Idempotent conjunctive combination of belief functions: Extending the minimum rule of possibility theory.

Sébastien Destercke, Didier Dubois

HAL Id: hal-00651875
https://hal.science/hal-00651875

Submitted on 14 Dec 2011

Idempotent Conjunctive Combination of Belief Functions: Extending the Minimum Rule of Possibility Theory

S. Destercke[a,*], D. Dubois[b]

[a]*Centre de cooperation internationale en recherche agronomique pour le developpement (CIRAD), UMR IATE, Campus Supagro, Montpellier, France*
[b]*Institut de Recherche en Informatique de Toulouse (IRIT), 118 Route de Narbonne, 31400 Toulouse*

Abstract

When conjunctively merging two belief functions concerning a single variable but coming from different sources, Dempster rule of combination is justified only when information sources can be considered as independent. When dependencies between sources are ill-known, it is usual to require the property of idempotence for the merging of belief functions, as this property captures the possible redundancy of dependent sources. To study idempotent merging, different strategies can be followed. One strategy is to rely on idempotent rules used in either more general or more specific frameworks and to study, respectively, their particularisation or extension to belief functions. In this paper, we study the feasibility of extending the idempotent fusion rule of possibility theory (the minimum) to belief functions. We first investigate how comparisons of information content, in the form of inclusion and least-commitment, can be exploited to relate idempotent merging in possibility theory to evidence theory. We reach the conclusion that unless we accept the idea that the result of the fusion process can be a family of belief functions, such an extension is not always possible. As handling such families seems impractical, we then turn our attention to a more quantitative criterion and consider those combinations that maximise the expected cardinality of the joint belief functions, among the least committed ones, taking advantage of the fact that the expected cardinality of a belief function only depends on its contour function.

Keywords: belief functions, least commitment, idempotence, ill-known dependencies, contour function

1. Introduction

To-date, there exist many fusion rules in the theory of belief functions [34]. When conjunctively merging sources (assuming that they are reliable), the most usual one is still Dempster's rule of combination [5], either normalised or not. When several sources deliver information over a common frame of discernment $\mathcal{V}$, combining belief functions by Dempster's rule is justified only when the sources can be assumed to be independent. However, such an assumption cannot always be made. Sometimes, a specific dependence structure between sources can be assumed (or known), and merging rules corresponding to such structures can then be used [11, 19, 31] (for example, an assumption of complete positive or negative correlation between the confidence levels in the correctness of sources).

However, assuming that the (in)dependence structure between sources is well-known is often unrealistic. In those cases, an alternative is to adopt a conservative approach when merging belief functions (i.e., by adding no more extra information nor assumption about source relationship in the combination process than necessary). Adopting such a cautious attitude is equivalent to applying the "least commitment principle", which informally states that one should never presuppose more beliefs than justified. This principle is

---

*Corresponding author
Email addresses:* `desterck@irit.fr` (S. Destercke), `dubois@irit.fr` (D. Dubois)

basic in the frameworks of possibility theory (minimal specificity), imprecise probability (the idea of natural extension) [36], and the Transferable Belief Model (TBM) [35]. It can be naturally exploited for the cautious merging of belief functions.

This cautious approach can be interpreted and used in different ways. For instance, Denoeux [8] proposes a cautious conjunctive rule of combination based on Smets canonical decomposition of a belief function [33]. Cattaneo [2] proposes to first consider the set of merged belief functions minimizing the resulting conflict (hence maximizing the result coherence) and then to select the most cautious one among them. Recently [10], we have proposed to define a belief function resulting from a cautious merging process by maximizing the expected cardinality of its focal sets. Although different, all these approaches agree on the fact that a cautious conjunctive merging should satisfy the property of idempotence, as this property ensures that the same information supplied by two possibly dependent sources will remain unchanged after merging (i.e., our beliefs are not modified if two possibly dependent sources provide the same information about a common event or variable).

There are three main strategies to investigate idempotent merging that makes sense in the belief function setting. The first one looks for idempotent rules that satisfy a certain number of desired properties and appear sensible in the framework of belief functions. This is the solution retained by Denoeux [8] and Cattaneo [2,3]. The second strategy relies on the natural idempotent rule consisting of intersecting sets of probabilities and tries to express it in the particular case of belief functions (Chateauneuf [4]). Finally, the third approach, explored in this paper, starts from the natural idempotent rule in a less general framework, possibility theory, trying to extend it to belief functions. If $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ stand for two belief functions, $\mathcal{P}_1, \mathcal{P}_2$ two convex sets of probabilities, and π_1, π_2 two possibility distributions, the three approaches are summarised in Figure 1 below.
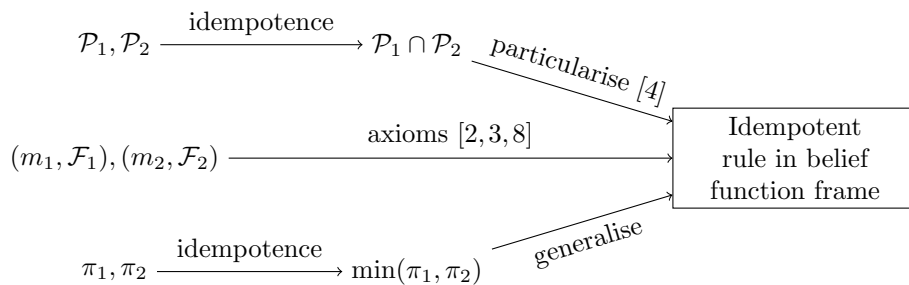

Figure 1: Search of idempotent merging rules

Viewing possibility distributions as contour functions of consonant belief functions [30], we propose various expressions of the principle of "least commitment" so as to extend the minimum rule of possibility theory to general belief functions (so that the latter be recovered when particularised to consonant belief functions). This is similar to the unnormalised Dempster rule (also called TBM conjunctive rule) that performs the product of contour functions (without maintaining consonance). Least commitment principles rely on the use of various inclusion relations that compare information content of belief functions. As this principle often leads to a solution consisting of sets of merged belief structures instead of a single one, the second part of the paper explores the maximisation of expected cardinality as a possible practical refinement that can help in the selection of one belief structure in this set.

Section 2 recalls basics of belief functions and defines conjunctive merging in this framework. Section 3 then studies to what extent the minimum rule of possibility theory can be extended to the framework of belief functions. The idea is to request that the contour function after merging be the minimum of the contour functions of the input belief functions, what we call the *strong idempotent contour function merging principle*. In Section 4, we are led to propose a weak version of the idempotent contour function merging principle, as the former condition turns out to be too strong. This part of the paper extends some previous preliminary results [9]. Section 5 then studies the maximisation of expected cardinality as a simple computational tool to select minimally committed merged belief structures. The notion of commensurate

2

belief functions is used to gain insight into the structure of focal element combinations allowing to reach a maximal expected cardinality. Finally, Section 6 compares the cautious approach considered in this paper with other propositions found in the literature.

2. Preliminaries

This section recalls basic notions of evidence theory needed in this paper, as well as links between belief functions and with other frameworks (possibility theory [13], convex sets of probabilities [23]). It also provides an overview of definitions of relative amount of precision based on generalised inclusions. Then we propose a general definition of conjunctive merging of belief functions, and we recall a result stating that the minimum rule of possibility theory obeys a minimal commitment principle in the theory of evidence, restricted to consonant belief functions.

In the whole paper, we consider that information pertains to a variable V taking its values on a finite space $\mathcal{V}$, with generic element denoted v.

2.1. Belief functions

Here, we assume that belief states are modelled by so-called basic belief assignments determining belief structures.

Definition 1 (Basic belief assignment). *A basic belief assignment (bba) is a function m from the power set $2^{\mathcal{V}}$ of $\mathcal{V}$ to $[0,1]$ such that $\sum_{E \subseteq \mathcal{V}} m(E) = 1$.*

Formally, a bba is a random set [25].[1] We denote by $\mathcal{M}_{\mathcal{V}}$ the set of bba's on $2^{\mathcal{V}}$. A set E such that $m(E) > 0$ is called a focal set. We denote by $\mathcal{F}$ the set of focal sets corresponding to bba m, and $(m, \mathcal{F})$ a belief structure. The value $m(E) > 0$ is the mass assigned to E. This value represents the probability that the statement $v \in E$ is a correct model of the available knowledge about v. It is the probability of knowing only that $v \in E$. In opposition to the probability of $A \subset \mathcal{V}, P(A)$, that refers to the occurrence of an event A, the quantity $m(E)$ refers to the arrival of the message stating $v \in E$. Shafer [30] assumes $m(\emptyset) = 0$, a condition ruling out inconsistent information from the bba, while Smets [35] does not make this assumption, $m(\emptyset)$ expressing the possibility of values outside $\mathcal{V}$.

Given a bba m, belief, plausibility and commonality functions of an event $E \subseteq \mathcal{V}$ are, respectively,

$$
\begin{aligned}
bel(A) &= \sum_{\emptyset \neq E \subseteq A} m(E), \\
pl(A) &= \sum_{E : A \cap E \neq \emptyset} m(E) = 1 - bel(E^c) - m(\emptyset), \\
q(A) &= \sum_{E \supseteq A} m(E).
\end{aligned}
$$

When a bba m is unnormalised, the implicability function of an event $E \subseteq \mathcal{V}$ is defined as

$$
b(A) = bel(A) + m(\emptyset) = 1 - pl(A^c),
$$

and it coincides with the belief function when m is normalised.

A belief function measures to what extent an event is directly supported by the available non-contradictory information, while a plausibility function measures the maximal amount of evidence that could support a

---

[1]Note that there are two interpretations of random sets, one where the focal sets are conjunctions of elements viewed as actual entities like regions in an area (Matheron [24]), and the other where they represent incomplete information about a precise but unknown element. We adopt the latter view here, along with Shafer and Smets.

given event. A commonality function measures the quantity of mass that may be re-allocated to a particular set from its supersets. The commonality function increases when larger focal sets receive greater mass assignments, hence the greater the commonality degrees, the less informative is the belief function. It can be shown [30] that any of the four representations, namely bbas, belief, plausibility and commonality functions contains the same amount of information. That is, from the (complete) knowledge of any of them, all the others can be retrieved by bijective transformations.

Note that in Shafer's seminal work [30], extensively taken over by Smets in his Transferable Belief Model [35], there are no references to any underlying probabilistic interpretation or framework. However, when $m(\emptyset) = 0$, a belief structure $(m, \mathcal{F})$ can also be mathematically represented by a non-empty convex set of probabilities [36] $\mathcal{P}_{(m,\mathcal{F})}$ such that $Bel(A)$ and $Pl(A)$ are coherent probability bounds:

$$\mathcal{P}_{(m,\mathcal{F})} = \{P | \forall A \subset \mathcal{V}, \ Bel(A) \leq P(A) \leq Pl(A)\}.$$

Namely, $Bel(A) = \inf_{P \in \mathcal{P}_{(m,\mathcal{F})}} P(A)$ and $Pl(A) = \sup_{P \in \mathcal{P}_{(m,\mathcal{F})}} P(A)$. Classical probability distributions are retrieved when only singletons receive positive masses. However for Walley as well as Smets there is no "real" probabilistic model underlying the probability bounds. The interpretation of Shafer-style upper and lower probabilities as an ill-known probabilistic model is closer to Dempster's view [5]

2.2. Possibility distributions and contour functions

A possibility distribution [13, 39] is a mapping $\pi : \mathcal{V} \to [0, 1]$. π is normalised when $\pi(v) = 1$ for at least one element $v \in \mathcal{V}$. It represents incomplete information about v, in the form of a fuzzy set of more or less plausible values for v. Two dual functions (respectively the possibility and necessity function) can be defined from π: $\Pi(A) = \sup_{v \in A} \pi(v)$ and $N(A) = 1 - \Pi(A^c)$. Their characteristic properties are that, for any pair $A, B \subseteq \mathcal{V}$, we have:

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)); \quad N(A \cap B) = \min(N(A), N(B)).$$

A contour function of a belief structure $(m, \mathcal{F})$ is defined as follows

Definition 2. *The contour function π_m of a belief structure $(m, \mathcal{F})$ is a mapping $\pi_{(m,\mathcal{F})} : \mathcal{V} \to [0, 1]$ such that, for any $v \in \mathcal{V}$,*

$$\pi_{(m,\mathcal{F})}(v) = \sum_{v \in E} m(E) = pl(\{v\}) = q(\{v\}),$$

with pl, q the plausibility and commonality functions[2] of $(m, \mathcal{F})$, respectively.

A belief structure $(m, \mathcal{F})$ is called consonant when its focal sets are completely ordered with respect to inclusion (that is, for any $A, B \in \mathcal{F}$, we have either $A \subset B$ or $B \subset A$). In this case, the plausibility and belief functions have the characteristic properties of, respectively, possibility and necessity functions, and the information contained in the consonant belief structure can be totally encoded by the possibility distribution identical to the contour function $\pi_{(m,\mathcal{F})}$.

Conversely, any possibility distribution defines a unique consonant belief structure whose associated plausibility (resp. belief) function is a possibility (resp. necessity) function. If π is a possibility distribution, and if $0 = \alpha_0 \leq \alpha_1 \leq \ldots \leq \ldots \alpha_M \leq 1$ is the (finite) set of distinct values assumed by π over $\mathcal{V}$, then the corresponding belief structure $(m_\pi, \mathcal{F}_\pi)$ has, for $i = 1, \ldots, M$, the following M focal sets:

$$\begin{cases} E_i = \{v \in X | \pi(v) \geq \alpha_i\} \\ m_\pi(E_i) = \alpha_i - \alpha_{i-1}, \end{cases} \tag{1}$$

with $m_\pi(E_i)$ the mass given to E_i, i.e. $\pi(v) = \sum_{v \in E} m_\pi(E)$.

