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KRZYSZTOF BOGDAN, BARTŁOMIEJ DYDA, AND TOMASZ LUKS

Abstract. We characterize conditional Hardy spaces of the Laplacian and the fractional Laplacian by using Hardy-Stein type identities.

1. Introduction

Let $D \subset \mathbb{R}^d$ be open and let $x_0 \in D$. For $p > 0$ and $0 < \alpha < 2$ we will consider the Hardy space $\mathcal{H}^p(D, \alpha)$ of the fractional Laplacian $\Delta^{\alpha/2}$,

$$\Delta^{\alpha/2} u(x) = \lim_{\eta \to 0^+} \int_{|y-x|>\eta} \mathcal{A} \, \frac{u(y) - u(x)}{|y-x|^{d+\alpha}} \, dy \,. \tag{1}$$

Here $\mathcal{A} = \Gamma((d+\alpha)/2)/(2^{-\alpha}\pi^{d/2}|\Gamma(-\alpha/2)|)$. $\mathcal{H}^p(D, \alpha)$ is defined as follows. Let X be the isotropic α-stable Lévy process ([9]), i.e. the symmetric Lévy process on $\mathbb{R}^d$ with the Lévy measure $\nu(dy) = \mathcal{A}|y|^{-d-\alpha} dy$ and zero Gaussian part ([9]). Let $\mathbb{E}_x$ be the expectation for X starting at $x \in \mathbb{R}^d$. We define $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$, the first exit time of X from D. A Borel function $u : \mathbb{R}^d \to \mathbb{R}$ is called α–harmonic on D if for every open U relatively compact in D (denoted $U \subset\subset D$) we have

$$u(x) = \mathbb{E}_x u(X_{\tau_U}) \,, \quad x \in U. \tag{2}$$

Here we assume that the expectation is absolutely convergent, in particular– finite. Equivalently, u is α–harmonic on D if u is twice continuously differentiable on D, $\int_{\mathbb{R}^d} |u(y)|(1+|y|)^{-d-\alpha} dy < \infty$, and

$$\Delta^{\alpha/2} u(x) = 0 \,, \quad x \in D \,. \tag{3}$$

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We refer to [8] and [12] for this characterization and a detailed discussion of α-harmonic functions, including structure theorems for nonnegative α-harmonic functions, and explicit formulas for the Green function, Poisson kernel and Martin kernel of $\Delta^{\alpha/2}$ for the ball. We also refer to remarks in [13, p. 120] explaining why the mean value property (2) is preferred over analogues of (3) for harmonic functions of more general Markov processes. The reader may verify, using (2) and the strong Markov property of X, that $\{u(X_{\tau_U})\}_{U\subset\subset D}$ is a martingale ordered by inclusion of sets U. In particular, $\mathbb{E}_x|u(X_{\tau_U})|^p$ is non-decreasing in U, if $p \geq 1$.

Definition 1. *Let $0 < p < \infty$. We write $u \in \mathcal{H}^p = \mathcal{H}^p(D,\alpha)$, if u is α-harmonic on D and*

$$\|u\|_p := \sup_{U\subset\subset D} \left(\mathbb{E}_{x_0}|u(X_{\tau_U})|^p\right)^{1/p} < \infty\,. \tag{4}$$

The finiteness condition does not depend on the choice of x_0, because $U \ni x \mapsto \mathbb{E}_x|u(X_{\tau_U})|^p$ satisfies Harnack inequality for arbitrary (Borel) function u ([12]). If $p \leq q$, then $\mathcal{H}^p \supset \mathcal{H}^q$.

We will say that nonnegative functions $f(u)$ and $g(u)$ are *comparable*, and write $f(u) \asymp g(u)$, if numbers $0 < c \leq C < \infty$ exist such that $cf(u) \leq g(u) \leq Cf(u)$ for every u.

The reader may notice that (4) is far from being explicit because it involves the distribution of X_{τ_U} for all $U \subset D$. The following result and the exact formula for $\|u\|_p$ given in (18) below may simplify the perspective. Let $G_D(x,y)$ be the Green function of $\Delta^{\alpha/2}$ for the Dirichlet problem on D ([12]).

Theorem 1. *If $1 < p < \infty$, then $\|u\|_p^p$ is* comparable *on $\mathcal{H}^p$ with*

$$|u(x_0)|^p + \int_D G_D(x_0,y) \int_{\mathbb{R}^d} \frac{[u(z)-u(y)]^2\ \big(\ |u(z)| \vee |u(y)|\ \big)^{p-2}}{|z-y|^{d+\alpha}}\, dz\, dy. \tag{5}$$

In fact, $u \in \mathcal{H}^p$ if and only if u is α-harmonic in D and the integral is finite.

Below we analogously decribe Hardy spaces $\mathcal{H}^p_h = \mathcal{H}^p_h(D,\alpha)$ related to

$$\Delta_h^{\alpha/2}(u) = \frac{1}{h}\Delta^{\alpha/2}(hu)\,. \tag{6}$$

Here h is a fixed function harmonic for $\Delta^{\alpha/2}$, positive on D and vanishing on D^c. The boundary behavior of functions in $\mathcal{H}^p_h$ is of considerable interest because it directly relates to *ratios* of α-harmonic functions. In fact, the above *h-transform* (6) of $\Delta^{\alpha/2}$ is implicit in the relative Fatou theorems studied in [29, 24, 10] and in the theory of conditional stable processes [8, 14].

We also give similar characterizations for Hardy spaces of the classical Laplacian Δ: formula (36) below is the celebrated Hardy-Stein identity but Theorem 3, which may be considered a *conditional* Hardy-Stein identity, is apparently new, and may be interesting for its own sake.

The paper is composed as follows. In Section 2 we observe the formula

$$\sup_{U\subset\subset D}\mathbb{E}_{x_0}u^2(X_{\tau_U}) = |u(x_0)|^2 + \int_D G_D(x_0,y)\int_{\mathbb{R}^d}\mathcal{A}\,\frac{[u(z)-u(y)]^2}{|z-y|^{d+\alpha}}\,dz\,dy \tag{7}$$

for the norm of $\mathcal{H}^2$, and we extend it in Lemma 4 and Theorem 1 to $\mathcal{H}^p$ for $p>1$. The conditional Hardy spaces $\mathcal{H}^p_h$ are characterized in Lemma 8, Theorem 2 and formula (35) in Section 3, see also Remark 1. In Section 4 we state the results for the Laplacian: formula (36) and Theorem 3. In Section 5 we describe the norm of the Hardy spaces in terms of the Krickeberg decomposition for $p\geq 1$, and we prove a classical Littlewood-Paley inequality.

Formula (7) and its modifications (18, 35, 37) below are the main subject of the paper, and they may be considered nonlocal or conditional extensions of the classical Hardy-Stein equality, for which we refer the reader to (36) in Section 4 and [39, 32, 33].

Our work was motivated by the notion of the quadratic variation of martingales, carré du champ, and the characterization of the classical and martingale Hardy and Bergman spaces ([17, 28, 34, 33, 41, 39, 27, 42]). The resulting technique should apply to Hardy spaces of operators and Markov processes more general than the fractional Laplacian and the isotropic stable Lévy process. The style of the presentation and the inclusion of both jump and continuous processes in the present paper is intended to clarify the methodology and indicate such extensions. Our development is analytic. In fact, the definitions of the Hardy spaces can be easily formulated analytically by using the harmonic measures of the Laplacian and the fractional Laplacian ([4, 26]). A clarifying comparison of the conditional and the non-conditional cases is made at the end of Section 4.

