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To cite this version:
Léo Agélas. On the regularity of solutions of the 3D Axisymmetric Navier-Stokes Equations with swirl. 18 pages. 2016. <hal-00649051v4>

HAL Id: hal-00649051
https://hal.archives-ouvertes.fr/hal-00649051v4
Submitted on 12 Oct 2016
On the regularity of solutions of the 3D Axisymmetric Navier-Stokes Equations with swirl

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June 1, 2016

Abstract

One of the most challenging questions in fluid dynamics is whether the incompressible three-

dimensional (3D) Navier-Stokes equations can develop a finite-time singularity from smooth and

bounded initial data. It is well-known that global regularity of the incompressible Navier-Stokes

equations is still wide open even in the axisymmetric case with general non-trivial swirl, al-

though this case appeared more tractable than the full three-dimensional problem due to special

features. In this paper, we prove that the blowup of the solutions of the 3D Navier-Stokes equa-
tions in the axisymmetric case with general non-trivial swirl can not occur at the time $T$ if the

scale-invariant quantity $\inf_{R>0} \sup_{t \in [0,T]} \| \Gamma(t) \mathbf{1}_{\{r \leq R\}} \|_{L^\infty}$ is sufficiently small, where $\Gamma = ru_\theta$. To get

our result, we use some results of recent works on the stabilizing effect of the convection term in

the 3D incompressible Navier-Stokes equations and the interaction between the swirling velocity

and the angular vorticity fields. We show also that our regularity criterion is less restrictive than

those involved in the recent papers.

Keywords  Navier-Stokes equations; 3D Axisymmetric flows; Regularity criterion

Mathematics Subject Classification  35Q30, 76D03, 76D05

1 Introduction

The study of the incompressible Navier-Stokes in three space dimensions has a long history. For

a long time ago, a global weak solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$ and $\nabla v \in L^2(\mathbb{R}^3 \times (0, \infty))^3$ to the Navier-Stokes equations (2)-(3) was built by Leray [34]. In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equation, and proved that, for every given $v_0 \in L^2(\mathbb{R}^3)\times \mathbb{R}^3$, there exists a global weak solution $u \in L^\infty([0, +\infty[, L^2(\mathbb{R}^3))^3 \cap L^2([0, +\infty[; H^1(\mathbb{R}^3))^3$. Hopf has

proved the existence of a global weak solution in the general case $\mathbb{R}^d$, $d \geq 2$, [23]. Several ways

are known to construct weak solution ([15, 18, 11]), but the regularity and the uniqueness of this

weak solution remained yet open in the general case, till now in spite of great efforts made (see

[10, 12, 41, 36, 7, 46, 52, 12, 24, 47, 48, 51, 1, 17]). In two dimensions, the existence of classical

solutions has been known for a long time ago (see [27, 37, 35, 49]). Thus a natural question, namely what can be said about the 3D axisymmetric flow, appears.

Axisymmetric flow is an important subject in fluid dynamics and has become standard textbook

material as a starting point of theoretical study for complicated flow patterns. Although the number of independent spatial variables is reduced by symmetry, some of the essential features

and complexities of generic 3D flows remain. For example, when the swirling velocity is nonzero,
there is a vorticity stretching term present.

The first results in the existence of classical solutions were obtained in the late sixties for 3D axisymmetric flow without swirl (see \[28\, 50\]) and later also in \[33\].

In the case of 3D axisymmetric flow with swirl, the question of finite time blow-up of solutions remained a challenging open problem in spite of tremendous efforts made (see \[6\, 42\, 43\, 8\, 9\, 21\, 45\, 22\, 29\, 16\, 30\, 26\]).

In several recent papers (\[19\, 20\, 21\, 22\]), two systems of equations are proposed in order to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier-Stokes and Euler equations. By exploiting the special structure of the nonlinearity of the equations, the authors prove the global regularity of the three-dimensional Navier-Stokes equations for a family of initial data.

Furthermore, in more recent activities, regularity results for axisymmetric solutions of the 3D Navier-Stokes equations are obtained under the assumption that some scale-invariant quantities remain finite (but not necessarily small).

Indeed in \[9\, 8\] it was proven that suitable axially symmetric solutions bounded by \(C r^{-1+\epsilon}(t_0 - t)^{-\frac{3}{2}}\) with \(0 \leq \epsilon \leq 1\) are smooth at time \(t_0\), here \(r\) is the distance from a point \(x\) to the \(z\)-axis.

Similar results were also obtained in \[29\, 30\] and a local version in \[45\].

In \[31\] it was proven that there exists a constant \(C > 1\) such that if there exists \(R \in [0, \frac{3}{2}]\) such that \(\sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty} \leq C_1 \ln R^{-\frac{3}{2}}\) then the solutions of the 3D Navier-Stokes equations in the axisymmetric case with general non-trivial swirl and a viscosity \(\nu\) of one can not blow up at the time \(T\), where \(\Gamma(x, t) = \Gamma u_\theta(r, t)\), here \(u_\theta\) is the swirl component of \(u\) and \(r = |x'|\) with \(x' \in \mathbb{R}^2\) such that \(x \equiv (x', z) \in \mathbb{R}^3\).

Later in \[53\], the previous result have been improved in the sense that if there exists \(R \in [0, \frac{1}{2}]\) such that \(\sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty} \leq C_1 \ln R^{-\frac{3}{2}}\) then the solutions of the 3D Navier-Stokes equations in the axisymmetric case with general non-trivial swirl and a viscosity \(\nu\) of one can not blow up at the time \(T\).

In this paper, from our Theorem 5.1, we obtain that the blowup of the solutions of the 3D Navier-Stokes equations in the axisymmetric case with general non-trivial swirl and a viscosity \(\nu\) of one can not occur at the time \(T\) if the scale-invariant quantity \(\inf_{R > 0} \sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty}\) is smaller than a certain absolute constant.

We draw attention to the fact that our regularity criterion is less restrictive than those involved in \[8\, 9\, 29\, 30\, 45\], indeed under their assumptions we infer that \(\Gamma(x, t)\) is Hölder continuous at \((r, t) \equiv (0, T)\) uniformly (see section 5 in \[9\], Theorem 3.1 in \[8\], see also Theorem 1.1 for \[30\]).

Then, for any \(\epsilon > 0\) there exist \(t_\epsilon \in [0, T]\) and \(R_\epsilon > 0\) such that for all \(t \in [t_\epsilon, T]\) and \(0 < R \leq R_\epsilon\), \(\|\Gamma(t) - \Gamma(t_\epsilon)\|_{L^\infty} \leq \frac{\epsilon}{2}\) and by setting \(\tilde{R}_\epsilon = \frac{\epsilon}{2(1 + \|u\|_{L^\infty([0, T])}}\) we get that for all \(t \in [0, t_\epsilon]\) and for all \(0 < R \leq \tilde{R}_\epsilon\), \(\|\Gamma(t)1_{r \leq R}\|_{L^\infty} \leq \tilde{R}_\epsilon\|u(t)\|_{L^\infty} \leq \frac{\epsilon}{2}\). Then by taking \(\tilde{R}_\epsilon = \min\{R_\epsilon, \tilde{R}_\epsilon\}\), we infer that for all \(t \in [0, T]\), \(\|\Gamma(t)1_{r \leq R_\epsilon}\|_{L^\infty} \leq \epsilon\) and then we infer that for any \(\epsilon > 0\), \(\inf_{R > 0} \sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty} \leq \epsilon\) which means that \(\inf_{R > 0} \sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty} = 0\).

Then, we conclude that the regularity criteria involved in \[8\, 9\, 29\, 30\, 45\] imply that \(\inf_{R > 0} \sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty} = 0\) which prove that their regularity criteria are more restrictive than our criterion.

We draw also attention to the fact that our regularity criterion is less restrictive than those involved in \[31\, 53\] since to get non blowup of the solutions, we require only that there exists \(R > 0\) such that \(\sup_{t \in [0, T]} \|\Gamma(t)1_{r \leq R}\|_{L^\infty} \leq \gamma_0\) where \(\gamma_0 > 0\) is an absolute constant.