For any belief structure $(m, \mathcal{F})$, the contour function can be seen as a (possibly subnormalised) possibility distribution, and is a trace of the whole belief structure $(m, \mathcal{F})$ restricted to singletons. Except when $(m, \mathcal{F})$ is consonant, the contour function represents only part of the information contained in $(m, \mathcal{F})$, and the possibility measure $\Pi_{(m,\mathcal{F})}$ built from $\pi_{(m,\mathcal{F})}$ is an inner approximation to the belief function ($\Pi_{(m,\mathcal{F})} \leq pl$) [16]. The contour function is therefore only a summary, easier to manipulate than the whole random set.

---

[2]Note that the equality between pl and q on singletons always holds.

2.3. Inclusion and information orderings between belief functions

Inclusion relationships are natural tools to compare the informative contents of set-valued uncertainty representations. Recall that a sure statement of the form $v \in E$ can be represented by the belief structure $(m, \{E\})$ such that $m(E) = 1$ or by the possibility distribution $\pi(v) = 1$ if $v \in E$, zero otherwise. Let $E_1, E_2 \subseteq \mathcal{V}$ be two sets, $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ the corresponding belief structures and π_1, π_2 the corresponding possibility distributions. In this special case, the following expressions are equivalent:

- $E_1 \subseteq E_2$,

- $\forall A \subseteq \mathcal{V},\ pl_1(A) \leq pl_2(A)$,

- $\forall A \subseteq \mathcal{V},\ b_1(A) \geq b_2(A)$

- $\forall A \subseteq \mathcal{V},\ q_1(A) \leq q_2(A)$,

- $\forall v \in \mathcal{V},\ \pi_1(v) \leq \pi_2(v)$.

When working with general belief structures, these inequalities are no longer equivalent [12], and they lead to the definitions of so-called *x-inclusions*, with $x \in \{pl, bel, q, s, \pi\}$.

Definition 3 (pl-inclusion). *A belief structure $(m_1, \mathcal{F}_1)$ defined on $\mathcal{V}$ is said to be* pl-included *in another belief structure $(m_2, \mathcal{F}_2)$ defined on $\mathcal{V}$ if and only if, for all $A \subseteq \mathcal{V}$,*

$$pl_1(A) \leq pl_2(A)$$

and this relation is denoted by $(m_1, \mathcal{F}_1) \sqsubseteq_{pl} (m_2, \mathcal{F}_2)$ and by $(m_1, \mathcal{F}_1) \sqsubset_{pl} (m_2, \mathcal{F}_2)$ if the above inequality is strict for at least one event.

When m_1 and m_2 are normalised, we also have that $(m_1, \mathcal{F}_1)$ is pl-included in $(m_2, \mathcal{F}_2)$ if and only if $\mathcal{P}_{(m_1, \mathcal{F}_1)} \subseteq \mathcal{P}_{(m_2, \mathcal{F}_2)}$, thus the notion of *pl*-inclusion is in agreement with the interpretation of belief functions as probability bounds. Note that, due to the duality relation between plausibility and belief functions, *pl*-inclusion is equivalent to *b*-inclusion (and to *bel*-inclusion when bbas are normalised).

Definition 4 (q-inclusion). *A belief structure $(m_1, \mathcal{F}_1)$ defined on $\mathcal{V}$ is said to be* q-included *in another belief structure $(m_2, \mathcal{F}_2)$ defined on $\mathcal{V}$ if and only if, for all $A \subseteq \mathcal{V}$,*

$$q_1(A) \leq q_2(A)$$

and this relation is denoted by $(m_1, \mathcal{F}_1) \sqsubseteq_q (m_2, \mathcal{F}_2)$ and by $(m_1, \mathcal{F}_1) \sqsubset_q (m_2, \mathcal{F}_2)$ if the above inequality is strict for at least one event.

As the commonality function is greater when greater masses are given to larger sets, the notion of *q*-inclusion also comes down to comparing informative contents, but neither does it imply nor is it implied by the *pl*-inclusion [12]. Also, existing results tend to show that the notion of *q*-inclusion is more natural when working within the TBM interpretation [10, 18].

The next definition is the direct extension of set-inclusion to random sets:

Definition 5 (s-inclusion). *A belief structure $(m_1, \mathcal{F}_1)$ defined on $\mathcal{V}$ with $\mathcal{F}_1 = \{E_1, \ldots, E_q\}$ is said to be* s-included *in another belief structure $(m_2, \mathcal{F}_2)$ defined on $\mathcal{V}$ with $\mathcal{F}_2 = \{E'_1, \ldots, E'_p\}$ if and only if there exists a non-negative matrix $G = [g_{ij}]$ such that*

$$\text{for } j = 1, \ldots, p, \quad \sum_{i=1}^{q} g_{ij} = 1,$$

$$g_{ij} > 0 \Rightarrow E_i \subseteq E'_j,$$

$$\text{for } i = 1, \ldots, q, \quad \sum_{j=1}^{p} m_2(E'_j) g_{ij} = m_1(E_i).$$

This relation is denoted by $(m_1, \mathcal{F}_1) \sqsubseteq_s (m_2, \mathcal{F}_2)$ and by $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2)$ if there is at least a pair i, j such that $g_{ij} > 0$ and $E_i \subset E_j$.

The term g_{ij} is the proportion of the focal set E'_j that "flows down" to focal set E_i. In other words, $(m_1, \mathcal{F}_1)$ is s-included in $(m_2, \mathcal{F}_2)$ if the mass of any focal set E'_j of $(m_2, \mathcal{F}_2)$ can be redistributed among subsets of E_j in $(m_1, \mathcal{F}_1)$.

Definition 6 (π-inclusion). *A belief structure $(m_1, \mathcal{F}_1)$ defined on $\mathcal{V}$ is said to be π-included in another belief structure $(m_2, \mathcal{F}_2)$ defined on $\mathcal{V}$ if and only if, for all $v \in \mathcal{V}$,*

$$\pi_{(m_1, \mathcal{F}_1)}(v) \leq \pi_{(m_2, \mathcal{F}_2)}(v)$$

and this relation is denoted $(m_1, \mathcal{F}_1) \sqsubseteq_\pi (m_2, \mathcal{F}_2)$ and by $(m_1, \mathcal{F}_1) \sqsubset_\pi (m_2, \mathcal{F}_2)$ if the above inequality is strict for at least one element.

The notion of π-inclusion is the extension to general belief structures of the notion of specificity between possibility distributions (a possibility distribution π_1 is more specific, or included in another possibility distribution π_2 if $\pi_1 \leq \pi_2$).

Since notions of inclusion allow to compare informative contents, we will also say, when $(m_1, \mathcal{F}_1) \sqsubseteq_x (m_2, \mathcal{F}_2)$ $((m_1, \mathcal{F}_1) \sqsubset_x (m_2, \mathcal{F}_2))$ with $x \in \{pl, q, s, \pi\}$, that $(m_1, \mathcal{F}_1)$ is (strictly) more x-committed than $(m_2, \mathcal{F}_2)$. The following implications hold between these notions of inclusion [12]:

$$(m_1, \mathcal{F}_1) \sqsubseteq_s (m_2, \mathcal{F}_2) \Rightarrow \left\{ \begin{array}{c} (m_1, \mathcal{F}_1) \sqsubseteq_{pl} (m_2, \mathcal{F}_2) \\ (m_1, \mathcal{F}_1) \sqsubseteq_q (m_2, \mathcal{F}_2) \end{array} \right\} \Rightarrow (m_1, \mathcal{F}_1) \sqsubseteq_\pi (m_2, \mathcal{F}_2). \tag{2}$$

As classical inclusion does with crisp sets, each of these notions induces a partial ordering between elements of $\mathcal{M}_\mathcal{V}$. Note that the relation $\sqsubseteq_\pi$ only induces a partial pre-order (i.e., we can have $(m_1, \mathcal{F}_1) \sqsubseteq_\pi (m_2, \mathcal{F}_2)$ and $(m_2, \mathcal{F}_2) \sqsubseteq_\pi (m_1, \mathcal{F}_1)$ with $(m_1, \mathcal{F}_1) \neq (m_2, \mathcal{F}_2)$), while the others induce partial orders (i.e., they are antisymmetric). This is due to the fact that the notion of π-inclusion is based on the contour function that does not, in general, contain all the information contained in a belief structure. Note that, since notions of pl, q and s-inclusion are antisymmetric, we also have

$$(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2) \Rightarrow \left\{ \begin{array}{l} (m_1, \mathcal{F}_1) \sqsubset_{pl} (m_2, \mathcal{F}_2) \\ (m_1, \mathcal{F}_1) \sqsubset_q (m_2, \mathcal{F}_2) \end{array} \right. \tag{3}$$

The following example illustrates the fact that π-inclusion not being antisymmetric, we can have strict q-inclusion and pl-inclusion in opposite directions while having equality for these two functions on singletons. In fact, it is obvious that

$$(m_1, \mathcal{F}_1) \sqsubset_{pl} (m_2, \mathcal{F}_2) \text{ and } (m_2, \mathcal{F}_2) \sqsubset_q (m_1, \mathcal{F}_1) \text{ imply } \pi_{(m_1, \mathcal{F}_1)} = \pi_{(m_2, \mathcal{F}_2)}. \tag{4}$$

Example 1. *Consider the two belief structures $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ on the domain $\mathcal{V} = \{v_1, v_2, v_3\}$*

$(m_1, \mathcal{F}_1)$		$(m_2, \mathcal{F}_2)$	
Focal sets	*Mass*	*Focal sets*	*Mass*
$E_{11} = \{v_2\}$	*0.5*	$E_{21} = \{v_2, v_3\}$	*0.5*
$E_{12} = \{v_1, v_2, v_3\}$	*0.5*	$E_{22} = \{v_1, v_2\}$	*0.5*

These two random sets have the same contour function, while $(m_1, \mathcal{F}_1) \sqsubset_{pl} (m_2, \mathcal{F}_2)$ and $(m_2, \mathcal{F}_2) \sqsubset_q (m_1, \mathcal{F}_1)$. And $\pi_{(m_1, \mathcal{F}_1)} = \pi_{(m_2, \mathcal{F}_2)}$.

Finally, we can prove the following result:

Proposition 1. $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2) \Rightarrow (m_1, \mathcal{F}_1) \sqsubset_\pi (m_2, \mathcal{F}_2)$.

Proof. Note that as $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2)$, $m_1(E_i) = \sum_{j: E_i \subseteq F_j} g_{ij} m_2(F_j)$ where $\sum_{j: E_i \subseteq F_j} g_{ij} = 1$, since g_{ij} is the proportion of $m_1(E_i)$ flowing to F_j. Now,

$$\pi_1(v) = \sum_{v \in E_i} m_1(E_i) = \sum_{v \in E_i} m_1(E_i) \sum_{j: E_i \subseteq F_j} g_{ij} = \sum_{i,j: v \in E_i \subseteq F_j} m_1(E_i) g_{ij}$$

Likewise

$$\pi_2(v) = \sum_{v \in F_j} m_2(F_j) = \sum_{j:v \in F_j} \sum_{i:E_i \subseteq F_j} g_{ij} m_1(E_i) = \sum_{i,j:v \in F_j, E_i \subseteq F_j} m_1(E_i) g_{ij}$$

$$= \sum_{i,j:v \in E_i \subseteq F_j} m_1(E_i) g_{ij} + \sum_{i,j:v \in F_j \setminus E_i, E_i \subseteq F_j} m_1(E_i) g_{ij}$$

$$= \pi_1(v) + \sum_{i,j:v \in F_j \setminus E_i, E_i \subseteq F_j} m_1(E_i) g_{ij}$$

The second term is positive, due to $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2), \exists i, j E_i \subseteq F_j, v \in F_j \setminus E_i$ with $g_{ij} > 0$. $\qquad \square$

If belief structures are consonant, then all above x-inclusions reduce to the same definition (that is, π-inclusion).

Recently, Denoeux [8] has introduced yet other information orderings between belief functions, namely the w-inclusion and v-inclusion, based on Smets canonical decomposition [33]. They also induce partial orders between belief functions. Given a belief structure $(m, \mathcal{F})$ for which $m(\mathcal{V}) \neq 0$, the w-transform assigns the value $w(A) \in (0, +\infty)$ to each subset $A \subseteq \mathcal{V}$, such that

$$w(A) = \prod_{B \supseteq A} q(B)^{\mathbb{C}(B) - \mathbb{C}(A) + 1} \tag{5}$$

with $\mathbb{C}(E)$ the cardinality of set E. A belief structure $(m_1, \mathcal{F}_1)$ is said to be w-included in another belief structure $(m_2, \mathcal{F}_2)$ if $w_1(A) \leq w_2(A)$ for any $A \subseteq \mathcal{V}$. This is a stronger notion than the s-inclusion: If a bba is less w-committed than another one, then it is a specialisation thereof.

