A generalization of Fermat-Catalan conjecture

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An algebraic-analytic approach of Fermat-Catalan conjecture

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Sommaire

Nous commençons avec l'équation de Beal (ou Fermat-Catalan) $U^{c+2} = X^{a+2}+Y^{b+2}$ puis nous definissons deux équations qui en sont conséquentes. Après en avoir déduit quelques résultats, nous généralisons l'approche à toutes les équations de Fermat-Catalan ; ce qui nous permet de relier le problème au théorème de Matyasevitch. En effet, nos investigations vont nous amener à établir des égalités qui ne s'expliquent que par l'indécidabilité de certaines conjectures inhérentes aux équations diophantiennes. Notre approche nous permet alors également de proposer une nouvelle conjecture concernant les équations de Fermat-Catalan généralisées. Nous proposons, d'autre part, une démonstration de la conjecture de BeaL. (MSC=11D04) Mots clés : Fermat-Catalan ; Equations diophantiennes ; Analyse ; Suites ; Séries ; Séries de Fourier ; Conjecture.

Abstract

We begin with Beal equation (or Fermat-Catalan) $U^{c+2} = X^{a+2} + Y^{b+2}$ and establish two equivalent equations. We generalize the approach to all Fermat-Catalan equations which allows us to relate the problem to Matyasevich theorem. Our approach will lead us to propose a new conjecture concerning Fermat-Catalan equations. We propose also a proof of Beal conjecture. (MSC=11D04) Keywords : Fermat-Catalan ; Diophantine equations ; Analysis ; Series ; Fourier series ; Conjecture.

The approach

Let

$$u = U^{2(c+2)}$$

$$x = U^{c+2}X^{a+2}$$

$$y = U^{c+2}Y^{b+2}$$

$$z = X^{a+2}Y^{b+2}$$

We have

$$u = U^{2(c+2)} = U^{c+2}(X^{a+2} + Y^{b+2}) = x + y \qquad (1)$$

And

$$\frac{1}{z} = \frac{1}{X^{a+2}Y^{b+2}} = \frac{U^{2(c+2)}}{U^{c+2}X^{a+2}U^{c+2}Y^{b+2}} = \frac{u}{xy} = \frac{x+y}{xy} = \frac{1}{x} + \frac{1}{y} \qquad (2)$$

Lemma 1 If U, X, Y, a, b, c are integers verifying $U^{c+2} = X^{a+2} + Y^{b+2}$, then u, x, y, z as defined higher verify simultaneously (1) and (2) which follow

$$u = x + y \qquad (1)$$

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y} \qquad (2)$$

Or

$$z = X^{a+2}Y^{b+2}$$

$$u = (X^{a+2} + Y^{b+2})^2$$

$$x = X^{a+2}(X^{a+2} + Y^{b+2})$$

$$y = Y^{b+2}(X^{a+2} + Y^{b+2})$$

Let us define the sequences whose first terms are

$$x_1 = x$$

And

$$y_1 = y$$

But $\forall x_1, y_1$, $\exists z_1$ verifying

$$\frac{1}{z_1} = \frac{1}{x_1} + \frac{1}{y_1}$$

And

$$z_1 = \frac{xy}{x+y} = z$$

Or

$$(x_1 + y_1)z_1 = x_1 y_1$$

And

$$x_1(y_1 - z_1) = z_1 y_1$$

Let us pose

$$y_2 = y_1 - z_1 = \frac{z_1 y_1}{x_1}$$

And

$$y_1(x_1 - z_1) = z_1 x_1$$

Also

$$x_2 = x_1 - z_1 = \frac{z_1 x_1}{y_1}$$

So, we have

$$x_2 y_2 = z_1^2$$

Or

$$x_1 = x_2 + z_1 = x_2 + \sqrt{x_2 y_2}$$

And

$$y_1 = y_2 + z_1 = y_2 + \sqrt{x_2 y_2}$$

With

$$u_1 = u = (x_1 + y_1) = (\sqrt{x_2} + \sqrt{y_2})^2 > x_2 + y_2 > 0$$

$(x_1 + y_1)$ integer

$$x_1 = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) > x_2 > 0$$

x_1 integer

$$y_1 = \sqrt{y_2}(\sqrt{x_2} + \sqrt{y_2}) > y_2 > 0$$

y_1 integer

$$z_1 = \frac{x_1 y_1}{x_1 + y_1} = X^{a+2} Y^{b+2} = \sqrt{x_2 y_2} > z_2 = \frac{x_2 y_2}{x_2 + y_2} > 0$$

z_2 rational
Because
$\forall x_2, y_2$ rational, $\exists z_2$ rational which verifies

$$\frac{1}{z_2} = \frac{1}{x_2} + \frac{1}{y_2}$$

Until infinity. For i

$$(x_i + y_i) = (\sqrt{x_{i+1}} + \sqrt{y_{i+1}})^2 > x_{i+1} + y_{i+1} > 0$$

And $x_i + y_i$ rational for $i > 2$

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) > x_{i+1} > 0$$

x_i rational for $i > 2$

$$y_i = \sqrt{y_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) > y_{i+1} > 0$$

y_i rational for $i > 2$

$$z_i = \frac{x_i y_i}{x_i + y_i} = \sqrt{x_{i+1} y_{i+1}} > z_{i+1} = \frac{x_{i+1} y_{i+1}}{x_{i+1} + y_{i+1}} > 0$$

z_i rational for $i > 1$ and also, of course

$$\frac{1}{z_{i+1}} = \frac{1}{x_{i+1}} + \frac{1}{y_{i+1}}$$

Lemma 2 The expressions of the sequences are

$$(x_i + y_i) = (\sqrt{x_{i+1}} + \sqrt{y_{i+1}})^2$$

$$x_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \qquad (3)$$

$$y_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-1} \qquad (4)$$

Prof of lemma 2 We prove it by induction, we have

$$x = \sqrt{x_2}(\sqrt{x_2} + \sqrt{y_2}) = \sqrt{x_2}(x+y)^{\frac{1}{2}}$$

$$x_2 = \frac{x^2}{x+y}$$

Also

$$y_2 = \frac{y^2}{x+y}$$

The lemma is verified for $i = 2$. Let us suppose it verified for i, from (3) and (4)

$$x_i = \sqrt{x_{i+1}}(\sqrt{x_{i+1}} + \sqrt{y_{i+1}}) = \sqrt{x_{i+1}}(x_i + y_i)^{\frac{1}{2}}$$

Which means

$$x_{i+1} = x_i^2(x_i + y_i)^{-1}$$

But the lemma is verified for i, it leads us to

$$x_{i+1} = x^{2^i} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})^{-2} (x^{2^{i-1}} + y^{2^{i-1}})^{-1} \prod_{j=0}^{j=i-2} (x^{2^j} + y^{2^j})$$

$$= x^{2^i} \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

The lemma is verified for $i+1$, the proof is the same for y_i, we deduce

$$x_i = x^{(2^{i-1})} \prod_{j=0}^{j=i-2} (x^{(2^j)} + y^{(2^j)})^{-1}$$

And for y_i

$$y_i = y^{(2^{i-1})} \prod_{j=0}^{j=i-2} (x^{(2^j)} + y^{(2^j)})^{-1}$$

$$\forall i > 1$$

But

$$x - y = (x^{(2^{i-1})} - y^{(2^{i-1})}) \prod_{j=0}^{j=i-2} (x^{(2^j)} + y^{(2^j)})^{-1} \tag{5}$$

And

$$\prod_{j=0}^{j=i-2} (x^{(2^j)} + y^{(2^j)}) = \frac{(x^{(2^{i-1})} - y^{(2^{i-1})})}{x - y}$$

Lemma 3 This lemma is not necessary, we establish it because it will allow us to meet generalised Fermat numbers $x^{2^{i-1}}$. We must suppose here that

$$x \neq y$$

or

$$x_i \neq y_i$$

$$x_i = \frac{x^{(2^{i-1})}}{x^{(2^{i-1})} - y^{(2^{i-1})}}(x - y) = U^{c+2} \frac{X^{(a+2)2^{i-1}}}{X^{(a+2)2^{i-1}} - Y^{(b+2)2^{i-1}}}(X^{a+2} - Y^{b+2})$$

And

$$y_i = \frac{y^{(2^{i-1})}}{x^{(2^{i-1})} - y^{(2^{i-1})}}(x - y) = U^{c+2} \frac{Y^{(b+2)2^{i-1}}}{X^{(a+2)2^{i-1}} - Y^{(b+2)2^{i-1}}}(X^{a+2} - Y^{b+2})$$

$$u_i = x_i + y_i = U^{c+2} \frac{X^{(a+2)2^{i-1}} + Y^{(b+2)2^{i-1}}}{X^{(a+2)2^{i-1}} - Y^{(b+2)2^{i-1}}}(X^{a+2} - Y^{b+2})$$

Lemma 4

$$x_i - y_i = x - y$$

From lemma 3, this lemma is evident, and also for $x = y$, we have

$$x_i - y_i = x_{i+1} - y_{i+1} = x - y$$

Lemma 5 We have supposed $x - y \neq 0$. A priori nothing allows to say that x is différent or equal to y. Nonetheless, our investigations leaded ud to a strange result, which is that $x = y$, without any condition on x and y. Why this impossible result ? We think about Matyasavitch theorem. All diophantine equations have not solutions and the conjectures linked to those equations are not all decidable. But, the sequences established here are available for all Fermat-Catalan equations, even for the following one $kU^n = k_1X_1^{n_1} + k_2X_2^{n_2} + ... + k_iX_i^{n_i}$. This equation resumes all Fermat-Catalan equations. Nowadays, we do not know when there are solutions and when there are not. But, if we pose

$$u = k^2U^{2n}$$

$$x = kU^n(kU^n - k_jX_j^{n_j})$$

$$y = kU^nk_jX_j^{n_j}$$

$$z = k_jX_j^{n_j}(kU^n - k_jX_j^{n_j})$$

we find lemma (1) :

$$u = x + y$$

and

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$$

If there is an undecidability, those sequences should lead to an impossibility. The impossibuility is $xy(x - y) = 0$ for all x, y. We will prove that $xy(x - y) = 0$ formally. We have a lot of proofs. Here are some. THe first utilises series and particularly Fourier series. Effectively, as

$$x_i = \frac{x^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y)$$

And

$$y_i = \frac{y^{2^{i-1}}}{x^{2^{i-1}} - y^{2^{i-1}}}(x - y)$$

We can say

$$x \geq y \Rightarrow \lim_{i \longrightarrow \infty}(x_i) = x - y, \qquad \lim_{i \longrightarrow \infty}(y_i) = 0$$

And

$$y \geq x \Rightarrow \lim_{i \longrightarrow \infty}(y_i) = y - x, \qquad \lim_{i \longrightarrow \infty}(x_i) = 0$$

The series

As we have seen

$$\sqrt{x_i y_i} = y_{i-1} - y_i = x_{i-1} - x_i$$

Or

$$x_i - x_{i+1} = \sqrt{x_{i+1} y_{i+1}}$$

$$x_{i-1} - x_i = \sqrt{x_i y_i}$$

$$...$$

$$x_1 - x_2 = x - x_2 = \sqrt{x_2 y_2}$$

Telescopic series

$$\sum_{j=2}^{j=i+1} (\sqrt{x_j y_j}) = x - x_2 + x_2 - x_3 + ... + x_i - x_{i+1} = x - x_{i+1}$$

And the limit

$$\sum_{j=2}^{j=\infty} (\sqrt{x_j y_j}) = \lim_{i \longrightarrow \infty} (x - x_{i+1})$$

If $x \geq y$

$$\sum_{j=2}^{j=\infty} (\sqrt{x_j y_j}) = \lim_{i \longrightarrow \infty} (x - x_{i+1}) = x - (x - y) = y$$