Moreover, our criterion is bounded by \(\|\Gamma_0\|_{L^\infty}\) thanks to (17), this feature eases the numerical detection of potential blowup of the solutions.
To obtain this result, we have been able to show the following energy estimate on $[0,T]$:

$$
\frac{d}{dt} \left( \frac{1}{2} \| u_1(t) \|^2 + \frac{1}{2} \| \omega_1(t) \|^2 \right) + \left( \frac{8}{9} \right) \left( C(1 + \| \Gamma(t) \| \chi_{\{r \leq R\}} \| L^\infty(\mathbb{R})) \right) \| \nabla |u_1(t)|^2 \|^2 
+ \frac{1}{2} \| \nabla \omega_1(t) \|^2 \leq C \left( \frac{\| \Gamma_{0} \| L^\infty(\mathbb{R})}{R^2} \right) \left( 1 + \| \Gamma_{0} \| L^\infty(\mathbb{R}) \right) \| u_1(t) \|_3^3,
$$

where $u_1 = \frac{u_0}{r}$, $\omega_1 = \frac{\partial \theta}{r}$ and $\Gamma = ru_0$. Then, the paper is organized as follows:

- In section 2, we recall some results known about the solutions of Navier-Stokes equations.
- In section 3, we introduce the 3D axisymmetric incompressible Navier-Stokes equations with some known results.
- In section 4, we recall some estimates on $\Gamma$.
- In section 5, we obtain an estimate on $\| u_1(t) \|_3^3 + \| \omega_1(t) \|_3^2$ in Lemma 5.4 by showing inequality (1) and then we obtain our Theorem 5.1.

First, we give some notations.

Some notations : For any $m \in \mathbb{N}^*$ function $\varphi$ defined on $\mathbb{R}^m \times [0, +\infty[$, for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on $\mathbb{R}^m$ by $x \mapsto \varphi(x,t)$. For any vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we denote by $|x|$ the norm defined by $|x| = \sqrt{\sum_{i=1}^{3} x_i^2}$. For any axisymmetric function $f$ defined on $\mathbb{R}^3$, for the sake of simplicity, the value $f(x)$ with $x = (x, y, z) \in \mathbb{R}^3$ is denoted using coordinates cylindrical, $f(r,z)$ with $r = \sqrt{x^2 + y^2}$. For any $d \geq 1$, $\Omega \subset \mathbb{R}^d$, we denote by $C^\infty_0(\Omega)$ (resp $C_0(\Omega)$) the space constituted by all infinitely differentiable (resp continuous) functions with compact support in $\Omega$. For any $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, we denote by $\chi_\Omega$, the function defined on $\mathbb{R}^d$, by $\chi_{\Omega}(x) = 1$ for all $x \in \Omega$ and 0 elsewhere. For any $R > 0$, we denote by $\chi_{\{r \leq R\}}$ (resp $\chi_{\{r \geq R\}}$) the function defined on $\mathbb{R}_+ \times \mathbb{R}$ such that for all $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$, $\chi_{\{r \leq R\}}(r, z) = 1$ (resp $\chi_{\{r \geq R\}}(r, z) = 1$) for all $r \leq R$ and 0 elsewhere. The symbol $\int_S$ denotes the integral over $\mathbb{R}^3$ equal using cylindrical coordinates to $\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \ldots r \ dr \ dz \ dr$. For any $q > 1$, the norm in $L^q(\mathbb{R}^3)$ will be denoted by $\| \cdot \|_{L^q}$ and also $\| \cdot \|_q$. We denote $A \lesssim B$, the estimate $A \leq C B$ where $C > 0$ is an absolute constant.

2 Local regularity of solution of Navier-Stokes equation

In this section, we deal with the main result on local regularity of Navier-Stokes equations in its general form.

Consider the Navier-Stokes equations,

$$
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p &= 0, \\
\nabla \cdot u &= 0,
\end{align*}
$$

in which $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$, $p = p(x,t) \in \mathbb{R}$ and $\nu > 0$ denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x, t) \in \mathbb{R}^3 \times [0, \infty)$ and $\nu > 0$ the viscosity of the fluid,

with initial conditions,

$$
u(x,0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3.
$$
Without loss of generality, in what follows, we assume that $\nu = 1$.

Assuming $u_0 \in H^m(\mathbb{R}^3)$ for a given $m \geq 1$, thanks to the results obtained in [34], Theorem 3.5 in [25], Lemma 5.6 [11], Theorem 6.1 [12] or the results obtained in [18], we get that there exists a time $T > 0$ such that there exists an unique solution $u \in C([0,T];H^m(\mathbb{R}^3))^3 \cap L^2([0,T]; H^{m+1}(\mathbb{R}^3))^3$ to the Navier-Stokes Equations (2)-(3). Due to the regularity of solution of Navier-Stokes equation, $u \in C([0,T];H^m(\mathbb{R}^3))^3$ and thanks to the results obtained in [44], [36], we get the energy equality, in other words, for all $t \in [0,T]$,.

$$\|u(t)\|_{L^2(\mathbb{R}^3)^3}^2 + 2 \int_0^t \|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)^3} = \|u_0\|_{L^2(\mathbb{R}^3)^3}^2. \tag{4}$$

Moreover if $u \not\in C([0,T];H^m(\mathbb{R}^3))^3$, then thanks to the results obtained in [34], Theorem 6.1 [11], Lemma 6.2 [12], we infer that,

$$\limsup_{t \to T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)^3} = +\infty, \tag{5}$$

and thanks to Theorem 3.1.1 in [3], we have also,

$$\limsup_{t \to T} \|\omega(t)\|_{L^2(\mathbb{R}^3)^3} = +\infty, \tag{6}$$

where $\omega = \nabla \times u$ is the vorticity of $u$.

Moreover up to the initial time, the solution of Navier-Stokes equation is smooth, $u \in C^\infty(\mathbb{R}^3 \times [0,T])$ (see Theorem 3 and 4 in [18], see also Lemma 5.6 and Theorem 5.2 in [11]). We denote by $\omega_0 = \nabla \times u_0$ the vorticity of $u_0$.

3 Axisymmetric flows

By an axisymmetric solution of the Navier-Stokes equations, we mean a solution of the equations of the form

$$u(x, y, z, t) = u_r(r, z, t)e_r + u_\theta(r, z, t)e_\theta + u_z(r, z, t)e_z.$$

in the cylindrical coordinate system, where we used the basis

$$e_r = \left(\frac{x}{r}, \frac{y}{r}, 0\right), \quad e_\theta = \left(-\frac{y}{r}, \frac{x}{r}, 0\right), \quad e_z = (0, 0, 1) \quad \text{and} \quad r = \sqrt{x^2 + y^2}.$$ 

In the above expression, $u_\theta$ is called the swirl component of the velocity field $u$. For the axisymmetric solutions, we can rewrite the equations (2) as follows:

$$\begin{cases} 
\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} = \mathcal{L} u_\theta - \frac{u_r u_\theta}{r}, \\
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = \mathcal{L} u_r + \frac{u_\theta^2}{r} + \partial_z p, \\
\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = \Delta u_z + \partial_z p, \\
\partial_t (ru_r) + \partial_z (ru_z) = 0. 
\end{cases} \tag{7}$$

For the axisymmetric vector field $u$, we can compute the vorticity $\omega = \nabla \times u$ as follows,

$$\omega = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z,$$

where $\omega_r = - (u_\theta)_z$, $\omega_\theta = (u_r)_z - (u_z)_r$ and $\omega_z = \frac{1}{r}(ru_\theta)_r$.

Moreover, the vorticity components satisfy:

$$\begin{cases} 
\frac{\partial \omega_r}{\partial t} + u_r \frac{\partial \omega_r}{\partial r} + u_z \frac{\partial \omega_r}{\partial z} = \mathcal{L} \omega_r + \frac{u_r \omega_\theta}{r} - \frac{2}{r} u_\theta \omega_r, \\
\frac{\partial \omega_\theta}{\partial t} + u_r \frac{\partial \omega_\theta}{\partial r} + u_z \frac{\partial \omega_\theta}{\partial z} = \mathcal{L} \omega_\theta + \frac{u_\theta \omega_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial z}, \\
\frac{\partial \omega_z}{\partial t} + u_r \frac{\partial \omega_z}{\partial r} + u_z \frac{\partial \omega_z}{\partial z} = \Delta \omega_z + \frac{1}{r} \frac{\partial u_\theta}{\partial z} - \frac{1}{r} \frac{\partial u_\theta}{\partial r}. \tag{8} 
\end{cases}$$
The operator \( \mathcal{L} \) and \( \Delta \) is defined by:

\[
\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2, \\
\mathcal{L} = \left( \Delta - \frac{1}{r^2} \right).
\]  

\( (9) \)

One can derive evolution equations for \((u_\theta, \omega, \psi_\theta)\) which completely determine the evolution of the three-dimensional axisymmetric Navier-Stokes equations \((7)\) once the initial condition is given (see e.g. \([10], [6]\)):

\[
\begin{align*}
\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} &= \mathcal{L} u_\theta - \frac{u_r u_\theta}{r} + \frac{1}{r} \frac{\partial u_\theta^2}{\partial r} + \frac{u_r}{r} \omega_\theta, \\
\frac{\partial \omega_\theta}{\partial t} + u_r \frac{\partial \omega_\theta}{\partial r} + u_z \frac{\partial \omega_\theta}{\partial z} &= \mathcal{L} \omega_\theta + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{u_r}{r} \omega_\theta,
\end{align*}
\]

\( (10) \)

where \(u_r\) and \(u_z\) can be expressed in terms of the angular component of the stream function \(\psi_\theta\) as follows:

\[
u_r = -\frac{\partial \psi_\theta}{\partial z}, \quad u_z = \frac{1}{r} \frac{\partial (r \psi_\theta)}{\partial r}.
\]

\( (11) \)

We note that the incompressibility condition implies that

\[
\partial_r (ru_r) + \partial_z (ru_z) = 0.
\]

\( (12) \)

In \([39]\), there are shown the equivalence between the systems of equations \((2)\) and \((10)-(12)\), to mention their main result, we introduce some spaces with the same notations as in \([39]\).