When $(m, \mathcal{F})$ is normalised, consonant and has π for contour function, if we note $\pi_k = \pi(v_k)$ with a ranking of elements $\mathcal{V} = \{v_1, \ldots, v_K\}$ such that $1 = \pi_1 \geq \pi_2 \geq \ldots \geq \pi_K$, and $A_k = \{v_1, \ldots, v_k\}$, its w-transform is

$$w(A) = \begin{cases} \frac{\pi_{k+1}}{\pi_k}, & A = A_k, 1 \leq k < K, \\ 1, & otherwise. \end{cases}$$

We will not consider these orderings here, for the reason that the notion of w-inclusion does not reduce to the notion of π-inclusion when considering only consonant random sets, as the next example shows.

Example 2. *Consider the two possibility distributions π_1, π_2 on space $\mathcal{V} = \{v_1, v_2, v_3, v_4\}$ summarized in the following table*

	π_1					π_2		
v_1	v_2	v_3	v_4		v_1	v_2	v_3	v_4
0.1	*0.5*	*0.6*	*1*		*0.15*	*0.8*	*1*	*1*

we do have $\pi_1 < \pi_2$ but $w_1(\{v_1, v_2\}) = {}^5/_6 > w_2(\{v_1, v_2\}) = {}^8/_{10}$ and $w_1(\{v_1, v_2, v_3\}) = {}^6/_{10} < w_2(\{v_1, v_2, v_3\}) = 1$, hence the corresponding random sets are w-incomparable. The intuition justifying the idea that (according to w-inclusion) π_1 is not more informative than π_2 in this example remains unclear as of now.

As all these notions induce partial orders between belief structures, it is sometimes desirable (for example, when one has to select a single least-specific belief structure among a set of such structures) to use additional criteria inducing complete ordering between belief structures. One such criterion, that is used in other approaches to the cautious merging of belief function [10, 18], is the expected cardinality of a belief structure $(m, \mathcal{F})$, that we denote by $\mathbb{C}(m, \mathcal{F})$ and whose value is

$$\mathbb{C}(m, \mathcal{F}) = \sum_{E \in \mathcal{F}} m(E) \mathbb{C}(E).$$

It is also equal to the cardinality of the contour function $\pi_{(m,\mathcal{F})}$ [16], that is

$$\mathbb{C}(m, \mathcal{F}) = \sum_{v \in \mathcal{V}} \pi_{(m,\mathcal{F})}(v). \tag{6}$$

We use the same notation for set cardinalities and belief structure expected cardinalities, since equation (6) reduces to $\mathbb{C}(m, \mathcal{F}) = \mathbb{C}(E)$ if $m(E) = 1$. Note that other information measures exist [2, 22], but since our aim is to generalize idempotent merging coming from possibility theory, expected cardinality appears to be the best choice, due to the relation between the expected cardinality and the contour function (Equality (6)). We can thus define the notion of cardinality-based specificity:

Definition 7 ($\mathbb{C}$-specificity). *A belief structure $(m_1, \mathcal{F}_1)$ defined on $\mathcal{V}$ is said to be more $\mathbb{C}$-specific than another belief structure $(m_2, \mathcal{F}_2)$ defined on $\mathcal{V}$ if and only if we have the inequality*

$$\mathbb{C}(m_1, \mathcal{F}_1) \leq \mathbb{C}(m_2, \mathcal{F}_2)$$

and this relation is denoted by $(m_1, \mathcal{F}_1) \sqsubseteq_{\mathbb{C}} (m_2, \mathcal{F}_2)$ and by $(m_1, \mathcal{F}_1) \sqsubset_{\mathbb{C}} (m_2, \mathcal{F}_2)$ if the above inequality is strict.

The following lemma also indicates that the pre-order between belief structures induced by $\mathbb{C}$-specificity is in agreement (and therefore coherent) with the other inclusion notions used in this paper.

Lemma 1. *Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two random sets. Then, the following implications holds:*

  I $(m_1, \mathcal{F}_1) \sqsubset_{\pi} (m_2, \mathcal{F}_2) \Rightarrow (m_1, \mathcal{F}_1) \sqsubset_{\mathbb{C}} (m_2, \mathcal{F}_2)$
  II $(m_1, \mathcal{F}_1) \sqsubset_{s} (m_2, \mathcal{F}_2) \Rightarrow (m_1, \mathcal{F}_1) \sqsubset_{\mathbb{C}} (m_2, \mathcal{F}_2)$
 III $(m_1, \mathcal{F}_1) \sqsubset_{pl} (m_2, \mathcal{F}_2) \Rightarrow (m_1, \mathcal{F}_1) \sqsubseteq_{\mathbb{C}} (m_2, \mathcal{F}_2)$
 IV $(m_1, \mathcal{F}_1) \sqsubset_{q} (m_2, \mathcal{F}_2) \Rightarrow (m_1, \mathcal{F}_1) \sqsubseteq_{\mathbb{C}} (m_2, \mathcal{F}_2)$

Proof.   I Immediate, since $\pi_{(m_1,\mathcal{F}_1)} < \pi_{(m_2,\mathcal{F}_2)}$ implies the same strict inequality between $\mathbb{C}(m_1, \mathcal{F}_1)$ and $\mathbb{C}(m_2, \mathcal{F}_2)$ (see Eq. (6))

  II From Proposition 1 strict s-inclusion implies strict π-inclusion. Then simply use the previous item.

 III The implication is immediate, given Eq (6) and the fact that $pl_1 \leq pl_2$. Example 3 indicates that two strictly pl-included belief structures can have equal cardinality.

 IV Immediate, with the same arguments applied to q-inclusion.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

2.4. Conjunctive merging and least commitment

Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two normalised belief structures defined on $\mathcal{V}$, supplied by two, not necessarily independent, sources (e.g., two experts potentially sharing some common opinions, two physical models based on similar equations). We define a belief structure $(m_\cap, \mathcal{F}_\cap)$ resulting from a conjunctive merging of $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ as the result of the following procedure [10]:

1. A joint mass (we use boldface $\mathbf{m}$ to tell joint mass and classical mass structures apart) distribution $\mathbf{m}$ is built on $\mathcal{V}^2$, having focal sets of the form $E \times F$ where $E \in \mathcal{F}_1, F \in \mathcal{F}_2$ and preserving m_1, m_2 as marginals. It means that

$$\forall E \in \mathcal{F}_1, \ m_1(E) = \sum_{F \in \mathcal{F}_2} \mathbf{m}(E, F), \tag{7}$$

$$\forall F \in \mathcal{F}_2, \ m_2(F) = \sum_{E \in \mathcal{F}_1} \mathbf{m}(E, F).$$

2. Each joint mass $\mathbf{m}(E, F)$ is allocated to the subset $E \cap F$ in the final belief structure $m_\cap$.

We call a merging rule satisfying these two conditions *conjunctive*[3], and denote by $\mathcal{M}_{12}$ the set of conjunctively merged belief structures. This approach to defining conjunctive rules appears intuitive [2, 11, 19], and is well in agreement with Dempster's seminal work, since a joint mass $\mathbf{m}(E, F)$ can be interpreted as a joint probability expressing a particular dependence structure between the sources. The marginal equations express the idea that the information provided by each source can be retrieved from the richer information that includes a representation of their mutual dependence. Once this principle is granted, the combination rule extends the set-theoretic conjunction because we assign the joint mass to the conjunction of focal sets.[4] Not every belief structure $(m_\cap, \mathcal{F}_\cap)$ obtained by conjunctive merging is normalised (i.e., one may get $m(\emptyset) \neq 0$). In this paper, unless stated otherwise, we do not assume that a conjunctively merged belief structure has to be normalised. Not requiring normalization means that we may admit that a belief structure can be self-contradictory to a certain extent, reflecting the level of conflict between sources. We also do not renormalise such belief structures, because, after renormalisation, they would no longer satisfy Eq. (7)[5]. Moreover, renormalisation is usually not required when working with possibility distributions.

By construction, a belief structure $(m_\cap, \mathcal{F}_\cap)$ on $\mathcal{V}$ obtained by a conjunctive merging rule is a specialisation of both $(m_1, \mathcal{F}_1)$ and $(m_2, \mathcal{F}_2)$, and $\mathcal{M}_{12}$ is a subset of all belief structures that are specialisations of both $(m_1, \mathcal{F}_1)$ and $(m_2, \mathcal{F}_2)$, that is

$$\mathcal{M}_{12} \subseteq \{m \in \mathcal{M}_\mathcal{V} | i = 1, 2, \ m \sqsubseteq_s m_i\},$$

with the inclusion being usually strict[6]. There are three possible situations for the content of set $\mathcal{M}_{12}$:

1. $\mathcal{M}_{12}$ only contains normalised belief functions. It means that $\forall E \in \mathcal{F}_1, F \in \mathcal{F}_2, E \cap F \neq \emptyset$. The two bbas are said to be *logically consistent*.
2. $\mathcal{M}_{12}$ contains both subnormalised and normalised bbas. It means that $\exists E, F, E \cap F = \emptyset$ and that the marginal-preservation Equations (7) have solutions which allocate zero mass $m(E, F)$ to such $E \times F$. The two bbas are said to be *non-conflicting*.
3. $\mathcal{M}_{12}$ contains only subnormalised belief functions. This situation is equivalent to having $\mathcal{P}_{(m_1, \mathcal{F}_1)} \cap \mathcal{P}_{(m_2, \mathcal{F}_2)} = \emptyset$. The two bbas are said to be *conflicting*.

Chateauneuf [4] has shown that being non-conflicting or logically consistent is a sufficient and necessary condition for $\mathcal{P}_{(m_1, \mathcal{F}_1)} \cap \mathcal{P}_{(m_2, \mathcal{F}_2)}$ to be non-empty, and that for each subset $A \subseteq \mathcal{V}$, the lower probability $\underline{P}(A)$ of $\mathcal{P}_{(m_1, \mathcal{F}_1)} \cap \mathcal{P}_{(m_2, \mathcal{F}_2)}$ is equal to the least belief degree $bel(A)$ on A induced by all normalised belief structures in $\mathcal{M}_{12}$. On the contrary, case 1 is a stronger form of logical consistency: it means that whatever the meaning of the message $v \in E$ conveyed by one source is, it is logically consistent with the meaning of the message $v \in F$ supplied by the other source. Case 2 corresponds to a form of probabilistic consistency. The conjunctive rule consisting in allocating positive joint mass only to non-empty sets of the form $E \cap F$ presupposes sources are reliable, hence cannot contradict each other, which means $\mathbf{m}(E, F) = 0$ whenever $E \cap F = \emptyset$. But then the dependence between the sources is not known, which leaves several possible joint mass functions.

When sources are considered as mutually independent, the TBM conjunctive rule consists of merging belief structures inside $\mathcal{M}_{12}$, using the product of masses (i.e., $\mathbf{m}(E, F) = m_1(E) \cdot m_2(F)$ in Equations (7)) for the joint mass. This assumption may sound especially natural in the case of logically consistent bbas; indeed the result of merging by the TBM conjunctive rule of combination is then normalised. In case 2,

---

[3]A disjunctive merging rule can be defined likewise, changing $\cap$ into $\cup$.

[4]There are other ways of defining conjunctive merging, such as Denoeux's w-rule [8], but these rules are not based on bbas (only on the weight function w). How to reinterpret them in terms of the above two-stepped process has not yet been investigated (as far as we know).

[5]This is the reason why Dempster rule of conditioning is criticized by subjectivists in imprecise probability theory as being incoherent. In fact, renormalisation corresponds to a belief revision step, wich precisely consists in breaking away from previous beliefs.

[6]Suppose, for example, the marginals are empty belief structures $m_1(\mathcal{V}) = m_2(\mathcal{V}) = 1$. The set of specialisation of both of them contains every possible belief structure, while only the empty one can be reached by the conjunctive merging defined in this paper as only $(m_\cap, \mathcal{F}_\cap) = (m_1, \mathcal{F}_1) = (m_2, \mathcal{F}_2)$ is then possible.

$m_\cap(\emptyset) > 0$ measures the level of conflict between the sources, the assumption of their reliability being all the more questionable as $m_\cap(\emptyset)$ is large. Renormalising $m_\cap$ like Shafer [30] suggests comes down to restoring the assumption that sources are reliable, and considering the apparent conflict as a kind of noise one has to get rid of. Merging then leads to correcting the available information rather than just exploiting it.

When sources cannot be considered as independent and the dependence structure between them is not well-known, a common practice is to use the principle of least-commitment to build the merged belief structure. That is, to adopt a cautious attitude. Let us denote by $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ the set of all maximal elements inside $\mathcal{M}_{12}$ with respect to x-inclusion order, with $x \in \{s, pl, q, \pi, \mathbb{C}\}$. The least-commitment principle then often consists in choosing a given type of x-inclusion and picking a particular element inside $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ according to a number of desired additional properties. Interestingly, applying the principle of least-commitment implies that the corresponding conjunctive rule is idempotent [8]. Indeed, if $m_1 = m_2 = m$, the least x-specific belief function in $\mathcal{M}_{12}$ obviously has mass function m, whatever x.

2.5. The minimum rule of possibility theory

If π_1, π_2 denote two possibility distributions, the standard conjunctive idempotent merging between these two distributions is the pointwise minimum [14]:

$$\pi_{1 \wedge 2}(v) = \min(\pi_1(v), \pi_2(v)), \forall v \in \mathcal{V}.$$

It can also be seen as the most cautious result, as the minimum is the most conservative of all t-norms [21].

Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be normalised consonant belief structures corresponding to possibility distributions π_1, π_2. In this case, it is known [19] that the consonant belief structure corresponding to $\min(\pi_1, \pi_2)$ lies inside $\mathcal{M}_{12}$. It assumes some dependency between focal sets. Let $0 = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_M = 1$ be the distinct values taken by both π_1, π_2 over $\mathcal{V}$, then $\min(\pi_1, \pi_2)$ corresponds to the conjunctively merged belief structure $(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})$ that, for $i = 1, \ldots, M$, has focal sets

$$\begin{cases} E_i = E_{i,1} \cap E_{i,2}, \\ m_{1 \wedge 2}(E_i) = \alpha_i - \alpha_{i-1}, \end{cases} \tag{8}$$

with $E_{i,j} = \{v | \pi_j(v) \geq \alpha_i\}$.

Smets and colleagues [17, 18], have shown the following result:

Proposition 2. *The least q-committed belief structure in $\mathcal{M}_{12}$ is unique. It is the consonant belief structure whose contour function is $\min(\pi_1, \pi_2)$ (i.e., $\mathcal{M}_{12}^{\sqsubseteq \mathbf{q}}$ is reduced to this belief structure).*

Proof. To see it, just notice that for consonant belief structures with contour function π, the commonality function is of the form $Q(A) = \min_{v \in A} \pi(v)$. So $Q_{1 \wedge 2} = \min(Q_1, Q_2)$ is the commonality function of the unique consonant belief structure with contour function $\min(\pi_1, \pi_2)$. If a belief structure $(m, \mathcal{F})$ is such that $Q \leq Q_1$ and $Q \leq Q_2$, where $(m_1, \mathcal{F}_1)$ and $(m_2, \mathcal{F}_2)$ are consonant, $Q \leq \min(Q_1, Q_2)$ so that the least q-committed belief structure in $\mathcal{M}_{12}$ is the consonant belief structure $(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})$ with commonality function $\min(Q_1, Q_2)$. So, $\mathcal{M}_{12}^{\sqsubseteq \mathbf{q}} = \{(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})\}$. $\qquad \square$

This consonant merged belief structure is thus least π-committed inside $\mathcal{M}_{12}$, and it is also one of the s-least committed inside $\mathcal{M}_{12}$ (i.e., it is among the elements of $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$). The next example, which completes Example 1, indicates that, when merging consonant belief structures, none of $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$ or $\mathcal{M}_{12}^{\sqsubseteq \pi}$ is necessarily reduced to a single element.

Example 3. *Consider the possibility distributions π, ρ, expressed as belief structures $(m_\pi, \mathcal{F}_\pi), (m_\rho, \mathcal{F}_\rho)$.*

	$(m_\pi, \mathcal{F}_\pi)$		$(m_\rho, \mathcal{F}_\rho)$	
Focal sets	*Mass*		*Focal sets*	*Mass*
$E_1 = \{v_0, v_1, v_2\}$	*0.5*		$F_1 = \{v_2, v_3\}$	*0.5*
$E_2 = \{v_0, v_1, v_2, v_3\}$	*0.5*		$F_2 = \{v_1, v_2, v_3, v_4\}$	*0.5*

The two belief structures $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ of Example 1 can be obtained by conjunctively merging these two marginal belief structures. Namely, $(m_1, \mathcal{F}_1) = \{(E_1 \cap F_1, 0.5), (E_2 \cap F_2, 0.5)\}$ is the consonant belief structure corresponding to $\min(\pi, \rho)$, while the non-consonant $(m_2, \mathcal{F}_2) = \{(E_1 \cap F_2, 0.5), (E_2 \cap F_1, 0.5)\}$ has the same contour function. None of these two belief structures is s-included in the other, while we do have $(m_2, \mathcal{F}_2) \sqsubset_q (m_1, \mathcal{F}_1)$.

Note that the consonant merged belief structure obtained using the minimum rule on consonant belief structures may be subnormalised, except when there is at least one element v in $\mathcal{V}$ such that $\pi_1(v) = \pi_2(v) = 1$ (that is, $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ are logically consistent).

Since idempotence is a natural property to satisfy when cautiously merging multiple sources of information, it is natural to link idempotent rules coming from other frameworks to the merging of belief structures. Chateauneuf [4] has already explored the links between belief structures and the intersection of the corresponding probability sets. In the rest of the paper, we explore the links between the conjunctive merging of belief structure defined above and the minimum rule of possibility theory, investigating under which conditions the latter can be extended to belief structures.

3. The Strong Idempotent Contour Function Merging Principle (SICFMP)

Now, let us consider two general random sets $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ and their respective contour functions $\pi_{(m_1, \mathcal{F}_1)}, \pi_{(m_2, \mathcal{F}_2)}$. First notice the following property:

Proposition 3 (s-covering)**.** *Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two belief structures. Then, the following inequality holds for any $v \in \mathcal{V}$:*

$$\max_{(m, \mathcal{F}) \in \mathcal{M}_{12}} \pi_{(m, \mathcal{F})}(v) \leq \min(\pi_{(m_1, \mathcal{F}_1)}(v), \pi_{(m_2, \mathcal{F}_2)}(v)). \tag{9}$$

Proof. Since any element $(m, \mathcal{F})$ in $\mathcal{M}_{12}$ is s-included in $(m_1, \mathcal{F}_1)$ and $(m_2, \mathcal{F}_2)$, and s-inclusion implies π-inclusion (Equation (2)), $\pi_{(m, \mathcal{F})}(v) \leq \min(\pi_{(m_1, \mathcal{F}_1)}(v), \pi_{(m_2, \mathcal{F}_2)}(v))$, for any $x \in \mathcal{V}$ and any $(m, \mathcal{F}) \in \mathcal{M}_{12}$. Since this is true for all elements of $\mathcal{M}_{12}$, this is enough to prove (9). $\square$

It is known [26] that the same inequality holds for sets of probabilities, since, given two such sets $\mathcal{P}_1, \mathcal{P}_2$, their respective upper probabilities $\overline{P}_1, \overline{P}_2$, their intersection $\mathcal{P}_1 \cap \mathcal{P}_2$ and the induced upper probability $\overline{P}_{12}$, we have, for all events $A \subseteq \mathcal{V}$, $\overline{P}_{12}(A) \leq \min(\overline{P}_1(A), \overline{P}_2(A))$[7]. To extend the idempotent rule of possibility theory to the non-consonant case, it makes sense to require that inequality (9) become an equality.

Definition 8 (Strong idempotent contour function merging principle (SICFMP))**.** *Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two belief structures and $\mathcal{M}_{12}$ the set of conjunctively merged belief structures. Then, an element $(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})$ in $\mathcal{M}_{12}$ is said to satisfy the strong idempotent contour function merging principle if, for any $v \in \mathcal{V}$,*

$$\pi_{(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})}(v) = \min(\pi_{(m_1, \mathcal{F}_1)}(v), \pi_{(m_2, \mathcal{F}_2)}(v)), \tag{10}$$

$\pi_{(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})}$ being the contour function of $(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})$.

That is, we require that the selected merged belief structure should have a contour function equal to the minimum of the two marginal contour functions. Note that the minimum rule is retrieved if both $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ are consonant.

Let us first assume that the SICFMP can be satisfied when combining two belief structures. In this case, a merging rule satisfying the SICFMP also satisfies idempotence.

Proposition 4 (idempotence)**.** *Let $(m_1, \mathcal{F}_1) = (m_2, \mathcal{F}_2) = (m, \mathcal{F})$ be two identical belief structures. Then, the unique element in $\mathcal{M}_{12}$ satisfying Equation (10) is*

$$(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2}) = (m, \mathcal{F}).$$

---

[7]Note that the present situation is a bit different, since merging of probability sets does not allow for subnormalised belief structures, while we do allow for such belief structures here.

Proof. That $(m, \mathcal{F})$ is a solution of Equation (10) is immediate, since for any $v \in \mathcal{V}$ we have $pl_1(\{v\}) = pl_2(\{v\}) = pl(\{v\})$. We must now show that any other belief structure in $\mathcal{M}_{12}$ is not a solution to (10). Consider a belief structure $(m', \mathcal{F}')$ that is a specialization of $(m, \mathcal{F})$ and is not $(m, \mathcal{F})$. Then $(m', \mathcal{F}') \sqsubset_s (m, \mathcal{F})$. Hence by Proposition 1, $(m', \mathcal{F}') \sqsubset_\pi (m, \mathcal{F})$, hence the contour function of $(m', \mathcal{F}')$ cannot be expressed by Equation (10). $\square$

The SICFMP is therefore a sufficient condition to ensure that a merging rule is idempotent. It also satisfies the following stronger property.

Proposition 5 (*s*-coherence). *Let $(m_1, \mathcal{F}_1)$ be strictly s-included in $(m_2, \mathcal{F}_2)$, that is $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2)$. Then, the unique element in $\mathcal{M}_{12}$ satisfying Equation (10) is*

$$(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2}) = (m_1, \mathcal{F}_1).$$

Proof. If $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2)$, this means that $pl_1(\{v\}) \leq pl_2(\{v\})$ for all $v \in \mathcal{V}$, with a strict inequality for at least one element. Second, if $(m_1, \mathcal{F}_1) \sqsubset_s (m_2, \mathcal{F}_2)$, then $(m_1, \mathcal{F}_1)$ is a specialisation both of itself and $(m_2, \mathcal{F}_2)$, hence it is in $\mathcal{M}_{12}$. Consequently, it is a solution of Equation (10). To show that it is the unique solution inside $\mathcal{M}_{12}$, we can advocate a similar argument as in the previous proof. $\square$

This indicates that satisfying the SICFMP is coherent with the notion of specialisation, that is the notion of inclusion that looks the most sensible when extending possibilistic inclusion to the belief function framework. The next example shows that Proposition 5 does not extend to the notions of *pl*- and *q*-inclusions.

Example 4. *The two belief structures in Example 1 have the same contour function but one of them is strictly pl-included in the other and the other is strictly q-included in the first (and they are s-incomparable). Clearly, there are two (consonant) s-least committed belief structures resulting from conjunctive merging in $\mathcal{M}_{12}$, the one obtained as $\{(E_{11} \cap E_{21}, 0.5), (E_{12} \cap E_{22}, 0.5)\} = \{(\{v_2\}, 0.5), (\{v_1, v_2\}, 0.5)\}$ and the other as $\{(E_{11} \cap E_{22}, 0.5), (E_{12} \cap E_{21}, 0.5)\} = \{(\{v_2\}, 0.5), (\{v_2, v_3\}, 0.5)\}$. None of them satisfies the SICFMP nor is equal to any of the marginal belief structures.*

We now proceed to show that satisfying the SICFMP is too demanding for general belief functions. Actually, general necessary and sufficient conditions under which the merged bba has a contour function satisfying the SICFMP were found by Dubois and Prade [15]. Namely let $(m, \mathcal{F}) \in \mathcal{M}_{12}$, and let $\mathbf{m}(E_i, F_j)$ be the joint mass obtained in the first merging step. The contour function of $(m, \mathcal{F})$ is such that

$$\pi_{(m, \mathcal{F})}(v) = \sum_{v \in E_i \cap F_j} \mathbf{m}(E_i, F_j) = \pi_{(m_1, \mathcal{F}_1)}(v) - \sum_{v \in E_i \cap F_j^c} \mathbf{m}(E_i, F_j).$$

Hence

$$\min(\pi_{(m_1, \mathcal{F}_1)}(v), \pi_{(m_2, \mathcal{F}_2)}(v)) = \pi_{(m, \mathcal{F})}(v) + \min(\sum_{v \in E_i \cap F_j^c} \mathbf{m}(E_i, F_j), \sum_{v \in E_i^c \cap F_j} \mathbf{m}(E_i, F_j)).$$

So, the minimum rule is recovered if and only if $\forall v \in \mathcal{V}$, one of $\sum_{v \in E_i \cap F_j^c} m(E_i, F_j)$ or $\sum_{v \in E_i^c \cap F_j} m(E_i, F_j)$ is equal to 0. For each $v \in \mathcal{V}$, it comes down to enforcing $m(E_i, F_j) = 0$ either for all i, j such that $v \in E_i \cap F_j^c$, or for all i, j such that $v \in E_i^c \cap F_j$.

Example 5. *Let $\mathcal{V} = \{a, b, c\}$. Define*
$(m_1, \mathcal{F}_1) : m_1(\{a\}) = 0.2; \quad m_1(\{a, b\}) = 0.1; \quad m_1(\{a, c\}) = 0.3; \quad m_1(\{b, c\}) = 0.3; \quad m_1(\{a, b, c\}) = 0.1.$
$(m_2, \mathcal{F}_2) : m_2(\{a\}) = 0.3; \quad m_2(\{a, b\}) = 0.4; \quad m_2(\{a, b, c\}) = 0.3.$

We can decide to let $\mathbf{m}(E_i, F_j) = 0$ for $a, b \in E_i \cap F_j^c$ and $c \in E_i^c \cap F_j$. Then, for b, it enforces $\mathbf{m}(\{a, b\}, \{a\}) = \mathbf{m}(\{b, c\}, \{a\}) = \mathbf{m}(\{a, b, c\}, \{a\}) = 0$, for c, $\mathbf{m}(\{a\}, \{a, b, c\}) = \mathbf{m}(\{b, c\}, \{a, b, c\}) = 0$, but it creates no such constraint for a (as it is present in every focal element of m_2). The following joint mass provides a solution to the marginal equations (where entries $0_b, 0_c$ are enforced by the SICFMP):

$\mathbf{m}(E_i, F_j)$	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$
$\{a\}$	0.1	0.1	0_c
$\{a, b\}$	0_b	0.1	0_c
$\{a, c\}$	0.2	0	0.1
$\{b, c\}$	0_b	0.2	0.1
$\{a, b, c\}$	0_b	0	0.1

It is easy to check that the SICFMP holds.

