 + d\tau_1$ the oscillator undergoes a stick motion defined by
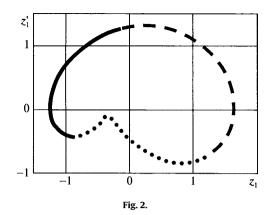
$$Z(t) = \Gamma(t - \tau - d\tau - \tau_1 - d\tau_1)Z(\tau + d\tau + \tau_1 + d\tau_1) \tag{4.12}$$

The motion terminates at $t = T + dT(dT = d\tau + d\tau_1 + d\tau_2)$, when the perturbed motion intersects the initial map Σ_0:

$$dZ_f = Z(T + dT) - Z(T) = \Gamma(\tau_2 + d\tau_2)(Z_c + dZ_c) - \Gamma(\tau_2)Z_c \tag{4.13}$$

The components of the vector dZ_f are given by formulae (4.5) with the subscript b replaced by f.
Taking into account definition (4.2) of the map Σ_0, we have

$$x_f = y_f, \quad w_f = 0 \tag{4.14}$$

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Fig. 2.

From (4.13) and (4.14), we deduce

$$\begin{pmatrix} x_f \\ v_f \end{pmatrix} = J_f \begin{pmatrix} x_c \\ y_c \\ v_c \end{pmatrix}, \quad J_f = \begin{pmatrix} \gamma_{11}(1-\kappa_7) & \gamma_{12}+\kappa_7(1-\gamma_{12}) & \gamma_{13}+\kappa_7(\gamma_{23}-\gamma_{13}) \\ \gamma_{31}-\kappa_8\gamma_{11} & \gamma_{32}+\kappa_8(1-\gamma_{12}) & \gamma_{11}+\kappa_8(\gamma_{23}-\gamma_{13}) \end{pmatrix}$$

$$\Gamma(\tau_2) = (\gamma_{ij}), \quad i,j = 1,2,3; \quad \kappa_7 = \frac{z'_{10}}{z'_{10}-V}, \quad \kappa_8 = \frac{\chi z_{20}-z_{10}}{z'_{10}-V}$$

$$z_{i0} = z_i(0), \quad i = 1,2, \quad z'_{10} = z'_1(0) \tag{4.15}$$

The final perturbations are obtained in terms of the initial ones by the linear relation

$$\begin{pmatrix} x_f \\ v_f \end{pmatrix} = J \begin{pmatrix} x \\ v \end{pmatrix}, \quad J = J_f J_c J_b \tag{4.16}$$

The stability of the periodic solution depends on the eigenvalues of the Jacobian matrix J.

5. Numerical results

Periodicity condition (3.6) gives the equations

$$(N_1 - I)z_0 + N_2 z'_0 + 2l_1 d_0 = 0, \quad N_3 z_0 + (N_4 - I)z'_0 + 2l_2 d_0 = 0 \tag{5.1}$$

$$N \equiv \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix} = \Gamma(\tau_2)H(\tau_1)H(\tau) = \Gamma(\tau_2)H(\tau_1+\tau), \quad l \equiv \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \Gamma(\tau_2)L(\tau_1)$$

This system provides the values of z_0 and z'_0 in terms of τ, τ_1, τ_2. Substituting these results into condition (3.1) and the first condition of (3.7), we obtain three equations for determining τ, τ_1, τ_2. However, in order to obtain a realistic solution, the corresponding orbit must meet condition (3.2) for $t=\tau$ and constraint (2.7) for $\tau+\tau_1 < t < \tau+\tau_1+\tau_2$. Moreover, it is easy to prove that the solution only depends on the ratio u_r/V (or u_s/V), leading to the assumption $V=1$.

For the following data set T:

$$\chi = 0.2, \quad \eta = 4, \quad V = 1, \quad u_r = 0.6178, \quad u_s = 0.0724$$

we obtain

$$\tau = 4.5, \quad \tau_1 = 2.717, \quad \tau_2 = 3.0536, \quad z_{10} = 1.1844, \quad z_{20} = 3.9116, \quad z'_{10} = -0.8075$$

In the phase portraits of the system (Figs. 2 and 3), the solid curves are related to the overshooting part of the motion, the dashed curves are related to stick motion and the dotted curves show the slip part of it.

Constraints (2.7):

$$F_1 \equiv \chi(z_2 - z_1) + u_r + u_s > 0, \quad F_2 \equiv \chi(z_2 - z_1) - u_r + u_s \le 0,$$
$$0 < t' < \tau_2 \quad (t' = t - \tau - \tau_1)$$

are shown in Fig. 4.

This solution is (conditionally) stable, the eigenvalues of the related Jacobian matrix J are $\lambda_{1,2} = 0.1705 \pm i0.0821$.

Several other solutions can be obtained for another set of data, involving stable or unstable orbits. However, this new kind of periodic orbits is obtained for a rather high ratio u_r/u_s; the solution shown in Figs. 2 and 3 relates[5] to the ratio $u_r/u_s = 8.54$.
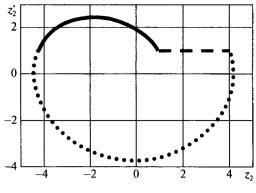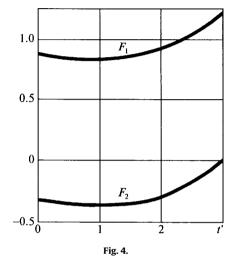
Fig. 3.

Fig. 4.

6. Concluding remarks

A new set of periodic orbits for a two-degree-of-freedom system excited by dry friction has been obtained. In the past, this kind of periodic orbits has only been observed for friction characteristics more complex than Coulomb's one. Moreover, we have proved that this so called "overshooting" phenomenon is impossible for a one-degree-of-freedom system excited by dry Coulomb friction. It has been demonstrated that the behaviour of a multi-degree-of-freedom system differs quite substantially from the behaviour of a one-degree-of-freedom system.

Acknowledgement

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