Denote by \(C^k_s(\mathbb{R}^3, \mathbb{R}^3)\) the axisymmetric divergence free subspace of \(C^k\) vector fields:

\[
C^k_s(\mathbb{R}^3, \mathbb{R}^3) = \{ \mathbf{u} \in C^k(\mathbb{R}^3, \mathbb{R}^3) \mid \partial_\theta u_z = \partial_\theta u_r = \partial_\theta u_\theta = 0, \nabla \cdot \mathbf{u} = 0 \}.
\]

Thanks to Lemma 2 (see also Lemma 2’ in \([39]\), we have,

\[
C^k_s(\mathbb{R}^3, \mathbb{R}^3) = \{ u \theta + \nabla \times (\psi_\theta) \mid u \in C^k_s(\mathbb{R}_+ \times \mathbb{R}), \psi \in C^{k+1}_s(\mathbb{R}_+ \times \mathbb{R}) \},
\]

where \(C^k_s(\mathbb{R}_+ \times \mathbb{R})\) is the function space defined by

\[
C^k_s(\mathbb{R}_+ \times \mathbb{R}) = \{ f(r, z) \in C^k(\mathbb{R}_+ \times \mathbb{R}) \mid \partial_r^{2j} f(0^+, z) = 0, \ 0 \leq 2j \leq k \}.
\]

We can now define the Sobolev spaces for axisymmetric solenoidal vector fields:

\[
\dot{H}^1_s(\mathbb{R}_+ \times \mathbb{R}) = \text{Completions of } C^1_s(\mathbb{R}_+ \times \mathbb{R}) \cap C_0(\mathbb{R}_+ \times \mathbb{R}) \text{ with respect to } \| \cdot \|_{\dot{H}^1_s(\mathbb{R}_+ \times \mathbb{R})},
\]

\[
H^k_s(\mathbb{R}_+ \times \mathbb{R}) = \text{Completions of } C^k_s(\mathbb{R}_+ \times \mathbb{R}) \cap C_0(\mathbb{R}_+ \times \mathbb{R}) \text{ with respect to } \| \cdot \|_{H^k_s(\mathbb{R}_+ \times \mathbb{R})},
\]

where \(C_0\) denotes the space of compactly supported functions.

As mentioned in \([22]\) and proved in \([39]\), any smooth solution of the 3D axisymmetric Navier-Stokes equations must satisfy the following compatibility condition at \(r = 0\):

\[
u_\theta(0, z, t) = \psi_\theta(0, z, t) = \omega_\theta(0, z, t) = 0.
\]

\( (13) \)

More precisely, we have the following result, thanks to Lemma 8, Theorem 4 and Corollary 3 in \([39]\),

**Theorem 3.1.** If \(u_\theta \in H^k(\mathbb{R}^3)^3\) is an axisymmetric solenoidal vector field with \(k \geq 1\), then there exists \(u_\theta, \psi_\theta \in \dot{H}^1_s(\mathbb{R}_+ \times \mathbb{R})\) with \(\mathcal{L} \psi_\theta, u_\theta \in H^{-1}_s(\mathbb{R}_+ \times \mathbb{R})\) such that \(u_\theta = u_{\theta, 0} e_\theta + \nabla \times (\psi_\theta e_\theta)\) and there exists a time \(T > 0\) such that \(u = u_{\theta, 0} e_\theta + \nabla \times (\psi_\theta e_\theta)\) corresponds to the unique strong solution to the Navier-Stokes equations \((2)\) in the class \(C([0, T]; H^k(\mathbb{R}^3)^3)\) where \((u_\theta, \psi_\theta, \omega_\theta)\) is solution to \((10)-(12)\) for the initial data \((u_{\theta, 0}, \psi_{\theta, 0}, -\mathcal{L} \psi_{\theta, 0})\) and satisfies,

\[
\begin{align*}
\psi_\theta &\in C([0, T]; H^{k+1}_s(\mathbb{R}_+ \times \mathbb{R})), \\
u_\theta &\in C([0, T]; H^k_s(\mathbb{R}_+ \times \mathbb{R})), \\
\omega_\theta &\in C([0, T]; H^{k-1}_s(\mathbb{R}_+ \times \mathbb{R})).
\end{align*}
\]
4 Estimates for axisymmetric solution

In this section, we recall some estimates on the quantity $\Gamma = ru_\theta$. For this, it is assumed that $u_0 \in H^m$ is a axisymmetric solenoidal vector field, with $m \geq 2$, then Theorem 3.1 holds and there exists a time $T > 0$ such that there exists an unique strong solution $u$ to the Navier-Stokes equations (2) which belongs to $C([0, T]; H^m(\mathbb{R}^3)) \cap L^2([0, T]; H^{m+1}(\mathbb{R}^3))$ with $m \geq 2$ (see Section 2).

A special feature of the axisymmetric Navier-Stokes equations is that the quantity $\Gamma = ru_\theta$ satisfies an parabolic equation on $[0, T]$ with singular drift terms:

$$\left(\partial_t + b \cdot \nabla - \Delta + \frac{2}{r} \partial_r\right) \Gamma = 0$$

with boundary conditions,

$$\Gamma|_{r=0} = 0,$$  

with initial conditions,

$$\Gamma(x, 0) = \Gamma_0(x) \text{ for a.e } x \in \mathbb{R}^3,$$

where, $\Gamma_0 = ru_{0, \theta}$, $b = u_r e_r + u_z e_z$, $b \cdot \nabla = u_r \partial_r + u_z \partial_z$ and $\text{div } b = 0$.

Note that in equation (14), the convection term has absorbed the term $\frac{ru_{\theta}}{r}$ in the first equation (10), which highlights the stabilizing effect of the convection.

We remark also that $\Gamma$ enjoys the maximal principle. Indeed thanks to inequality (4.6) in [42] (see also Proposition 1 in [6]), we have for all $q \in [2, \infty]$, for all $t \in [0, T]$,

$$\|\Gamma(t)\|_{L^q(\mathbb{R}^3)} \leq \|\Gamma_0\|_{L^q(\mathbb{R}^3)}.$$  

5 Global regularity

In this section, we assume that $u_0 \in H^m$ is an axisymmetric solenoidal vector field, with $m \geq 2$ and $\Gamma_0 = ru_{0, \theta} \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, then Theorem 3.1 holds and there exists a time $T > 0$ such that there exists an unique strong solution $u$ to the Navier-Stokes equations (2) which belongs to $C([0, T]; H^m(\mathbb{R}^3)) \cap L^2([0, T]; H^{m+1}(\mathbb{R}^3))$ (see Section 2). This section is devoted to the proof of Theorem 5.1. The proof of our Theorem is obtained in three steps:

- First, thanks to the convection term, we eliminate an annoying term in (10), $\frac{ru_r}{r} \omega_\theta$, by using the change of unknowns from $(u_\theta, \psi_\theta, \omega_\theta)$ to $(u_1, \psi_1, \omega_1)$ (see (18)).

- Second, thanks to Lemmata 5.2 and 5.3 , we establish in Lemma 5.4 a dynamic control of $\|u_1(t)\|_3^2 + \|\omega_1(t)\|_2^2$ which reveals a dynamic interaction between the angular velocity and the angular vorticity fields.

- Third, using this dynamic control, we obtain the proof of our Theorem 5.1.