As there is a choice between two options for each element v of $\mathcal{V}$ there are at most $2^{\mathbb{C}(\mathcal{V})}$ possible sets of constraints of the form $\mathbf{m}(E_i, F_j) = 0$ on top of marginal constraints, so as to define a belief structure. Not all of these problems will have solutions, and even less if we restrict to normalised resulting belief structures (enforcing moreover $\mathbf{m}(E_i, F_j) = 0$ whenever $E_i, F_j = \emptyset$).

Also, verifying that the problem has a solution is difficult to check in practice. However, there may be specific cases where the problem always have solutions. This is why, in the following, we separately consider the cases of logically consistent (situation 1), non-conflicting (situation 2) or conflicting (situation 3) marginal belief structures, as well as require the conjunctively merged belief structure to be normalised or not. Let us first explore the most specific case, that is the one where marginal belief structures are logically consistent (note that, in this case, all conjunctively merged belief structures are normalised). The next example indicates that the SICFMP cannot always be satisfied in this restrictive case.

Example 6. *Let us consider again the two belief structures $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ of Example 1 as our marginal belief structures. They are logically consistent, and if there is a belief structure $(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})$ in $\mathcal{M}_{12}$ that can satisfy SICFMP, this belief structure should have the contour function of both $(m_1, \mathcal{F}_1)$ and $(m_2, \mathcal{F}_2)$:*

$$pl_{1 \wedge 2}(v_1) = 0.5 \quad pl_{1 \wedge 2}(v_2) = 1 \quad pl_{1 \wedge 2}(v_3) = 0.5,$$

hence its expected cardinality $\mathbb{C}(m_{1 \wedge 2}, \mathcal{F}_{1 \wedge 2})$ should be equal to 2.

As expected cardinality is a linear function, as well as the constraints described by Eq. (7), we can easily search for the maximal expected cardinality that a conjunctively merged belief structure can attain, given its marginals. The linear programming problem corresponding to our example is

$$\max m_{1 \cap 2}(\{v_2\}) + 2.m_{1 \cap 2}(\{v_2, v_3\}) + m_{1 \cap 2}(\{v_2\}) + 2.m_{1 \cap 2}(\{v_1, v_2\})$$

under the constraints (maginalisation, normalisation):

$$\begin{cases} m_{1 \cap 2}(\{v_2\}) + m_{1 \cap 2}(\{v_2, v_3\}) = 0.5 \\ m_{1 \cap 2}(\{v_2\}) + m_{1 \cap 2}(\{v_1, v_2\}) = 0.5 \\ m_{1 \cap 2}(\{v_2\}) + m_{1 \cap 2}(\{v_2\}) = 0.5 \\ m_{1 \cap 2}(\{v_2, v_3\}) + m_{1 \cap 2}(\{v_1, v_2\}) = 0.5 \\ m_{1 \cap 2}(\{v_2\}) + m_{1 \cap 2}(\{v_2, v_3\}) + m_{1 \cap 2}(\{v_2\}) + m_{1 \cap 2}(\{v_1, v_2\}) = 1 \end{cases}$$

The maximal value of expected cardinality in the above problem is 1.5, and is given, for example, by $m_{1 \cap 2}(\{v_2, v_3\}) = 0.5$, $m_{1 \cap 2}(\{v_2\}) = 0.5$. This maximal expected cardinality is less than the expected cardinality that a conjunctively merged belief structure satisfying the SICFMP should reach.

This counter-example indicates that, when the conjunctively merged belief structure has to be normalised, the SICFMP cannot be always satisfied. The two next examples indicate that the SICFMP cannot always be satisfied independently of whether the marginal belief structures are conflicting or not.

Example 7. *Let us consider the space $\mathcal{V} = \{v_1, v_2, v_3\}$ and the two non-conflicting (but not logically consistent) marginal belief structures $(m_1, \mathcal{F}_1)$, $(m_2, \mathcal{F}_2)$ summarized in the table below.*

Set	$\{v_1\}$	$\{v_2\}$	$\{v_3\}$	$\{v_1, v_2\}$	$\{v_1, v_3\}$	$\{v_2, v_3\}$	$\mathcal{V}$
m_1	0.3	0	0	0	0	0.4	0.3
m_2	0.2	0.1	0.1	0.2	0.2	0.1	0.1

13

Constraints / Situation	Consonance	$m_{1\cap2}(\emptyset) = 0$	unconstrained
Logically consistent	$\checkmark$	$\times$	$\times$
Non-conflicting	$\checkmark$	$\times$	$\times$
Conflicting	$\checkmark$	N.A.	$\times$

Table 1: Satisfiability of SICFMP given $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$. $\checkmark$: always satisfiable. $\times$: not always satisfiable. N.A.: Not Applicable

Their contour functions and their minimum are summarized in the next table.

Element $v_i \in \mathcal{V}$	$\pi_1(v_i)$	$\pi_2(v_i)$	$\min(\pi_1, \pi_2)$
v_1	0.6	0.7	0.6
v_2	0.7	0.5	0.5
v_3	0.7	0.5	0.5

The expected cardinality of $\min(\pi_1, \pi_2)$ is 1.6. However, solving the linear program computing the maximal expected cardinality of the elements of $\mathcal{M}_{12}$ gives 1.5 as solution (consider, for example, the conjunctively merged belief structure such that $m(\{v_1\}) = 0.3$, $m(\{v_2\}) = m(\{v_1, v_3\}) = 0.2$, $m(\{v_3\}) = m(\{v_2, v_3\}) = m(\mathcal{V}) = 0.1$). Therefore, there is no element in $\mathcal{M}_{12}$ satisfying the SICFMP for this example.

Example 8. *Let us then consider the two conflicting random sets $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ summarised below.*

$(m_1, \mathcal{F}_1)$		$(m_2, \mathcal{F}_2)$	
Focal sets	*Mass*	*Focal sets*	*Mass*
$E_{11} = \{v_2\}$	*0.5*	$E_{21} = \{v_1 v_2, v_3\}$	*0.5*
$E_{12} = \{v_3\}$	*0.5*	$E_{22} = \{v_1\}$	*0.5*

Their contour functions and their minimum are summarized in the next table.

Element $v_i \in X$	$\pi_1(v_i)$	$\pi_2(v_i)$	$\min(\pi_1, \pi_2)$
v_1	0	1	0
v_2	0.5	0.5	0.5
v_3	0.5	0.5	0.5

This time, the expected cardinality of $\min(\pi_1, \pi_2)$ is 1, while the maximum expected cardinality reachable by an element of $\mathcal{M}_{12}$ is 0.5 (by distributing $m_2(\{v_1 v_2, v_3\})$ to either v_2 or v_3.

Table 1 summarizes in which cases the SICFMP can always be satisfied. Results in this section indicates that the SICFMP, even if it extends the cautious merging of possibility distributions to belief structures, is too strong a requirement to be satisfied in general. A possible alternative is to search for subsets of conjunctively merged belief structures *jointly* satisfying the idempotent contour function merging principle, thus working with sets of belief functions rather than with a single one. This goes in the same line as proposals made by other authors in order to deal with situations where dependencies between, or exact features of, belief structures are not precisely known [1, 6, 37]. Such an alternative is explored in the next section.

4. The Weak Idempotent Contour Function Merging Principle (WICFMP)

In this section, we still assume that we start from marginal belief structures coming from sources whose dependencies are ill-known. While we still require the result of their conjunctive merging to coincide on singletons with the minimum of the contour functions π_1, π_2, we no longer require that the result of the merging be a single belief structure.

Definition 9 (WICFMP)**.** *Consider two belief structures* $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ *and* $\mathcal{M}_{12}$ *the set of conjunctively merged belief structures. Then, a subset* $\mathcal{M} \subseteq \mathcal{M}_{12}$ *is said to satisfy the weak idempotent contour function merging principle if, for any* $v \in \mathcal{V}$,

$$\max_{(m, \mathcal{F}) \in \mathcal{M}} \pi_m(v) = \min(\pi_{m_1}(v), \pi_{m_2}(v)), \tag{11}$$

The subset $\mathcal{M}$ is sufficient to extend the cautious merging of possibility theory to the framework of belief functions. The selection of joint masses **m** in the above definition can be seen as describing different possible dependence structures between the two marginals. Note that any merged belief structure that satisfies the SICFMP also satisfies the WICFMP. In the following, we look for subsets of $\mathcal{M}_{12}$ that always satisfy the WICFMP.

4.1. Subsets of normalised merged belief functions

A first subset of conjunctively merged belief structures that is interesting to explore is the one containing only normalised merged belief structures in $\mathcal{M}_{12}$ (that is, all $(m, \mathcal{F}) \in \mathcal{M}_{12}$ such that $m(\emptyset) = 0$). Given the link between this set and the intersection of induced probability sets $\mathcal{P}_1, \mathcal{P}_2$, we will denote it by $\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$.

Note that linear programming techniques can be used to check that a subset of merged belief functions satisfies the WICFMP, as long as constraints imposed on belief structures in the subset are linear. A linear program can then be written for each $v \in \mathcal{V}$, to check whether Eq. (11) is satisfied.

As considering a subset of conjunctively merged belief structures is less constraining than selecting only one of them, there will be some cases for which the SICFMP cannot be satisfied, while the WICFMP will be, even if we restrict ourselves to normalised belief structures. For instance, it is impossible to satisfy the SICFMP by merging the belief functions of Example 1 but the two consonant belief structures of Example 4 obtained from this merging jointly satisfy the WICFMP. Nevertheless, the next example shows that there are cases where the WICFMP cannot be satisfied even when we consider the subset $\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$ on belief structures that are logically inconsistent but non-conflicting.

Example 9. *Consider the two marginal belief structure* $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ *on* $\mathcal{V} = \{v_1, v_2, v_3\}$ *such that*

$$m_1(\{v_1\}) = 0.5; \quad m_1(\{v_1, v_2, v_3\}) = 0.5,$$

$$m_2(\{v_1, v_2\}) = 0.5; \quad m_2(\{v_3\}) = 0.5.$$

The minimum of contour functions $\pi_{\min} = \min(\pi_1, \pi_2)$ *is given by* $\pi_{\min}(v_i) = 0.5$ *for* $i = 1, 2, 3$. *The only merged bba* m_{12} *to be in* $\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$ *is*

$$m_{12}(\{v_1\}) = 0.5; \quad m_{12}(\{v_3\}) = 0.5,$$

for which $\pi_{12}(v_2) = 0 < 0.5$.

Another interesting aspect of the above example is that the element v_2 is considered as impossible by the intersection of sets of probabilities, while both sources consider v_2 as somewhat possible. It indicates that requiring a normalised result by ensuring probabilistic consistency (or coherence in the sense of Walley, i.e., $m(\emptyset) = 0$) while conjunctively merging uncertain information can be, in some situations, questioned, and a partially inconsistent result be preferred so as to preserve the apparent agreement between sources regarding the plausibility of some values.

4.2. Subsets of s-least committed merged belief structures

Another possible solution is to consider a subset coherent with the least commitment principle. That is, given two belief structures $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$, we consider the subsets $\mathcal{M}_{12}^{\sqsubseteq_{\mathbf{x}}}$, with $x \in \{s, pl, q, \pi\}$. Recall that

$$\mathcal{M}_{12}^{\sqsubseteq_{\mathbf{x}}} = \{(m, \mathcal{F}) \in \mathcal{M}_{12} | \nexists (m, \mathcal{F})' \in \mathcal{M}_{12}, (m, \mathcal{F}) \sqsubset_x (m, \mathcal{F})'\}.$$

The following proposition shows that the subset of s-least committed belief structures in $\mathcal{M}_{12}$ always satisfies the WICFMP.

Proposition 6. *Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two marginal belief structures on $\mathcal{V}$. Then, the subset $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$ satisfies the WICFMP, in the sense that*

$$\max_{(m, \mathcal{F}) \in \mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}} \pi_{(m, \mathcal{F})}(v) = \min(\pi_1(v), \pi_2(v)),$$

with $\pi_1, \pi_2, \pi_{(m, \mathcal{F})}$ the contour functions of, respectively, $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2), (m, \mathcal{F})$.

Proof. To prove this proposition, we will simply show that for any $v \in \mathcal{V}$ there is at least one merged belief structure $(m_v, \mathcal{F}_v)$ in $\mathcal{M}_{12}$ such that $\pi_{(m_v, \mathcal{F}_v)}(v) = \min(\pi_1(v), \pi_2(v))$. If this is true, then, either this merged belief structure is s-least committed, and the problem is solved, or there is a belief structure $(m_v, \mathcal{F}_v)'$ s-less committed than $(m_v, \mathcal{F}_v)$ in $\mathcal{M}_{12}$. There is at least one s-least committed element (i.e., an element of $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$) among such s-less committed belief structures $(m_v, \mathcal{F}_v)'$. However, any belief structure $(m_v, \mathcal{F}_v)'$ is such that

$$\min(\pi_1(v), \pi_2(v)) \geq \pi_{(m_v, \mathcal{F}_v)'}(v) \geq \pi_{(m_v, \mathcal{F}_v)}(v) = \min(\pi_1(v), \pi_2(v)),$$

the first inequality following from Proposition 3 and the second from the definition of s-inclusion, the third equality holding by assumption.