2. Characterization of $\mathcal{H}^p$

Consider open $U\subset\subset D$ and a real-valued function $\phi:\ \mathbb{R}^d\to\mathbb{R}$ which is C^2 in a neighborhood of $\overline{U}$ and satisfies $\int_{\mathbb{R}^d}|\phi(y)|(1+|y|)^{-d-\alpha}<\infty$. Then $\Delta^{\alpha/2}\phi$ is bounded on $\overline{U}$, and for every $x\in\mathbb{R}^d$ we have

$$\phi(x) = \mathbb{E}_x\phi(X_{\tau_U}) - \int_U G_U(x,y)\Delta^{\alpha/2}\phi(y)\,dy\,. \tag{8}$$

Indeed, the difference of both sides of (8) vanishes continuously on U^c and annihilates $\Delta^{\alpha/2}$ on U. Thus the difference is zero on U by the maximum principle for $\Delta^{\alpha/2}$, see [8, Lemma 3.8 and the proof of Theorem 3.9], or [11, Appendix]. Alternatively, (8) may be obtained from Dynkin's formula.

Lemma 1. *If u is α-harmonic on D and $U \subset\subset D$, then*

$$\mathbb{E}_{x_0} u^2(X_{\tau_U}) = u(x_0)^2 + \int_U G_U(x_0,y) \int_{\mathbb{R}^d} \mathcal{A}\, \frac{[u(z)-u(y)]^2}{|z-y|^{d+\alpha}}\, dz\, dy\,. \tag{9}$$

Proof. If $\int_{\mathbb{R}^d} u(y)^2(1+|y|)^{-d-\alpha} = \infty$, then $\int_{\mathbb{R}^d}[u(z)-u(y)]^2/|z-y|^{d+\alpha}\, dz = \infty$ for every y. Also $\mathbb{E}_{x_0} u^2(X_{\tau_U}) = \infty$, because the distribution of X_{τ_U} has density function bounded below by a multiple of $(1+|y|)^{-d-\alpha}$ in the complement of the neighborhood of $\overline{U}$ ([5]). Therefore in what follows we may assume that $\int_{\mathbb{R}^d} u(y)^2(1+|y|)^{-d-\alpha} < \infty$. Since u^2 is smooth in a neighborhood of $\overline{U}$, $\Delta^{\alpha/2}(u^2)$ is bounded on $\overline{U}$. By (8) with $\phi = u^2$, for $x \in \mathbb{R}^d$ we have

$$\mathbb{E}_x u^2(X_{\tau_U}) = u^2(x) + \int_U G_U(x,y)\Delta^{\alpha/2}(u^2)(y)\, dy\,. \tag{10}$$

For $y \in \overline{U}$, $z \in \mathbb{R}^d$, we have $u^2(z)-u^2(y)-2u(y)[u(z)-u(y)] = [u(z)-u(y))]^2$. Since $\Delta^{\alpha/2}u(y) = 0$, we have

$$\begin{aligned}
\Delta^{\alpha/2}u^2(y) &= \Delta^{\alpha/2}u^2(y) - 2u(y)\Delta^{\alpha/2}u(y) \\
&= \lim_{\eta\to 0^+} \int\limits_{\{z\in\mathbb{R}^d\colon |z-y|>\eta\}} \mathcal{A}\, \frac{u^2(z)-u^2(y)-2u(y)[u(z)-u(y)]}{|z-y|^{d+\alpha}}\, dz \\
&= \int_{\mathbb{R}^d} \mathcal{A}\, \frac{[u(z)-u(y)]^2}{|z-y|^{d+\alpha}}\, dz\,,
\end{aligned}$$

and (9) follows. □

Recall that $G_U(x,y) \uparrow G_D(x,y)$ as $U \uparrow D$. By the monotone convergence theorem we obtain the description of $\mathcal{H}^2$ aforementioned in Introduction.

Corollary 2. *If u is α-harmonic in D, then (7) holds.*

We conclude that $\mathcal{H}^2$ consists of precisely all those functions α-harmonic on D for which the quadratic form on the right hand side of (7) is finite.

We will now consider (general) $p > 1$. We note that $x \mapsto |x|^p$ is convex on $\mathbb{R}$, with the derivative $pa|a|^{p-2}$ at $x = a$. For $a, b \in \mathbb{C}$ we let

$$F(a,b) = |b|^p - |a|^p - p\overline{a}|a|^{p-2}(b-a)\,, \tag{11}$$

a second order Taylor remainder. We have $F(a,b) = |b|^p$ if $a = 0$ and $F(a,b) = (p-1)|a|^p$ if $b = 0$. By convexity, $F(a,b) \geq 0$ for $a, b \in \mathbb{R}$.

Example 1. For (even) $p = 2, 4, \dots$ and $a, b \in \mathbb{R}$, we have

$$F(a,b) = b^p - a^p - pa^{p-1}(b-a) = (b-a)^2 \sum_{k=0}^{p-2} (k+1) b^{p-2-k} a^k.$$

For $p \in (1,2)$ we will use a certain approximation procedure. Let ε, b and a be real numbers, and define

$$F_\varepsilon(a,b) = \operatorname{Re} F(a+i\varepsilon, b+i\varepsilon) = |b+i\varepsilon|^p - |a+i\varepsilon|^p - pa|a+i\varepsilon|^{p-2}(b-a)\,. \tag{12}$$

Of course, $F_\varepsilon(a,b) \to F(a,b)$ as $\varepsilon \to 0$.

Lemma 3. *For every $p > 1$, we have*

$$F(a,b) \asymp (b-a)^2(|b| \vee |a|)^{p-2}, \quad a, b \in \mathbb{R}. \tag{13}$$

If $p \in (1,2)$, then uniformly in $\varepsilon, a, b \in \mathbb{R}$, we have

$$F_\varepsilon(a,b) \le c(b-a)^2(|b| \vee |a|)^{p-2}. \tag{14}$$

Proof. We denote $K(a,b) = (b-a)^2(|b| \vee |a|)^{p-2}$. For every $k \in \mathbb{R}$, $F(ka,kb) = |k|^p F(a,b)$ and $K(ka,kb) = |k|^p K(a,b)$. If $a = 0$, then (13) becomes equality, hence we may assume that $a \neq 0$, in fact – that $a = 1$. Let $f(b) = F(1,b) = |b|^p - 1 - p(b-1)$. We will prove that

$$c_p(b-1)^2(|b| \vee 1)^{p-2} \le f(b) \le C_p(b-1)^2(|b| \vee 1)^{p-2}. \tag{15}$$

Since $f(1) = f'(1) = 0$ and $f''(y) = p(p-1)|y|^{p-2}$, we obtain

$$f(b) = p(p-1)\int_1^b \int_1^x |y|^{p-2}\,dy\,dx = p(p-1)\int_1^b |y|^{p-2}(b-y)\,dy\,. \tag{16}$$