We re-write $u_\theta$ and $\psi_\theta$ as follows:

$$u_\theta(r, z, t) = ru_1(r, z, t), \quad \omega_\theta(r, z, t) = ru_1(r, z, t), \quad \psi_\theta(r, z, t) = r\psi_1(r, z, t).$$  

(18)

Since $m \geq 2$, then $u \in C([0, T]; H^2(\mathbb{R}^3))^3 \cap L^2([0, T]; H^3(\mathbb{R}^3))^3$ and thanks to Lemmata 3-6 in [42], we deduce that,

$$u_1 \in C([0, T]; H^1(\mathbb{R}^3))$$

$$\omega_1 \in C([0, T]; L^2(\mathbb{R}^3)) \cap L^2([0, T]; H^1(\mathbb{R}^3)).$$  

(19)

Thanks to (19) and Lemma 1 in [22] used firstly with $u = \psi_1$, $f = \omega_1$, secondly with $u = \psi_1$, $f = \omega_1$ and using the same choice of the weight $w$ as in Lemma 2 ([22]), we get,

$$\psi_1 \in C([0, T]; H^2(\mathbb{R}^3)) \cap L^2([0, T]; H^3(\mathbb{R}^3)).$$  

(20)
As in [21], from (10), we derive the following equivalent system for \((u_1, \omega_1, \psi_1)\):

\[
\begin{align*}
\partial_t u_1 + u_r \partial_r u_1 + u_z \partial_z u_1 &= 2u_1 \partial_z \psi_1(t) + \left( \partial^2_r u_1 + \partial_z^2 u_1 + \frac{3}{r} \partial_r u_1 \right), \\
\partial_t \omega_1 + u_r \partial_r \omega_1 + u_z \partial_z \omega_1 &= \partial_z (u_1)^2 + \left( \partial^2_r \omega_1 + \partial_z^2 \omega_1 + \frac{3}{r} \partial_r \omega_1 \right), \\
- \left( \partial^2_r \psi_1 + \partial_z^2 \psi_1 + \frac{3}{r} \partial_r \psi_1 \right) &= \omega_1
\end{align*}
\]  

(21)

where,

\[
u_r = -r \frac{\partial \psi_1}{\partial z}, \quad \nu_z = \frac{1}{r} \frac{\partial (r^2 \psi_1)}{\partial r}
\]  

(22)

Note that in the new system (21), the convection term has absorbed one of the vortex-stretching terms \(\frac{u_r \omega_\theta}{r}\), which originally appears in the second equation of (10). In some sense, the convection term has already stabilized one of the potentially destabilized vortex-stretching terms in the above reformulation.

To obtain the proof of the crucial Lemma 5.4, we use Lemma 5.2 and Lemma 5.3. Lemma 5.2 depends on Lemma 5.1 which is an immediate consequence of CKN-type inequalities proved in [5].

**Lemma 5.1.** There exists a constant \(C > 0\) such that for all \(v \in C^\infty_0([0, +\infty[ \setminus \{0\})\) and \(\alpha > \frac{1}{2}\), we have,

\[
\int_0^\infty |v(r)|^2 r^{2(\alpha-1)} \, dr \leq C \int_0^\infty |v'(r)|^2 r^{2\alpha} \, dr.
\]

Here is the proof of Lemma 5.2.

**Lemma 5.2.** There exists a constant \(C > 0\) such that for all \(t \in [0, T]\) and for all \(R > 0\), we have,

\[
\|u_1(t)\|^4 \leq C \|\Gamma(t)\|_{L^\infty} \|\nabla \|u_1(t)\|_L^2\|^2 + \frac{C \|\Gamma_0\|_{L^\infty}}{R^2} \|u_1(t)\|^3
\]

**Proof.** Let \(R > 0\). Consider the cut-off function \(\zeta\) defined on \(\mathbb{R}_+\) for which \(0 \leq \zeta \leq 1\), \(\zeta = 1\) on \([0, \frac{R}{2}]\), \(\text{supp} \zeta \subset [0, 1]\). Now, we consider the rescaled cut-off function \(\zeta_R\) defined on \(\mathbb{R}_+\) by \(\zeta_R(r) = \zeta\left(\frac{r}{R}\right)\). For any \(x \in \mathbb{R}^3\), we write \(x\) under the form \(x = (x', z)\) where \(x' \in \mathbb{R}^2\). Then, we have,

\[
\int_{\mathbb{R}^3} |u_1(t)|^4 \, dx = \int_{\mathbb{R}^3} (|u_1(x, t)| \zeta_R(|x'|) + |u_1(x, t)| (1 - \zeta_R(|x'|)))^4 \, dx \\
\leq 4 \int_{\mathbb{R}^3} (|u_1(x, t)| \zeta_R(|x'|))^4 \, dx + 4 \int_{\mathbb{R}^3} (|u_1(x, t)| (1 - \zeta_R(|x'|)))^4 \, dx \\
\leq 4 \int_{\mathbb{R}^3} (|u_1(x, t)| \zeta_R(|x'|))^4 \, dx + 4 \int_{\mathbb{R}^3} |u_1(x, t)|^4 \chi_{\{|x'| \geq \frac{R}{2}\}} \, dx.
\]

With \(r = |x'|\), we recall that \(\Gamma = ru_\theta\) and \(\Gamma_0 = ru_\theta(0)\), we notice that \(r^2 u_1(t) = \Gamma(t)\), then

\[
|u_1(t)|^4 \chi_{\{|x'| \geq \frac{R}{2}\}} \leq \frac{4 \|\Gamma\|_{L^\infty}}{R^2} |u_1(t)|^3
\]

and thanks to (17), we obtain,

\[
|u_1(t)|^4 \chi_{\{|r| \geq \frac{R}{2}\}} \leq \frac{4 \|\Gamma_0\|_{L^\infty}}{R^2} |u_1(t)|^3.
\]

Therefore, we deduce for all \(t \in [0, T]\),

\[
\int_{\mathbb{R}^3} |u_1(t)|^4 \, dx \leq 4 \int_{\mathbb{R}^3} (|u_1(x, t)| \zeta_R(|x'|))^4 \, dx + \frac{16 \|\Gamma_0\|_{L^\infty}}{R^2} \int_{\mathbb{R}^3} |u_1(t)|^3.
\]  

(23)
We consider $u_1(x, t)$ under the form $u_1(r, z, t)$, then we have,

$$\int (|u_1(x, t)|_{\xi}^2)^4 = 2\pi \int_0^\infty \int_0^\infty |u_1(r, z, t)|^4 \zeta_r^4 r dr dz.$$  

For a.e $z \in \mathbb{R}$, thanks to Lemma 5.1 used with $v = (|u_1(\cdot, z, t)|_{\xi}^2)^{\alpha}$ and $\alpha = \frac{3}{2}$, we obtain,

$$\int_0^\infty (|u_1(r, z, t)|_{\xi}^2)^4 r dr \leq C \int_0^\infty |\partial_r(|u_1(r, z, t)|_{\xi}^2)|^2 r^3 dr$$

$$= 4C \int_0^\infty (|u_1(r, z, t)|_{\xi}^2)^2 |\partial_r(|u_1(r, z, t)|_{\xi}^2)|^2 r^3 dr$$

$$\leq 4C \int_0^\infty |\Gamma(r, z, t)||u_1(r, z, t)||\partial_r(|u_1(r, z, t)|_{\xi}^2)|^2 r dr$$

$$\leq 8C \int_0^\infty |\Gamma(r, z, t)||u_1(r, z, t)||u_1(r, z, t)||_2\zeta_r^2(r)^2 + \zeta_r^2(r)^2 |\partial_r|u_1(r, z, t)||^2 r dr$$

$$\leq 8C \int_0^\infty |\Gamma(r, z, t)| \left( |u_1(r, z, t)|^3 \frac{\zeta_r^2(r)^2}{R^2} + \frac{4}{9} \chi_{|r| \leq R} |\partial_r|u_1(r, z, t)||^2 \right) r dr.$$  

(24)

Thanks to Inequality (17), then from (24), we infer that there exists a constant $C_1 > 0$ such that,

$$\int_0^\infty (|u_1(r, z, t)|_{\xi}^2)^4 r dr \leq C_1 \left( \frac{\Gamma_0}{R^2} \right) \int_0^\infty |u_1(r, z, t)|^3 r dr + C_1 \|\Gamma(t)\chi_{|r| \leq R}\|_{L^\infty} \int_0^\infty |\partial_r|u_1(r, z, t)||^2 r dr.$$  

Therefore, we obtain,

$$\int_{\mathbb{R}^3} (|u_1(x, t)|_{\xi}^2)^4 dx \leq C_1 \left( \frac{\Gamma_0}{R^2} \right) \int_{\mathbb{R}^3} |u_1(t)|^3 + C_1 \|\Gamma(t)\chi_{|r| \leq R}\|_{L^\infty} \int_{\mathbb{R}^3} |\nabla u_1(t)|^2.$$  

(25)

Then, using (23) and (25), we conclude the proof.

To prove Lemma 5.4, the main Lemma in this section, we need Lemma 5.3.