If $x \leq y$

$$\sum_{j=2}^{j=\infty} (\sqrt{x_j y_j}) = \lim_{i \longrightarrow \infty} (x - x_{i+1}) = x$$

The applications of the series

Let us suppose firstly $x \geq y$. We always have

$$\sum_{j=2}^{j=i} ((-1)^j \sqrt{x_j y_j}) = x - x_2 - (x_2 - x_3) + (x_3 - x_4) - ... + (-1)^i (x_{i-1} - x_i)$$

$$= x - 2x_2 + 2x_3 - ... + 2(-1)^{i-1} x_{i-1} + (-1)^{i+1} x_i$$

$$= 2 \sum_{j=2}^{j=i-1} ((-1)^{j+1} x_j) + x + (-1)^{i+1} x_i$$

$$= 2 \sum_{j=1}^{j=i} ((-1)^{j+1} x_j) - x - (-1)^{i+1} x_i$$

$$= \sum_{j=2}^{j=i-1} ((-1)^{j+1} x_j) + \sum_{j=1}^{j=i} ((-1)^{j+1} x_j)$$

$$= 2 \sum_{j=2}^{j=i-1} ((-1)^{j+1} y_j) + y + (-1)^{i+1} y_i$$

$$= 2\sum_{j=1}^{j=i}((-1)^{j+1}y_j) - y - (-1)^{i+1}y_i$$

$$= \sum_{j=2}^{j=i-1}((-1)^{j+1}y_j) + \sum_{j=1}^{j=i}((-1)^{j+1}y_j)$$

Or

$$2\sum_{j=1}^{j=i}((-1)^{j+1}x_j) = \sum_{j=2}^{j=i}((-1)^{j}\sqrt{x_j y_j}) + x + (-1)^{i+1}x_i$$

And

$$2\sum_{j=1}^{j=i}((-1)^{j+1}y_j) = \sum_{j=2}^{j=i}((-1)^{j}\sqrt{x_j y_j}) + y + (-1)^{i+1}y_i$$

As we do not know the limit of $(-1)^{i+1}x_i$, we can say that

$$\sum_{j=1}^{j=\infty}((-1)^{j}x_j)$$

is perhapse not convergent. But

$$\sum_{j=2}^{j=\infty}((-1)^{j}\sqrt{x_j y_j})$$

is absolutely convergent. As y_i tends to zero in the infinity, thus

$$\sum_{j=1}^{j=\infty}((-1)^{j}y_j)$$

is convergent. The limit of

$$\sum_{j=2}^{j=i}((-1)^{j}\sqrt{x_j y_j}) = 2\sum_{j=1}^{j=i}((-1)^{j+1}y_j) - y - (-1)^{i+1}y_i$$

$$= 2\sum_{j=1}^{j=i}((-1)^{j+1}x_j) - x - (-1)^{i+1}x_i$$

And $\sum_{j=1}^{j=\infty}((-1)^{j+1}x_j)$ is convergent ! It implies, as we said and we have several proofs, that

$$\lim_{i\longrightarrow\infty}(x_i) = x - y = 0$$

It is confirmed by the fact that the limit of the general term of the series is equal to zero. Let us prove it ! We have

$$\sum_{k=1}^{k=2m}((-1)^{k+1}x_k e^{-\frac{k}{\sqrt{2m}}})$$

$$= xe^{-\frac{1}{\sqrt{2m}}} - x_2e^{-\frac{2}{\sqrt{2m}}} + x_3e^{-\frac{3}{\sqrt{2m}}} - ... + (-1)^{2m+1}x_{2m}e^{-\frac{2m}{\sqrt{2m}}}$$

$$= xe^{-\frac{2}{\sqrt{2m}}} + x(e^{-\frac{1}{\sqrt{2m}}} - e^{-\frac{2}{\sqrt{2m}}}) - x_2e^{-\frac{2}{\sqrt{2m}}} + x_3e^{-\frac{4}{\sqrt{2m}}} + x_3(e^{-\frac{3}{\sqrt{2m}}} - e^{-\frac{4}{\sqrt{2m}}}) - x_4e^{-\frac{4}{\sqrt{2m}}} + ... - x_{2m}e^{-\frac{2m}{\sqrt{2m}}}$$

$$= xe^{-\frac{2}{\sqrt{2m}}}(e^{\frac{1}{\sqrt{2m}}} - 1) + x_3e^{-\frac{4}{\sqrt{2m}}}(e^{\frac{1}{\sqrt{2m}}} - 1) + ... + x_{2m-1}e^{-\frac{2m}{\sqrt{2m}}}(e^{\frac{1}{\sqrt{2m}}} - 1) +$$

$$+ (x - x_2)e^{-\frac{2}{\sqrt{2m}}} + (x_3 - x_4)e^{-\frac{4}{\sqrt{2m}}} + ... + (x_{2m-1} - x_{2m})e^{-\frac{2m}{\sqrt{2m}}}$$

$$= (e^{\frac{1}{\sqrt{2m}}} - 1)(xe^{-\frac{2}{\sqrt{2m}}} + x_3e^{-\frac{4}{\sqrt{2m}}} + ... + x_{2m-1}e^{-\sqrt{2m}}) + (\sqrt{x_2y_2}e^{-\frac{2}{\sqrt{2m}}} + \sqrt{x_4y_4}e^{-\frac{4}{\sqrt{2m}}} + ... + \sqrt{x_{2m}y_{2m}}e^{-\frac{2m}{\sqrt{2m}}})$$

$$= (e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k}{\sqrt{2m}}}) + \sum_{k=1}^{k=m}(\sqrt{x_{2k}y_{2k}}e^{-\frac{2k}{\sqrt{2m}}})$$

And also

$$\sum_{k=1}^{k=2m}((-1)^{k+1}y_ke^{-\frac{k}{\sqrt{2m}}})$$

$$= ye^{-\frac{1}{\sqrt{2m}}} - y_2e^{-\frac{2}{\sqrt{2m}}} + y_3e^{-\frac{3}{\sqrt{2m}}} - ... + (-1)^{2m+1}y_{2m}e^{-\frac{2m}{\sqrt{2m}}}$$

$$= (e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k}{\sqrt{2m}}}) + \sum_{k=1}^{k=m}(\sqrt{x_{2k}y_{2k}}e^{-\frac{2k}{\sqrt{2m}}})$$

But

$$(e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k}{\sqrt{2m}}}) = S$$

We have

$$(e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k+1}{\sqrt{2m}}}) < S < (e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k-1}{\sqrt{2m}}})$$

$$(e^{\frac{1}{\sqrt{2m}}} - 1)e^{-\frac{3}{\sqrt{2m}}}\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}}) < S < (e^{\frac{1}{\sqrt{2m}}} - 1)e^{-\frac{1}{\sqrt{2m}}}\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}})$$

Thus

$$\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}})) = \lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}} - 1)y + (e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=2}^{k=m}(y_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}}))$$

$$= \lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=2}^{k=m}(y_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}})) = \lim_{m\longrightarrow\infty}(S)$$

We have

$$(e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=p}^{k=m}(y_{2k-1}e^{-\frac{2k-p}{\sqrt{2m}}}) = A$$

And

$$(e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=p}^{k=m}(y_{2k-1}e^{-\frac{2k-p+1}{\sqrt{2m}}}) < A < (e^{\frac{1}{\sqrt{2m}}} - 1)\sum_{k=p}^{k=m}(y_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}})$$

Or

$$(e^{\frac{1}{\sqrt{2m}}}-1)e^{-\frac{2}{\sqrt{2m}}}\sum_{k=p}^{k=m}(y_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}})<A<(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(y_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}})$$

Hence

$$\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(y_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)y_{2p-1}e^{-\frac{p-1}{\sqrt{2m}}}+(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p+1}^{k=m}(y_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p+1}^{k=m}(y_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}}))=\lim_{m\longrightarrow\infty}(A)=\lim_{m\longrightarrow\infty}(S)$$

Thus

$$p=m\Rightarrow\lim_{m\longrightarrow\infty}(A)=\lim_{m\longrightarrow\infty}(S)=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)y_{2m-1}e^{-\frac{m}{\sqrt{2m}}})=0$$

Consequently

$$\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=1}^{k=m}(y_{2k-1}e^{-\frac{2k}{\sqrt{2m}}}))=0$$

By the same process

$$(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k}{\sqrt{2m}}})=S$$

We have

$$(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k+1}{\sqrt{2m}}})<S<(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k-1}{\sqrt{2m}}})$$

$$(e^{\frac{1}{\sqrt{2m}}}-1)e^{-\frac{3}{\sqrt{2m}}}\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}})<S<(e^{\frac{1}{\sqrt{2m}}}-1)e^{-\frac{1}{\sqrt{2m}}}\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}})$$

Thus

$$\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}}))=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)x+(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=2}^{k=m}(x_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=2}^{k=m}(x_{2k-1}e^{-\frac{2k-2}{\sqrt{2m}}}))=\lim_{m\longrightarrow\infty}(S)$$

We have

$$(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(x_{2k-1}e^{-\frac{2k-p}{\sqrt{2m}}})=A$$

And

$$(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(x_{2k-1}e^{-\frac{2k-p+1}{\sqrt{2m}}})<A<(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(x_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}})$$

Or

$$(e^{\frac{1}{\sqrt{2m}}}-1)e^{-\frac{2}{\sqrt{2m}}}\sum_{k=p}^{k=m}(x_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}})<A<(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(x_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}})$$

Hence

$$\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p}^{k=m}(x_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)x_{2p-1}e^{-\frac{p-1}{\sqrt{2m}}}+(e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p+1}^{k=m}(x_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=p+1}^{k=m}(x_{2k-1}e^{-\frac{2k-p-1}{\sqrt{2m}}}))=\lim_{m\longrightarrow\infty}(A)=\lim_{m\longrightarrow\infty}(S)$$

Thus

$$p=m\Rightarrow\lim_{m\longrightarrow\infty}(A)=\lim_{m\longrightarrow\infty}(S)=\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)x_{2m-1}e^{-\frac{m}{\sqrt{2m}}})=0$$

Consequently

$$\lim_{m\longrightarrow\infty}((e^{\frac{1}{\sqrt{2m}}}-1)\sum_{k=1}^{k=m}(x_{2k-1}e^{-\frac{2k}{\sqrt{2m}}}))=0$$

We deduce

$$0<\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}((-1)^{k+1}x_ke^{-\frac{k}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}((-1)^{k+1}y_ke^{-\frac{k}{\sqrt{2m}}}))$$

$$=\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=m}(\sqrt{x_{2k}y_{2k}}e^{-\frac{2k}{\sqrt{2m}}}))<\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=m}(\sqrt{x_{2k}y_{2k}}))$$

$$<\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}(\sqrt{x_ky_k}))=y$$

Thus

$$\lim_{m\longrightarrow\infty}(\sum_{k=1}^{k=2m}((-1)^{k+1}(x_k-y_k)e^{-\frac{k}{\sqrt{2m}}}))=0$$

$$=\lim_{m\longrightarrow\infty}((x-y)\sum_{k=1}^{k=2m}(e^{-\frac{k}{\sqrt{2m}}}))=\lim_{m\longrightarrow\infty}((x-y)e^{-\frac{1}{\sqrt{2m}}}\frac{1-e^{-\sqrt{2m}}}{1+e^{-\frac{1}{\sqrt{2m}}}})=\frac{x-y}{2}=0$$