The first integral is over a simplex of area $(b-1)^2/2$, and it is a monotone function of the simplex (as ordered by inclusion). For b close to 1 the integral is comparable to $(b-1)^2$. For large $|b|$ the (second) integral is comparable to $|b|^p$. This proves (15), hence (13). We will now prove (14) for $p \in (1,2)$. Let $f_\varepsilon(b) = F_\varepsilon(1,b) = (b^2+\varepsilon)^{p/2} - (1+\varepsilon^2)^{p/2} - p(1+\varepsilon^2)^{(p-2)/2}(b-1)$. We have $f_\varepsilon(1) = f_\varepsilon'(1) = 0$ and

$$f_\varepsilon''(b) = (b^2+\varepsilon^2)^{(p-4)/2} p[b^2(p-1)+\varepsilon^2] \asymp (b^2+\varepsilon^2)^{(p-2)/2} \le |b|^{p-2}.$$

Thus, uniformly in ε,

$$f_\varepsilon(b) \le c\int_1^b \int_1^x |y|^{p-2}\,dy\,dx = c\int_1^b (b-y)|y|^{p-2}\,dy \asymp (b-1)^2(|b| \vee 1)^{p-2}.$$

If $a \neq 0$, then $F_\varepsilon(a,b) = |a|^p f_{\varepsilon/a}(b/a) \le c|a|^p(b/a-1)^2(|b/a| \vee 1)^{p-2}$. If $a = 0$, then $F_\varepsilon(a,b) = (b^2+\varepsilon^2)^{p/2} - |\varepsilon|^p \asymp |\varepsilon|^p[(b/\varepsilon)^2 \wedge |b/\varepsilon|^p] \le |b|^p$. The proof of (14) is complete. □

Lemma 4. *If u is α-harmonic in D, $U \subset\subset D$, and $p > 1$, then*

$$\mathbb{E}_{x_0}|u(X_{\tau_U})|^p = |u(x_0)|^p + \int_U G_U(x_0, y) \int_{\mathbb{R}^d} \mathcal{A}\, \frac{F(u(y), u(z))}{|z-y|^{d+\alpha}}\, dz\, dy\,. \tag{17}$$

Proof. We proceed as in Lemma 1. In particular we will assume that $\int_{\mathbb{R}^d} |u(y)|^p(1+|y|)^{-d-\alpha} < \infty$, for otherwise both sides of (17) are infinite. We will first consider the case of $p \geq 2$ and apply (8) to $\phi = |u|^p \in C^2(D)$. For $y \in D$ we have $\Delta^{\alpha/2}u(y) = 0$, hence

$$\begin{aligned}
\Delta^{\alpha/2}|u|^p(y) &= \Delta^{\alpha/2}|u|^p(y) - pu(y)|u(y)|^{p-2}\Delta^{\alpha/2}u(y)\\
&= \lim_{\eta\to 0^+} \int\limits_{\{z\in\mathbb{R}^d\colon |z-y|>\eta\}} \mathcal{A}\, \frac{|u(z)|^p - |u(y)|^p - pu(y)|u(y)|^{p-2}[u(z)-u(y)]}{|z-y|^{d+\alpha}}\, dz\\
&= \int_{\mathbb{R}^d} \mathcal{A}\, \frac{F(u(y), u(z))}{|z-y|^{d+\alpha}}\, dz.
\end{aligned}$$

This and (8) yield (17) for $p \geq 2$. We now consider $1 < p < 2$. We note that $|u + i\varepsilon|^p \in C^2(D)$. As in the first part of the proof,

$$\begin{aligned}
\Delta^{\alpha/2}|u+i\varepsilon|^p(y) &= \Delta^{\alpha/2}|u+i\varepsilon|^p(y) - pu(y)|u(y)+i\varepsilon|^{p-2}\Delta^{\alpha/2}u(y)\\
&= \int_{\mathbb{R}^d} \mathcal{A}\, \frac{F_\varepsilon(u(y), u(z))}{|z-y|^{d+\alpha}}\, dz,
\end{aligned}$$

hence

$$\mathbb{E}_{x_0}|u(X_{\tau_U}) + i\varepsilon|^p = |u(x_0) + i\varepsilon|^p + \int_U G_U(x_0, y) \int_{\mathbb{R}^d} \mathcal{A}\, \frac{F_\varepsilon(u(y), u(z))}{|z-y|^{d+\alpha}}\, dz\, dy.$$

We let $\varepsilon \to 0$. By Fatou's lemma, (14), (13) and dominated convergence, we obtain (17). $\square$

Proof of Theorem 1. Lemma 3, Lemma 4 and monotone convergence imply the comparability of $\|u\|_p^p$ and (5) with the same constants as in (15), under the mere assumption that u be α-harmonic on D. In fact,

$$\|u\|_p^p = |u(x_0)|^p + \int_D G_D(x_0, y) \int_{\mathbb{R}^d} \mathcal{A}\, \frac{F(u(y), u(z))}{|z-y|^{d+\alpha}}\, dz\, dy\,. \tag{18}$$

$\square$

We note that in many cases the exact asymptotics of G_D is known. For instance, if D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and $d > \alpha$, then

$$G_D(x_0, y) \asymp \delta_D(y)^{\alpha/2}|y - x_0|^{\alpha-d}\ ,$$

where $\delta_D(y) := \mathrm{dist}(y, D^c)$. Here we refer the reader to [22], also for the case of Lipschitz domains.

Recall the definition of F_ε, (12), and the fact that $F_0 = F$ of (11). Before moving to conditional Hardy spaces we will record the following observation.

Lemma 5. *For every $p > 1$ and a_1, a_2, b_1, b_2, $\varepsilon \in \mathbb{R}$, we have*

$$F_\varepsilon(a_1 \wedge a_2, b_1 \wedge b_2) \le F_\varepsilon(a_1, b_1) \vee F_\varepsilon(a_2, b_2), \tag{19}$$

$$F_\varepsilon(a_1 \vee a_2, b_1 \vee b_2) \le F_\varepsilon(a_1, b_1) \vee F_\varepsilon(a_2, b_2). \tag{20}$$

In particular, $F(a \wedge 1, b \wedge 1) \le F(a, b)$ and $F(a \vee (-1), b \vee (-1)) \le F(a, b)$, for all $a, b \in \mathbb{R}$. The latter also extends to $K(a, b) = (b - a)^2(|b| \vee |a|)^{p-2}$.

Proof. Let $\varepsilon \neq 0$. We claim that the function $b \mapsto F_\varepsilon(a, b)$ decreases on $(-\infty, a]$ and increases on $[a, \infty)$. To see this, we consider

$$\frac{\partial F_\varepsilon}{\partial b}(a, b) = pb(b^2 + \varepsilon^2)^{p/2-1} - pa(a^2 + \varepsilon^2)^{p/2-1} . \tag{21}$$

The function $h(x) = px(x^2+\varepsilon^2)^{p/2-1}$ has derivative $h'(x) = p(x^2+\varepsilon^2)^{p/2-2}(x^2(p-1) + \varepsilon^2) > 0$. If follows that the difference in (21) is positive if $b > a$ and negative if $b < a$. This proves our claim.