We now have to prove that it is possible to build a merged belief structure $(m_v, \mathcal{F}_v)$ for any given $v \in \mathcal{V}$ such that $\pi_{(m_v, \mathcal{F}_v)}(v) = \min(\pi_1(v), \pi_2(v))$. Without loss of generality, consider that, for v, $\pi_1(v) = \sum_{v \in E} m_1(E) \leq \sum_{v \in E'} m_2(E') = \pi_2(v)$. It is then always possible to transfer part of the masses $m_2(E'), v \in E' \in \mathcal{F}_2$ to subsets $E \cap E'$ containing v, so as to ensure $\sum_{v \in E \cap E'} m_v(E \times E') = \sum_{E \in \mathcal{F}_{v,1}} m_1(E)$, while respecting Eq. (7). The same reasoning can be applied for all $v \in \mathcal{V}$ to finish the proof. $\qquad \square$

Note that the proof of Proposition 6 also indicates that, in all cases, one can always satisfy the WICFMP by choosing a set of at most $n = \mathbb{C}(\mathcal{V})$ merged belief structures, each of them equal to the minimum of the marginal contour functions for one element $v \in \mathcal{V}$. However, "best" strategies to choose them remain to be defined. Another interesting result follows from Proposition 6.

Corollary 1. *Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two marginal belief structures on $\mathcal{V}$. Then, the subsets $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ for $x = \{pl, q, \pi\}$ satisfy the WICFMP, in the sense that*

$$\max_{(m, \mathcal{F}) \in \mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}} \pi_{(m, \mathcal{F})}(v) = \min(\pi_1(v), \pi_2(v)),$$

with $\pi_1, \pi_2, \pi_{(m, \mathcal{F})}$ the contour functions of, respectively, $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2), (m, \mathcal{F})$.

Proof. Given the implications between notions of inclusions of belief structures, any element in $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ with $x = \{pl, q, \pi\}$ is also in $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$. However, there are some elements of $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$ that are not in $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$. What we have to do is to show that, if one element is suppressed, then this element is of no use to satisfy Proposition 6. Let us consider two such elements m_1, m_2 in $\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$ (i.e., they are s-incomparable) and such that $m_1 \sqsubset_x m_2$, hence m_1 is not present in $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$. However, for any $x \in \{pl, q, \pi\}$, we do have (see Lemma 1)

$$m_1 \sqsubseteq_x m_2 \Rightarrow \pi_1 \leq \pi_2,$$

with π_1, π_2 the contour functions of m_1, m_2. $\pi_1 \leq \pi_2$ ensures that m_1 is of no use when taking the maximum of all contour functions to satisfy the WICFMP. $\qquad \square$

Corollary 2. *Let $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ be two marginal belief structures on $\mathcal{V}$. Then, if any of the subsets $\mathcal{M}_{12}^{\sqsubseteq \mathbf{x}}$ with $x = \{s, pl, q, \pi\}$ is reduced to a singleton $(m_x, \mathcal{F}_x)$, then this element satisfies the SICFMP.*

This is, for instance, the case with $\mathcal{M}_{12}^{\sqsubseteq \mathbf{q}}$ when both $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$ are consonant. As for SICFMP, Table 2 summarises for which subset of merged belief structures the WICFMP is always satisfiable.

However Corollary 1 is not valid for expected cardinality as shown by the following counterexample:

Example 10. *Consider the same marginal belief structures as in Example 8, except that the element v_3 is replaced by $\{v_3, v_4\}$, as summarized in the next table.*

Situation \ Subset	$\mathcal{M}_{\mathcal{P}_1 \cap \mathcal{P}_2}$	$\mathcal{M}_{12}^{\sqsubseteq \mathbf{s}}$	$\mathcal{M}_{12}^{\sqsubseteq \mathbf{pl}}$	$\mathcal{M}_{12}^{\sqsubseteq \mathbf{q}}$	$\mathcal{M}_{12}^{\sqsubseteq \pi}$
Logically consistent	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Non-conflicting	$\times$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
Conflicting	N.A.	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$

Table 2: Satisfiability of WICFMP given $(m_1, \mathcal{F}_1), (m_2, \mathcal{F}_2)$. $\checkmark$: always satisfiable. $\times$: not satisfiable in general. N.A.: Not Applicable

$(m_1, \mathcal{F}_1)$		$(m_2, \mathcal{F}_2)$	
Focal sets	*Mass*	*Focal sets*	*Mass*
$E_{11} = \{v_2\}$	*0.5*	$E_{21} = \{v_1 v_2, v_3, v_4\}$	*0.5*
$E_{12} = \{v_3, v_4\}$	*0.5*	$E_{22} = \{v_1\}$	*0.5*

The two possible idempotently merged belief structures allocate 0.5, respectively to $\{v_3, v_4\}$ or $\{v_2\}$ and both remain s-least specific. The former has a greater expected cardinality, and is the unique element having maximal expected cardinality, but it does not satisfy the WICFMP.

In fact, suppose that, by construction, there are two distinct least committed merged bba's m, m' in the sense of the π-ordering within $\mathcal{M}_{12}$, if any. They have contour functions π and π' that are such that $\exists v_1 \neq v_2 \in V, \pi(v_1) > \pi'(v_1)$ and $\pi'(v_2) > \pi(v_2)$. Assume they are also $\mathbb{C}$-least specific so that $\mathbb{C}(m) = \sum_{v \in V} \pi(v) = \mathbb{C}(m') = \sum_{v \in V} \pi'(v)$. Suppose V is changed into another frame of discernment W, a refinement of V where v_1 is changed into a subset V_1 and v_2 into a subset V_2 disjoint from V_1. Then, marginal mass functions m_1 and m_2 on V canonically induce mass functions on W, changing focal elements in 2^V into their expansions in 2^W, turning v_i into V_i. Clearly, the two distinct least committed merged bba's m, m' in the sense of the π-ordering become two distinct least committed merged bba's m_W, m'_W on W. Now their expected cardinalities are

$$\mathbb{C}(m_W) = \sum_{v \in V \setminus \{v_1, v_2\}} \pi(v) + \mathbb{C}(V_1)\pi(v_1) + \mathbb{C}(V_2)\pi(v_2)$$

and

$$\mathbb{C}(m'_W) = \sum_{v \in V \setminus \{v_1, v_2\}} \pi'(v) + \mathbb{C}(V_1)\pi'(v_1) + \mathbb{C}(V_2)\pi'(v_2).$$

In general, $\mathbb{C}(m_W) \neq \mathbb{C}(m'_W)$ so only one of them will be $\mathbb{C}$-least specific. As a consequence, the $\mathbb{C}$-least specific belief functions in $\mathcal{M}_{12}$ will not even satisfy the WICFMP.

5. A general method for constructing idempotent merging rules

We now suggest a constructive method to generate all merged belief structures that satisfy the marginal constraints (7). From this method, we can induce guidelines as to how general bbas should be combined to result in an idempotent merging rule and a $\mathbb{C}$-least specific bba. We start from a work of Dubois and Yager [19], where they show the existence of lot of idempotent rules that combine two bbas by using the concept of commensurate bbas.

5.1. Commensurate bba's

In the following, we slightly generalize the notion of bba and consider it as a relation between the power set of $\mathcal{V}$ and $[0, 1]$. In other words, a generalized bba may assign several weights to the same subset of $\mathcal{V}$.

Definition 10. *Let m be a bba with focal sets $\{E_1, \ldots, E_n\}$ and associated weights $\{m^1, \ldots, m^n\}$. A split of m is a bba m' with focal sets $\{A'_1, \ldots, A'_{n'}\}$ and associated weights $\{m'^1, \ldots, m'^{n'}\}$ s.t. $\sum_{A'_j = E_i} m'^j = m^i$*

In other words, a split is a new bba where the original weight given to a focal set is separated in smaller weights given to the same focal set, with the sum of weights given to a specific focal set being constant. Two generalized bbas m_1, m_2 are said to be equivalent if $pl_1(E) = pl_2(E)$ and $bel_1(E) = bel_2(E)$ $\forall E \subseteq \mathcal{V}$. If m_1 and m_2 are equivalent, it means that they are splits of the same regular bba [19]. In the following, a bba should be understood as a generalized one.

Definition 11. *Let m_1, m_2 be two bbas with respective focal sets $\{E_1, \ldots, E_n\}$, $\{F_1, \ldots, F_k\}$ and associated weights $\{m_1^1, \ldots, m_1^n\}$, $\{m_2^1, \ldots, m_2^k\}$. Then, m_1 and m_2 are said to be commensurate if $k = n$ and there is a permutation σ of $\{1, \ldots, n\}$ s.t. $m_1^j = m_2^{\sigma(i)}, \forall i = 1, \ldots, n$.*

Two bbas are commensurate if their distribution of weights over focal sets can be described by the same vector of numbers. In [19], Dubois and Yager propose an algorithm, given a prescribed ranking of focal sets on each side, that makes any two bbas commensurate by successive splitting. Based on this algorithm, they provide an idempotent rule $\bigoplus$ that allows to merge any two bbas. This merging rule is conjunctive and the result depends on the ranking of focal sets used in the commensuration algorithm, summarized as follows:

- Let m_1, m_2 be two bbas and $\{E_1, \ldots, E_n\}$, $\{F_1, \ldots, F_k\}$ the two sets of ordered focal sets with weights $\{m_1^1, \ldots, m_1^n\}$, $\{m_2^1, \ldots, m_2^k\}$

- By successive splitting of each bbas (m_1, m_2), build two generalised bbas $\{R_1^1, \ldots, R_1^l\}$ and $\{R_2^1, \ldots, R_2^l\}$ with weights $\{m_{R_1}^1, \ldots, m_{R_1}^l\}$, $\{m_{R_2}^1, \ldots, m_{R_2}^l\}$ s.t. $m_{R_1}^i = m_{R_2}^i$ and $\sum_{R_1^i = E_j} m_{R_1}^i = m_1^j$, $\sum_{R_2^i = F_j} m_{R_2}^i = m_2^j$.

- Algorithm results in two commensurate generalised bbas m_{R_1}, m_{R_2} that are, respectively, equivalent to the original bbas m_1, m_2.

Once this commensuration is done, the conjunctive rule $\bigoplus$ proposed by Dubois and Yager defines a merged bba $m_{12} \in \mathcal{M}_{12}$ with focal sets $\{R_{1 \oplus 2}^i = R_1^i \cap R_2^i, i = 1, \ldots, l\}$ and associated weights $\{m_{R_{1 \oplus 2}}^i = m_{R_1}^i = m_{R_2}^i, i = 1 \ldots, l\}$. The whole procedure is illustrated by the following example.

Example 11. *Commensuration*

	m_1		m_2			l	m_{R^l}	R_1^l	R_2^l	$R_{1 \oplus 2}^l$
E_1	.5	F_1	.6			1	.5	E_1	F_1	$E_1 \cap F_1$
E_2	.3	F_2	.2	$\rightarrow$		2	.1	E_2	F_1	$E_2 \cap F_1$
E_3	.2	F_3	.1			3	.2	E_2	F_2	$E_2 \cap F_2$
		F_4	.1			4	.1	E_3	F_3	$E_3 \cap F_3$
						5	.1	E_3	F_4	$E_3 \cap F_4$

From this example, it is easy to see that the final result crucially depends of the chosen rankings of the focal sets of m_1 and m_2. In fact, it can be shown that any conjunctively merged bba in $\mathcal{M}_{12}$ can be produced in this way.

Definition 12. *Two commensurate generalised bbas are said to be equi-commensurate if each of their focal sets has the same weight.*

Any two bbas m_1, m_2 can be made equi-commensurate. In our example, bbas can be made equi-commensurate by splitting the first line into five similar lines of weight 0.1 and the third line into two similar lines of weight 0.1. Every line then has weight 0.1, and applying Dubois and Yager's rule to these bbas yields a bba equivalent to the one obtained before equi-commensuration. Combining two equi-commensurate bbas $\{R_1^1, \ldots, R_1^l\}$, $\{R_2^1, \ldots, R_2^l\}$ by Dubois and Yager rule results in a bba s.t. every focal element in $\{R_{1 \oplus 2}^1, \ldots, R_{1 \oplus 2}^l\}$ has equal weight $m_{R_{1 \oplus 2}}$ (0.1 in our example). The resulting bba is still in $\mathcal{M}_{12}$.

Proposition 7. *Any merged bba in $\mathcal{M}_{12}$ can be approached as close as possible by means of Dubois and Yager rule using appropriate commensurate bbas equivalent to m_1 and m_2 and the two appropriate rankings of focal sets.*

Proof. We first assume masses (of marginal and merged bbas) are rational numbers that have an arbitrary finite number of decimals. Let $m \in \mathcal{M}_{12}$ be the merged bba we want to reach by using Dubois and Yager's rule. Let $m(E_i, F_j)$ be the mass allocated to $E_i \cap F_j$ in m. It is of the form $k_{12}(E_i, F_j) \times 10^{-n}$ where k_{12}, n are integers. By successive splitting followed by a reordering of elements R_1^j, we can always reach m. For instance, let k_R be equal to the greatest common divisor of all values $k_{12}(E_i, F_j)$. Then, $k_{12}(E_i, F_j) = q_{ij} \times k_R$, for an integer q_{ij}. Then, it suffices to re-order elements R_1^k by a re-ordering σ s.t. for q_{ij} of them, $R_1^k = E_i$ and $R_2^{\sigma(k)} = F_j$. Then, by applying Dubois and Yager's rule, we obtain the result m. Rational numbers being dense in reals, this means that we can always get as close as we want to any merged bba by considering a sufficiently large number of decimals. $\square$

5.2. A property of $\mathbb{C}$-least committed merging

Although Example 10 and the associated comment in the previous section show that maximising expected cardinality is not enough to respect the WICFMP, we see at least two reasons why it is interesting to study $\mathbb{C}$-least specific bbas and to retain this criterion to pick a single conjunctively merged belief structure:

- expected cardinality has a simple relation with contour functions, as Eq. (6) shows,

- if there is a merged belief structure in $\mathcal{M}_{12}$ satisfying the SCFMP, then it will be $\mathbb{C}$-least specific. As finding $\mathbb{C}$-least specific belief functions can be done by solving a unique linear program, searching for merged belief structures maximising the expected cardinality can be seen as an easy way to check that two marginal belief structures satisfy the SCFMP.

