Hybrid–mixed formulation for static–current flow problem compared with conventional formulation

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Hybrid-Mixed Formulation for Static-Current Flow Problem Compared with Conventional Formulation

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Abstract — The electric scalar potential formulation is traditionally used to determine the static-current flow. This formulation is not naturally compatible with the electric vector potential formulation (T-ϕ). This is explained by a calculation of current coming from the summation of static and eddy currents whose functional spaces are different. In this paper, we propose an alternative approach based on mixed elements to obtain the current densities in the space H(div) and compare them to the electric scalar potential formulation.

I. Introduction

A. *Static-current flow* ($\mathbf{J}_s$) *in the framework of magneto-dynamics modeling*

The solution of a magnetodynamic problem is usually performed in two steps. The first is to determine the static-current flow $\mathbf{J}_s$ in the massive conductor thanks to an electric scalar potential formulation (V). The second is to determine the eddy current $\mathbf{J}_e$ flow thanks to an electric vector potential formulation ($\mathbf{T}$-ϕ) [1]. The use of an electric scalar potential leads to several difficulties:

- the current density $\mathbf{J}_s$ provides only weakly local and global conservation,
- this solution procedure is not consistent with patterns suggested by Tonti's diagram [2],
- the current densities obtained are not in the same Hilbert space functional ($\mathbf{J}_s \in$ H(rot) and $\mathbf{J}_e \in$ H(div)) which possibly leads to inaccuracies.

To establish a more relevant process, it is necessary to focus on modeling the static-current flow problem.

B. *Static-current flow problem* ($\mathbf{J}_s$)

To solve a static current flow problem, it is necessary to ensure:

- Ohm's law and supply constraints (for the sake of simplicity these constraints are not represented in functionals),
- Maxwell's equation (Maxwell-Faraday's and current density divergence free) (1).

$$\mathrm{div}\mathbf{J}_s = 0 \quad , \quad \mathbf{curlE} = \mathbf{0} \qquad (1)$$

An energetic approach based on the constitutive relation error [3][4] allows to identify two dual approaches to determine the static-current flow. The solutions are obtained from pair of *independent* standard minimizations subject to Maxwell's equation constraints (1):

$$F^-(\mathbf{E}') = \int_\Omega \left(\int_0^{\mathbf{E}'} \mathbf{j}(\mathbf{e}) \cdot \partial\mathbf{e} \right) d^3x, \ \mathbf{E} = \arg \min_{\substack{\mathbf{curlE}'=0 \\ \mathbf{E}'\in H(curl)}} F^-(\mathbf{E}') \qquad (2)$$

$$F^+(\mathbf{J}'_s) = \int_\Omega \left(\int_0^{\mathbf{J}'} \mathbf{e}(\mathbf{j}) \cdot \partial\mathbf{j} \right) d^3x, \ \mathbf{J}_s = \arg \min_{\substack{\mathrm{div}\mathbf{J}'_s=0 \\ \mathbf{J}'_s\in H(div)}} F^+(\mathbf{J}'_s) \qquad (3)$$

Maxwell's equation constraints can be ensured:

- either strongly by the introduction of potentials (V,$\mathbf{T}_s$) thanks to Poincare's Lemma:

$$V = \arg \min_{V'\in H^1} F^-(V') \quad , \qquad \mathbf{E} = -\mathbf{grad}V \qquad (4)$$

$$\mathbf{T}_s = \arg \min_{\mathbf{T}'_s\in H(curl)} F^+(\mathbf{T}'_s), \qquad \mathbf{J}_s = \mathbf{curlT}'_s \qquad (5)$$

- or weakly by the introduction of Lagrange's multipliers and saddle point problems [5] of this Lagrangian functions:

$$L^-(\mathbf{E}', \mathbf{T}'_s) = F^-(\mathbf{E}') + \int_\Omega \mathbf{T}'_s \cdot \mathbf{curlE}' \, d^3x \qquad (6)$$

$$L^+(\mathbf{J}'_s, V') = F^+(\mathbf{J}'_s) + \int_\Omega V' \mathrm{div}\mathbf{J}'_s \, d^3x \qquad (7)$$

This method is called mixed method and can be used to solve electromagnetic problem [6].

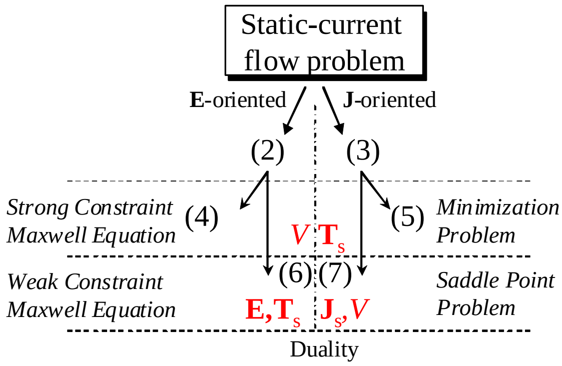

Fig. 1. Two dual approaches are exhibited and can be translated differently depending on the type of constraint.

