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Global Regularity of the 3D Axisymmetric Navier-Stokes Equations with swirl

Léo Agélas *
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Abstract

In this paper, we prove global regularity of the 3D axisymmetric Navier-Stokes equations with swirl. To our knowledge, our article answers to an open problem old of decades despite of great effort by many mathematicians and physicists. Our proof is based on recent works about the surprising stabilizing effect of the convection term in the three-dimensional incompressible Navier-Stokes (see [HLE],[HLI]) and the interaction between the angular velocity and the angular vorticity fields (see [HLL]). Our proof uses also some estimates on the angular velocity $u_\theta$, more precisely on $ru_\theta$, that has been so far neglected.

Keywords Navier-Stokes equations; Global existence

Mathematics Subject Classification 35Q30, 76D03, 76D05

1 Introduction

The study of the incompressible Navier-Stokes in three space dimensions has a long history. For a long time ago, a global weak solution $u \in L^\infty(0, \infty; L^2(\mathbb{R}^3))^3$ and $\nabla v \in L^2(\mathbb{R}^3 \times (0, \infty))^3$ to the Navier-stokes equations (2)-(3) was built by Leray [LER]. In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equation, and proved that, for every given $v_0 \in L^2(\mathbb{R}^3)$, there exists a global weak solution $u \in L^\infty([0, +\infty[; L^2(\mathbb{R}^3))^3 \cap L^2([0, \infty[; H^1(\mathbb{R}^3))^3$. Hopf has proved the existence of a global weak solution in the general case $\mathbb{R}^d$, $d \geq 2$, [HOP]. Several ways are known to construct weak solution ([GMA], [HEY], [GAL]), but the regularity and the uniqueness of this weak solution remained yet open in the general case, till now in spite of great efforts made (see [FLT],[GIGA], [MON],[LIO2],[GP], [SER], [WAHL], [GIGA], [ISS] and [VE]).

In two dimensions, the existence of classical solutions has been known for a long time ago (see [LAD], [LPRO], [LIO], [TEM]). Thus a natural question, namely what can be said about the axisymmetric flow, appears.

Axisymmetric flow is an important subject in fluid dynamics and has become standard textbook material as a starting point of theoretical study for complicated flow patterns. Although the number of independent spatial variables is reduced by symmetry, some of the essential features and complexities of generic three-dimensional (3D) flows remain. For example, when the swirling velocity is nonzero, there is a vorticity stretching term present.

The first results in the existence of classical solutions were obtained in the late sixties for axisymmetric flow without swirl (see [LAD2], [UY]) and later also in [LMNP].

In the case of axisymmetric flow with swirl, the question of finite time blow-up of solution remains a challenging open problem.

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In several recent papers ([HLE], [HLE2], [HLI], [HLL]), two systems of equations are proposed in order to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier-Stokes and Euler equations. By exploiting the special structure of the nonlinearity of the equations, the authors prove the global regularity of the three-dimensional Navier-Stokes equations for a family of initial data.

In this paper, we prove global regularity of the 3D axisymmetric Navier-Stokes Equations with swirl by showing a surprising energy estimates: for any $R > 0$,

$$\frac{d}{dt} \left( \|u_1(t)\|^2_2 + \|\omega_1(t)\|^2_2 + \|u_2(t)\|^2_2 + \|\omega_1(t)\|^2_2 \right) \leq C\left(\|\Gamma_0\|_{L^\infty} \|\nabla u_1(t)\|^2_2 + \frac{\|\Gamma_0\|_{L^\infty} \|u_1(t)\|^2_3}{R^2} \right),$$

where $u_1 = \frac{u_0}{r}$, $\omega_1 = \frac{\omega_0}{r}$ and $\Gamma = r\omega_0$. We show also that the term $\|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty}$ becomes small uniformly in time when $R$ tends to zero.

For this, in section 2, we recall some results known about solution of Navier-Stokes equations and in section 3, we introduce the 3D axisymmetric incompressible Navier-Stokes equations with the results known.