And

$$x-y=0$$

If $U^{c+2}X^{a+2}=U^{c+2}Y^{b+2}$ then $X=Y=Z=0$. With a second manner : Let

$$x_i(t)=\sum_{k=2}^{k=i-1}(x_ke^{-kt})=\sum_{k=2}^{k=i}(x_ke^{-kt})-x_ie^{-it}=x'_i(t)-x_ie^{-it}$$

$$y_i(t)=\sum_{k=2}^{k=i-1}(y_ke^{-kt})=\sum_{k=2}^{k=i}(y_ke^{-kt})-y_ie^{-it}=y'_i(t)-y_ie^{-it}$$

$$u_i(t)=\sum_{k=2}^{k=i}(\sqrt{x_ky_k}e^{-kt})=\sum_{k=2}^{k=i-1}((x_{k-1}-x_k)e^{-kt})$$

$$=xe^{-2t}+x_2(-e^{-2t}+e^{-3t})+x_3(-e^{-3t}+e^{-4t})+...+x_{i-1}(-e^{-(i-1)t}+e^{-it})-x_ie^{-it}$$

$$=xe^{-2t}+x_2e^{-2t}(-1+e^{-t})+x_3e^{-3t}(-1+e^{-t})+...+x_{i-1}e^{-(i-1)t}(-1+e^{-t})-x_ie^{-it}$$

$$=xe^{-2t}-x_ie^{-it}+(-1+e^{-t})(x_2e^{-2t}+x_3e^{-3t}+...+x_ie^{-(i-1)t})$$

$$=xe^{-2t}-x_ie^{-it}+(-1+e^{-t})x_i(t)=xe^{-2t}-x_ie^{-it}+(-1+e^{-t})x'_i(t)-(-1+e^{-t})x_ie^{-it}$$

$$=xe^{-2t}-x_ie^{-(i+1)t}+(-1+e^{-t})x'_i(t)$$

For all t, particularly

$$t=\frac{1}{\sqrt{i}}\Rightarrow\lim_{i\longrightarrow\infty}(u_i(\frac{1}{\sqrt{i}}))=\lim_{i\longrightarrow\infty}(\sum_{k=2}^{k=i}(\sqrt{x_ky_k}e^{-\frac{k}{\sqrt{i}}}))=\lim_{i\longrightarrow\infty}(\sum_{k=2}^{k=i}(\sqrt{x_ky_k}))=y$$

$$=x+\lim_{i\longrightarrow\infty}((-1+e^{-\frac{1}{\sqrt{i}}})\sum_{k=2}^{k=i-1}(x_ke^{-\frac{k}{\sqrt{i}}})-x_ie^{-\sqrt{i}})=x$$

$$=y+\lim_{i\longrightarrow\infty}((-1+e^{-\frac{1}{\sqrt{i}}})\sum_{k=2}^{k=i-1}(y_ke^{-\frac{k}{\sqrt{i}}})-y_ie^{-\sqrt{i}})=y$$

Let us recapitulate

$$x_i > x_{i+1}, \quad y_i > y_{i+1} \Rightarrow x = y$$

$$x_i = x_{i+1} = x_{i+1} + \sqrt{x_{i+1}y_{i+1}} \Rightarrow xy = 0$$

$$y_i = y_{i+1} = y_{i+1} + \sqrt{x_{i+1}y_{i+1}} \Rightarrow xy = 0$$
$$\Rightarrow xy(x-y) = 0$$

Another proof : We have, if we suppose $x \neq y$

$$xx_{i+1} - yy_{i+1} = (x-y)x_{i+1} + y(x_{i+1} - y_{i+1}) = (x-y)(x_{i+1}+y) = (x-y)(x+y_{i+1})$$

$$= (xx^{2^i} - yy^{2^i}) \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1}$$

$$= (x-y)(\sum_{j=0}^{j=2^i} (x^{2^i} \frac{y^j}{x^j}) \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1})$$

$$= (x-y)x_{i+1}(1 + \sum_{j=1}^{j=2^i} (\frac{y^j}{x^j})) = (x-y)(x_{i+1}+y) = (x-y)(y_{i+1}+x)$$

$$= (x-y)(\sum_{j=0}^{j=2^i} (y^{2^i} \frac{x^j}{y^j}) \prod_{j=0}^{j=i-1} (x^{2^j} + y^{2^j})^{-1})$$

$$= (x-y)y_{i+1}(1 + \sum_{j=1}^{j=2^i} (\frac{x^j}{y^j})) = (x-y)(y_{i+1}+x) = (x-y)(x_{i+1}+y)$$

We deduce

$$x = y_{i+1} \sum_{j=1}^{j=2^i} (\frac{x^j}{y^j}) = x - y + x_{i+1} \sum_{j=1}^{j=2^i} (\frac{y^j}{x^j})$$

$$y = x_{i+1} \sum_{j=1}^{j=2^i} (\frac{y^j}{x^j}) = y - x + y_{i+1} \sum_{j=1}^{j=2^i} (\frac{x^j}{y^j})$$

And

$$x - 2^i y_{i+1} = y_{i+1} \sum_{j=1}^{j=2^i} (\frac{x^j - y^j}{y^j}) = y_{i+1}(x-y) \sum_{j=0}^{j=2^i-1} (\frac{\sum_{k=0}^{k=j} (y^k x^{j-k})}{y^j})$$

$$x - 2^i x_{i+1} = x - y + x_{i+1} \sum_{j=1}^{j=2^i} (\frac{y^j - x^j}{x^j}) = x - y + x_{i+1}(y-x) \sum_{j=0}^{j=2^i-1} (\frac{\sum_{k=0}^{k=j} (x^k y^{i-k})}{x^j})$$

Also

$$y - 2^i x_{i+1} = x_{i+1} \sum_{j=1}^{j=2^i} (\frac{y^j - x^j}{x^j}) = x_{i+1}(y-x) \sum_{j=0}^{j=2^i-1} (\frac{\sum_{k=0}^{k=j} (x^k y^{j-k})}{x^j})$$

$$y-2^iy_{i+1}=y-x+y_{i+1}\sum_{j=1}^{j=2^i}(\frac{x^j-y^j}{y^j})=y-x+y_{i+1}(x-y)\sum_{j=0}^{j=2^i-1}(\frac{\sum_{k=0}^{k=j}(y^kx^{j-k})}{y^j})$$

If we sustract

$$x-y-2^i(y_{i+1}-x_{i+1})=(x-y)(1+2^i)$$

$$=(x-y)(\sum_{j=0}^{j=2^i-1}(\frac{y_{i+1}\sum_{k=0}^{k=j}(y^kx^{j-k})}{y^j}+\frac{x_{i+1}\sum_{k=0}^{k=j}(y^kx^{j-k})}{x^j}))$$

$$x-y-2^i(x_{i+1}-y_{i+1})=(y-x)(2^i-1)$$

$$=2(x-y)+(y-x)(\sum_{j=0}^{j=2^i-1}(\frac{x_{i+1}\sum_{k=0}^{k=j}(x^ky^{j-k})}{x^j}+\frac{y_{i+1}\sum_{k=0}^{k=j}(y^kx^{j-k})}{y^j}))$$

Hence

$$(x-y)2^{i+1}=2(y-x)+2(x-y)(\sum_{j=0}^{j=2^i-1}(\frac{x_{i+1}\sum_{k=0}^{k=j}(x^ky^{j-k})}{x^j}+\frac{y_{i+1}\sum_{k=0}^{k=j}(y^kx^{j-k})}{y^j}))$$

Or

$$2^i(x-y)=-(x-y)+(x-y)(\sum_{j=0}^{j=2^i-1}(\frac{x_{i+1}\sum_{k=0}^{k=j}(x^ky^{j-k})}{x^j}+\frac{y_{i+1}\sum_{k=0}^{k=j}(y^kx^{j-k})}{y^j}))$$

But

$$1+2^i\neq\sum_{j=0}^{j=2^i-1}(\frac{y_{i+1}\sum_{k=0}^{k=j}(x^ky^{j-k})}{y^j}+\frac{x_{i+1}\sum_{k=0}^{k=j}(x^{j-k}y^k)}{x^j})$$

And it is impossible, the hypothesis $x\neq y$ is false. Another proof, if $x\geq y$ let $d_i=\frac{y_i}{x_i}$, and for n great

$$(x-y)^{n+1}=\lim_{i\longrightarrow\infty}(x_i^{n+1})\geq 1\geq 0=\lim_{i\longrightarrow\infty}((x_i+1)y_i^n)$$

Thus

$$x_i^n(x_i^{n+1}-(x_i+1)y_i^n)\geq 0\geq -y_i^{2n+1}$$

Consequently

$$x_i^n(x_i^{n+1}-y_i^n)\geq y_i^n(x_i^{n+1}-y_i^{n+1})$$

And

$$x_i^{2n+1}\geq y_i^n(x_i^{n+1}-y_i^{n+1})+x_i^ny_i^n$$

Hence

$$\frac{x_i^n}{y_i^n}\geq\frac{x_i^{n+1}-y_i^{n+1}}{x_i^{n+1}}+\frac{1}{x_i}$$

We have

$$\frac{1}{d_i^n}\geq 1-d_i^{n+1}+\frac{1-d_i}{x-y}$$

Or if we suppose $d_i \neq 1$

$$\frac{1}{d_i^n(1-d_i)} \geq \frac{1-d_i^{n+1}}{1-d_i} + \frac{1}{x-y} = 1 + \frac{1}{d_i} + \frac{1}{d_i^2} + ... + \frac{1}{d_i^n} + \frac{1}{x-y}$$

And

$$\frac{1}{d_i^n(1-d_i)} - \frac{1}{d_i^n} = \frac{1}{d_i^{n-1}(1-d_i)} \geq 1 + \frac{1}{d_i} + \frac{1}{d_i^2} + ... + \frac{1}{d_i^{n-1}} + \frac{1}{x-y}$$

Until

$$\frac{1}{d_i(1-d_i)} \geq 1 + \frac{1}{d_i} + \frac{1}{x-y}$$

Which means

$$\frac{1}{d_i(1-d_i)} - \frac{1}{d_i} = \frac{1}{1-d_i} \geq 1 + \frac{1}{x-y}$$

Or

$$\frac{x_i}{x-y} \geq 1 + \frac{1}{x-y}$$

And

$$x_i - 1 \geq x - y = x_i - y_i \Rightarrow y_i \geq 1$$

Or

$$y_i = \frac{y_i}{x_i - y_i}(x-y) \geq 1 \Rightarrow y_i(x-y) \geq x - y \Rightarrow \lim_{i \longrightarrow \infty}(y_i(x-y)) = 0 \geq x - y \geq 0$$

Thus $x = y$. Another proof : We have

$$x_i - y_i + 1 \geq x_i \geq x_i - y_i$$

Hence

$$\frac{x_i - y_i + 1}{x_i y_i} \geq \frac{x_i}{x_i y_i}$$

And

$$\frac{1}{y_i} - \frac{1}{x_i} + \frac{1}{x_i y_i} \geq \frac{1}{y_i}$$

$$-\frac{1}{y_i} - \frac{1}{x_i} + \frac{2}{y_i} + \frac{1}{x_i y_i} \geq \frac{1}{y_i}$$

Or

$$-\frac{1}{z_i} + \frac{1}{x_i y_i} \geq \frac{-1}{y_i}$$

Or

$$\frac{1}{y_i} \geq \frac{1}{\sqrt{x_{i+1} y_{i+1}}} - \frac{1}{\sqrt{x_{i+1} y_{i+1}}(x_i + y_i)}$$

And

$$\frac{1}{\sqrt{y_{i+1}}(\sqrt{x_i + y_i})} \geq \frac{x_i + y_i - 1}{\sqrt{x_{i+1} y_{i+1}}}$$

It means

$$\frac{\sqrt{x_{i+1}}}{\sqrt{x_{i+1}} + \sqrt{y_{i+1}}} \geq x_i + y_i - 1$$

Or

$$\frac{2\sqrt{x_{i+1}}+\sqrt{y_{i+1}}}{\sqrt{x_{i+1}}+\sqrt{y_{i+1}}} \geq x_i + y_i$$

$\forall i > 1$ particularly in the infinty

$$\lim_{i \longrightarrow \infty}(2\sqrt{x_{i+1}}+\sqrt{y_{i+1}}) = 2\sqrt{x-y} \geq \lim_{i \longrightarrow \infty}(\sqrt{(x_i+y_i)^3}) = \sqrt{(x-y)^3} \geq 2\sqrt{x-y}$$