**Lemma 5.3.** There exists a constant $C > 0$ such that for all $f \in L^2(\mathbb{R}^2)$ radial function such that $|x|^2 f \in L^2(\mathbb{R}^2)$ and $g \in H^2(\mathbb{R}^2)$, we have,

$$\left| \int_{\mathbb{R}^2} fg \right| \leq C \left( \int_{\mathbb{R}^2} |x|^3 f(x)^2 dx \right)^{\frac{1}{2}} \|\Delta g\|_{L^2(\mathbb{R}^2)}.$$  

**Proof.** Since $f$ is a radial function, there exists $\zeta$ a real function on $\mathbb{R}^+$ such that for a.e $x \in \mathbb{R}^2$,

$$f(x) = \zeta(|x|),$$  

(26)

and using the change of variables with polar coordinates $x = (r \cos \theta, r \sin \theta)$, $r \in \mathbb{R}^+$ and $\theta \in [0, 2\pi]$, we obtain,

$$\|f\|_{L^2(\mathbb{R}^2)} = \sqrt{2\pi} \|\zeta(r)^{\frac{1}{2}}\|_{L^2(\mathbb{R}^+)}$$

$$\| |x|^2 f\|_{L^2(\mathbb{R}^2)} = \sqrt{2\pi} \|\zeta(r)^{\frac{3}{2}}\|_{L^2(\mathbb{R}^+)}.$$  

(27)

Let $K > 0$, $\zeta_K$ the real function defined on $\mathbb{R}^+$ by $\zeta_K(r) = \zeta(r)\chi_{[0 \leq r \leq K]}$ for all $r \geq 0$. We introduce also $\phi_K$ the real function defined on $\mathbb{R}^+$ for all $r > 0$ by,

$$\phi_K(r) = \int_0^r \frac{1}{r} \int_0^r \tau \zeta_K(\tau) d\tau d\rho.$$  

(28)

Using successively the fact that $\text{supp} \zeta_K \subset [0, K]$ and $|\zeta_K| \leq |\zeta|$, for all $\alpha \geq 0$ and for a.e $\tau > 0$, we get,

$$|\tau \zeta_K(\tau)| = \tau^{\frac{1}{2} - \alpha} |\tau^{\alpha + \frac{1}{2}} \zeta_K(\tau)| \leq \tau^{\frac{1}{2} - \alpha} K^{\alpha + \frac{1}{2}} |\zeta(\tau)| = \left( \frac{K^{\alpha + \frac{1}{2}}}{\tau^{\alpha}} \right) \left( \tau^{\frac{1}{2}} |\zeta(\tau)| \right).$$  

(29)
Using definition (28), inequality (29), Cauchy-Schwarz inequality and (27), we deduce that \( \phi_K \in C^1([0, +\infty]) \) and for all \( r > 0 \) and \( \alpha > \frac{1}{2} \),

\[
\begin{align*}
 r^{\alpha - \frac{1}{2}} |\phi_K(r)| & \leq \frac{K^{\alpha + \frac{1}{2}}}{(\alpha - \frac{1}{2})\sqrt{2\pi(2\alpha - 1)}} \|f\|_{L^2(\mathbb{R}^2)}, \\
 r^{\alpha + \frac{1}{2}} |\phi'_K(r)| & \leq \frac{K^{\alpha + \frac{3}{2}}}{\sqrt{2\pi(2\alpha - 1)}} \|f\|_{L^2(\mathbb{R}^2)}, \\
 r^\alpha |(r\phi'_K(r))'| & \leq K^{\alpha + \frac{1}{2}} |r\frac{\partial}{\partial r} \zeta(r)|.
\end{align*}
\]

(30)

Let us show that

\[
 r^{\frac{3}{2}} \phi_K \in L^2([0, +\infty]) \text{ and } r^{\frac{3}{2}} \phi'_K \in L^2([0, +\infty]).
\]

(31)

Using the first inequality of (30) with \( \alpha = \frac{3}{4} \) and \( \alpha = 2 \), we infer respectively that \( r^{\frac{3}{4}} |\phi_K(r)| \leq C_K \|f\|_{L^2(\mathbb{R}^2)} \) and \( r^{\frac{3}{2}} |\phi_K(r)| \leq C_K \|f\|_{L^2(\mathbb{R}^2)} \), where \( C_K > 0 \) is a real depending only on \( K \). Then, we get

\[
\begin{align*}
 \int_0^{+\infty} r \phi_K(r)^2 \, dr & = \int_0^1 r \phi_K(r)^2 \, dr + \int_1^{+\infty} r \phi_K(r)^2 \, dr \\
 & = \int_0^1 r^{\frac{3}{2}} (r^{\frac{3}{2}} \phi_K(r))^2 \, dr + \int_1^{+\infty} \frac{1}{r^2} (r^{\frac{3}{2}} \phi_K(r))^2 \, dr \\
 & \leq \frac{5}{3} C_K^2 \|f\|_{L^2(\mathbb{R}^2)^2}.
\end{align*}
\]

Therefore, we deduce that \( r^{\frac{3}{2}} \phi_K \in L^2([0, +\infty]) \). It remains to show that \( r^{\frac{3}{2}} \phi'_K \in L^2([0, +\infty]) \).

Using the second inequality of (30) with \( \alpha = \frac{3}{4} \) and \( \alpha = 2 \), we infer respectively that \( r^{\frac{3}{4}} |\phi'_K(r)| \leq \tilde{C}_K \|f\|_{L^2(\mathbb{R}^2)} \) and \( r^{\frac{3}{2}} |\phi'_K(r)| \leq \tilde{C}_K \|f\|_{L^2(\mathbb{R}^2)} \), where \( \tilde{C}_K > 0 \) is a real depending only on \( K \). Then, we get

\[
\begin{align*}
 \int_0^{+\infty} r^3 |\phi'_K(r)|^2 \, dr & = \int_0^1 r^3 |\phi'_K(r)|^2 \, dr + \int_1^{+\infty} r^3 |\phi'_K(r)|^2 \, dr \\
 & = \int_0^1 r^{\frac{5}{2}} (r^{\frac{5}{2}} \phi'_K(r))^2 \, dr + \int_1^{+\infty} \frac{1}{r^2} (r^{\frac{5}{2}} \phi'_K(r))^2 \, dr \\
 & \leq \frac{5}{3} \tilde{C}_K^2 \|f\|_{L^2(\mathbb{R}^2)^2}.
\end{align*}
\]

Therefore, we deduce that \( r^{\frac{3}{2}} \phi'_K \in L^2([0, +\infty]) \).

By using also the third inequality of (30) with \( \alpha = \frac{3}{2} \) and thanks to (27), we infer,

\[
 r^{\frac{3}{2}} (r \phi'_K)' \in L^2([0, +\infty]).
\]

(32)

Then thanks to (31) and (32), by using twice Lemma 5.1 with \( \alpha = \frac{3}{2} \), we deduce,

\[
\begin{align*}
 \|r^{\frac{1}{2}} \phi_K\|_{L^2(\mathbb{R}_+)} & \lesssim \|r^{\frac{3}{2}} \phi'_K\|_{L^2(\mathbb{R}_+)} \\
 & = \|r^{\frac{3}{2}} (r \phi'_K)\|_{L^2(\mathbb{R}_+)} \\
 & \lesssim \|r^{\frac{3}{2}} (r \phi'_K)'\|_{L^2(\mathbb{R}_+)} \\
 & = \|r^{\frac{5}{2}} \Delta \phi_K(r)\|_{L^2(\mathbb{R}_+)},
\end{align*}
\]

(33)

where for all \( r > 0 \), \( \Delta \phi_K(r) := \frac{1}{r} (r \phi'_K)' \). From (28), we notice,

\[
\tilde{\Delta} \phi_K(r) = \zeta_K(r).
\]

(34)
Then, from \((33)\), using \((34)\), we obtain,

\[
\|r^{\frac{1}{2}} \phi_K\|_{L^2(\mathbb{R}^+)} \lesssim \|r^{\frac{1}{2}} \nu_K(r)\|_{L^2(\mathbb{R}^+)} \leq \|r^{\frac{1}{2}} \zeta(r)\|_{L^2(\mathbb{R}^+)}. \tag{35}
\]

We introduce the radial function \(\Phi_K\) defined on \(\mathbb{R}^2\) by,

\[
\Phi_K(x) = \phi_K(|x|). \tag{36}
\]

Then, we get \(\Delta \Phi_K(x) = \tilde{\Delta} \phi_K(|x|)\) and thanks to \((34)\), we have

\[
\Delta \Phi_K(x) = \zeta_K(|x|) \chi_{\{|x| \leq K\}} = f(x) \chi_{\{|x| \leq K\}}.
\]