In section 4, we show that the term $\|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty}$ becomes small uniformly in time when $R$ tends to zero.

In section 5, we prove Inequality $(1)$ leading to global regularity.

First, we give some notations.

**Some notations**: For any $m \in \mathbb{N}^+$ function $\varphi$ defined on $\mathbb{R}^m \times [0, +\infty]$, for all $t \geq 0$, we denote by $\varphi(t)$ the function defined on $\mathbb{R}^m$ by $x \mapsto \varphi(x, t)$. For any vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we denote by $|x|$ the norm defined by $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. For any axisymmetric function $f$ defined on $\mathbb{R}^3$, for the sake of simplicity, the value $f(x)$ with $x = (x, y, z) \in \mathbb{R}^3$ is denoted using coordinates cylindrical, $f(r, z)$ with $r = \sqrt{x^2 + y^2}$. For any $d = 1, \Omega \subset \mathbb{R}^d$, we denote by $C^\infty_0(\Omega)$ (resp $C_0(\Omega)$) the space constituted by all infinitely differentiable (resp continuous) functions with compact support in $\Omega$. $BC$ denotes the class of bounded and continuous functions. We denote by $B(0, R)$ the open ball of $\mathbb{R}^3$ of center 0 and radius $R$ and we denote by $B(0, R)^c$ the set $\mathbb{R}^3 \setminus B(0, R)$. For any $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, we denote by $\chi_\Omega$, the function defined on $\mathbb{R}^d$, by $\chi_\Omega(x) = 1$ for all $x \in \Omega$ and 0 elsewhere. For any $R > 0$, we denote by $\chi_{\{r \leq R\}}$ (resp $\chi_{\{r \geq R\}}$) the function defined on $\mathbb{R}^3 \times \mathbb{R}$ such that for all $(r, z) \in \mathbb{R}_+ \times \mathbb{R}$, $\chi_{\{r \leq R\}}(r, z) = 1$ (resp $\chi_{\{r \geq R\}}(r, z) = 1$) for all $r \leq R$ and 0 elsewhere. We denote by $\nabla^2$ the hessian matrix operator,

$$\nabla^2 = \begin{pmatrix} \frac{\partial^2}{\partial x_i \partial x_j} \end{pmatrix}_{1 \leq i, j \leq 3}.$$ 

The symbol $\int$ denotes the integral over $\mathbb{R}^3$ equal using cylindrical coordinates to

$$\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} r d\theta dz dr.$$ 

For any $q > 1$, the norm in $L^q(\mathbb{R}^3)$ will be denoted by $\| \cdot \|_{L^q}$ and also $\| \cdot \|_q$.

**2 Local regularity of solution of Navier-Stokes equation**

In this section, we deal with the main result on local regularity of Navier-Stokes equations in its general form and we introduce, $T^*$, the maximal time existence of strong and smooth solution of Navier-Stokes equations.

Consider the Navier-Stokes equations,

$$\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \nabla p = 0, \\
\nabla \cdot u = 0,
\end{cases}$$

in which $u = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t)) \in \mathbb{R}^3$, $p = p(x, t) \in \mathbb{R}$ and $\nu > 0$ denote respectively the unknown velocity field, the scalar pressure function of the fluid at the point $(x, t) \in \mathbb{R}^3 \times [0, \infty[$ and
\( \nu > 0 \) the viscosity of the fluid,

with initial conditions,

\[ u(x,0) = u_0(x) \text{ for a.e } x \in \mathbb{R}^3. \]  \tag{3}

Without loss of generality, in what follows, we assume that \( \nu = 1 \).