Therefore

$$x - y = 0$$

Another proof : as

$$y_i^2 - x_i^2 = (y-x)(x_i+y_i) = (y_{i+1}-x_{i+1})(x_{i+1}+y_{i+1}+2\sqrt{x_{i+1}y_{i+1}})$$
$$= y_{i+1}^2 - x_{i+1}^2 + 2\sqrt{x_{i+1}y_{i+1}}(y_{i+1}-x_{i+1})$$

$\sqrt{y_{i+1}} = Z$ is solution of

$$Z^4 + 2\sqrt{x_{i+1}}Z^3 - 2\sqrt{x_{i+1}^3}Z - x_{i+1}^2 + x_i^2 - y_i^2 = 0$$

And $\sqrt{x_{i+1}} = Z'$ is solution of

$$Z'^4 + 2\sqrt{y_{i+1}}Z'^3 - 2\sqrt{y_{i+1}^3}Z' - y_{i+1}^2 + y_i^2 - x_i^2 = 0$$

And

$$Z^4 + 2Z'Z^3 - 2Z'^3Z - Z'^4 + x_i^2 - y_i^2 = 0$$

Also

$$Z'^4 + 2ZZ'^3 - 2Z^3Z' - Z^4 + y_i^2 - x_i^2 = 0$$

We pose

$$Z = u - \frac{Z'}{2}$$

But

$$(u-\frac{Z'}{2})^4 + 2Z'(u-\frac{Z'}{2})^3 - 2Z'^3(u-\frac{Z'}{2}) - Z'^4 + x_i^2 - y_i^2 = 0$$

And

$$u^4 + \frac{Z'^4}{16} - 2u^3Z' + \frac{3}{2}u^2Z'^2 +$$
$$-\frac{1}{2}uZ'^3 + 2Z'u^3 - 3u^2Z'^2 + \frac{3}{2}uZ'^3 - \frac{1}{4}Z'^4 +$$
$$-2Z'^3u + Z'^4 - Z'^4 + x_i^2 - y_i^2 = 0$$
$$= u^4 - (\frac{3}{2}Z'^2)u^2 - Z'^3u - \frac{3}{16}Z'^4 + x_i^2 - y_i^2 = 0$$

Or

$$(u^2 - \frac{3}{4}Z'^2)^2 - \frac{3}{4}Z'^4 - Z'^3u + x_i^2 - y_i^2 = 0 \quad (5)$$

And $Z' = v - \frac{Z}{2}$ leads to

$$(v-\frac{Z}{2})^4 + 2Z(v-\frac{Z}{2})^3 - 2Z^3(v-\frac{Z}{2}) - Z^4 + y_i^2 - x_i^2 = 0$$

Or

$$v^4+\frac{Z^4}{16}-2v^3Z+\frac{3}{2}v^2Z^2+$$
$$-\frac{1}{2}vZ^3+2Zv^3-3v^2Z^2+\frac{3}{2}vZ^3-\frac{1}{4}Z^4+$$
$$-2Z^3v+Z^4-Z^4+y_i^2-x_i^2=0$$
$$=v^4-(\frac{3}{2}Z^2)v^2-Z^3v-\frac{3}{16}Z^4+y_i^2-x_i^2=0$$

Or

$$(v^2-\frac{3}{4}Z^2)^2-\frac{3}{4}Z^4-Z^3v+y_i^2-x_i^2=0 \quad (6)$$

We add (5) and (6), we have

$$(u^2-\frac{3}{4}Z'^2)^2+(v^2-\frac{3}{4}Z^2)^2-\frac{3}{4}(Z^4+Z'^4)-Z'^3u-Z^3v=0$$

$$=(Z^2+\frac{Z'^2}{4}+ZZ'-\frac{3}{4}Z^2)^2+(Z'^2+\frac{Z^2}{4}+Z'Z-\frac{3}{4}Z'^2)^2-\frac{3}{4}(Z^4+Z'^4)-ZZ'^3-Z^3Z'-\frac{Z'^4+Z^4}{2}=0$$

$$=(\frac{Z^2+Z'^2}{4}+ZZ')^2+(\frac{Z^2+Z'^2}{4}+Z'Z)^2-\frac{5}{4}(Z^4+Z'^4)-ZZ'(Z^2+Z'^2)=0$$

$$=\frac{Z^4+Z'^4+2Z^2Z'^2}{8}+2Z^2Z'^2+ZZ'(Z^2+Z'^2)-\frac{5}{4}(Z^4+Z'^4)-ZZ'(Z^2+Z'^2)=0$$

$$=-\frac{9}{8}(Z^4+Z'^4)+\frac{9}{4}Z^2Z'^2=0$$

$$=-(Z^2-Z'^2)^2\frac{9}{8}=0$$

The solution is

$$Z^2-Z'^2=y_{i+1}-x_{i+1}=y-x=0$$

Another proof : we have

$$\sqrt{x_{i-1}}+\sqrt{x_i}=\frac{x_{i-1}-x_i}{\sqrt{x_{i-1}}-\sqrt{x_i}}=\frac{\sqrt{x_iy_i}}{\sqrt{x_{i-1}}-\sqrt{x_i}}$$

$$=\frac{1}{\sqrt{\frac{x_{i-1}}{x_iy_i}}-\sqrt{\frac{x_i}{x_iy_i}}}=\frac{1}{\sqrt{\frac{\sqrt{x_{i-1}+y_{i-1}}}{\sqrt{x_i}y_i}}-\frac{1}{\sqrt{y_i}}}$$

$$=\frac{1}{\frac{\sqrt[4]{x_{i-1}+y_{i-1}}-\sqrt[4]{x_i}}{\sqrt[4]{x_i}\sqrt{y_i}}}=\frac{\sqrt[4]{x_i}\sqrt{y_i}}{\sqrt[4]{x_{i-1}+y_{i-1}}-\frac{\sqrt{x_{i-1}}}{\sqrt[4]{x_{i-1}+y_{i-1}}}}$$

$$=\frac{\sqrt[4]{x_i}\sqrt{y_i}(\sqrt[4]{x_{i-1}+y_{i-1}})}{\sqrt{x_{i-1}+y_{i-1}}-\sqrt{x_{i-1}}}\leq\frac{\sqrt[4]{x_i}\sqrt{y_i}\sqrt[4]{x_{i-1}+y_{i-1}}}{\sqrt[4]{y_{i+1}}}$$

Because

$$\sqrt{x_{i-1}+y_{i-1}}-\sqrt{x_{i-1}}=\frac{y_{i-1}}{\sqrt{x_{i-1}+y_{i-1}}+\sqrt{x_{i-1}}}\geq\sqrt[4]{y_{i+1}}$$

$$y_{i-1}=\sqrt{y_i}\sqrt{x_{i-1}+y_{i-1}}\geq\sqrt[4]{y_{i+1}}(\sqrt{x_{i-1}+y_{i-1}}+\sqrt{x_{i-1}})=\frac{\sqrt{y_i}}{\sqrt[4]{x_i+y_i}}(\sqrt{x_{i-1}+y_{i-1}}+\sqrt{x_{i-1}})$$

And

$$\sqrt[4]{x_i+y_i}\sqrt{x_{i-1}+y_{i-1}}\geq\sqrt{x_{i-1}+y_{i-1}}+\sqrt{x_{i-1}}$$

Because

$$\sqrt[4]{x_i+y_i}=\frac{\sqrt[4]{x_{i-1}^2+y_{i-1}^2}}{\sqrt[4]{x_{i-1}+y_{i-1}}}\geq 1+\sqrt{\frac{x_{i-1}}{x_{i-1}+y_{i-1}}}$$

And

$$\sqrt[4]{x_{i-1}^2+y_{i-1}^2}\geq\sqrt[4]{x_{i-1}+y_{i-1}}+\sqrt[4]{x_{i-1}+y_{i-1}}\frac{\sqrt{x_{i-1}}}{\sqrt{x_{i-1}+y_{i-1}}}$$

We deduce

$$\sqrt{x_{i-1}}+\sqrt{x_i}\leq\frac{\sqrt[4]{x_i}\sqrt{y_i}}{\sqrt[4]{y_{i+1}}}=\frac{\sqrt[4]{x_i}\sqrt[4]{y_{i+1}}(\sqrt[4]{x_i+y_i})}{\sqrt[4]{y_{i+1}}}=\sqrt[4]{x_i}(\sqrt[4]{x_i+y_i})$$

In the infinity

$$0<\sqrt{x-y}\leq\lim_{i\longrightarrow\infty}(\sqrt{x_{i-1}}+\sqrt{x_i})=2\sqrt{x-y}\leq\lim_{i\longrightarrow\infty}(\sqrt[4]{x_i}\sqrt[4]{x_i+y_i})=\sqrt{x-y}$$

And

$$\sqrt{x-y}=2\sqrt{x-y}=0$$

Another proof : we have

$$x-y=(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})=x_i-y_i=(\sqrt{x_i}-\sqrt{y_i})(\sqrt{x_i}+\sqrt{y_i})$$

$$\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x_i}+\sqrt{y_i}}=\frac{\sqrt{x_i}-\sqrt{y_i}}{\sqrt{x}+\sqrt{y}}$$

$$=\lim_{i\longrightarrow\infty}(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x_i}+\sqrt{y_i}})$$

$$=\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x-y}}=\frac{\sqrt{x}-\sqrt{y}}{\sqrt{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}}$$

$$=\sqrt{\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}}$$

$$\sqrt{\sqrt{x}-\sqrt{y}}(\sqrt{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}-\sqrt{x_i}-\sqrt{y_i})=0$$

$$=\sqrt{\sqrt{x}-\sqrt{y}}(\sqrt{x-y}-\sqrt{x_i}-\sqrt{y_i})=0$$

Or

$$x-y=0$$

Another proof. Let

$$x_i=a_ix_{i+1}=x_{i+1}+\sqrt{x_{i+1}y_{i+1}}\Rightarrow a_i=1+\sqrt{\frac{y_{i+1}}{x_{i+1}}}=1+\frac{y_i}{x_i}$$

And

$$y_i = b_i y_{i+1} = y_{i+1} + \sqrt{x_{i+1}y_{i+1}} \Rightarrow b_i = 1 + \sqrt{\frac{x_{i+1}}{y_{i+1}}} = 1 + \frac{x_i}{y_i}$$