Finally the static-current flow problem can be treated by two dual approaches (**E** or **J**-oriented) and be available in two ways (Fig. 1) depending on the type of imposed constraint (weak or strong). It is interesting to emphasize that the Lagrange multipliers of a mixed formulation is equivalent to the potential of the dual formulation.

To address the problem presented in part A, it is necessary to promote the **J**-oriented approach (mixed or potential). $\mathbf{T}_s$ potential formulation (5) is interesting but requires the introduction of cut to add supply constraint [7]. Subsequently, the ($\mathbf{J}_s$,V) mixed formulation will be studied.

II. Static-Current Flow Modeling with Hybrid-Mixed Formulation

In the Mixed Finite Element (MFE) method two different Finite Element spaces are used for the current density $\mathbf{J}_s$ and for V the electrical scalar potential. The solution is sought by solving the saddle point problem (7) as follows:

$$\left(\mathbf{J}_s, V\right) = \arg\left(\inf_{\mathbf{J}'_s \in H(\mathrm{div})} \sup_{V \in L^2} F^{+}\left(\mathbf{J}'_s\right) + \int_{\Omega} V' \mathrm{div}\mathbf{J}'_s \, d^3x \right) \quad (9)$$

This one presents numerical difficulties because matrix system of linear equations is indefinite. To overcome this difficulty the constraint on the continuity of the normal component of $\mathbf{J}_s$ is relaxed by the introduction of a new Lagrange multiplier V_f associated with faces. A static condensation process allows to eliminate the mixed variables ($\mathbf{J}_s$,V) in favour of the hybrid Lagrange multipliers (V_f). This approach is called hybrid mixed method and will be used for comparison [8].

TABLE I
CHARACTERISTICS OF FORMULATION

Formulation	Electric scalar potential (V)	Hybrid-Mixed ($\mathbf{J}$,V,V_f)
Local current conservation	weakly	strongly
Normal current continuity	weakly	weakly
Global current conservation	weakly	weakly
Support for DoF	node	face
Shape function	V: Linear Lagrange	$\mathbf{J}$: Linear Raviart-Thomas V: constant by cell V_f: constant by face

III. Hybrid-Mixed and Scalar Potential Formulations Comparison

A. Preamble

A 2D L-shaped conductor with linear conductivity is studied and the modeling is performed with Matlab. The characteristics of each formulation are presented in Table I.

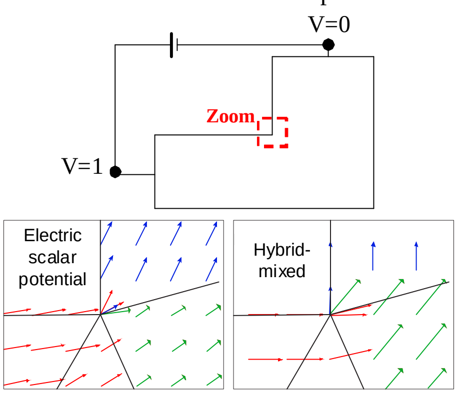

Fig. 2. Current density vectors for both formulations (electric scalar potential and hybrid-mixed formulation) in the vicinity of the right angle

B. Local comparison

The current density vectors are represented for each formulation in the vicinity of the right angle (Fig. 2). We note that only the electric scalar potential formulation presents current density vectors directed outward from geometry, where the natural condition is $\mathbf{J}\cdot\mathbf{n}$=0.

C. Global comparison

Joule losses are evaluated for several meshes and compared with the analytical solution (Fig. 3).

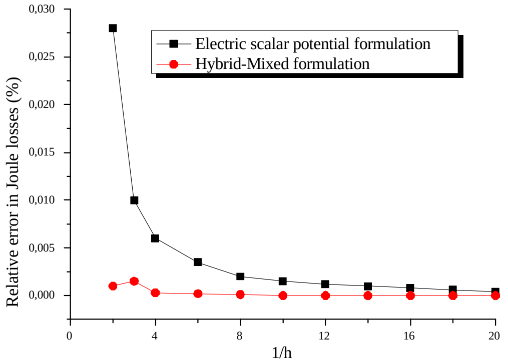

Fig. 3. Relative error in Joule losses in the conductor for both approaches (electric scalar potential and hybrid-mixed formulation) according to the characteristic size of mesh h

The electric scalar potential and hybrid-mixed formulations are consistent with the size of the mesh and the difference of convergence is explained by the order of shape functions.

The use of a hybrid-mixed formulation (7) to determine the static-current flow problem is suitable and the current density obtained is compatible with an eddy current determined by an electric vector potential formulation.

The implementation of hybrid-mixed formulation is under progress in the commercial soft Flux® and an improvement of the computation of currents is expected.

IV. References

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