Furthermore the function $a \mapsto F_\varepsilon(a, b)$ decreases on $(-\infty, b]$ and increases on $[b, \infty)$, as follows from calculating the derivative,

$$\frac{\partial F_\varepsilon}{\partial a}(a, b) = p(a - b)(a^2 + \varepsilon^2)^{p/2-2}(a^2(p - 1) + \varepsilon^2) . \tag{22}$$

We will now prove (19). If $b_1 \wedge b_2 = b_1$ and $a_1 \wedge a_2 = a_1$ (or $b_1 \wedge b_2 = b_2$ and $a_1 \wedge a_2 = a_2$), then (19) is trivial. If $b_1 \wedge b_2 = b_1$ and $a_1 \wedge a_2 = a_2$, then the monotonicity of F yields

$$\begin{aligned} F_\varepsilon(a_2, b_1) &\le F_\varepsilon(a_1, b_1), \quad \text{if } b_1 < a_2, \\ F_\varepsilon(a_2, b_1) &\le F_\varepsilon(a_2, b_2), \quad \text{if } b_1 \ge a_2. \end{aligned}$$

The case $b_1 \wedge b_2 = b_2$ and $a_1 \wedge a_2 = a_1$ obtains by renaming the arguments. This proves inequality (19). (20) follows from (19) and the identity

$$F_\varepsilon(-a, -b) = F_\varepsilon(a, b).$$

The case $\varepsilon = 0$ obtains by passing to the limit. When $a = b$, we have $F(a, b) = 0$, which yields the second last statement of the lemma. For K we obviously have $(b \wedge 1 - a \wedge 1)^2(|b \wedge 1| \vee |a \wedge 1|)^{p-2} \le (b - a)^2(|b| \vee |a|)^{p-2}$ and $(b \vee (-1) - a \vee (-1))^2(|b \vee (-1)| \vee |a \vee (-1)|)^{p-2} \le (b - a)^2(|b| \vee |a|)^{p-2}$. □

In passing we note that if the right-hand side of (17) is finite for u, then it is also finite (in fact–smaller) for $u \wedge 1$ and $u \vee (-1)$. The latter functions have smaller values and increments than u, a property defining *normal contractions* for Dirichlet forms ([19]).

3. Characterization of $\mathcal{H}_h^2$

The fractional Laplacian is a nonlocal operator and the corresponding stochastic process X has jumps. In consequence the definitions of α-harmonicity (2) and (3) involve the values of the function on the whole of D^c ([12]). This is somewhat unusual compared with the classical theory of the Laplacian and the Brownian motion, and attempts were made by various authors to ascribe genuine boundary conditions to such processes and functions ([3, 7, 21, 29]). One possibility is to study the boundary behavior of harmonic functions after an appropriate normalization. In this section we will focus on α-harmonic functions vanishing on D^c, so that D^c may be ignored, and we will use Doob's h-conditioning to normalize ([16]). Let h be harmonic and positive on D, and let h vanish on D^c. In particular h may be a Martin integral,

$$h(x) = \int M(x,z)\,\mu(dz)\,, \quad x \in D\,,$$

where $\mu \geq 0$ is a finite nonzero Borel measure on the set of the so-called accessible points of ∂D, and $M(\cdot, z)$ denotes the Martin kernel with the pole at $z \in \partial D$, see [12]. We like to note that if D is Lipschitz, then all the points of ∂D are accessible ([12]), while in the general case $\partial_M D \subset \partial D \cup \{\infty\}$. Doob's h-transform (6) yields a nontrivial modification of the potential theory of $\Delta^{\alpha/2}$. In particular, the ratio u/h is harmonic for $\Delta_h^{\alpha/2}$ if u is α-harmonic and vanishes on D^c (see below). This observation offers a convenient framework for studying growth and boundary behavior of such ratios.

We will consider the transition semigroup

$$P_t^h f(x) = \frac{1}{h(x)} \int p_D(t,x,y) f(y) h(y) dy\,. \tag{23}$$

Here $p_D(t,x,y) = \mathbb{E}_x\{t < \tau_D : \, p(t-\tau_D, X_{\tau_D}, y)\}$ is the time-homogeneous transition density of X *killed* on leaving D ([8]). The semigroup property of P_t^h follows directly from the Chapman-Kolmogorov equations for p_D,

$$\int_{\mathbb{R}^d} p_D(s,x,y) p_D(t,y,z) dy = p_D(s+t,x,z)\,.$$

By α-harmonicity and the optional stopping theorem, $\mathbb{E}_x h(X_{\tau_U \wedge t}) = h(x)$, if $x \in U \subset\subset D$. Letting $U \uparrow D$, by Fatou's lemma we obtain $\int p_D(t,x,y)h(y)dy = \mathbb{E}_x\{t < \tau_D : \, h(X_t)\} \leq h(x)$, i.e. P_t^h is subprobabilistic.

The conditional process is defined as the Markov process with the transition semigroup P^h, and it will be denoted by the same symbol X. We let $\mathbb{E}_x^h$ be the corresponding expectation for X starting at $x \in D$,

$$\mathbb{E}_x^h f(X_t) = \frac{1}{h(x)} \mathbb{E}_x[t < \tau_D; \, f(X_t)h(X_t)]\,,$$

see also [8]. A Borel function $r : D \to \mathbb{R}$ is h–harmonic (on D) if and only if for every open $U \subset\subset D$ we have

$$r(x) = \mathbb{E}_x^h r(X_{\tau_U}) = \frac{1}{h(x)} \mathbb{E}_x [X_{\tau_U} \in D; r(X_{\tau_U}) h(X_{\tau_U})] , \quad x \in U. \tag{24}$$

Here we assume that the expectation is absolutely convergent, in particular–finite. It is evident that r is h-harmonic if and only if $r = u/h$ on D, where u is *singular α-harmonic on D* ([12]), i.e. u is α-harmonic on D and $u = 0$ on D^c. We are interested in L^p integrability of u/h, since it amounts to *weighted* L^p integrability of u. The following definition is adapted from [30].

Definition 2. *For $0 < p < \infty$ we define $\mathcal{H}_h^p = \mathcal{H}_h^p(D, \alpha)$ as the class of all the functions $u : \mathbb{R}^d \to \mathbb{R}$, singular α-harmonic on D, and such that*

$$\|u\|_{\mathcal{H}_h^p}^p := \sup_{U \subset\subset D} \mathbb{E}_{x_0}^h \left| \frac{u(X_{\tau_U})}{h(X_{\tau_U})} \right|^p = \frac{1}{h(x_0)} \sup_{U \subset\subset D} \mathbb{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} < \infty ,$$

where $\mathbb{E}_{x_0} |u(X_{\tau_U})|^p / h(X_{\tau_U})^{p-1}$ means $\mathbb{E}_{x_0} \left[X_{\tau_U} \in D; |u(X_{\tau_U})|^p / h(X_{\tau_U})^{p-1} \right]$.

By Harnack inequality, $\mathcal{H}_h^p$ does not depend on the choice of $x_0 \in D$.