Or

$$a_i = 1 + \frac{a_i}{b_i} = 1 + \frac{y_i}{x_i}$$

And

$$b_i = 1 + \frac{b_i}{a_i} = 1 + \frac{x_i}{y_i}$$

With

$$a_i b_i = a_i + b_i$$

We have

$$x_i = b_i\sqrt{x_{i+1}y_{i+1}}, \quad y_i = a_i\sqrt{x_{i+1}y_{i+1}}$$

But

$$\frac{a}{b} = \frac{y}{x} = \frac{a^2+b}{b^2+a} = \frac{y(y+x_i)}{x(x+y_i)}$$

$$= \frac{y^2+yx_i}{x^2+xy_i} = \frac{y^2+by_2x_i}{x^2+ax_2y_i} = \frac{a^2+b}{b^2+a} = a-1$$

$$b^2y^2 + b^3y_2x_i + ay^2 + aby_2x_i = a^2x^2 + a^3x_2y_i + bx^2 + abx_2y_i$$

$$b^2yx_i + ay^2 + ayx_i = a^2xy_i + bx^2 + bxy_i$$

$$= ((a+b^2)x_i + ay)y = ((a^2+b)y_i + bx)x$$

$$\frac{y}{x} = \frac{(a^2+b)y_i + bx}{(b^2+a)x_i + ay} = \frac{a^2+b}{b^2+a}$$

$$(a^2+b)(b^2+a)y_i + b(b^2+a)x = (a^2+b)(b^2+a)x_i + a(a^2+b)y$$

$$(a^2+b)(b^2+a)(x_i - y_i) = (a^2+b)(b^2+a)(x-y) = b(b^2+a)x - a(a^2+b)y$$

$$= \frac{(b^2+a)^2}{a^2+b}ax - \frac{(a^2+b)^2}{b^2+a}by$$

$$= ax(\frac{(b^2+a)^3 - (a^2+b)^3}{(a^2+b)(b^2+a)})$$

$$(a^2+b)^2(b^2+a)^2(x-y) = (x+y)((b^2+a)^3 - (a^2+b)^3)$$

$$= (x+y)(b^2+a-a^2-b)((b^2+a)^2 + (a^2+b)(b^2+a) + (a^2+b)^2)$$

$$= (x+y)(b-a)(ab-1)((b^2+a)^2 + (b^2+a)(a^2+b) + (a^2+b)^2)$$

$$= (a+b)(x-y)(ab-1)((b^2+a)^2 + (b^2+a)(a^2+b) + (a^2+b)^2)$$

$$= (x-y)((ab-1)ab((b^2+a)^2 + (b^2+a)(a^2+b) + (a^2+b)^2)$$

And if

$$(a^2+b)(b^2+a)((a^2+b)(b^2+a) - ab(ab-1)) = ab(ab-1)((b^2+a)^2 + (a^2+b)^2)$$

$$= (a^2+b)(b^2+a)(b^3+a^3+2ab)$$

It means

$$(a^2+b)((b^2+a)(a^3+b^3+2ab)-2ab(ab-1))$$
$$=(b^2+a)(2ab(ab-1)-(a^2+b)(a^3+b^3+2ab))$$

Or

$$a((b^2+a)(a^3+b^3+2ab)-2ab(ab-1))$$
$$=b(2ab(ab-1)-(a^2+b)(a^3+b^3+2ab))$$

And

$$(a^3+b^3+2ab)(ab^2+a^2+a^2b+b^2)-2a^2b^2(ab-1)=0$$
$$=(a^3+b^3+2ab)(a^2b^2+a^2+b^2)-2a^2b^2(ab-1)=0$$
$$=(a^3+b^3+2ab)(2a^2b^2-2ab)-2a^2b^2(ab-1)=0$$
$$a^3+b^3+2ab-2ab=0=a^3+b^3$$

And it is impossible ! Thus

$$x-y=0$$

Another proof : If $x>y$, we have

$$a_ix_i=b_iy_i=x_i+y_i$$
$$\frac{2-a}{b_i-2}=\frac{y_i}{x},\quad \frac{2-a_i}{b-2}=\frac{y}{x_i}$$
$$\frac{x_ix}{yy_i}-1=\frac{(b-2)(b_i-2)}{(2-a)(2-a_i)}-1$$
$$=\frac{x_ix-y_iy}{y_iy}=\frac{b_ib-2b_i-2b-a_ia+2a+2a_i}{(2-a)(2-a_i)}$$
$$=\frac{(x-y)x_i+y(x_i-y_i)}{y_iy}=\frac{(x-y)(x_i+y)}{y_iy}=\frac{(b_i-a_i)b+a_i(b-a)+2(a-b)+2(a_i-b_i)}{(2-a)(2-a_i)}$$
$$=\frac{(a_i-b_i)(2-b)+(a-b)(2-a_i)}{(2-a)(2-a_i)}$$
$$=\frac{\sqrt{x_{i+1}y_{i+1}x_2y_2}((a_i-b_i)(2-b)+(a-b)(2-a_i))}{\sqrt{x_{i+1}y_{i+1}x_2y_2}(2-a)(2-a_i)}$$
$$=\frac{(y-x)(2\sqrt{x_2y_2}-x)+(y-x)(2\sqrt{x_{i+1}y_{i+1}}-y_i)}{(2\sqrt{x_2y_2}-y)(2\sqrt{x_{i+1}y_{i+1}}-y_i)}$$
$$=\frac{(y-x)(\sqrt{x_2}(\sqrt{y_2}-\sqrt{x_2})-\sqrt{y_{i+1}}(\sqrt{x_{i+1}}-\sqrt{y_{i+1}}))}{\sqrt{y_2}(\sqrt{x_2}-\sqrt{y_2})\sqrt{y_{i+1}}(\sqrt{x_{i+1}}-\sqrt{y_{i+1}})}$$
$$=(y-x)(\frac{-x}{y\sqrt{y_{i+1}}(\sqrt{x_{i+1}}-\sqrt{y_{i+1}})}-\frac{1}{\sqrt{y_2}(\sqrt{x_2}-\sqrt{y_2})})=(x-y)(\frac{x_i}{y_iy}+\frac{1}{y_i})$$

Or

$$(x-y)(\frac{x}{y\sqrt{y_{i+1}}(\sqrt{x_{i+1}}-\sqrt{y_{i+1}})}-\frac{x_i}{y\sqrt{y_{i+1}}(\sqrt{x_{i+1}}+\sqrt{y_{i+1}})}+\frac{1}{\sqrt{y_2}(\sqrt{x_2}-\sqrt{y_2})}-\frac{1}{\sqrt{y_{i+1}}(\sqrt{x_{i+1}}+\sqrt{y_{i+1}})})=0$$
$$=(x-y)(\frac{b_ix}{by_2(b_i-2)y_i}-\frac{x_i}{by_2b_iy_{i+1}}+\frac{b}{(b-2)y}-\frac{1}{y_i})=0$$

$$= (x-y)(\frac{b_i x}{yy_i(b_i-2)} - \frac{x_i}{yy_i} + \frac{b}{(b-2)y} - \frac{1}{y_i}) = 0$$

$$= (x-y)(\frac{b_i(b-2)x - (b-2)(b_i-2)x_i + b(b_i-2)y_i - (b-2)y(b_i-2)}{(b-2)y(b_i-2)y_i}) = 0$$

And

$$(x-y)((b-2)(b_i x - (b_i-2)x_i) + (b_i-2)(by_i - (b-2)y)) = 0$$

$$= (x-y)((b-2)(b_i(x-x_i) + 2x_i) + (b_i-2)(b(y_i-y) + 2y)) = 0$$

$$= (x-y)((b-2)(b_i(y-y_i) + 2x_i) + (b_i-2)(-b(y-y_i) + 2y)) = 0$$

$$= (x-y)((y-y_i)(b_i(b-2) - b(b_i-2)) + 2(b_i y - 2y + bx_i - 2x_i)) = 0$$

$$= 2(x-y)((y-y_i)(b-b_i) + (b_i(y-y_i) + b_i y_i - 2y + b(x_i-x) + bx - 2x_i)) = 0$$

$$= 2(x-y)((y-y_i)(b-b_i+b_i-b)+b_i y_i-2y+bx-2x_i) = 0 = 2(x-y)(x_i+y_i-2y+bx-2x_i)$$

$$= 2(x-y)(y_i-x_i-2y+bx) = 2(x-y)(-x-y+bx) = 2(x-y)(-by+bx) = 2b(x-y)^2 = 0$$

It means that x can not be different of y and

$$x - y = 0$$

Another proof : We have

$$\frac{a_i b_{i+1}}{b_i a_{i+1}} = \frac{b_i}{a_i}$$

$$= \frac{b_{i+1} + \sqrt{a_{i+1}b_{i+1}}}{a_{i+1} + \sqrt{a_{i+1}b_{i+1}}}$$

$$= \frac{b_{i+1}y_{i+2} + \sqrt{a_{i+1}b_{i+1}y_{i+2}^2}}{a_{i+1}y_{i+2} + \sqrt{a_{i+1}b_{i+1}y_{i+2}^2}}$$

$$= \frac{y_{i+1} + a_{i+1}\sqrt{x_{i+2}y_{i+2}}}{(a_{i+1}-1)b_{i+1}y_{i+2} + a_{i+1}\sqrt{x_{i+2}y_{i+2}}}$$

$$= \frac{2y_{i+1}}{(a_{i+1}-1)y_{i+1} + y_{i+1}} = \frac{2}{a_{i+1}} = b_i - 1$$

Thus

$$2 = a_{i+1}(b_i - 1) = \sqrt{a_{i+1}b_{i+1}}$$

Or

$$\sqrt{a_{i+1}b_{i+1}} - 2 = 0 = a_i b_i - 4 = \frac{(x-y)^2}{x_i y_i} = 0$$

Another proof : If we make the hypothesis that $x \neq y$, $\forall x_1 = p_1, y_1 = q_1$, $\exists r_1$ verifying

$$\frac{1}{r_1} = \frac{1}{q_1} - \frac{1}{p_1}$$

And

$$r_1 = \frac{xy}{x-y} = r$$

Or

$$(p_1 - q_1)r_1 = p_1 q_1$$

And

$$p_1(r_1 - q_1) = r_1 q_1$$

Let us pose

$$q_2 = r_1 - q_1 = \frac{r_1 q_1}{p_1}$$

And

$$q_1(r_1 + p_1) = r_1 p_1$$

Also

$$p_2 = r_1 + p_1 = \frac{r_1 p_1}{q_1}$$

So, we have

$$p_2 q_2 = r_1^2$$

Or

$$p_1 = p_2 - r_1 = p_2 - \sqrt{p_2 q_2}$$

And

$$q_1 = r_1 - q_2 = -q_2 + \sqrt{p_2 q_2}$$

With

$$w_1 = w = (p_1 - q_1) = (\sqrt{p_2} - \sqrt{q_2})^2$$

$(p_1 - q_1)$ integer

$$p_1 = \sqrt{p_2}(\sqrt{p_2} - \sqrt{q_2})$$

p_1 integer

$$q_1 = \sqrt{q_2}(\sqrt{p_2} - \sqrt{q_2})$$

q_1 integer

$$r_1 = \frac{x_1 y_1}{x_1 - y_1} = \sqrt{p_2 q_2}$$

r_2 rational
Because
$\forall p_2, q_2$ rational, $\exists r_2$ rational which verifies

$$\frac{1}{r_2} = \frac{1}{q_2} - \frac{1}{p_2}$$

Until infinity. For i

$$(p_i - q_i) = (\sqrt{p_{i+1}} - \sqrt{q_{i+1}})^2$$

And $p_i - q_i$ rational for $i > 2$

$$p_i = \sqrt{p_{i+1}}(\sqrt{p_{i+1}} - \sqrt{q_{i+1}})$$

p_i rational for $i > 2$

$$q_i = \sqrt{q_{i+1}}(\sqrt{p_{i+1}} - \sqrt{q_{i+1}})$$

q_i rational for $i > 2$

$$r_i = \frac{p_i q_i}{p_i - q_i} = \sqrt{p_{i+1} q_{i+1}}$$

r_i rational for $i > 1$ and also, of course

$$\frac{1}{r_{i+1}} = \frac{1}{q_{i+1}} - \frac{1}{p_{i+1}}$$

Lemma a The expressions of the sequences are

$$(p_i - q_i) = (\sqrt{p_{i+1}} - \sqrt{q_{i+1}})^2$$

$$p_i = x^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} - y^{2^j})^{-1} \tag{5}$$

$$q_i = y^{2^{i-1}} \prod_{j=0}^{j=i-2} (x^{2^j} - y^{2^j})^{-1} \tag{6}$$

Prof of lemma a We prove it by induction, we have

$$p = \sqrt{p_2}(\sqrt{p_2} - \sqrt{q_2}) = \sqrt{p_2}(x-y)^{\frac{1}{2}}$$

$$p_2 = \frac{x^2}{x-y}$$

Also

$$q_2 = \frac{y^2}{x-y}$$

The lemma is verified for $i = 2$. Let us suppose it verified for i, from (5) and (6)

$$p_i = \sqrt{p_{i+1}}(\sqrt{p_{i+1}} - \sqrt{q_{i+1}}) = \sqrt{p_{i+1}}(p_i - q_i)^{\frac{1}{2}}$$