Assuming \( u_0 \in H^m(\mathbb{R}^3) \) for a given \( m \geq 1 \), thanks to the results obtained in [LER], Theorem 3.5 in [KP], Lemma 5.6 [GAL], Theorem 6.1 [GIGA] or the results obtained in [HEY], we get that there exists a maximal time of existence strictly positive \( T^* > 0 \) such that there exists an unique solution \( u \in C([0,T^*]; H^m(\mathbb{R}^3))^3 \cap L^2([0,T^*]; H^{m+2}(\mathbb{R}^3))^3 \) to the Navier-Stokes Equations (2)-(3). Due to the regularity of solution of Navier-Stokes equation, \( u \in C([0,T^*]; H^m(\mathbb{R}^3))^3 \) and thanks to the results obtained in [POP], [LIO2], we get the energy equality, in other words, for all \( t \in [0,T^*[, \)

\[ \|u(t)\|^2_{L^2(\mathbb{R}^3)^3} + \frac{2}{t} \int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^3)^3} = \|u_0\|^2_{L^2(\mathbb{R}^3)}. \]  \tag{4}

Thanks to the results obtained in [LER], Theorem 6.1 [GAL], Lemma 6.2 [GIGA], if this maximal time \( T^* \) is finite, then we have,

\[ \lim_{t \to T^*} \|\nabla u(t)\|_{L^2(\mathbb{R}^3)^3} = +\infty, \]  \tag{5}

and thanks to Theorem 3.1.1 in [CHE], we have also,

\[ \lim_{t \to T^*} \|\omega(t)\|_{L^2(\mathbb{R}^3)^3} = +\infty, \]  \tag{6}

where \( \omega = \nabla \times u \) is the vorticity of \( u \).

Moreover up to the initial time, the solution of Navier-Stokes equation is smooth, \( u \in C^\infty(\mathbb{R}^3 \times [0,T^*]) \) (see Theorem 3 and 4 in [HEY], see also Lemma 5.6 and Theorem 5.2 in [GAL]). Thanks to (4), we get the following estimate (for the proof see Appendix A and (43) in [CONST]), for all \( t \in [0,T^*[), \)

\[ \int_0^t \|u(s)\|_{L^\infty} \, ds \leq C \left( \|u_0\|^2_{L^2} + \sqrt{T} \right), \]  \tag{7}

where \( C > 0 \) is a constant.

We denote by \( \omega_0 = \nabla \times u_0 \) the vorticity of \( u_0 \).

Thanks to Inequality (20) in [CFE], if \( \omega_0 \in L^1 \) then we get for all \( t \in [0,T^*[ , \)

\[ \|\omega(t)\|_{L^1} \leq \Omega_1, \]  \tag{8}

where,

\[ \Omega_1 = \|\omega_0\|_{L^1} + 2\|u_0\|^2_{L^2}. \]

In what follows, \( T^* \) denotes the maximal time existence of the unique strong solution of Navier-Stokes equations (2) in the class of regularity \( C([0,T^*]; H^m(\mathbb{R}^3))^3 \cap L^2([0,T^*]; H^{m+2}(\mathbb{R}^3))^3 \) for a given integer \( m \geq 1 \), this time satisfies (5) and (6) if it is finite.

We are concerned in this paper with the global regularity of axisymmetric weak Leray-Hopf solutions of the Navier-Stokes equations that means to prove \( T^* = \infty \).

Thus, initially, we assume that \( T^* < \infty \) and the solution of Navier-Stokes equations will be considered on the interval \([0,T^*[. \)

3 Axisymmetric flows

By an axisymmetric solution of the Navier-Stokes equations, we mean a solution of the equations of the form

\[ u(x,y,z,t) = u_r(r,z,t)e_r + u_\theta(r,z,t)e_\theta + u_z(r,z,t)e_z. \]
in the cylindrical coordinate system, where we used the basis
\[ c_r = \left( \frac{x}{r}, \frac{y}{r}, 0 \right), \quad c_\theta = \left( \frac{y}{r}, -\frac{x}{r}, 0 \right), \quad c_z = (0, 0, 1) \] and \( r = \sqrt{x^2 + y^2} \)