Remark 1. Note that the elements of this $\mathcal{H}_h^p$ are α-harmonic, rather than h-harmonic. In view of Definition 2, the genuine conditional Hardy space of $\Delta^{\alpha/2}$ is $\frac{1}{h}\mathcal{H}_h^p = \{u/h : \ u \in \mathcal{H}_h^p\}$, with the norm $\|u/h\| = \|u\|_{\mathcal{H}_h^p}$. $\mathcal{H}_h^p$ is more of a weighted Hardy space of $\Delta^{\alpha/2}$, but it is convenient to call it the *conditional Hardy space*, too. Below we will focus on calculating $\|u\|_{\mathcal{H}_h^p}$, which yields a description of both spaces.

Lemma 6. *If u is singular α-harmonic on D and $U \subset\subset D$, then*

$$\mathbb{E}_{x_0} \frac{u^2(X_{\tau_U})}{h(X_{\tau_U})} = \frac{u(x_0)^2}{h(x_0)} + \int_U G_U(x_0, y) \int_{\mathbb{R}^d} \mathcal{A} \left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)} \right]^2 \frac{h(z) dz dy}{|z-y|^{d+\alpha}} . \tag{25}$$

Proof. Let $y \in D$. For arbitrary $z \in \mathbb{R}^d$ we have

$$\begin{aligned} &\frac{u^2(z)}{h(z)} - \frac{u^2(y)}{h(y)} - 2\frac{u(y)}{h(y)}(u(z) - u(y)) + \frac{u^2(y)}{h^2(y)}(h(z) - h(y)) \\ &= \left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)} \right]^2 h(z) . \end{aligned} \tag{26}$$

By (26) and α-harmonicity,

$$\begin{aligned} \Delta^{\alpha/2} \left(\frac{u^2}{h} \right)(y) &= \Delta^{\alpha/2} \left(\frac{u^2}{h} \right)(y) - 2\frac{u(y)}{h(y)} \Delta^{\alpha/2} u(y) + \frac{u^2(y)}{h^2(y)} \Delta^{\alpha/2} h(y) \\ &= \int_{\mathbb{R}^d} \mathcal{A} \left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)} \right]^2 |z-y|^{-d-\alpha} h(z) \, dz . \end{aligned} \tag{27}$$

Noteworthy, the integrand is nonnegative. Following (10), for u^2/h we get

$$\mathbb{E}_{x_0}\frac{u^2(X_{\tau_U})}{h(X_{\tau_U})} = \frac{u(x_0)^2}{h(x_0)} + \int_U G_U(x_0,y)\Delta^{\alpha/2}\left(\frac{u^2}{h}\right)(y)\,.$$

By using (27) we obtain (25). □

We can interpret (25) in terms of h-conditioning and $r = u/h$ as follows,

$$\mathbb{E}^h_{x_0} r(X_{\tau_U})^2 = r(x_0)^2 + \int_U \frac{G_U(x_0,y)}{h(x_0)h(y)}\int_{\mathbb{R}^d}\mathcal{A}\frac{[r(z)-r(y)]^2}{|z-y|^{d+\alpha}}\,\frac{h(z)}{h(y)}\,dz\,h^2(y)dy\,. \tag{28}$$

This is an analogue of (7), and also indicates the general situation. For $p > 1$ we will consider the expressions of the form

$$F\Big(\frac{a}{s},\frac{b}{t}\Big)\,, \quad a,b\in\mathbb{C},\ s,t>0\,,$$

see (11). If $1 < p < 2$, then we will need an approximation based on

$$F_\varepsilon\Big(\frac{a}{s},\frac{b}{t}\Big)\,, \quad a,b\in\mathbb{C},\ s,t>0,\ \varepsilon\in\mathbb{R},$$

where F_ε is defined in (12). By Lemma 3 we have that $F(a/s,b/t)\geq 0$,

$$F(\frac{a}{s},\frac{b}{t}) \asymp \Big(\frac{b}{t}-\frac{a}{s}\Big)^2\Big(\frac{|b|}{t}\vee\frac{|a|}{s}\Big)^{p-2}, \quad a,b\in\mathbb{R},\ s,t>0\,, \tag{29}$$

and uniformly in $\varepsilon\in\mathbb{R}$ we have

$$F_\varepsilon(\frac{a}{s},\frac{b}{t}) \leq c\Big(\frac{b}{t}-\frac{a}{s}\Big)^2\Big(\frac{|b|}{t}\vee\frac{|a|}{s}\Big)^{p-2}, \quad a,b\in\mathbb{R}\,,\ s,t>0\,, \tag{30}$$

and the comparisons in (29) hold with the constants c_p and C_p of (15).

Lemma 7. *For $p > 1$, $a,b\in\mathbb{C}$ and $s,t>0$, we have*

$$F(\frac{a}{s},\frac{b}{t}) = \frac{|b|^p}{t^p} - \frac{|a|^p}{ts^{p-1}} - \frac{p|a|^{p-2}\overline{a}(b-a)}{ts^{p-1}} + \frac{(p-1)|a|^p(t-s)}{ts^p}\,. \tag{31}$$

Proof. The equality is straightforward to verify, and is left to the reader. □

The homogeneity seen on the left-hand side of (31) is an interesting feature for the right-hand side of (31). We also like to note that $tF(a/s,b/t)$ is the remainder in the first order Taylor's expansion of $(a,s)\mapsto |a|^p/s^{p-1}$ at (b,t) and, of course, $F_\varepsilon(a/s,b/t)\to F(a/s,b/t)$ as $\varepsilon\to 0$.

Lemma 8. *If u is singular α-harmonic on D, $U\subset\subset D$ and $p>1$, then*

$$\mathbb{E}_{x_0}\frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0)|^p}{h(x_0)^{p-1}} + \int_U G_U(x_0,y)\int_{\mathbb{R}^d} F\left(\frac{u(y)}{h(y)},\frac{u(z)}{h(z)}\right)\,\mathcal{A}\,\frac{h(z)dzdy}{|z-y|^{d+\alpha}}\,.$$

Proof. If $p \geq 2$, then $|u|^p/h^{p-1} \in C^2(D)$. By (10),

$$\mathbb{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0)|^p}{h(x_0)^{p-1}} + \int_U G_U(x_0,y)\Delta^{\alpha/2}\left(\frac{|u|^p}{h^{p-1}}\right)(y)\,dy. \tag{32}$$

By α-harmonicity of h and u, and by Lemma 7,

$$\begin{aligned}
\Delta^{\alpha/2}\left(\frac{|u|^p}{h^{p-1}}\right)(y) &= \Delta^{\alpha/2}\left(\frac{|u|^p}{h^{p-1}}\right)(y) - \frac{p|u(y)|^{p-2}u(y)}{h(y)^{p-1}}\Delta^{\alpha/2}u(y) \\
&\qquad + \frac{(p-1)|u(y)|^p}{h(y)^p}\Delta^{\alpha/2}h(y) \\
&= \lim_{\eta\to 0^+} \int\limits_{\{z\in\mathbb{R}^d\colon |z-y|>\eta\}} \mathcal{A}\left[\frac{|u(z)|^p}{h(z)^{p-1}} - \frac{|u(y)|^p}{h(y)^{p-1}} - \frac{p|u(y)|^{p-2}u(y)}{h(y)^{p-1}}(u(z)-u(y))\right. \\
&\qquad \left. + \frac{(p-1)|u(y)|^p}{h(y)^p}(h(z)-h(y))\right]|z-y|^{-d-\alpha}\,dz \\
&= \int_{\mathbb{R}^d} h(z)\,F(u(y)/h(y), u(z)/h(z))\ \mathcal{A}\,|z-y|^{-d-\alpha}dz\,, \qquad (33)
\end{aligned}$$