Which means

$$p_{i+1} = p_i^2(p_i - q_i)^{-1}$$

But the lemma is verified for i, it leads us to

$$p_{i+1} = x^{2^i}\prod_{j=0}^{j=i-2}(x^{2^j} - y^{2^j})^{-2}(x^{2^{i-1}} - y^{2^{i-1}})^{-1}\prod_{j=0}^{j=i-2}(x^{2^j} - y^{2^j})$$

$$= x^{2^i}\prod_{j=0}^{j=i-1}(x^{2^j} - y^{2^j})^{-1}$$

The lemma is verified for $i+1$, the proof is the same for q_i, we deduce

$$p_i = x^{(2^{i-1})}\prod_{j=0}^{j=i-2}(x^{(2^j)} - y^{(2^j)})^{-1}$$

And for q_i

$$q_i = y^{(2^{i-1})}\prod_{j=0}^{j=i-2}(x^{(2^j)} - y^{(2^j)})^{-1}$$

But

$$\frac{p_i}{q_i} = \frac{x_i}{y_i} = \sqrt{\frac{x_{i+1}}{y_{i+1}}} = \sqrt{\frac{p_{i+1}}{q_{i+1}}}$$

Or

$$\frac{p_i}{x_i} = \frac{q_i}{y_i}$$

Thus

$$p_i y_i = q_i x_i$$

But, let

$$p_i x_i - q_i y_i = q_i x_i(\frac{p_i}{q_i} - \frac{y_i}{x_i}) = q_i x_i(\frac{x_i}{y_i} - \frac{y_i}{x_i}) = q_i x_i(\frac{x_i^2 - y_i^2}{x_i y_i})$$

We deduce

$$x_i(p_i - q_i\frac{x_i^2}{y_i}) = y_i(q_i - q_iy_i)$$

$$\frac{x_i}{y_i} = \frac{q_i(1-y_i)}{p_i - q_i\frac{x_i^2}{y_i}} = \frac{q_i(1-y_i)}{p_i - p_ix_i} = \frac{q_i(1-y_i)}{p_i(1-x_i)} = \frac{y_i(1-y_i)}{x_i(1-x_i)}$$

Thus

$$\frac{x_i^2}{y_i^2} = \frac{1-y_i}{1-x_i}$$

Or

$$x_i^2 - x_i^3 - y_i^2 + y_i^3 = 0 = (x-y)(x_i + y_i - x_i^2 - y_i^2 - x_iy_i)$$

And

$$x - y = 0$$

Another proof : we have

$$x_2 = \frac{x^2}{x+y} = \frac{x^2}{x^2-y^2}(x-y), \quad y_2 = \frac{y^2}{x+y} = \frac{x^2}{x^2-y^2}(x-y)$$

And

$$p_2 = \frac{x^2}{x-y} = \frac{x^2}{x^2-y^2}(x+y), \quad q_2 = \frac{y^2}{x-y} = \frac{y^2}{x^2-y^2}(x+y)$$

But

$$(p_2+x_2)(x-y) = \frac{2x^3}{x+y} = 2x_2x, \quad (q_2+y_2)(x-y) = \frac{2y^3}{x+y} = 2y_2y$$

And

$$(p_2 - q_2 + x_2 - y_2)(x-y) = (x+y+x-y)(x-y) = 2x(x-y)$$

$$= 2x_2x - 2y_2y = 2(x_2-y_2)x + 2y_2(x-y) = 2(x-y)(x+y_2)$$

Hence, once again, x can not be different of y

$$2(x-y)(x+y_2-x) = 0 = y_2(x-y)$$

Another proof : we have

$$x+y = p_2-q_2 = p_3+q_3-2\sqrt{p_3q_3} = p_4-q_4-2\sqrt{p_3q_3} = ... = p_{2k}-q_{2k}-2\sqrt{p_3q_3}-2\sqrt{p_5q_5}-...-2\sqrt{p_{2k-1}q_{2k-1}}$$

$$= p_{2k+1} + q_{2k+1} - 2\sqrt{p_3q_3} - 2\sqrt{p_5q_5} - ... - 2\sqrt{p_{2k+1}q_{2k+1}}$$

$$= p_{2k} - q_{2k} + 2(p_2 - p_3) + 2(p_4 - p_5) + ... + 2(p_{2k-2} - p_{2k-1})$$

$$= p_{2k} - q_{2k} - 2(q_3 + q_2) - 2(q_5 + q_4) + ... - 2(q_{2k-1} + q_{2k-2})$$

$$= p_{2k+1} + q_{2k+1} + 2(p_2 - p_3) + 2(p_4 - p_5) + ... + 2(p_{2k} - p_{2k+1})$$

$$= q_{2k+1} - p_{2k+1} + 2p_2 - 2p_3 + 2p_4 - 2p_5 + ... + 2p_{2k}$$

$$= p_{2k+1} + q_{2k+1} - 2(q_3 + q_2) - 2(q_5 + q_4) + ... - 2(q_{2k+1} + q_{2k})$$

$$= -q_{2k+1} + p_{2k+1} - 2q_2 - 2q_3 - 2q_4 - 2q_5 ... - 2q_{2k}$$

Thus

$$2p_2 + 2q_2 - 2p_3 + 2q_3 + 2p_4 + 2q_4 - ... + 2p_{2k-2} + 2q_{2k-2} - 2p_{2k-1} + 2q_{2k-1} = 0$$

And

$$2q_{2k+1} - 2p_{2k+1} + 2p_2 + 2q_2 - 2p_3 + 2q_3 + 2p_4 + 2q_4 - ... - 2p_{2k-1} + 2q_{2k-1} + 2p_{2k} - 2q_{2k} = 0$$

$$= 2q_{2k+1} - 2p_{2k+1} + 2p_{2k} - 2q_{2k} = 2q_{2k+1} - 2p_{2k+1} + 2p_{2k+1} + 2q_{2k+1} - 4\sqrt{p_{2k+1}q_{2k+1}} = 0$$

$$= 4q_{2k+1} - 4\sqrt{p_{2k+1}q_{2k+1}} = 0 = 4q_{2k+1}(1 - \sqrt{\frac{p_{2k+1}}{q_{2k+1}}})$$

$$= 4q_{2k+1}(1 - \frac{x_{2k}}{y_{2k}}) = 4q_{2k+1}(\frac{y-x}{y_{2k}}) = 0$$

And it is impossible, the initial hypothesis ($x \neq y$) is false ! There are too proofs, the doubt is not allowed. Let us prove Beal conjecture, now ! This conjecture stipulates that Beal equation $z^{c+2} = x^{a+2} + y^{b+2}$ is impossible for $a > 0$ and $b > 0$ and $c > 0$ and x, y, z coprime. Let us prove that there are solutions only for $abc = 0$.

Proof of Beal conjecture

Beal equation is $z^{c+2} = x^{a+2} + y^{b+2}$. With x, y and z coprime and greater than zero. z is odd and one of x and y is even. Let x odd. Let firstly $b \geq a$. We have

$$2(a_1S_1 + a_2S_2) = (a_1 + a_2)(S_1 + S_2) + (a_1 - a_2)(S_1 - S_2) \quad (7)$$

But

$$z^{c+2} = x^{a+2} + y^{b+2} = xx^{a+1} + yy^{b+1}$$

And (7)

$$2(x^{a+2} + y^{b+2}) = (x + y)(x^{a+1} + y^{b+1}) + (x - y)(x^{a+1} - y^{b+1})$$

If $a = 0$, then

$$2z^{c+2} = 2(x^2 + y^{b+2}) = (x + y)(x + y^{b+1}) + (x - y)(x - y^{b+1})$$

Our approach will end here (as we will see). For $c = 0$

$$2y^{b+2} = 2(z^2 - x^{a+2}) = (z - x)(z + x^{a+1}) + (z + x)(z - x^{a+1})$$

THe calculus ends here. Thus, for $a > 0$, $b > 0$, $c > 0$, with (7)

$$(x + y)(x^{a+1} + y^{b+1}) = \frac{1}{2}((x + y)^2(x^a + y^b) + (x + y)(x - y)(x^a - y^b))$$

$$\frac{1}{2}(x + y)^2(x^a + y^b) = \frac{1}{2^2}((x + y)^3(x^{a-1} + y^{b-1}) + (x + y)^2(x - y)(x^{a-1} - y^{b-1}))$$

Until a

$$\frac{1}{2^{a-1}}(x+y)^a(x^2+y^{b-a+2})=\frac{1}{2^a}((x+y)^{a+1}(x+y^{b-a+1})+(x+y)^a(x-y)(x-y^{b-a+1}))$$

We add and simplify

$$2(x^{a+2}+y^{b+2})$$
$$=\frac{1}{2^a}(x+y)^{a+1}(x+y^{b-a+1})+(x-y)\sum_{m=0}^{m=a}(\frac{(x+y)^m}{2^m}(x^{a+1-m}-y^{b+1-m})))$$

If we multiply by 2^a

$$2^{a+1}(x^{a+2}+y^{b+2})=2^{a+1}z^{c+2}$$
$$=(x+y)^{a+1}(x+y^{b-a+1})+(x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m(x^{a+1-m}-y^{b+1-m}))$$

$2^{a+1}z^{c+2}$ is like $(x+y)^{a+1}(x+y^{b-a+1}+(2m+1)(x-y)$, we deduce

$$2^{a+1}z^{c+2}=2^{a+1}(x^{a+2}+y^{b+2})=(x+y)^{a+1}(x+y^{b-a+1})+(2m+1)(x-y)$$

But x, y and z are coprimes, z is odd and x or y is odd. Let x odd, thus y is even and we will study the case $a\geq b$. But

$$2(x^{a+2}-y^{b+2})=(x-y)(x^{a+1}+y^{b+1})+(x+y)(x^{a+1}-y^{b+1})$$

$$(x+y)(x^{a+1}-y^{b+1})=\frac{1}{2}((x+y)(x-y)(x^a+y^b)+(x+y)^2(x^a-y^b))$$

$$\frac{1}{2}(x+y)^2(x^a-y^b)=\frac{1}{2^2}((x+y)^2(x-y)(x^{a-1}+y^{b-1})+(x+y)^3(x^{a-1}-y^{b-1}))$$

Until a

$$\frac{1}{2^{a-1}}(x+y)^a(x^2-y^{b-a+2})=\frac{1}{2^a}((x+y)^a(x-y)(x+y^{b-a+1})+(x+y)^{a+1}(x-y^{b-a+1}))$$

We add and simplify

$$2(x^{a+2}-y^{b+2})=\frac{1}{2^a}(x+y)^{a+1}(x-y^{b-a+1})+(x-y)\sum_{m=0}^{m=a}(\frac{(x+y)^m}{2^m}(y^{b+1-m}+x^{a+1-m})))$$

We multiply by 2^a

$$2^{a+1}(x^{a+2}-y^{b+2})=(x+y)^{a+1}(x-y^{b-a+1})+(x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m(y^{b+1-m}+x^{a+1-m}))$$

We remember

$$2^{a+1}(x^{a+2}+y^{b+2}) = (x+y)^{a+1}(x+y^{b-a+1})+(x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m(-y^{b+1-m}+x^{a+1-m}))$$

We deduce

$$2^{a+2}x^{a+2} = 2x(x+y)^{a+1} + 2(x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m x^{a+1-m})$$

And

$$2^{a+2}y^{b+2} = 2y^{b-a+1}(x+y)^{a+1} - 2(x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m y^{b+1-m})$$

Thus

$$2^{a+1}x^{a+1} = (x+y)^{a+1} + (x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m x^{a-m})$$

$$= (x+y)^{a+1} + (x-y)(2^a x^a + 2^{a-1}(x+y)x^{a-1} + ... + 2(x+y)^{a-1}x + (x+y)^a)$$