Let us now show that, in order to maximise cardinality by using commensurate bbas, chosen rankings should be extensions of the partial ordering induced by inclusion (i.e., $E_i < E_j$ if $E_i \subset E_j$), which is the central notion of consonant bbas. This is due to the following result:

Lemma 2. *Let A, B, C, D be four sets s.t. $A \subseteq B$ and $C \subseteq D$. Then, we have the following inequality*

$$\mathbb{C}(A \cap D) + \mathbb{C}(B \cap C) \leq \mathbb{C}(A \cap C) + \mathbb{C}(B \cap D) \tag{12}$$

Proof. From the assumption, the inequality $\mathbb{C}((B \setminus A) \cap C) \leq \mathbb{C}((B \setminus A) \cap D)$ holds. Then consider the following equivalent inequalities:

$$
\begin{aligned}
\mathbb{C}((B \setminus A) \cap C) + \mathbb{C}(A \cap C) &\leq \mathbb{C}(A \cap C) + \mathbb{C}((B \setminus A) \cap D) \\
\mathbb{C}(B \cap C) &\leq \mathbb{C}(A \cap C) + \mathbb{C}((B \setminus A) \cap D) \\
\mathbb{C}(A \cap D) + \mathbb{C}(B \cap C) &\leq \mathbb{C}(A \cap C) + \mathbb{C}(A \cap D) + \mathbb{C}((B \setminus A) \cap D) \\
\mathbb{C}(A \cap D) + \mathbb{C}(B \cap C) &\leq \mathbb{C}(A \cap C) + \mathbb{C}(B \cap D)
\end{aligned}
$$

hence the inequality (12) is true. $\square$

When using equi-commensurate bbas, masses in the formula of expected cardinality can be factorized, and expected cardinality then becomes

$$\mathbb{C}(m_{m_{R_1 \oplus 2}}, \mathcal{F}_{m_{R_1 \oplus 2}}) = m_{R_1 \oplus 2} \sum_{i=1}^{l} \mathbb{C}(R_{1 \oplus 2}^i) = m_{R_1 \oplus 2} \sum_{i=1}^{l} \mathbb{C}(R_1^i \cap R_2^i),$$

where $m_{R_1 \oplus 2}$ is the smallest interval length enabling mass equi-commensuration. We are now ready to prove the following proposition

Proposition 8. *If $m \in \mathcal{M}_{12}$ is $\mathbb{C}$-least specific, there exists an idempotent conjunctive merging rule $\bigwedge$ constructing m by the commensuration method, s.t. focal sets are ranked on each side in agreement with the partial order of inclusion.*

Proof. Suppose $\widehat{m}_{12} \in \mathcal{M}_{12}$ is minimally committed for expected cardinality. It can be obtained by commensuration. Let m_{R_1}, m_{R_2} be the two equi-commensurate bbas with n elements each derived from the two original bbas m_1, m_2. Suppose that the rankings used display four focal sets $R_1^i, R_1^j, R_2^i, R_2^j$, $i < j$, such that $R_1^i \supset R_1^j$ and $R_2^i \subseteq R_2^j$. By Lemma 2, $\mathbb{C}(R_1^j \cap R_2^j) + \mathbb{C}(R_1^i \cap R_2^i) \leq \mathbb{C}(R_1^j \cap R_2^i) + \mathbb{C}(R_1^i \cap R_2^j)$. Hence, if we permute focal sets R_1^i, R_1^j before applying Dubois and Yager's merging rule, we end up with a merged bba $m_{R'_{1 \oplus 2}}$ s.t. $\mathbb{C}(m_{R_{1 \oplus 2}}, \mathcal{F}_{R_{1 \oplus 2}}) \leq \mathbb{C}(m_{R'_{1 \oplus 2}}, \mathcal{F}_{R'_{1 \oplus 2}})$. Such a permutation can be done (iteratively) for any collection of focal sets satisfying Lemma 2 conditions, each time increasing expected cardinality. If there are no such collection, then the ranking of focal sets of m_{R_1}, m_{R_2} is in agreement with the partial order of inclusion. Since any merged bba can be reached by splitting m_1, m_2 and by inducing the proper ranking of focal sets of the resulting bbas m_{R_1}, m_{R_2}, any merged bba $\widehat{m}_{12} \in \mathcal{M}_{12}$ maximizing expected cardinality can be reached by Dubois and Yager's rule, using rankings of focal sets in accordance with the inclusion ordering. $\qquad\square$

However, ranking focal sets in accordance with inclusion is neither sufficient nor the only way of maximizing expected cardinality when merging two given bbas, as shown by the following examples.

Example 12. *Let m_1, m_2 be two bbas of the space $\mathcal{V} = \{v_1, v_2, v_3\}$. Let $m_1(E_1 = \{v_1, v_2\}) = 0.5, m_1(E_2 = \{v_1, v_2, v_3\}) = 0.5$ be the two focal sets of m_1 and $m_2(F_1 = \{v_1, v_2\}) = 0.2, m_2(F_2 = \{v_2\}) = 0.3, m_2(F_3 = \{v_1, v_2, v_3\}) = 0.5$ be the focal sets of m_2. The following table shows the result of Dubois and Yager's merging rule after commensuration:*

l	m_{R^l}	R_1^l	R_2^l	$R_{1 \oplus 2}^l$
1	*.2*	E_1	F_1	$E_1 \cap F_1 = \{v_1, v_2\}$
2	*.3*	E_1	F_2	$E_1 \cap F_2 = \{v_2\}$
3	*.5*	E_2	F_3	$E_2 \cap F_3 = \{v_1, v_2, v_3\}$

Although focal sets F_i are not ordered by inclusion ($F_1 \supset F_2$), the result maximizes expected cardinality (the result is m_2, which is a specialization of m_1). This shows that the technique based on proposition 8 is not necessary. Nevertheless, the same result is obtained by using order F_2, F_1, F_3, as any $\mathbb{C}$-least specific bbas can be by the technique based on Proposition 8.

Now, consider the same bba m_1 and another bba m_2 s.t. $m_2(F_1 = \{v_2\}) = 0.3$, $m_2(F_2 = \{v_2, v_3\}) = 0.3$, $m_2(F_3 = \{v_1, v_2\}) = 0.1$, $m_2(F_4 = \{v_1, v_2, v_3\}) = 0.3$. m_2 is no longer a specialization of m_1, and the ranking F_1, F_2, F_3, F_4 is one of the two possible extensions of the partial order induced by inclusion. The result of Dubois and Yager's rule gives us:

l	m_{R^l}	R_1^l	R_2^l	$R_{1 \oplus 2}^l$
1	*.2*	E_1	F_1	$E_1 \cap F_1 = \{v_2\}$
2	*.3*	E_1	F_2	$E_1 \cap F_2 = \{v_2\}$
3	*.1*	E_2	F_2	$E_2 \cap F_2 = \{v_2, v_3\}$
4	*.1*	E_2	F_3	$E_2 \cap F_3 = \{v_1, v_2\}$
5	*.3*	E_2	F_4	$E_1 \cap F_4 = \{v_1, v_2, v_3\}$

and the expected cardinality of the merged bba is 1.8. *If, instead of the ranking F_1, F_2, F_3, F_4, we choose the order F_1, F_3, F_2, F_4 (i.e., the other extension of the partial order induced by inclusion), applying Dubois and Yager's rule gives us a merged bba of expected cardinality* 2.0, *which is greater than the previous one. Hence, we see that proposition 8 is not sufficient in general to reach maximal cardinality. Thus, proposition 8 gives us guidelines for combining belief functions so as to maximise cardinality, but further conditions should be stated to select the proper total orderings of focal sets.*

Finally it is clear that any conjunctive merging rule that ranks focal sets in agreement with the inclusion ordering prior to commensuration is an extension of the minimum rule of possibility theory : the latter can be precisely obtained by following this method, since in the consonant case the inclusion ordering is linear.

6. Related Works

We can distinguish three main kinds of alternative approaches in the literature that deal with combination of belief functions when independence (or distinctness) between sources cannot be assumed:

- the first and most common one consists in selecting, among the belief functions in $\mathcal{M}_{12}$, one of the least-committed one according to one of the ordering considered in Section 2.3, possibly requiring the rule to satisfy additional properties. Denoeux [8] and Cattaneo [2] proposals, as well as our proposition of maximising expected cardinality, fall into this category;

- the second one consists in studying how cautious and idempotent rules originating from other uncertainty theories can be adapted to the belief function setting. Chateauneuf [4] has studied how intersection of sets of probabilities inducing belief functions could be interpreted in terms of bba, while studying the extension of the minimum possibilistic rule was the central topic of this paper. This approach has been already discussed in previous sections;

- the third one, investigated by Smets [32] and more recently by Kallel et al. [20], consists in identifying the precise dependence structure between sources and then to combine the marginal belief structures accordingly. This approach is quite different from the two others, in which one the dependence structure is considered to be ill-known.

6.1. Minimizing conflict

When two bbas are not logically consistent (i.e., there are focal elements E_i, F_j for which $E_i \cap F_j = \emptyset$), a conjunctively merged bba that maximizes expected cardinality may not, in general, minimize conflict (i.e., $m \in \mathcal{M}_{12}$ s.t. $m(\emptyset)$ is minimal). This is illustrated by the following example:

Example 13. *Consider the two following possibility distributions* π_1, π_2, *expressed as belief structures* m_1, m_2

$\pi_1 = m_1$		$\pi_2 = m_2$	
Focal sets	*Mass*	*Focal sets*	*Mass*
$\{v_1, v_2\}$	*0.5*	$\{v_4\}$	*0.5*
$\{v_0, v_1, v_2, v_3, v_4\}$	*0.5*	$\{v_2, v_3, v_4, v_5, v_6\}$	*0.5*

And the following table shows the result of applying the minimum possibilistic rule (thus maximising expected cardinality or selecting the q-least committed bba in $\mathcal{M}_{12}$*) and the unnormalised Dempster rule of combination*

$Min(\pi_1, \pi_2)$		unnormalised Dempster's rule			
Focal sets	*Mass*	*Focal sets*	*Mass*	*Focal sets*	*Mass*
$\{v_2, v_3, v_4\}$	*0.5*	$\{v_2\}$	*0.25*	$\{v_2, v_3, v_4\}$	*0.25*
$\emptyset$	*0.5*	$\{v_4\}$	*0.25*	$\emptyset$	*0.25*

With Dempster rule, conflict value is 0.25 and expected cardinality is 1.25, while with the minimum, the conflict value is 0.5 and expected cardinality is 1.5.

Provided one considers that minimizing the conflict is as desirable as finding a least-committed way of merging the information, this can problematic. A possible alternative is then to find $m \in \mathcal{M}_{12}$ that is least-committed among those for which $m(\emptyset)$ is minimal. This problem was studied by Cattaneo in [2]. Cattaneo proposes to find the merged bba $m \in \mathcal{M}_{12}$ that maximizes the following function:

$$F(m) = m(\emptyset)f(0) + (1 - m(\emptyset)) \sum_{A \neq \emptyset} m(A)f(|A|) \tag{13}$$

with f s.t. $f(0) \leq -|\mathcal{V}|$ and $f(n) = \log_2 n$ for all $n > 2$. In the above equation, $m(\emptyset)f(0)$ can be seen as a penalty given to the evaluation of the merged belief when conflict appears, while the second part of the right-hand side of equation (13) is equivalent to expected cardinality where $|A|$ is replaced by $\log_2(|A|)$

(more generally, we can replace $|A|$ by any non-decreasing function $f(|A|)$ from $\mathbb{N}$ to $\mathbb{R}$). A similar strategy (penalizing the appearance of conflict) could thus be adopted, along with expected cardinality (or with any function $f(|A|)$).

Now, the claim that a cautious conjunctive rule should give a merged bba where the conflict is minimized is questionable. This is shown by our small example 13, where minimizing the conflict, by assigning zero mass to empty intersections while respecting the marginals, produces the bba $m(\{v_2\}) = 0.5, m(\{v_4\}) = 0.5$, which is the only probability distribution distribution in $\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2}$. Indeed, this bba is the most precise result possible, and its informational content is clearly more precise (i.e., less cautious) than the bba corresponding to $\min(\pi_1, \pi_2)$.