see (31). This gives the result for $p \geq 2$. If $1 < p < 2$ then we argue as follows. Let $\varepsilon \to 0$ and consider $u + i\varepsilon h$ in place of u in (32). By α-harmonicity, $\Delta^{\alpha/2}\left(|u+i\varepsilon h|^p h^{1-p}\right)(y)$ equals

$$\begin{aligned}
&\lim_{\eta\to 0^+} \int\limits_{\{z\in\mathbb{R}^d\colon |z-y|>\eta\}} \mathcal{A}\left[\frac{|u(z)+i\varepsilon h(z)|^p}{h(z)^{p-1}} - \frac{|u(y)+i\varepsilon h(z)|^p}{h(y)^{p-1}}\right. \\
&\quad - \frac{p|u(y)+i\varepsilon h(y)|^{p-2}u(y)}{h(y)^{p-1}}(u(z)-u(y)) \\
&\quad \left. + \frac{(p-1)|u(y)+i\varepsilon h(y)|^p}{h(y)^p}(h(z)-h(y))\right]|z-y|^{-d-\alpha}\,dz \\
&= \int_{\mathbb{R}^d} h(z)\,F_\varepsilon(u(y)/h(y), u(z)/h(z))\ \mathcal{A}\,|z-y|^{-d-\alpha}dz\,. \qquad (34)
\end{aligned}$$

We then use Lemma 7, Fatou's lemma, and dominated convergence. $\square$

Theorem 2. *Let $1 < p < \infty$. For u singular α-harmonic on D, $\|u\|^p_{\mathcal{H}^p_h}$ is comparable with*

$$\frac{|u(x_0)|^p}{h(x_0)^p} + \int_D \frac{G_D(x_0,y)}{h(x_0)h(y)} \int_{\mathbb{R}^d} \left(\frac{|u(z)|}{h(z)} \vee \frac{|u(y)|}{h(y)}\right)^{p-2}\left[\frac{u(z)}{h(z)} - \frac{u(y)}{h(y)}\right]^2 \frac{h(z)dz\,h^2(y)dy}{h(y)|z-y|^{d+\alpha}}\,.$$

Proof. The result follows from Lemma 8 and (29). In fact,

$$\|u\|^p_{\mathcal{H}^p_h} = \frac{|u(x_0)|^p}{h(x_0)^p} + \int_U \frac{G_U(x_0,y)}{h(x_0)h(y)} \int_{\mathbb{R}^d} F\left(\frac{u(y)}{h(y)}, \frac{u(z)}{h(z)}\right)\mathcal{A}\,\frac{h(z)dz\,h^2(y)dy}{h(y)|z-y|^{d+\alpha}}\,. \tag{35}$$

□

We remark in passing that for $h \equiv 1$ we obtain $\mathcal{H}^p_h = \mathcal{H}^p$. To rigorously state this observation, one should discuss conditioning by functions h with nontrivial values on D^c ([12]). We will not embark on this endeavor, instead in the next section we will fully discuss the conditional Hardy spaces of a local operator, in which case the values of h on D^c are irrelevant.

4. Classical Hardy spaces

We will describe the Hardy spaces and the conditional Hardy spaces of harmonic functions of the Laplacian $\Delta = \sum_{j=1}^d \partial^2/\partial x_j^2$. The former case has been widely studied in the literature, mainly for the ball and the half-space, but also for smooth and Lipschitz domains, see [1, 25, 36, 37, 23]. The characterization of the Hardy spaces in terms of quadratic functions appeared in [35] and [43] for harmonic functions on the half-space in $\mathbb{R}^d$. The case of D being the unit ball was studied in detail in [40, 32]. For more general domains in $\mathbb{R}^d$ see [41, 37, 23].

Throughout this section we will additionally assume that D is connected, i.e. it is a domain. For $0 < p < \infty$, the classical Hardy space $H^p(D)$ may be defined as the family of all those functions u on D which are harmonic on D (i.e. $u \in C^2(D)$ and $\Delta u(x) = 0$ for $x \in D$) and satisfy

$$\|u\|_{H^p} := \sup_{U \subset\subset D} \left(\mathbb{E}_{x_0} |u(W_{\tau_U})|^p \right)^{1/p} < \infty.$$

Here W is the Brownian motion on $\mathbb{R}^d$ and $\tau_U = \inf\{t \geq 0 : \; W_t \notin D\}$. For a positive harmonic function h on D and $0 < p < \infty$ we will consider the space $H^p_h(D)$ of all those functions u harmonic on D which satisfy

$$\|u\|^p_{H^p_h} := \sup_{U \subset\subset D} \mathbb{E}^h_{x_0} \left| \frac{u(W_{\tau_U})}{h(W_{\tau_U})} \right|^p = \frac{1}{h(x_0)} \sup_{U \subset\subset D} \mathbb{E}_{x_0} \frac{|u(W_{\tau_U})|^p}{h(W_{\tau_U})^{p-1}} < \infty,$$

where $\mathbb{E}^h_x$ is the expectation for the conditional Brownian motion (compare Section 3 or see [16]). Let G_D be the classical Green function of D for Δ. If $1 < p < \infty$ and u is harmonic on D, then the following Hardy-Stein identity holds

$$\|u\|^p_{H^p} = |u(x_0)|^p + p(p-1) \int_D G_D(x_0, y) |u(y)|^{p-2} |\nabla u(y)|^2 dy. \tag{36}$$

The identity (36) obtains by taking $h \equiv 1$ in the next theorem. (36) generalizes [40, Lemma 1] and [32, Theorem 4.3], where the formula was given for the ball in $\mathbb{R}^d$, see also [38]. We note that (36) is implicit in [41, Lemma 6], but apparently the identity did not receive enough attention for general domains. We also note that in many cases the exact asymptotics of G_D is known, and we obtain explicit estimates. For instance, if D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and $d \geq 3$, then $G_D(x_0, y) \asymp \delta_D(y)|y - x_0|^{2-d}$, where $\delta_D(y) := \mathrm{dist}(y, D^c)$, see [6], also for the case of Lipschitz domains.

Theorem 3. *If $1 < p < \infty$ and u is harmonic on D, then*

$$\|u\|^p_{H^p_h} = \frac{|u(x_0)|^p}{h(x_0)^p} + p(p-1) \int_D \frac{G_D(x_0, y)}{h(x_0)h(y)} \left|\frac{u(y)}{h(y)}\right|^{p-2} \left|\nabla \frac{u}{h}(y)\right|^2 h^2(y)dy. \tag{37}$$

The remainder of this section is devoted to the proof of Theorem 3. The reader interested mostly in (36) is encouraged to carry out similar but simpler calculations for $h \equiv 1$ and $p > 2$. We note that (37) is quite more general than (36) because usually u/h is not harmonic. The same remark concerns (39, 40) for general h as opposed to (39, 40) for $h = 1$, which is a classical result ([36, VII.3]). We start with the following well-known Green-type equality. Consider open $U \subset\subset D$ and a real-valued function $\phi :\ \mathbb{R}^d \to \mathbb{R}$ which is C^2 in a neighborhood of $\overline{U}$. Then $\Delta\phi$ is bounded on $\overline{U}$, and for every $x \in D$,

$$\phi(x) = \mathbb{E}_x \phi(W_{\tau_U}) - \int_U G_U(x, y)\Delta\phi(y)\, dy\,, \tag{38}$$

see, e.g., [18, p. 133] for the proof.