And

$$2^{a+1}y^{a+1} = (x+y)^{a+1} - (x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m y^{a-m})$$

We deduce

$$2^{a+1}(x^{a+1} - y^{a+1}) = (x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m(x^{a-m}+y^{a-m}))$$

Or

$$2^{2a+1}(x^{2a+1} - y^{2a+1}) = (x-y)\sum_{m=0}^{m=2a}(2^{2a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

And

$$2^{a+1}(\alpha x^{a+1}-\beta y^{a+1}) = (\alpha-\beta)(x+y)^{a+1}+(x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m(\alpha x^{a-m}+\beta y^{a-m}))$$

$\forall(\alpha,\beta)$, particularly $\alpha = x^a$, $\beta = y^a$, thus

$$2^{a+1}(x^{2a+1} - y^{2a+1})$$

$$= (x^a - y^a)(x+y)^{a+1} + (x-y)\sum_{m=0}^{m=a}(2^{a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

And

$$2^{2a+1}(x^{2a+1} - y^{2a+1})$$

$$= 2^a(x^a - y^a)(x+y)^{a+1} + (x-y)\sum_{m=0}^{m=a}(2^{2a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

$$= (x-y)\sum_{m=0}^{m=2a}(2^{2a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

And

$$2^a(x^a-y^a)(x+y)^{a+1}$$

$$= (x-y)(\sum_{m=0}^{m=a}((x+y)^m(2^{2a-m})(x^{2a-m}+y^{2a-m}-(x^{2a-m}+y^{2a-m})))+$$

$$+(x-y)\sum_{m=a+1}^{m=2a}(2^{2a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

$$= (x-y)\sum_{m=a+1}^{m=2a}(2^{2a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

It means

$$(x^a-y^a)(x+y)^{a+1} = (x-y)\sum_{m=a+1}^{m=2a}(2^{a-m}(x+y)^m(x^{2a-m}+y^{2a-m}))$$

Or

$$(x+y)^{a+1}(x^a-y^a-\frac{1}{2}(x-y)(x^{a-1}+y^{a-1}))$$

$$= (x-y)(x+y)^{a+2}\sum_{m=a+2}^{m=2a}(2^{a-m}(x+y)^{m-a-2}(x^{2a-m}+y^{2a-m}))$$

$$= (x+y)^{a+1}\frac{1}{2}(-xy^{a-1}+yx^{a-1}+x^a-y^a)$$

$$= (x+y)^{a+1}xy\frac{1}{2}(x^{a-2}-y^{a-2})+(x+y)^{a+1}\frac{1}{2}(x^a-y^a)$$

Thus

$$xy(x^{a-2}-y^{a-2}) = (x-y)(x+y)\sum_{m=a+2}^{m=2a}(2^{a+1-m}(x+y)^{m-a-2}(x^{2a-m}+y^{2a-m}))-(x^a-y^a)$$

$$= (x-y)(x+y)\sum_{m=a+2}^{m=2a}(2^{a+1-m}(x+y)^{m-a-2}(x^{2a-m}+y^{2a-m}))+$$

$$-(x-y)\sum_{m=a+1}^{m=2a}(2^{a-m}(x+y)^{m-a-1}(x^{2a-m}+y^{2a-m}))$$

$$= -(x-y)\frac{1}{2}(x^{a-1}+y^{a-1})+(x-y)\sum_{m=a+2}^{m=2a}(2^{a-m}(x+y)^{m-a-1}(x^{2a-m}+y^{2a-m}))$$

It means that

$$2^{a-2}xy(x^{a-2}-y^{a-2})$$

$$= -2^{a-3}(x-y)(x^{a-1}+y^{a-1})+(x-y)\sum_{m=a+2}^{m=2a}(2^{2a-2-m}(x+y)^{m-a-1}(x^{2a-m}+y^{2a-m}))$$

$$= -2^{a-3}(x-y)(x^{a-1}+y^{a-1})+(x-y)(x+y)\sum_{m=a+2}^{m=2a}(2^{2a-2-m}(x+y)^{m-a-2}(x^{2a-m}+y^{2a-m}))$$

$$= (x-y)xy\sum_{m=0}^{m=a-3}(2^{a-3-m}(x+y)^{m}(x^{a-3-m}+y^{a-3-m}))$$

$$= (x-y)xy(2^{a-3})(x^{a-3}+y^{a-3})+(x-y)xy(x+y)\sum_{m=1}^{m=a-3}(2^{a-3-m}(x+y)^{m-1}(x^{a-3-m}+y^{a-3-m}))$$

Or

$$2^{a-3}(x-y)(xy(x^{a-3}+y^{a-3})+x^{a-1}+y^{a-1})$$

$$= (x-y)(x+y)(\sum_{m=a+2}^{m=2a}(2^{2a-2-m}(x+y)^{m-a-2}(x^{2a-m}+y^{2a-m}))+$$

$$-xy\sum_{m=1}^{m=a-3}(2^{a-3-m}(x+y)^{m-1}(x^{a-3-m}+y^{a-3-m})))$$

Thus

$$B = 2^{a-1}(x-y)(xy(x^{a-3}+y^{a-3})+x^{a-1}+y^{a-1})$$

$$= (x-y)(x+y)(\sum_{m=a+2}^{m=2a}(2^{2a-m}(x+y)^{m-a-2}(x^{2a-m}+y^{2a-m}))+$$

$$-xy\sum_{m=1}^{m=a-3}(2^{a-1-m}(x+y)^{m-1}(x^{a-3-m}+y^{a-3-m}))) = (x-y)(x+y)A$$

B is of the form $(x-y)(x+y)A$, where A is an integer. Thus $x+y$ divises the expression B and it is impossible because x and y are coprime, but for $a=2$! Let $a=2$, hence

$$2^{a-2}xy(x^{a-2}-y^{a-2}) = 0$$

$$= -2^{a-3}(x-y)(x^{a-1}+y^{a-1})+(x-y)\sum_{m=a+2}^{m=2a}(2^{2a-2-m}(x+y)^{m-a-1}(x^{2a-m}+y^{2a-m}))$$

$$= -2^{-1}(x-y)(x+y)+2(x-y)(x+y) = \frac{3}{2}(x^2-y^2)$$

And it is also impossible ! The case $a \geq b$ is similar. We resume, Beal equation (or Fermat-Catalan) is impossible for $a > 0$ and $b > 0$ nd $c > 0$ and x, y, z coprime ($z^{c+2} = x^{a+2} - y^{b+2}$ is also impossible !). For $abc = 0$, we have

$$1^m + 2^3 = 3^2$$

$$2^5 + 7^2 = 3^4$$

$$13^2 + 7^3 = 2^9$$

$$2^7 + 17^3 = 71^2$$

$$3^5 + 11^4 = 122^2$$

$$33^8 + 1549034^2 = 15613^3$$

$$1414^3 + 2213459^2 = 65^7$$

$$9262^3 + 15312283^2 = 113^7$$

$$17^7 + 76271^3 = 21063928^2$$

$$43^8 + 96222^3 = 30042907^2$$

Another application is

$$U^n = X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}$$

X_j, X_k coprime $\forall j, k, j \neq k$ The conjecture that we propose is that there is no solutuion for $n > i(i-1)$ and $n_j > i(i-1)$, $\forall j \in \{1, 2, ..., i\}$. The approach is the same than higher. If X_i, $(i \geq 2)$, n_i, U, n, $X_1, X_2, ..., X_i$ coprime and integers and positive, then $X_a X_b = 0$, or $X_a^{n_a} = X_b^{n_b}$ $\forall a, b = 1, 2, ..., i$

$$n_k > i(i-1), \forall k = 1, 2, ..., i, n > i(i-1)$$

For the equation (e) which follows

$$X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i} = U^n$$

when $n \leq i(i-1), n_k \leq i(i-1), k = 1, 2, ..., i$ there are solutions, for example : $i = 2$ has

$$3^2 + 4^2 = 5^2$$

$i = 3$ hss

$$3^3 + 4^3 + 5^3 = 6^3$$

And

$$95800^4 + 217519^4 + 414560^4 = 422481^4$$

$i = 4$ has

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

etc... We Suppose (e) verified and X_k coprime, $k \in \{1, 2, ..., i\}; i \geq 2$, effectively

$$x_k = U^{(i-1)n} X_k^{n_k}$$

$$k = 1, 2, ..., i$$

And

$$u = U^{in}$$

And

$$v = X_1^{n_1} X_2^{n_2} ... X_i^{n_i}$$

Or

x_k,$k = 1, 2, ...i$, u, v verify (8)

$$x_1 + x_2 + ... + x_i = U^{(i-1)n}(X_1^{n_1} + X_2^{n_2} + ... + X_i^{n_i}) = U^{in} = u$$

And

(9)

$$\frac{1}{v} = \frac{1}{X_1^{n_1}X_2^{n_2}...X_i^{n_i}} = \frac{U^{i(i-1)n}}{U^{(i-1)n}X_1^{n_1}U^{(i-1)n}X_2^{n_2}...U^{(i-1)n}X_i^{n_i}} = \frac{u^{(i-1)}}{x_1x_2...x_i}$$

THe first terms of the sequences are

$$x_{k,0} = x_k$$

$$u_0 = u$$

$$v_0 = v$$

And

$$x_{k,1} = x_k^i(x_1 + x_2 + ... + x_i)^{-(i-1)}$$
$$k = 1, 2, ..., i$$

Which implies

$$u = x_1 + x_2 + ... + x_i = (x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + ... + x_{i,1}^{\frac{1}{i}})^i > u_1 > 1$$

And

$$x_{k,0} = x_k = x_{k,1}^{\frac{1}{i}}(x_1 + x_2 + ... + x_i)^{\frac{(i-1)}{i}}$$
$$= x_{k,1}^{\frac{1}{i}}(x_{1,1}^{\frac{1}{i}} + x_{2,1}^{\frac{1}{i}} + ... + x_{i,1}^{\frac{1}{i}})^{(i-1)} > x_{k,1} > 0$$

And

$$v = \frac{x_{1,0}x_{2,0}...x_{i,0}}{u^{(i-1)}} = x_{1,1}^{\frac{1}{i}}x_{2,1}^{\frac{1}{i}}...x_{i,1}^{\frac{1}{i}} > v_1 = \frac{x_{1,1}x_{2,1}...x_{i,1}}{u_1^{(i-1)}} > 0$$

until infinity.

$$u_j = x_{1,j} + x_{2,j} + ... + x_{i,j} = (x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^i > u_{j+1} > 1$$

And

$$x_{k,j} = x_{k,j+1}^{\frac{1}{i}}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{(i-1)} > x_{k,j+1} > 0$$

And

$$k = 1, 2, ..., i$$

And

$$v_j = \frac{x_{1,j}x_{2,j}...x_{i,j}}{u_j^{(i-1)}} = x_{1,j+1}^{\frac{1}{i}}x_{2,j+1}^{\frac{1}{i}}...x_{i,j+1}^{\frac{1}{i}} > v_{j+1} = \frac{x_{1,j+1}x_{2,j+1}...x_{i,j+1}}{u_{j+1}^{(i-1)}} > 0$$

$x_{k,j}$, v_j, u_j are positives $\forall j > 1$, $\forall k = 1, 2, ..., i$.