In the above expression, \( u_\theta \) is called the swirl component of the velocity field \( u \). For the axisymmetric solutions, we can rewrite Equations (2) as follows:

\[
C \quad \text{Denote by} \quad \text{their main result, we introduce some spaces with the same notations as in \cite{LW2}.
}

In \cite{LW2}, there are shown the equivalence between the system \( s \) of equations (2) and (11)-(13), to mention one can derive evolution equations for \( (u_\theta, \omega, \psi) \) which completely determine the evolution of the three-dimensional axisymmetric Navier-Stokes equations (9) once the initial condition is given (see e.g. \cite{MB}, \cite{CL}):

\[
\text{The operator} \quad L \quad \text{and} \quad \Delta \quad \text{is defined by:}
\]

\[
\Delta = \partial_r^2 + \frac{1}{r^2} \partial_r + \partial_z^2,
\]

\[
L = \left( \Delta - \frac{1}{r^2} \right).
\]

One can derive evolution equations for \( (u_\theta, \omega, \psi) \) which completely determine the evolution of the three-dimensional axisymmetric Navier-Stokes equations (9) once the initial condition is given (see e.g. \cite{MB}, \cite{CL}):

\[
\text{where} \quad u_r \quad \text{and} \quad u_z \quad \text{can be expressed in terms of the angular component of the stream function} \quad \psi \quad \text{as follows:}
\]

\[
u_r = -\frac{\partial \psi}{\partial z}, \quad \nu_z = \frac{1}{r} \frac{\partial (r \nu \psi)}{\partial r},
\]

\[
\text{We note that the incompressibility condition implies that}
\]

\[
\partial_r (ru_r) + \partial_z (ru_z) = 0.
\]

In \cite{LW2}, there are shown the equivalence between the systems of equations (2) and (11)-(13), to mention their main result, we introduce some spaces with the same notations as in \cite{LW2}.

Denote by \( C^k \) the axisymmetric divergence free subspace of \( C^k \) vector fields:

\[
C^k_c(\mathbb{R}^3, \mathbb{R}^3) = \{ u \in C^k(\mathbb{R}^3, \mathbb{R}^3) \mid \partial_r u_r = \partial_\theta u_r = \partial_\theta u_\theta = 0, \quad \nabla \cdot u = 0 \}.
\]
Thanks to Lemma 2 (see also Lemma 2’) in [LW2], we have,

\[ C^k_s(\mathbb{R}^3, \mathbb{R}^3) = \{ u e_\theta + \nabla \times (\psi e_\theta) \mid u \in C^k_s(\mathbb{R}^+ \times \mathbb{R}), \psi \in C^{k+1}_s(\mathbb{R}^+ \times \mathbb{R}) \}, \]

where \( C^k_s(\mathbb{R}^+ \times \mathbb{R}) \) is the function space defined by,

\[ C^k_s(\mathbb{R}^+ \times \mathbb{R}) = \{ f(r, z) \in C^k(\mathbb{R}^+ \times \mathbb{R}) \mid \partial^j_r f(0^+, z) = 0, \ 0 \leq 2j \leq k \}. \]

We can now define the Sobolev spaces for axisymmetric solenoidal vector fields :

\[ H^k_s(\mathbb{R}^+ \times \mathbb{R}) = \text{Completion of } C^k_s(\mathbb{R}^+ \times \mathbb{R}) \cap C_0(\mathbb{R}^+ \times \mathbb{R}) \text{ with respect to } \| \cdot \|_{H^1(\mathbb{R}^+ \times \mathbb{R})}, \]

\[ H^k(\mathbb{R}^+ \times \mathbb{R}) = \text{Completion of } C^k_s(\mathbb{R}^+ \times \mathbb{R}) \cap C_0(\mathbb{R}^+ \times \mathbb{R}) \text{ with respect to } \| \cdot \|_{H^k(\mathbb{R}^+ \times \mathbb{R})}, \]

where \( C_0 \) denotes the space of compactly supported functions.