Lemma 9. *Let $\varepsilon \neq 0$ and let u be harmonic on D. We have*

$$\Delta\left[\left(\frac{u^2}{h^2} + \varepsilon^2\right)^{p/2} h\right] = p\left(\frac{u^2}{h^2} + \varepsilon^2\right)^{(p-4)/2} \left[(p-1)\frac{u^2}{h^2} + \varepsilon^2\right]\left|\nabla \frac{u}{h}\right|^2 . \tag{39}$$

If $u \neq 0$ or $p \geq 2$, then

$$\Delta\left(\frac{|u|^p}{h^{p-1}}\right) = p(p-1)\left|\frac{u}{h}\right|^{p-2}\left|\nabla\frac{u}{h}\right|^2 h\ . \tag{40}$$

Proof. Denote $u_i = \partial u/\partial x_i$, $h_i = \partial h/\partial x_i$, $u_{ii} = \partial^2 u/\partial x_i^2$ and $h_{ii} = \partial^2 h/\partial x_i^2$, $i = 1, \ldots, d$. The lemma results from straightforward calculations based on the following observations:

$$\begin{aligned}
&\nabla|u|^p = \nabla\big(u^2\big)^{p/2} = p|u|^{p-2}u\nabla u\ , \qquad \text{if } p \geq 2 \text{ or } u \neq 0\ ,\\
&\frac{\partial^2}{\partial x_i^2}|u|^p = p(p-1)|u|^{p-2}u_i^2 + p|u|^{p-2}uu_{ii}\ , \qquad \text{if } p \geq 2 \text{ or } u \neq 0\ ,\\
&\nabla h^{1-p} = (1-p)h^{-p}\nabla h\ ,\\
&\Delta(fg) = f\Delta g + 2\nabla f \circ \nabla g + g\Delta f\ .
\end{aligned}$$

This yields (40) if $p \geq 2$ or $u(x) \neq 0$ at the point x where the derivatives are calculated (and so $|u|^p h^{1-p}$ is of class C^2 there). To prove (39) we let $\varepsilon \neq 0$, denote $f(x) = u^2/h^2 + \varepsilon^2$, and use a few more identities:

$$\nabla \frac{u}{h} = \frac{\nabla u}{h} - \frac{u\nabla h}{h^2}, \quad \nabla\left(\frac{u}{h}\right)^2 = 2\frac{u}{h}\nabla\frac{u}{h},$$

$$\Delta\left(\frac{u}{h}\right)^2 = \frac{2|\nabla u|^2}{h^2} - \frac{8u\nabla u \circ \nabla h}{h^3} + \frac{6u^2|\nabla h|^2}{h^4},$$

$$\nabla f^{p/2} = \frac{p}{2} f^{p/2-1} \nabla\left(\frac{u}{h}\right)^2,$$

$$\Delta f^{p/2} = \frac{p(p-2)}{4} f^{p/2-2}\left|\nabla\left(\frac{u}{h}\right)^2\right|^2 + \frac{p}{2} f^{p/2-1}\Delta\left(\frac{u}{h}\right)^2,$$

$$\Delta\left(f^{p/2}h\right) = \frac{p(p-2)}{4} f^{p/2-2}\left|\nabla\left(\frac{u}{h}\right)^2\right|^2 h + p f^{p/2-1}\left|\nabla\frac{u}{h}\right|^2 h.$$

□

Noteworthy, we obtained nonnegative expressions in (39) and (40). Also, if $\varepsilon \to 0$, then $\Delta[(u^2/h^2 + \varepsilon^2)^{p/2}h] \to \Delta(|u|^p h^{1-p})$ almost everywhere on D.

Lemma 10. *If u is harmonic on D, $U \subset\subset D$ and $p > 1$, then*

$$\mathbb{E}_{x_0} \frac{|u(X_{\tau_U})|^p}{h(X_{\tau_U})^{p-1}} = \frac{|u(x_0)|^p}{h(x_0)^{p-1}} + p(p-1)\int_U G_U(x_0,y)\left|\frac{u(y)}{h(y)}\right|^{p-2}\left|\nabla\frac{u}{h}(y)\right|^2 h(y)dy.$$

Proof. For $p \geq 2$ we have $|u|^p h^{1-p} \in C^2(D)$ and the result follows from (38) and Lemma 9. If $1 < p < 2$, then we consider $u + i\varepsilon h$ in place of u and we let $\varepsilon \to 0$. By (39), (38), Fatou's lemma and dominated convergence we obtain the result. □

Proof of Theorem 3. The result follows from Lemma 10 and monotone convergence, after dividing by $h(x_0)$ and rearranging the integrand. □

We observe very close similarities between the Hardy-Stein identities and conditional Hardy-Stein identities discussed in this paper. Specifically, functions u and u/h undergo the same transformation under the integral sign. In each case we see the Green function (and jump kernels in the non-local case) appropriate for the given operator, and in the conditional case, $h^2(y)dy$ appears as a natural reference measure. We remark in passing that the framework of conditional semigroups (23) should be convenient for such calculations in more general settings.

5. Further results

We will now discuss the structure of $\mathcal{H}^p$. We start with $p = 1$. The following is a counterpart of the theorem of Krickeberg for martingales ([15]), and an extension of [30, Theorem 1], where the result was proved for singular α-harmonic functions on bounded Lipschitz domains.

Lemma 11. *Let $u \in \mathcal{H}^1$. There exist nonnegative functions f and g which are α-harmonic on D, satisfy $u = f - g$ and uniquely minimize $f(x_0) + g(x_0)$. In fact, $f(x_0) + g(x_0) = \|u\|_1$. If u is singular α-harmonic on D, then so are f and g. If $1 \le p < \infty$ and $u \in \mathcal{H}^p$, then $\|u\|_p^p = \|f\|_p^p + \|g\|_p^p$.*

Proof. Let U_n be open, $U_n \subset\subset U_{n+1}$ for $n = 1, 2 \dots$ and $\bigcup_n U_n = D$. Let $\tau_n = \tau_{U_n}$. We have

$$\|u\|_1 = \lim_{n\to\infty} \mathbb{E}_{x_0} |u(X_{\tau_n})| < \infty \ .$$

Let $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. For $n = 1, 2, \dots$, we define

$$f_n(x) = \mathbb{E}_x u^+(X_{\tau_n}) \ , \quad g_n(x) = \mathbb{E}_x u^-(X_{\tau_n}) \ , \quad x \in \mathbb{R}^d \ .$$