Since, we have,

\[
\left| \int_{\mathbb{R}^2} fg \right| = \left| \int_{\mathbb{R}^2} f(x) \chi_{\{|x| \leq K\}} g(x) \, dx + \int_{\mathbb{R}^2} f(x) \chi_{\{|x| > K\}} g(x) \, dx \right| 
\leq \left| \int_{\mathbb{R}^2} f(x) \chi_{\{|x| \leq K\}} g(x) \, dx \right| + \left| \int_{\mathbb{R}^2} f(x) \chi_{\{|x| > K\}} g(x) \, dx \right|. \tag{37}
\]

Then, we deduce,

\[
\left| \int_{\mathbb{R}^2} fg \right| \leq \left| \int_{\mathbb{R}^2} \Delta \Phi_K(x) g(x) \, dx \right| + \left| \int_{\mathbb{R}^2} f(x) \chi_{\{|x| > K\}} g(x) \, dx \right|. \tag{38}
\]

For the first term at the right hand side of inequality \((37)\), using integration by parts and thanks to Cauchy-Schwarz inequality, we get,

\[
\left| \int_{\mathbb{R}^2} \Delta \Phi_K(x) g(x) \, dx \right| = \left| \int_{\mathbb{R}^2} \Phi_K(x) \Delta g(x) \, dx \right| \leq \|\Phi_K\|_{L^2(\mathbb{R}^2)} \|\Delta g\|_{L^2(\mathbb{R}^2)}. \tag{39}
\]

Using the change of variables with polar coordinates, from \((36)\), we observe,

\[
\|\Phi_K\|_{L^2(\mathbb{R}^2)} = \sqrt{2\pi} \|\phi_K(r) r^{\frac{1}{2}}\|_{L^2(\mathbb{R}^+)}, \tag{40}
\]

then thanks to \((35)\) and \((27)\), we deduce,

\[
\|\Phi_K\|_{L^2(\mathbb{R}^2)} \lesssim \|x^2 f\|_{L^2(\mathbb{R}^2)}. \tag{41}
\]

Then, using \((39)\), from \((38)\), we deduce,

\[
\left| \int_{\mathbb{R}^2} \Delta \Phi_K(x) g(x) \, dx \right| \lesssim \|x^2 f\|_{L^2(\mathbb{R}^2)} \|\Delta g\|_{L^2(\mathbb{R}^2)}. \tag{42}
\]

For the second term at the right hand side of inequality \((37)\), thanks to Cauchy-Schwarz inequality, we obtain,

\[
\left| \int_{\mathbb{R}^2} f(x) \chi_{\{|x| > K\}} g(x) \, dx \right| \leq \|f\|_{L^2(\{x \in \mathbb{R}^2, |x| > K\})} \|g\|_{L^2(\mathbb{R}^2)}. \tag{43}
\]

Using \((40)\) and \((41)\), from \((37)\), we obtain,

\[
\left| \int_{\mathbb{R}^2} fg \right| \lesssim \|x^2 f\|_{L^2(\mathbb{R}^2)} \|\Delta g\|_{L^2(\mathbb{R}^2)} \tag{44}
\]

Since \(f \in L^2(\mathbb{R}^2)\), then \(\|f\|_{L^2(\{x \in \mathbb{R}^2, |x| > K\})} \to 0\) as \(K \to \infty\). Then, taking the limit in inequality \((44)\) as \(K \to \infty\), we obtain,

\[
\left| \int_{\mathbb{R}^2} fg \right| \lesssim \|x^2 f\|_{L^2(\mathbb{R}^2)} \|\Delta g\|_{L^2(\mathbb{R}^2)},
\]

which concludes the proof.
Now, we turn to the proof of the main Lemma of this section.

**Lemma 5.4.** There exist two absolute constants $\gamma_0 > 0$ and $C > 0$ such that if there exists $R > 0$ such that,

$$\sup_{t \in [0,T]} \| \Gamma(t) \chi_{\{ r \leq R \}} \|_{L^\infty} \leq \gamma_0,$$

then we get that for all $t \in [0,T]$,

$$\frac{1}{3} \| u_1(t) \|_3^3 + \frac{1}{2} \| \omega(t) \|_2^2 \leq \left( \frac{1}{3} \| u_1(0) \|_3^3 + \frac{1}{2} \| \omega(0) \|_2^2 \right) \exp \left( 3C \frac{\| \Gamma_0 \|_{L^\infty}}{R^2} (1 + \| \Gamma_0 \|_{L^\infty}) \right) t. $$

**Proof.** We multiply the first equation of (21) by $u_1(t) |u_1(t)|$, integrate it over $\mathbb{R}^3$, use the incompressibility condition (12) and integration by parts, to obtain for all $t \in [0,T]$,

$$\frac{1}{3} \frac{d}{dt} \| u_1(t) \|_3^3 + \frac{8}{9} \int | \nabla |u_1(t)|^2 | \nabla \psi_1(t) | - \frac{2}{3} \int \| u_1(0, z, t) \|_3^3 dz = 2 \int |u_1(t)|^3 \partial_z \psi_1(t).$$

(43)

Note that, in order to treat the convex term, we have integrated by parts and the boundary integrals have vanished at $r = 0$ due to the fact that $u_0(0, z, t) = 0$, while near $r = \infty$ due to the standard density argument.

We observe,

$$\int |u_1(t)|^3 \partial_z \psi_1(t) = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |u_1(x', z, t)|^3 \partial_z \psi_1(x', z, t) dx' \right) dz. $$

(44)

Thanks to (19), (20) and Lemma 5.3, there exists a constant $C_0 > 0$ such that for a.e $z \in \mathbb{R}$,

$$\int_{\mathbb{R}^2} |u_1(x', z, t)|^3 \partial_z \psi_1(x', z, t) dx' \leq C_0 \| x' |u_1|^3 (\cdot, z, t) \|_{L^2(\mathbb{R})} \| \nabla x' (\partial_z \psi_1) (\cdot, z, t) \|_{L^2(\mathbb{R})}. $$

(45)

From (44), thanks to (45) and Cauchy-Schwarz inequality, we get,

$$\int |u_1(t)|^3 \partial_z \psi_1(t) \leq C_0 \left( \int_{\mathbb{R}} | x' |u_1|^3 (\cdot, z, t) \|_{L^2(\mathbb{R})} dz \right)^\frac{1}{2} \left( \int_{\mathbb{R}} \| x' \|_{L^2(\mathbb{R})} \| \nabla x' (\partial_z \psi_1) (\cdot, z, t) \|_{L^2(\mathbb{R})} dz \right)^\frac{1}{2} = C_0 \| x' |u_1|^3 \|_{L^2(\mathbb{R})} \| \nabla x' \partial_z \psi_1(t) \|_{L^2(\mathbb{R})}. $$

(46)

Thanks to Lemma 1 in [22] used with $u = \partial_z \psi_1(t)$, $f = \partial_z \omega_1(t)$ and using the same choice of the weight $w$ as in Lemma 2 of [22], we deduce that there exists a constant $C_1 > 0$ such that for all $t \in [0,T]$,

$$\int | \nabla^2 \partial_z \psi_1(t) |^2 \leq C_1 \int | \partial_z \omega_1(t) |^2.$$ 

(47)

Then, thanks to (46) and (47), we deduce that there exists a real $C_2 > 0$ such that for all $t \in [0,T]$,

$$2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq C_2 \| x' |u_1|^3 \|_{L^2(\mathbb{R})} \left( \int | \partial_z \omega_1(t) |^2 \right)^\frac{1}{2}. $$

(48)

Recalling $\Gamma = r u_0$, with $r = |x'|$, we notice that $|x'|^2 u_1(t) = \Gamma(t)$, then $|x'|^2 |u_1(t)|^3 = |\Gamma(t)| |u_1(t)|^2$, then from (48), we obtain,

$$2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq C_2 \| \Gamma(t) \|_1 |u_1(t)|^2 \| \partial_z \omega_1(t) \|_2 \leq C_2 \| \Gamma(t) \|_1 |u_1(t)|^2 \| \partial_z \omega_1(t) \|_2 \leq C_2 \| \Gamma(t) \|_1 |u_1(t)|^2 \| \partial_z \omega_1(t) \|_2 \leq C_2 \| \Gamma(t) \|_1 |u_1(t)|^2 + \frac{1}{4} \| \partial_z \omega_1(t) \|_2^2.$$ 