As mentioned in [HLL] and proved in [LW2], any smooth solution of the 3D axisymmetric Navier-Stokes equations must satisfy the following compatibility condition at \( r = 0 \) :

\[ u_0(0, z, t) = \psi_0(0, z, t) = \omega_0(0, z, t) = 0. \] (14)

More precisely, we have the following result, thanks to Lemma 8, Theorem 4 and Corollary 3 in [LW2],

**Theorem 3.1** If \( u_0 \in H^k(\mathbb{R}^3)^3 \) is an axisymmetric solenoidal vector field with \( k \geq 1 \), then there exists \( u_{0, \theta} \in H^1_s(\mathbb{R}^+ \times \mathbb{R}), \psi_{0, \theta} \in H^1_s(\mathbb{R}^+ \times \mathbb{R}) \) with \( \mathcal{L}\psi_{0, \theta} \in H^{k-1}_s(\mathbb{R}^+ \times \mathbb{R}) \) such that \( u_0 = u_{0, \theta} e_\theta + \nabla \times (\psi_{0, \theta} e_\theta) \) and \( u = u_{0, \theta} e_\theta + \nabla \times (\psi_0 e_\theta) \) corresponds to the unique strong solution to the Navier-Stokes equations (2) in the class \( C([0, T^*]: H^k(\mathbb{R}^3)^3) \) where \( (u_0, \psi_0, \omega_0) \) is solution to (11)-(13) for the initial data \( (u_{0, \theta}, \psi_{0, \theta}, -\mathcal{L}\psi_{0, \theta}) \) and satisfies,

\[ \psi_0 \in C([0, T^*]; H^{k+1}_s(\mathbb{R}^+ \times \mathbb{R})), \]

\[ u_0 \in C([0, T^*]; H^k_s(\mathbb{R}^+ \times \mathbb{R})), \]

\[ \omega_0 \in C([0, T^*]; H^{k-1}_s(\mathbb{R}^+ \times \mathbb{R})). \]

### 4 Estimate for axisymmetric solution

Let us introduce \( \Gamma = ru_\theta \), then \( \Gamma \) solves,

\[ \left( \partial_t + b \cdot \nabla - \Delta + \frac{2}{r} \partial_r \right) \Gamma = 0 \] (15)

where, \( b = u_r e_r + u_z e_z, b \cdot \nabla = u_r \partial_r + u_z \partial_z \), and \( \text{div} b = 0. \)

In this section, it is assumed that \( u_0 \in H^m \) is a axisymmetric solenoidal vector field, with \( m \geq 2 \), then Theorem 3.1 holds and we assume also that \( \Gamma_0 = ru_{0, \theta} \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \), and \( \omega_0 \in L^1(\mathbb{R}^3) \).

This section is devoted to the proof of a Lemma 4.8 which states that for all \( \varepsilon > 0 \), there exists \( R > 0 \), such that for all \( t \in [0, T^*] \), \( \| \Gamma(t) \chi_{\{r \leq R\}} \|_{L^\infty} \leq \varepsilon \). The main difficulty lies on the behaviour of \( \Gamma(t) \) at the times near \( t = T^* \), where singularities can appear at the time \( T^* \).

The proof of Lemma 4.8 is obtained in three steps :

- First, we prove that the trace of \( \Gamma(t) \) vanishes on \( r = 0 \) until the time \( t = T^* \), for this we prove in Lemma 4.4 that \( \Gamma \in C([0, T^*]; H^1(\mathbb{R}^3)) \) thanks to Lemmata 4.1, 4.2, 4.3, then we obtain Lemma 4.6 which states that \( \Gamma \in C([0, T^*]; H^1(\mathbb{R}^3)) \) and its trace on \( r = 0 \) vanishes.