Lemma L (P) is the expression :

$$x_{k,j} = x_k^{i^j}(\prod_{l=0}^{l=j-1} x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

Proof of lemma L by induction ! For $j = 1$ it is verified : see $x_{k,1}$, u_1 and v_1, we suppose (P) true, but with u_j it means with (P), that

$$x_{k,j} = x_{k,j+1}^{\frac{1}{i}}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{3}} + ... + x_{i,j+1}^{\frac{1}{i}})^{(i-1)}$$

But

$$x_{k,j+1}^{\frac{1}{i}} = x_{k,j}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{-(i-1)}$$

And

$$x_{k,j+1} = x_{k,j}^{i}(x_{1,j+1}^{\frac{1}{i}} + x_{2,j+1}^{\frac{1}{i}} + ... + x_{i,j+1}^{\frac{1}{i}})^{-(i-1)i} = x_{k,j}^{i}(x_{1,j} + x_{2,j} + ... + x_{i,j})^{-(i-1)}$$

$$= x_k^{i^{j+1}} \prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-i(i-1)}(x_1^{i^j} + x_2^{i^j} + ... + x_i^{i^j})^{-(i-1)}.$$

$$\prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{(i-1)^2}$$

$$= x_k^{i^{j+1}} \prod_{l=0}^{l=j} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

Hence

$$x_{k,j} = x_k^{i^j} \prod_{l=0}^{l=j-1} (x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

The solution $\forall k \geq 2$ is

$$X_k = X_m$$

for all k, m. Let

$$u = U^{2n}$$

$$x = U^n X_k^{n_k}$$

$$y = U^n(U^n - X_k^{n_k})$$

$$z = X_k^{n_k}(U^n - X_k^{n_k})$$

Thus

$$u = x + y$$

And

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$$

Llemma 1. Its solution is

$$U = X_k = 0; \forall k \in \{1, 2, ..., i\}$$

The conjecture is that generalized Fermat-Catalan equations have no solutions for $n > i(i-1)$ and $n_k > i(i-1)$. There does not exist an explicit formula for the solutions : for some i, it is i, for other $i+1$, but there are solutions in $n \leq i(i-1)$ or $n_k \leq i(i-1)$. Why did we have proposed $n \leq i(i-1)$, $n_k \leq i(i-1)$? Let

$$n = n_k = i(i-1)$$

The formula become

$$x_{k,j} = x_k^{i^j}(\prod_{l=0}^{l=j-1} x_1^{i^l} + x_2^{i^l} + ... + x_i^{i^l})^{-(i-1)}$$

$$= U^{ni^j} X_k^{n_k i^j}(\prod_{l=0}^{l=j-1} (U^{ni^l} X_1^{n_1 i^l} + U^{ni^l} X_2^{n_2 i^l} + ... + U^{ni^l} X_i^{n_i i^l}))^{-(i-1)}$$

$$= U^{i(i-1)i^j} X_k^{i(i-1)i^j}(\prod_{l=0}^{l=j-1} (U^{i(i-1)i^l} X_1^{i(i-1)i^l} + ... + U^{i(i-1)i^l} X_i^{i(i-1)i^l}))^{-(i-1)}$$

$$= U^{(i-1)i^{j+1}} X_k^{(i-1)i^{j+1}}(\prod_{l=0}^{l=j-1} (U^{(i-1)i^{l+1}} X_1^{(i-1)i^{l+1}} + ... + U^{(i-1)i^{l+1}} X_i^{(i-1)i^{l+1}}))^{-(i-1)}$$

$$= U^{(i-1)i^{j+1}} X_k^{(i-1)i^{j+1}}(\prod_{l=1}^{l=j} (U^{(i-1)i^l} X_1^{(i-1)i^l} + ... + U^{(i-1)i^l} X_i^{(i-1)i^l}))^{-(i-1)}$$

$$= (U^{i^{j+1}} X_k^{i^{j+1}}(\prod_{l=1}^{l=j} (U^{(i-1)i^l} X_1^{(i-1)i^l} + ... + U^{(i-1)i^l} X_i^{(i-1)i^l}))^{-1})^{i-1}$$

It is the expression of $x_{k,j+1}$ of the exponent $i-1$. If there are solutions for $i-1$, there will be solutions for $i(i-1)$. It is not the case of i, because of the exponent $i-1$ in the formula. Now, let

$$x_{i+1}^{n_{i+1}} = x_1^{n_1} + ... + x_i^{n_i}$$

We will prove now definitely that this equation is undecidable for some i. For this, we will try to find an impossibility. Let

$$z = x_m^{n_m} + x_k^{n_k}$$

With x_m, x_k and z greater than zero. Let firstly $n_k \geq n_m$. We have

$$2(a_1 S_1 + a_2 S_2) = (a_1 + a_2)(S_1 + S_2) + (a_1 - a_2)(S_1 - S_2) \quad (7)$$

But

$$z = x_m^{n_m} + x_k^{n_k} = x_m x_m^{n_m - 1} + x_k x_k^{n_k - 1}$$

And (7)

$$2(x_m^{n_m} + x_k^{n_k}) = (x_m + x_k)(x_m^{n_m - 1} + x_k^{n_k - 1}) + (x_m - x_k)(x_m^{n_m - 1} - x_k^{n_k - 1})$$

If $n_m - 1 = 1$, then

$$2z = (x_m + x_k)(x_m + x_k^{n_k - 1}) + (x_m - x_k)(x_m - x_k^{n_k - 1})$$

Our approach will end here (as we will see). Thus, for $n_m - 1 > 1$, with (7)

$$(x_m+x_k)(x_m^{n_1-1}+x_k^{n_k-1}) = \frac{1}{2}((x_m+x_k)^2(x_m^{n_m-2}+x_k^{n_k-2})+(x_m+x_k)(x_m-x_k)(x_m^{n_m-2}-x_k^{n_k-2}))$$

$$\frac{1}{2}(x_m+x_k)^2(x_m^{n_m-2}+x_k^{n_k-2}) = \frac{1}{2^2}((x_m+x_k)^3(x_m^{n_m-3}+x_k^{n_k-3})+(x_m+x_k)^2(x_m-x_k)(x_m^{n_m-3}-x_k^{n_k-3}))$$

Until $n_m - 2$

$$\frac{1}{2^{n_m-3}}(x_m + x_k)^{n_m-2}(x_m^2 + x_k^{n_k-n_m+2})$$

$$= \frac{1}{2^{n_m-2}}((x_m+x_k)^{n_m-1}(x_m+x_k^{n_k-n_m+1})+(x_m+x_k)^{n_m-2}(x_m-x_k)(x_m-x_k^{n_k-n_m+1}))$$

We add and simplify

$$2z = \frac{1}{2^{n_m-2}}(x_m + x_k)^{n_m-1}(x_m + x_k^{n_k-n_m+1})+$$

$$+(x_m - x_k)\sum_{m'=0}^{m'=n_m-2}(\frac{(x_m + x_k)^{m'}}{2^{m'}}(x_m^{n_m-1-m'} - x_k^{n_k-1-m'})))$$

If we multiply by 2^{n_m-2}

$$2^{n_m-1}z = (x_m + x_k)^{n_1-1}(x_m + x_k^{n_k-n_m+1})+$$

$$+(x_m - x_k)\sum_{m'=0}^{m'=n_m-2}(2^{n_m-2-m'}(x_m + x_k)^{m'}(x_m^{n_m-1-m'} - x_k^{n_k-1-m'}))$$

But

$$2(x_m^{n_m} - x_k^{n_k}) = (x_m - x_k)(x_m^{n_m-1} + x_k^{n_k-1}) + (x_m + x_k)(x_m^{n_m-1} - x_k^{n_k-1})$$

$$(x_m+x_k)(x_m^{n_m-1}-x_k^{n_k-1}) = \frac{1}{2}((x_m+x_k)(x_m-x_k)(x_m^{n_m-1}+x_k^{n_k-2})+(x_m+x_k)^2(x_m^{n_m-2}-x_k^{n_k-2}))$$

$$\frac{1}{2}(x_m+x_k)^2(x_m^{n_m-2}-x_k^{n_k-2}) = \frac{1}{2^2}((x_m+x_k)^2(x_m-x_k)(x_m^{n_m-3}+x_k^{n_k-3})+(x_m+x_k)^3(x_m^{n_m-3}-x_k^{n_k-3}))$$

Until $n_m - 2$

$$\frac{1}{2^{n_m-3}}(x_m + x_k)^{n_m-2}(x_m^2 - x_k^{n_k-n_m+2})$$

$$= \frac{1}{2^{n_m-2}}((x_m+x_k)^{n_m-2}(x_m-x_k)(x_m+x_k^{n_k-n_m+1})+(x_m+x_k)^{n_m-1}(x_m-x_k^{n_k-n_m+1}))$$

We add and simplify

$$2(x_m^{n_m} - x_k^{n_k}) = \frac{1}{2^{n_m-2}}(x_m + x_k)^{n_m-1}(x_m - x_k^{n_k-n_m+1}) +$$
$$+(x_m - x_k) \sum_{m'=0}^{m'=n_m-2} (\frac{(x_m + x_k)^{m'}}{2^{m'}}(x_k^{n_k-1-m'} + x_m^{n_m-1-m'})))$$

We multiply by 2^{n_m-2}

$$2^{n_m-1}(x_m^{n_m} - x_k^{n_k})$$
$$= (x_m+x_k)^{n_m-1}(x_m-x_k^{n_k-n_m+1})+(x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}(x_k^{n_k-1-m'} + x_m^{n_m-1-m'}))$$

We remember

$$2^{n_m-1}(x_m^{n_m} + x_k^{n_k})$$
$$= (x_m+x_k)^{n_m-1}(x_m+x_k^{n_k-n_m+1})+(x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}(-x_k^{n_k-1-m'} + x_m^{n_m-1-m'}))$$

We deduce

$$2^{n_m}x_m^{n_m} = 2x_m(x_m+x_k)^{n_m-1}+2(x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}x_m^{n_m-1-m'})$$

And

$$2^{n_m}x_k^{n_k} = 2x_k^{n_k-n_m+1}(x_m+x_k)^{n_m-1}-2(x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}x_k^{n_k-1-m'})$$

Thus

$$2^{n_m-1}x_m^{n_m-1} = (x_m+x_k)^{n_m-1}+(x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}x_m^{n_m-2-m'})$$

And

$$2^{n_m-1}x_k^{n_m-1} = (x_m+x_k)^{n_m-1}-(x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}x_k^{n_m-2-m'})$$

Hence

$$2^{n_m-1}(x_m^{n_m-1}-x_k^{n_m-1}) = (x_m-x_k) \sum_{m'=0}^{m'=n_m-2} (2^{n_m-2-m'}(x_m + x_k)^{m'}(x_m^{n_m-2-m'} + x_k^{n_m-2-m'}))$$

And by the same process than for $x = x_m$, $y = x_k$, $n_m = a+2$, $n_k = b+2$, we prove that $x_k = x_m$. We resume, generalized Fermat-Catalan equation is impossible for $n > 2$ and $n_k > 2$, it leads to $x_m = x_k$ for which the sequences have already leaded ud.

Some other exceptions For the equation

$$kU^n = X^n + Y^n$$

for some k integers like 7, there are solutions, for others like 2, there are not. If

$$u = (kU^n)^2$$

$$x = kU^n X^n$$

$$y = kU^n Y^n$$

$$z = X^n Y^n$$

Thus

$$u = kU^n(X^n + Y^n) = x + y$$

$$\frac{1}{z} = \frac{k^2U^{2n}}{kU^nX^nkU^nY^n} = \frac{u}{xy} = \frac{1}{x} + \frac{1}{y}$$

If

$$u = U^{2n}$$

$$x = U^n X^n$$

$$y = U^n Y^n$$

$$z = X^n Y^n$$

And

$$u = U^{2n} = \frac{kU^n(X^n + y^n)}{k^2} = \frac{x+y}{k}$$

And

$$\frac{1}{z} = \frac{U^{2n}}{U^nX^nU^nY^n} = \frac{u}{xy} = \frac{x+y}{kxy} = \frac{1}{kx} + \frac{1}{ky}$$

Imcompatible equations !

Conclusion This approach of Fermat-Catalan equations allowed to make an allusion to Matyasevich theorem and to generalize the Fermat-Catalan equations to a new conjecture. We have proved that Beal equation has no solution for $a > 0$ and $b > 0$ and $c > 0$ with x, y, z are coprime. There are solutions for $abc = 0$.

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