(49)
Further, we have

\[
\| \|\Gamma(t)\| u_1(t)\|^2 \|_2 = \int_{\mathbb{R}^3} |\Gamma(x, t)|^2 |u_1(x, t)|^4 \, dx
\]

\[
= \int_{\mathbb{R}^3} |\Gamma(x, t)\chi_{\{|x'| \leq R\}}|^2 |u_1(x, t)|^4 \, dx \, + \int_{\mathbb{R}^3} |\Gamma(x, t)\chi_{\{|x'| > R\}}|^2 |u_1(x, t)|^4 \, dx
\]

\[
\leq \int_{\mathbb{R}^3} |\Gamma(x, t)\chi_{\{|x'| \leq R\}}|^2 |u_1(x, t)|^4 \, dx + \frac{\|\Gamma_0\|_{L^\infty}^2}{R^2} \int_{\mathbb{R}^3} |u_1(x, t)|^3 \, dx,
\]

where for the last inequality we have used the fact that

\[
|\Gamma(x, t)\chi_{\{|x'| > R\}}|^2 |u_1(x, t)|^4 \leq \frac{\|\Gamma(t)\|_{L^\infty}^3}{R^2} |u_1(x, t)|^3
\]

\[
\leq \frac{\|\Gamma_0\|_{L^\infty}^3}{R^2} |u_1(x, t)|^3 (\text{ thanks to } (17)).
\]

Then, from (49), we obtain

\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq C_2^2 \|\Gamma(t)\chi_{\{|x'| \leq R\}}\|_{L^\infty} |u_1(t)|^4 + C_2^2 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} |u_1(t)|^3 + \frac{1}{4} \|\partial_z \omega_1(t)\|^2.
\]

Using (50), from (43), we deduce that for all \( t \in [0, T] \),

\[
\frac{1}{3} \frac{d}{dt} \|u_1(t)\|^3 + \frac{8}{9} \|\nabla u_1(t)\|^2 \|\omega_1(t)\|^2 \leq \frac{1}{4} \|\partial_z \omega_1(t)\|_2^2 + C_2^2 \|\Gamma(t)\chi_{\{|x'| \leq R\}}\|_{L^\infty} |u_1(t)|^4 + C_2^2 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} |u_1(t)|^3.
\]

We multiply the first equation of (21) by \( \omega_1(t) \), integrate it over \( \mathbb{R}^3 \), use the incompressibility condition (12), then we obtain for all \( t \in [0, T] \),

\[
\frac{1}{2} \frac{d}{dt} \|\omega_1(t)\|^2 + \int \|\nabla \omega_1(t)\|^2 + \int_{-\infty}^{\infty} |\omega_1(0, z, t)|^2 \, dz = \int \omega_1(t) \partial_z (u_1(t)^2).
\]

By using integration by parts, Cauchy-Schwarz inequality and Young inequality, we deduce that for all \( t \in [0, T] \),

\[
\int \omega_1(t) \partial_z (u_1(t)^2) = -\int \partial_z \omega_1(t) u_1(t)^2 \leq \|\partial_z \omega_1(t)\|_2 \|u_1(t)\|_4^2 \leq \frac{1}{4} \|\partial_z \omega_1(t)\|_2^2 + \|u_1(t)\|_4^4.
\]

Using (53), from (52), we obtain for all \( t \in [0, T] \),

\[
\frac{1}{2} \frac{d}{dt} \|\omega_1(t)\|^2 + \frac{3}{4} \|\nabla \omega_1(t)\|^2 \leq \|u_1(t)\|_4^4.
\]

We sum inequalities (51) and (54), then, we obtain for all \( t \in [0, T] \),

\[
\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) + \frac{8}{9} \|\nabla u_1(t)\|^2 \|\omega_1(t)\|^2 \leq (1 + C_2^2 \|\Gamma(t)\chi_{\{|x'| \leq R\}}\|_{L^\infty}^2) |u_1(t)|^4 + C_2^2 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} |u_1(t)|^3.
\]

Thanks to Lemma 5.2 and inequality (17), from (55), we deduce that there exists a constant \( C_3 > 0 \) such that for all \( t \in [0, T] \),

\[
\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) + \left( \frac{8}{9} - C_3(1 + \|\Gamma(t)\chi_{\{|x'| \leq R\}}\|_{L^\infty}^2)\|\nabla u_1(t)\|^2 \right) \|\nabla \omega_1(t)\|^2 \leq C_3 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} \left( 1 + \|\Gamma_0\|_{L^\infty}^2 \right) |u_1(t)|^3.
\]
Let us introduce the unique constant $\gamma_0 > 0$ satisfying $C_3(1 + \gamma_0^2)\gamma_0 = \frac{8}{9}$. Since the real-valued function $y \mapsto C_3(1 + y^2)\gamma_0$ is nondecreasing, then under the assumption that there exists $R > 0$ such that for any $t \in [0, T]$, $\|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty} \leq \gamma_0$, we get

$$C_3(1 + \|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^2})^2 \leq \frac{8}{9},$$

and from (56) we deduce that for all $t \in [0, T]$,

$$\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) \leq C_3^2 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} (1 + \|\Gamma_0\|_{L^\infty}^2) \|u_1(t)\|_3^3,$$  \hspace{1cm}(57)

which implies that for all $t \in [0, T]$,

$$\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) \leq 3C_3 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} (1 + \|\Gamma_0\|_{L^\infty}^2) \left( \frac{1}{3} \|u_1(0)\|_3^3 + \frac{1}{2} \|\omega_1(0)\|_2^2 \right).$$ \hspace{1cm}(58)

Then thanks to Gronwall inequality, we deduce that for all $t \in [0, T]$,

$$\frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \leq \left( \frac{1}{3} \|u_1(0)\|_3^3 + \frac{1}{2} \|\omega_1(0)\|_2^2 \right) \exp \left( 3C_3 \frac{\|\Gamma_0\|_{L^\infty}}{R^2} (1 + \|\Gamma_0\|_{L^\infty}^2) t \right),$$

which concludes the proof.

Now, we finish with our main result.

**Theorem 5.1.** Let $u_0 \in H^m(\mathbb{R}^3)$ axisymmetric solenoidal vector field, with $m \geq 2$ with $\Gamma_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Let $T > 0$ be such that there exists $u \in C([0, T]; H^m(\mathbb{R}^3)) \cap L^2([0, T]; H^{m+1}(\mathbb{R}^3))$ solution to the Navier-Stokes equations (2) for the initial data $u_0$. If $u \notin C([0, T], H^m(\mathbb{R}^3))$ then we get,

\[ \inf_{R > 0} \sup_{t \in [0, T]} \|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty} \geq \gamma_0, \]

where $\gamma_0 > 0$ is the absolute constant involved in Lemma 5.4.

**Proof.** To get the proof, we assume first that $\inf_{R > 0} \sup_{t \in [0, T]} \|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty} < \gamma_0$, then there exists $R > 0$ such that

\[ \sup_{t \in [0, T]} \|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty} \leq \gamma_0. \]

We derive first an estimate of $\omega_\theta \in L^\infty L^2$. Thanks to Lemma 5.4, we get that there exists a constant $C > 0$ such that for all $t \in [0, T]$,

\[ \frac{1}{3} \frac{\|u_\theta(t)\|_r^3}{r} + \frac{1}{2} \frac{\|\omega_\theta(t)\|_r^2}{r} \leq \left( \frac{1}{3} \frac{\|u_\theta(0)\|_r^3}{r} + \frac{1}{2} \frac{\|\omega_\theta(0)\|_r^2}{r} \right) \exp \left( 3C \frac{\|\Gamma_0\|_{L^\infty}}{R^2} (1 + \|\Gamma_0\|_{L^\infty}^2) T \right) =: Q_0. \]