- Second, this result combined with the results obtained on partial regularity of suitable weak solutions to the Three-Dimensional Navier-Stokes equations both in [CKN] and [LS] yield to the continuity of \( \Gamma \) until the time \( t = T^* \), more precisely on \( \mathbb{R}^3 \times [0, T^*] \).
Thanks to Proposition 1 in [CL], we have for all \( q \in [2, \infty] \), for all \( t \in [0, T^*] \),

\[
\|\Gamma(t)\|_{L^q(\mathbb{R}^3)} \leq \|\Gamma_0\|_{L^q(\mathbb{R}^3)}.
\]

**Lemma 4.1** There exists a constant \( C > 0 \) such that for all \( (s, t) \in [0, T^*[ \times [0, T^*] \),

\[
\|\Gamma(t) - \Gamma(s)\|_{L^2} \leq \|K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0\|_{L^2} + \|u_0\|_{L^2}(2C\|\Gamma_0\|_{L^\infty} + 1)\sqrt{t - s},
\]

where \( K \) is the heat kernel given by,

\[
K(t, x) = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{3/2}} \quad \text{for all } x \in \mathbb{R}^3 \text{ and } t > 0.
\]

**Proof.** We consider the integral form of equation (15), then we get for all \( t \in [0, T^*] \),

\[
\Gamma(t) = K(t) \ast \Gamma_0 - \int_0^t K(t - \tau) \ast \left( b(\tau) \cdot \nabla \Gamma(\tau) + \frac{2}{r} \partial_r \Gamma(\tau) \right) d\tau.
\]

Then, we deduce that for all \( (s, t) \in [0, T^*[ \times [0, T^*] \),

\[
\Gamma(t) - \Gamma(s) = K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0 - \int_s^t K(t - \tau) \ast \left( b(\tau) \cdot \nabla \Gamma(\tau) + \frac{2}{r} \partial_r \Gamma(\tau) \right) d\tau.
\]

Since \( \text{div} \ b(\tau) = 0 \), then we get \( b(\tau) \cdot \nabla \Gamma(\tau) = \nabla \cdot (b(\tau) \Gamma(\tau)) \) and we deduce,

\[
\Gamma(t) - \Gamma(s) = K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0 - \int_s^t \nabla K(t - \tau) \ast (b(\tau) \Gamma(\tau)) d\tau - \int_s^t K(t - \tau) \ast \frac{2}{r} \partial_r \Gamma(\tau) d\tau.
\]

Taking the \( L^2 \) norm, we deduce,

\[
\|\Gamma(t) - \Gamma(s)\|_{L^2} \leq \|K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0\|_{L^2} + \int_s^t \|\nabla K(t - \tau)\|_{L^1} \|b(\tau)\|_{L^2} \|\Gamma(\tau)\|_{L^2} d\tau + \int_s^t \|K(t - \tau)\|_{L^1} \left\|\frac{2}{r} \partial_r \Gamma(\tau)\right\|_{L^2} d\tau.
\]

Since there exists a constant \( C > 0 \) such that for all \( \sigma \in [0, T^*] \), \( \|\nabla K(\sigma)\|_{L^1} \leq \frac{C}{\sqrt{\sigma}} \) and we have also \( \|K(\sigma)\|_{L^1} = 1 \). Hence, we obtain,

\[
\|\Gamma(t) - \Gamma(s)\|_{L^2} \leq \|K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0\|_{L^2} + \int_s^t \frac{C}{\sqrt{t - \tau}} \|b(\tau)\|_{L^2} \|\Gamma(\tau)\|_{L^2} d\tau + \int_s^t \left\|\frac{2}{r} \partial_r \Gamma(\tau)\right\|_{L^2} d\tau.
\]