6.2. Least commitment based on the weight function

Given weight functions w_1, w_2 originating from m_1, m_2 and computed with Eq.(5), Denoeux proposes to apply the following cautious rule to weight functions:

$$w_{12}(A) = \min(w_1(A), w_2(A)), \forall A \neq \mathcal{V}.$$

and he shows that it produces the weight function of the least w-committed merged bba among those that are more w-committed than both marginals m_1 and m_2. Note that our approach only requires the result to be more s-committed than m_1 and m_2. which is a weaker condition than to be more w-committed. Now, let us compare Denoeux's rule vs. the maximisation of expected cardinality to the following example (Example 2 of [7]).

Example 14. : *Consider $\mathcal{V} = \{a, b, c\}$, m_1 defined by $m_1(\{a, b\}) = 0.3$, $m_1(\{b, c\}) = 0.5$, $m_1(\mathcal{V}) = 0.2$; m_2 defined by $m_2(\{b\}) = 0.3$, $m_2(\{b, c\}) = 0.4$, $m_2(\mathcal{V}) = 0.3$. Results of both rules are given in the following table*

Denoeux's rule (m^D)				Max. Exp. Card. rule (m^C)			
Focal Sets	Mass	Focal Sets	Mass	Focal Sets	Mass	Focal Sets	Mass
$\{b\}$	0.6	$\{b, c\}$	0.2	$\{b\}$	0.3	$\mathcal{V}$	0.2
$\{a, b\}$	0.12	$\mathcal{V}$	0.08	$\{b, c\}$	0.5		

In this example, our conjunctive cautious rule yields a merged bba m^C that is s-less committed (and hence has a greater expected cardinality) than m^D obtained with Denoeux's rule (note that $m^D \notin \mathcal{M}_{12}$, i.e., it is not a conjunctive merging rule obtained by the two-stepped procedure of Sec. 2). Nevertheless, the merged bba obtained by maximizing expected cardinality is not comparable in the sense of the w-ordering with any of the three other bbas (m_1, m_2, m^D), nor does it fulfil Denoeux's condition of being more w-committed than m_1 and m_2. The cautious w-merging of possibility distributions does not reduce to the minimum rule either. Thus, the two approaches are at odds. As it seems, using the w-ordering allows to easily find a unique least-committed element. See [8] for a more detailed discussion on these issues.

In his paper, Denoeux generalizes both the TBM conjunctive rule and his cautious rule with triangular norms [21]. However, the set of non-dogmatic belief functions equipped with the TBM conjunctive rule forms a group, as is the product of positive w-numbers. So the relevant setting for generalizing the product of weight functions seems to be the one of uninorms [38], that is, non-decreasing semi-group operations on the unit interval whose identity lies strictly between 0 and 1. But the minimum is not a uninorm on the positive real line. It is the greatest t-norm on $[0, 1]$, in particular, greater than product, and this property is in agreement with minimal commitment of contour functions. But the minimum rule no longer dominates the product on the positive real line, so that the bridge between Denoeux's idempotent rule and the idea of minimal commitment is not obvious beyond the w-ordering. See [27, 28] for developments along these lines.

6.3. Identifying dependency structures

The last approach consists in identifying the existing dependency structure between the two sources, that is to construct the precise joint structure (see Eq.(7)) to be used in the conjunctive merging of belief functions. In his proposal, Smets [32] considers that, given two marginal bbas m_1, m_2, their dependence

structure (that he calls correlation) can be represented as a bba m_0 representing the corpus of common knowledge shared by m_1 and m_2.

Provided one also knows the joint conjunctive belief structure underlying m_1, m_2, say $m_{1 \wedge 2}$, the commonality function q_0 of the bba m_0 representing the dependency structure is given, for any event $A \subseteq \mathcal{V}$, by the following formula:

$$q_0(A) = \frac{q_1(A)q_2(A)}{q_{1 \wedge 2}(A)}, \tag{14}$$

with $q_1, q_2, q_{1 \wedge 2}$ the commonality functions of, respectively, m_1, m_2 and $m_{1 \wedge 2}$. As emphasized by Smets, constructing m_0 without knowing $m_{1 \wedge 2}$ would require an "in-depth comparison of the origin of the pieces of evidence that have induced m_1 and m_2".

Recently, Kallel et al. [20] have proposed an approach to computing the bba m_0 from the sole knowledge of m_1 and m_2. In order to do so, they propose to replace the bba $m_{1 \wedge 2}$ in Eq. 14 respectively by the bba given by Denoeux's cautious rule when m_1, m_2 are not consonant and by the result of the minimum possibilistic rule when they are consonant (recall that this is the q-least committed).

From a theoretical point of view, this proposition is questionable: indeed, if no evidence is available, even if choosing a least-committed bba as the proper unknown joint conjunctive belief structure sounds natural, exploiting the correlation information computed from this assumption in further computations is debatable. And, at the theoretical level, switching from a q-least committed solution to a w-least committed joint structure according to whether the information is consonant or not is not consistent with the fact that the w-ordering based idempotent rule is not in agreement with possibility theory.

A more (theoretically) convincing approach to measure source dependency and identify a parametrised conjunctive rule from it is given by Quost et al. [29], who propose to use correlation and distance measures, similarly to what is done in statistics when measuring data correlations. The drawback of this approach is that it requires data from which the dependence can be inferred. It cannot be applied to situations where such data are unavailable e.g., different experts providing information.

7. Conclusion

In this paper, we have studied the link between the idempotent minimum rule, used in possibility theory to cautiously merge possibility distributions, and the more general framework of belief structures, by trying to extend the minimum rule to the latter. In order to achieve such an extension, we have proposed two principles, respectively the strong and weak idempotent contour function merging principles. These principles require that the contour function of the belief structure after merging be equal to the minimum of the original contour functions. Our results indicate that the strong version of this principle cannot be always satisfied by the resulting selected least committed belief structure. However it is relatively easy to satisfy the weaker version of the idempotent contour function merging principle, provided that the result of the merging could be a set of belief structures. In the latter case, restricting to least π-committed merged belief structures appears to be a good solution.

At the theoretical level, it becomes clear that there is no canonical idempotent merging rule in the non-consonant case, and that further assumptions on the dependence structure between sources are needed to select the proper combination mode if a single merged belief structure is to be selected. These results tend to confirm the claim that sets of belief structures should be used in place of a single one, particularly when dependencies are ill-known. This is in agreement with similar treatments done with precise probabilities when dependencies between variables are not known [37]. Section 4.1 also indicates that restricting ourselves to normalised merged belief functions may be too restrictive if we want to comply with the minimum rule. This indicates that requiring coherence is also perhaps too strong a requirement in some situations. This is in agreement with the Transferable Belief Model [35] and the open world assumption, where subnormalised belief structures are authorised.

From a practical standpoint, our results are incomplete, as they do not lead to an easy-to-use cautious merging rule extending the idempotent possibilistic rule. Nevertheless the paper provides clear guidelines in the form of additional zero-mass constraints on the joint belief structures needed if the merging process

is to satisfy the minimum rule. However, there is a need for computational methods that generate sets of x-least committed belief functions with $x \in \{s, p, q, \pi\}$. The commensuration method exploiting the partial inclusion order of focal elements, and the suitable use of the zero-mass constraints may be helpful to alleviate the computational burden. It is easier to obtain the subset of least committed merged bbas with maximal expected cardinality, as they are easy to compute by linear programming, even if they do not always jointly satisfy the WICFMP. However there is a need to develop practical methods that allow us to retrieve the set of x-least committed merged bbas (with $x \in \{s, p, q, \pi\}$) from the information provided by the marginals m_1, m_2. A solution could come from the exploration of geometrical properties of the set $\mathcal{M}_{12}$ of results of conjunctive combinations. As this set happens to be a convex polytope, it would be worth characterising the nature of its extreme points, notably in terms of x-least commitment.

Acknowledgements

References

[1] Th. Augustin, *Generalized basic probability assignments*, Int. J. of General Systems **34** (2005), no. 4, 451–463.

[2] M. Cattaneo, *Combining belief functions issued from dependent sources.*, Proc. third international symposium on imprecise probabilities and their application (isipta'03), 2003, pp. 133–147.

[3] Marco E. G. V. Cattaneo, *Belief functions combination without the assumption of independence of the information sources*, Int. J. Approx. Reasoning **52** (2011), no. 3, 299–315.

[4] Alain Chateauneuf, *Combination of compatible belief functions and relation of specificity*, Advances in the dempster-shafer theory of evidence, 1994, pp. 97–114.

[5] A.P. Dempster, *Upper and lower probabilities induced by a multivalued mapping*, Annals of Mathematical Statistics **38** (1967), 325–339.

[6] T. Denoeux, *Reasoning with imprecise belief structures*, Int. J. of Approximate Reasoning **20** (1999), 79–111.

[7] T. Denoeux, *Constructing belief functions from sample data using multinomial confidence regions*, Int. J. of Approximate Reasoning **42** (2006), 228–252.

[8] T. Denoeux, *Conjunctive and disjunctive combination of belief functions induced by non-distinct bodies of evidence*, Artificial Intelligence **172** (2008), 234–264.

[9] S. Destercke and D. Dubois, *Can the minimum rule of possibility theory be extended to belief functions?*, Ecsqaru, 2009, pp. 299–310.

[10] S. Destercke, D. Dubois, and E. Chojnacki, *Cautious conjunctive merging of belief functions*, Ecsqaru, 2007, pp. 332–343.

[11] D. Dubois and H. Prade, *On the unicity of the dempster rule of combination*, Int. J. of Intelligent Systems **1** (1986), 133–142.

[12] D. Dubois and H. Prade, *A set-theoretic view on belief functions: logical operations and approximations by fuzzy sets*, I. J. of General Systems **12** (1986), 193–226.

[13] D. Dubois and H. Prade, *Possibility theory: An approach to computerized processing of uncertainty*, Plenum Press, New York, 1988.

[14] D. Dubois and H. Prade, *Representation and combination of uncertainty with belief functions and possibility measures*, Computational Intelligence **4** (1988), 244–264.

[15] D. Dubois and H. Prade, *Fuzzy sets, probability and measurement*, European J. of Operational Research **40** (1989), 135–154.

[16] D. Dubois and H. Prade, *Consonant approximations of belief functions*, Int. J. of Approximate reasoning **4** (1990), 419–449.

[17] D. Dubois, H. Prade, and P. Smets, *New semantics for quantitative possibility theory*, Isipta'01, proceedings of the second international symposium on imprecise probabilities and their applications, 2001.

[18] D. Dubois, H. Prade, and P. Smets, *A definition of subjective possibility*, Int. J. of Approximate Reasoning **48** (2008), 352–364.

[19] D. Dubois and R.R. Yager, *Fuzzy set connectives as combination of belief structures*, Information Sciences **66** (1992), 245–275.

[20] A. Kallel and S. Le Hegarat-Mascle, *Combination of partially non-distinct beliefs: The cautious-adaptive rule*, Int. J. of Approximate Reasoning **50** (2009), 1000–1021.

[21] E.P. Klement, R. Mesiar, and E. Pap, *Triangular norms*, Kluwer Academic Publisher, Dordrecht, 2000.

[22] G.J. Klir, *Uncertainty and information measures for imprecise probabilities : An overview*, Proc. 1st international symposium on imprecise probabilities and their applications, 1999.

[23] I. Levi, *The enterprise of knowledge*, MIT Press, London, 1980.

[24] G. Matheron, *Random sets and integral geometry*, Wiley, New York, 1975.

[25] Ilya Molchanov, *Theory of random sets*, Springer, London, 2005.

[26] S. Moral and J. Sagrado, *Aggregation of imprecise probabilities*, Aggregation and fusion of imperfect information, 1997, pp. 162–188.

[27] F. Pichon and T. Denoeux, *The unnormalized dempster's rule of combination: a new justification from the least commitment principle and some extensions.*, J. of Automated Reasoning **45** (2010), no. 1, 61–87.

[28] Frédéric Pichon and T. Denoeux, *T-norm and uninorm-based combination of belief functions*, Proceedings of the annual meeting of the north american fuzzy information processing society, nafips 2008, new york, n.y., 2008.

[29] B. Quost, T. Denoeux, and M.H. Masson, *Adapting a combination rule to non-independent information sources*, Proceedings of ipmu'08, 2008, pp. 448–455.

[30] G. Shafer, *A mathematical theory of evidence*, Princeton University Press, New Jersey, 1976.

[31] P. Smets, *Combining nondistinct evidences*, Int. conf. of the north american fuzzy information processing society (nafips'86), 1986, pp. 544–549.

[32] P. Smets, *The concept of distinct evidence*, Proceedings of the 4th conference on information processing and management of uncertainty in knowledge-based systems, ipmu 92, 1992july, pp. 789–794.

[33] P. Smets, *The canonical decomposition of a weighted belief*, Proc. int. joint. conf. on artificial intelligence, 1995, pp. 1896–1901.

[34] P. Smets, *Analyzing the combination of conflicting belief functions*, Information Fusion **8** (2006), 387–412.

[35] P. Smets and R. Kennes, *The transferable belief model*, Artificial Intelligence **66** (1994), 191–234.

[36] P. Walley, *Statistical reasoning with imprecise probabilities*, Chapman and Hall, New York, 1991.

[37] R.C. Williamson and T. Downs, *Probabilistic arithmetic i : Numerical methods for calculating convolutions and dependency bounds*, I. J. of Approximate Reasoning **4** (1990), 8–158.

[38] R. R. Yager and A. Rybalov, *Uninorm aggregation operators*, Fuzzy Sets and Systems **80** (1996), 111–120.

[39] L.A. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems **1** (1978), 3–28.