We multiply the first equation of (8) by $\omega_\theta$ and integrate it over $\mathbb{R}^3$. Then, we have for all $t \in [0, T]$,

\[ \frac{1}{2} \frac{d}{dt} \frac{\|\omega_\theta(t)\|_r^2}{r} + \int |\nabla \omega_\theta(t)|^2 + \frac{\omega_\theta(t)}{r} \omega_\theta(t) \omega_\theta(t) = \int \frac{u_\theta(t)}{r} \omega_\theta(t) - 2 \int \frac{u_\theta(t)}{r} \omega_\theta(t) \omega_\theta(t). \]
On one hand, we have,
\[
\int \frac{u_r(t)}{r} \omega_\theta^2(t) \leq \|u_r(t)\omega_\theta(t)\|_2 \left\| \frac{\omega_\theta(t)}{r} \right\|_2^2 \leq \|u_r(t)\|_6 \|\omega_\theta(t)\|_3 \left\| \frac{\omega_\theta(t)}{r} \right\|_2^2 \leq \|u(t)\|_6 \|\omega_\theta(t)\|_3^2 \left\| \frac{\omega_\theta(t)}{r} \right\|_2^2 \leq C^2 \|\nabla u(t)\|_2^2 \left\| \frac{\omega_\theta(t)}{r} \right\|_2^2 \leq C^2 \|\nabla u(t)\|_2^2 \left\| \frac{\omega_\theta(t)}{r} \right\|_2^2 + \frac{1}{4} \left\| \nabla \omega_\theta(t) \right\|_L^2_2, \tag{61}
\]
where, we have used the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ with $C > 0$ a constant and Young inequality. On the other hand, we have,
\[
-2 \int \frac{u_\theta(t)}{r} \omega_r(t) \omega_\theta(t) \leq 2 \|\omega_\theta(t)\| \left\| \frac{u_\theta(t)}{r} \right\|_3 \leq 2 \|\omega_\theta(t)\|_6 \left\| \frac{u_\theta(t)}{r} \right\|_3 \leq C \|\nabla \omega_\theta(t)\|_2 \left\| \nabla u(t) \right\|_2 \left\| \frac{u_\theta(t)}{r} \right\|_3 \leq C^2 \|\nabla u(t)\|_2^3 \left\| \frac{u_\theta(t)}{r} \right\|_3^2 + \frac{1}{4} \left\| \nabla \omega_\theta(t) \right\|_L^2_2, \tag{62}
\]
Then, using (61) and (62), from (60), we deduce for all $t \in [0, T]$,
\[
\frac{1}{2} \frac{d}{dt} \|\omega_\theta(t)\|_2^2 + \frac{1}{2} \int |\nabla \omega_\theta(t)|^2 + \int \left| \frac{\omega_\theta(t)}{r} \right|^2 \leq C^2 \|\nabla u(t)\|_2^2 \left( \left\| \frac{\omega_\theta(t)}{r} \right\|_2^4 + \left\| \frac{u_\theta(t)}{r} \right\|_3^2 \right), \tag{63}
\]
which implies that for all $t \in [0, T]$,
\[
\frac{1}{2} \frac{d}{dt} \|\omega_\theta(t)\|_2^2 \leq C^2 \|\nabla u(t)\|_2^2 \left( \left\| \frac{\omega_\theta(t)}{r} \right\|_2^4 + \left\| \frac{u_\theta(t)}{r} \right\|_3^2 \right) \leq C^2 C_1 \|\nabla u(t)\|_2^2 Q_0^2, \tag{64}
\]
where for the last inequality we have used (59) with $C_1 > 0$ a constant. After an integration over $[0, t]$ of inequality (64), we deduce that for all $t \in [0, T]$,
\[
\|\omega_\theta(t)\|_2^2 \leq \|\omega_\theta(0)\|_2^2 + 2 C_1 Q_0^2 \int_0^t \|\nabla u(s)\|_2^2 ds \leq \|\omega_\theta(0)\|_2^2 + C_1 Q_0^2 \|u_0\|_2^2 =: \Omega_0, \tag{65}
\]
where we have used energy equality (4).

Now, we multiply the second equation of (8) by $\omega_r$, the third equation of (8) by $\omega_z$, integrate them over $\mathbb{R}^3$ and sum the equations obtained, then we get for all $t \in [0, T]$,
\[
\frac{1}{2} \frac{d}{dt} \left( \|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2 \right) + \int \left( |\nabla \omega_r(t)|^2 + |\nabla \omega_z(t)|^2 + \left| \frac{\omega_r(t)}{r} \right|^2 \right) = \int \left( \partial_r u_r(t) \omega_r^2(t) + (\partial_z u_r(t) + \partial_r u_z(t)) \omega_r(t) \omega_z(t) + \partial_z u_z(t) \omega_z^2(t) \right). \tag{66}
\]
Thanks to Lemma 2 in [6] and Theorem 3.1.1 in [3], we deduce that there exists a constant $C_2 > 0$ such that for all $t \in [0, T]$,
\[
\|\nabla u_r(t)\|_2 \leq C_2 \|\omega_\theta(t)\|_2 \tag{67}
\]
\[
\|\nabla u_z(t)\|_2 \leq C_2 \|\omega_\theta(t)\|_2.
\]

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Furthermore, thanks to Cauchy-Schwarz inequality and Young inequality, we have for all $t \in [0, T]$, 
\[
\|\omega_r(t)\omega_z(t)\|_2 \leq \|\omega_r(t)\|_4 \|\omega_z(t)\|_4 \\
\leq \frac{1}{2} \|\omega_r(t)\|^2 + \frac{1}{2} \|\omega_z(t)\|^2. \tag{68}
\]
From (66), using Cauchy-Schwarz inequality, (67) and (68), we deduce that there exists a constant $C_3 > 0$ such that for all $t \in [0, T]$, 
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2) + \int \left( \|\nabla \omega_r(t)\|^2 + \|\nabla \omega_z(t)\|^2 + \left| \frac{\omega_r(t)}{r} \right|^2 \right) \]
\[
\leq C_3(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2). \tag{69}
\]

Thanks to Interpolation inequality, Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, we deduce that there exists a constant $C_4 > 0$ such for all $t \in [0, T]$, 
\[
\|\omega_r(t)\|_4 \leq \|\omega_r(t)\|_2^{\frac{1}{2}} \|\omega_r(t)\|_6^{\frac{1}{6}} \\
\leq C_4\|\omega_r(t)\|_2^{\frac{1}{2}} \|\nabla \omega_r(t)\|_2^{\frac{3}{2}},
\]
and also, 
\[
\|\omega_z(t)\|_4 \leq C_4\|\omega_z(t)\|_2^{\frac{1}{2}} \|\nabla \omega_z(t)\|_2^{\frac{3}{2}}.
\]

Then, from (69), we deduce that there exists a constant $C_5 > 0$ such that for all $t \in [0, T]$, 
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2) + \int \left( \|\nabla \omega_r(t)\|^2 + \|\nabla \omega_z(t)\|^2 + \left| \frac{\omega_r(t)}{r} \right|^2 \right) \]
\[
\leq C_5(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2). \tag{70}
\]

Thanks to Young inequality, there exists a constant $C_6 > 0$ such for all $t \in [0, T]$, 
\[
C_5\|\omega_r(t)\|_2(\|\omega_r(t)\|_2^{\frac{2}{3}} \|\nabla \omega_r(t)\|_2^{\frac{1}{3}}) \leq C_6\|\omega_r(t)\|_2^{\frac{2}{3}} \|\omega_r(t)\|_2^{\frac{1}{3}} + \frac{1}{2} \|\nabla \omega_r(t)\|_2^2,
\]
and also, 
\[
C_5\|\omega_r(t)\|_2(\|\omega_z(t)\|_2^{\frac{2}{3}} \|\nabla \omega_z(t)\|_2^{\frac{1}{3}}) \leq C_6\|\omega_r(t)\|_2^{\frac{2}{3}} \|\omega_z(t)\|_2^{\frac{1}{3}} + \frac{1}{2} \|\nabla \omega_z(t)\|_2^2.
\]

Therefore, from (70), we deduce that for all $t \in [0, T]$, 
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2) + \frac{1}{2} \int \left( \|\nabla \omega_r(t)\|^2 + \|\nabla \omega_z(t)\|^2 + \left| \frac{\omega_r(t)}{r} \right|^2 \right) \]
\[
\leq C_6\|\omega_r(t)\|_2^2(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2), \tag{71}
\]

which implies, 
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2) \leq C_6\|\omega_r(t)\|_2^2(\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2). \tag{72}
\]

Then, we integrate (72) over $[0, t]$ and we obtain that for all $t \in [0, T]$, 
\[
\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2 \leq \|\omega_r(0)\|_2^2 + \|\omega_z(0)\|_2^2 + 2C_6 \int_0^t \|\omega_r(s)\|_2^2(\|\omega_r(s)\|_2^2 + \|\omega_z(s)\|_2^2) ds. \tag{73}
\]

Then, thanks to (65) and (4), from (73), we deduce that for all $t \in [0, T]$, 
\[
\|\omega_r(t)\|_2^2 + \|\omega_z(t)\|_2^2 \leq \|\omega_r(0)\|_2^2 + \|\omega_z(0)\|_2^2 + 2C_6\Omega_0^2 \|\nu_0\|_2^2. \tag{74}
\]
Then, thanks to (65) and (74), we deduce that
\[ \limsup_{t \to T} \| \omega(t) \|_2 < +\infty. \]
However since \( u \not\in C([0,T], H^m(\mathbb{R}^3)) \) with \( m \geq 2 \), then (6) holds and we thus infer a contradiction with (6).
Therefore we obtain that for any \( R > 0 \),
\[ \sup_{t \in [0,T]} \| \Gamma(t) \chi_{\{r \leq R\}} \|_{L^\infty} \geq \gamma_0, \]
which concludes the proof.

References