We have also \( \|b(\tau)\|_{L^2} \leq \|b(\tau)\|_{L^\infty} \), \( \|b(\tau)\|_{L^\infty} \leq \|u(\tau)\|_{L^2} \) and thanks to (22), from (21), we get,

\[
\|\Gamma(t) - \Gamma(s)\|_{L^2} \leq \|K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0\|_{L^2} + \int_s^t \frac{C}{\sqrt{t - \tau}} \|u(\tau)\|_{L^2} \|\Gamma(\tau)\|_{L^\infty} d\tau + 2 \int_s^t \|\nabla u(\tau)\|_{L^2} d\tau.
\]
Using Cauchy-Schwarz inequality, we have
\[ \left( \int_s^t \| \nabla u(\tau) \|_{L^2} \ d\tau \right)^{\frac{1}{2}} \leq \left( \int_s^t \| \nabla u(\tau) \|_{L^2}^2 \ d\tau \right)^{\frac{1}{2}} \sqrt{t-s} \]
and thanks to Energy equality (4), inequality (16), we get for all \((s,t) \in [0,T^*] \times [0,T^*] \),
\[ \| \Gamma(t) - \Gamma(s) \|_{L^2} \leq \| K(t) \ast \Gamma_0 - K(s) \ast \Gamma_0 \|_{L^2} + \| u_0 \|_{L^2} (2C\| \Gamma_0 \|_{L^{\infty}} + 1) \sqrt{t-s}, \] (24)
which conclude the proof. 

\[ \square \]

**Lemma 4.2** There exists a constant \( C > 0 \) such that for all \( t \in [0,T^*] \),
\[ \| \nabla \Gamma(t) \|_{L^2}^2 + \int_0^t \| \Delta \Gamma(s) \|_{L^2}^2 \ ds \leq \| \nabla \Gamma_0 \|_{L^2}^2 + \| u_0 \|_{L^2}^2 (C^2\| \Gamma_0 \|_{L^{\infty}} + 1). \]

**Proof.** We multiply Equation (15) by \(-\Delta \Gamma(t)\) for any \( t \in [0,T^*] \), integrate it on \( \mathbb{R}^3 \) and use integration by parts, then we obtain,
\[ \frac{1}{2} \frac{d}{dt} \| \nabla \Gamma(t) \|_{L^2}^2 + \| \Delta \Gamma(t) \|_{L^2}^2 = \int \frac{1}{r} \partial_r \Gamma(t) \ \Delta \Gamma(t) + \int b(t) \cdot \nabla \Gamma(t) \ \Delta \Gamma(t). \] (25)
Let us estimate the two terms at the right-hand side of Equality (25). Using the fact that \( \omega_z = \frac{1}{r} \partial_r \Gamma \) and Cauchy-Schwarz inequality, we observe,
\[ \left| \int \frac{1}{r} \partial_r \Gamma(t) \ \Delta \Gamma(t) \right| \leq \| \omega_z(t) \|_{L^2} \| \Delta \Gamma(t) \|_{L^2}. \] (26)
Thanks to Young inequality, we deduce,
\[ \left| \int \frac{1}{r} \partial_r \Gamma(t) \ \Delta \Gamma(t) \right| \leq \| \omega_z(t) \|_{L^2}^2 + \frac{1}{4} \| \Delta \Gamma(t) \|_{L^2}^2. \] (27)
It remains to estimate the second term at the right-hand side of Equality (25). Using an integration by parts, we get,
\[ \int b(t) \cdot \nabla \Gamma(t) \ \Delta \Gamma(t) = - \int (\nabla b(t) \nabla \Gamma(t)) \cdot \nabla \Gamma(t) - \int (b(t) \cdot \nabla) \nabla \Gamma(t) \cdot \nabla \Gamma(t). \]
Since \( \text{div} b(t) = 0 \), then we obtain,
\[ \int (b(t) \cdot \nabla) \nabla \Gamma(t) \cdot \nabla \Gamma(t) = 0, \]
and therefore,
\[ \int b(t) \cdot \nabla \Gamma(t) \ \Delta \Gamma(t) = - \int (\nabla b(t) \nabla \Gamma(t)) \cdot \nabla \Gamma(t). \] (28)
Thanks to Cauchy-Schwarz inequality and Gagliardo-Nirenberg inequality, we deduce that there exists a constant \( C > 0 \) such that,
\[ \left| \int (\nabla b(t) \nabla \Gamma(t)) \cdot \nabla \Gamma(t) \right| \leq \| \nabla b(t) \|_{L^2} \| \nabla \Gamma(t) \|_{L^2} \]
\[ \leq C \| \nabla b(t) \|_{L^2} \| \Gamma(t) \|_{L^{\infty}} \| \nabla^2 \Gamma(t) \|_{L^2}, \]
Since for all \( f \in H^2(\mathbb{R}^3) \), \( \| \nabla^2 f \|_{L^2} = \| \Delta f \|_{L^2} \), moreover we have \( \| \nabla b(t) \|_{L^2} \leq \| \nabla u(t) \|_{L^2} \), then we infer,
\[ \left| \int (\nabla b(t) \nabla \Gamma(t)) \cdot \nabla \Gamma(t) \right| \leq C \| \nabla u(t) \|_{L^2} \| \Gamma(t) \|_{L^{\infty}} \| \Delta \Gamma(t) \|_{L^2} \]
\[ \leq C^2 \| \nabla u(t) \|_{L^2}^2 \| \Gamma(t) \|_{L^{\infty}}^2 + \frac{1}{4} \| \Delta \Gamma(t) \|_{L^2}^2. \] (29)
Thanks to (29), from (28), we get,