Obviously, functions f_n and g_n are nonnegative on $\mathbb{R}^d$, and finite and α-harmonic on U_n. We have $u = f_n - g_n$. Since $\tau_n \le \tau_{n+1}$, for every $x \in \mathbb{R}^d$,

$$f_n(x) = \mathbb{E}_x \left[\mathbb{E}_{X_{\tau_n}} u(X_{\tau_{n+1}}) \ ; \ u(X_{\tau_n}) > 0\right] \le \mathbb{E}_x \left[\mathbb{E}_{X_{\tau_n}} u^+(X_{\tau_{n+1}})\right] = f_{n+1}(x) \ ,$$

and $g_n(x) \le g_{n+1}$. We let $f(x) = \lim f_n(x)$ and $g(x) = \lim g_n(x)$. By the monotone convergence theorem, the mean value property (2) holds for f and g. We obtain

$$f(x_0) + g(x_0) = \lim_{n\to\infty} \mathbb{E}_{x_0} |u(X_{\tau_n})| = \|u\|_1 < \infty \ .$$

In view of Harnack inequality we conclude that f and g are finite, hence α-harmonic on D. Also, $u = f - g$. If u vanishes on D, then so do f and g. For the uniqueness, we observe that if $\tilde{f}, \tilde{g} \ge 0$ are α-harmonic on D, and $u = \tilde{f} - \tilde{g}$, then $-\tilde{g} \le u \le \tilde{f}$, hence $f \le \tilde{f}$ and $g \le \tilde{g}$ by the construction of f and g. Therefore $f(x_0) + g(x_0) \le \tilde{f}(x_0) + \tilde{g}(x_0)$, and equality holds if and only if $f(x_0) = \tilde{f}(x_0)$ and $g(x_0) = \tilde{g}(x_0)$, henceforth $f = \tilde{f}$ and $g = \tilde{g}$.

Let $p > 1$ and suppose that $u \in \mathcal{H}^p \subset \mathcal{H}^1$. By Jensen's inequality,

$$f_n(x)^p \le \mathbb{E}_x u^+(X_{\tau_n})^p, \quad g_n(x)^p \le \mathbb{E}_x u^-(X_{\tau_n})^p,$$

hence

$$f_n(x)^p + g_n(x)^p \le \mathbb{E}_x |u(X_{\tau_n})|^p.$$

For $m < n$ we have

$$\mathbb{E}_{x_0}(f_n(X_{\tau_m})^p + g_n(X_{\tau_m})^p) \le \mathbb{E}_{x_0}\mathbb{E}_{X_{\tau_m}} |u(X_{\tau_n})|^p = \mathbb{E}_{x_0} |u(X_{\tau_n})|^p.$$

Letting $n \to \infty$, we get

$$\mathbb{E}_{x_0}(f(X_{\tau_m})^p + g(X_{\tau_m})^p) \le \|u\|_p^p.$$

Hence $\|f\|_p^p + \|g\|_p^p \le \|u\|_p^p$. On the other hand, $f, g \ge 0$, hence

$$\|u\|_p^p = \lim_{n\to\infty} \mathbb{E}_{x_0} |f(X_{\tau_n}) - g(X_{\tau_n})|^p$$

$$\leq \lim_{n\to\infty} \mathbb{E}_{x_0}(f(X_{\tau_n})^p + g(X_{\tau_n})^p) = \|f\|_p^p + \|g\|_p^p.$$

The proof is complete. □

We note that $\|u\|_p^p = \|f\|_p^p + \|g\|_p^p$ has a trivial analogue for L^p spaces.

In view of Lemma 11 and the representation of positive α-harmonic functions [12, Theorem 3], each $u \in \mathcal{H}^1$ has a unique integral representation

$$u(x) = \int_{\partial_M D} M_D(x,z)\mu(dz) + \int_{D^c} f(y)P_D(x,y)dy \ , \quad x \in D \ , \tag{41}$$

where μ is a (finite) signed measure on the Martin boundary of D, M_D is the Martin kernel of D, P_D is the Poisson kernel of D, and the *function* f satisfies $\int_{D^c} |f(y)|P_D(x_0,y)dy < \infty$, see [12].

Lemma 12. *Let $u \in \mathcal{H}^1_h$. There are nonnegative functions $f, g \in \mathcal{H}^1_h$ which satisfy $u = f - g$ and uniquely minimize $f(x_0)+g(x_0)$. In fact, $f(x_0)+g(x_0) = \|u\|_{\mathcal{H}^1_h} h(x_0)$. If $1 \leq p < \infty$ and $u \in \mathcal{H}^p_h$, then $\|u\|^p_{\mathcal{H}^p_h} = \|f\|^p_{\mathcal{H}^p_h} + \|g\|^p_{\mathcal{H}^p_h}$.*

Proof. If $u \in \mathcal{H}^1_h$, then u is singular α-harmonic on D, $u \in \mathcal{H}^1$ and $\|u\|_{\mathcal{H}^1_h} = h(x_0)^{-1}\|u\|_1$ (conditioning is trivial for $p = 1$). By Lemma 11, u has the Krickeberg decomposition $u = f - g$, and f, g are nonnegative and singular α-harmonic on D. In particular $\|f\|_{\mathcal{H}^1_h} = f(x_0)/h(x_0)$ and $\|g\|_{\mathcal{H}^1_h} = g(x_0)/h(x_0)$ are finite. The reader may easily verify the rest of the statement of the lemma, following the previous proof and using the conditional expectation $\mathbb{E}^h$. □

Remark 2. *Analogues of Lemma 11 and Lemma 12 are true for the classical Hardy spaces $H^p(D)$ and $H^p_h(D)$ for connected D.*

As an application of (36) we give a short proof of the following Littlewood-Paley type inequality (see [31], were the result was given for the ball in $\mathbb{R}^2$). Recall the notation $\delta_D(y) = \mathrm{dist}(y, D^c)$.

Proposition 13. *Consider a domain $D \subset \mathbb{R}^d$, and let $p \geq 2$. There is a constant $c > 0$ such that for every function u harmonic on D,*

$$\|u\|^p_{H^p} - |u(x_0)|^p \geq p(p-1)d^{2-p}2^{1-p}\int_D G_D(x_0,y)\delta_D(y)^{p-2}|\nabla u(y)|^p dy \ .$$

Proof. We may assume that $\|u\|_{H^p} < \infty$. In view of Lemma 11 and Remark 2, $u = f - g$, where f, g are positive and harmonic on D and $\|u\|^p_{H^p} = \|f\|^p_{H^p} + \|g\|^p_{H^p}$. Clearly, $|u(x_0)|^p \leq f(x_0)^p + g(x_0)^p$, hence $\|u\|^p_{H^p} - |u(x_0)|^p \geq \|f\|^p_{H^p} - |f(x_0)|^p + \|g\|^p_{H^p} - |g(x_0)|^p$. Furthermore, by Jensen's inequality,

$$|\nabla u|^p \leq 2^{p-1}(|\nabla f|^p + |\nabla g|^p) \ .$$

Recall the following gradient estimate for the nonnegative harmonic function f,

$$f(x) \geq |\nabla f(x)|\delta_D(x)/d \ , \quad x \in D \ ,$$

([20, Exercise 2.13], see also [2]). Here d is the dimension. By (36),

$$\|f\|^p_{H^p} - |f(x_0)|^p = p(p-1)\int_D G_D(x_0,y)|f(y)|^{p-2}|\nabla f(y)|^2 dy$$
$$\geq p(p-1)d^{2-p}\int_D G_D(x_0,y)\delta_D(y)^{p-2}|\nabla f(y)|^p dy\ ,$$

and a similar estimate holds for g. □

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