\[ \left| \int b(t) \cdot \nabla \Gamma(t) \Delta \Gamma(t) \right| \leq C^2 \| \nabla u(t) \|_{L^2}^2 \| \Gamma(t) \|_{L^\infty}^2 + \frac{1}{4} \| \Delta \Gamma(t) \|_{L^2}^2. \]  

(30)

Using (30), (27) and since \| \omega_z \|_{L^2} \leq \| \nabla u(t) \|_{L^2}, from (25), we obtain,

\[ \frac{1}{2} \frac{d}{dt} \| \nabla \Gamma(t) \|_{L^2}^2 + \frac{1}{2} \| \Delta \Gamma(t) \|_{L^2}^2 \leq \| \nabla u(t) \|_{L^2}^2 (C^2 \| \Gamma(t) \|_{L^\infty}^2 + 1). \]  

(31)

We integrate inequality (31) over \([0, t]\) with \(t \in [0, T^*]\) and we obtain,

\[ \| \nabla \Gamma(t) \|_{L^2}^2 + \int_0^t \| \Delta \Gamma(s) \|_{L^2}^2 \, ds \leq \| \nabla \Gamma_0 \|_{L^2}^2 + \int_0^t \| \nabla u(s) \|_{L^2}^2 (2C^2 \| \Gamma(s) \|_{L^\infty}^2 + 2) \, ds. \]  

(32)

Thanks to (16) and (4), from (32), we deduce that for all \(t \in [0, T^*],\)

\[ \| \nabla \Gamma(t) \|_{L^2}^2 + \int_0^t \| \Delta \Gamma(s) \|_{L^2}^2 \, ds \leq \| \nabla \Gamma_0 \|_{L^2}^2 + \| \varphi \|_{L^2}^2 (2C^2 \| \Gamma_0 \|_{L^\infty}^2 + 1), \]  

(33)

which conclude the proof.

\[ \square \]

**Lemma 4.3** There exists a constant \(C > 0\) such that for all \((s, t) \in [0, T^*] \times [0, T^*],\)

\[ \| \nabla \Gamma(t) - \nabla \Gamma(s) \|_{L^1} \leq \| K(t) \star \nabla \Gamma_0 - K(s) \star \nabla \Gamma_0 \|_{L^1} + C \sqrt{t-s} \left( \| \varphi \|_{L^2} \| \nabla \Gamma_0 \|_{L^2} + 2\Omega_1 \right), \]

where \(K\) is the heat kernel given by,

\[ K(t, x) = e^{-\frac{|x|^2}{4\pi t^3}} \]  

for all \(x \in \mathbb{R}^3\) and \(t > 0.\)

**Proof.** From (17) and since \(\omega_z = \frac{1}{r} \partial_r \Gamma,\) we get for all \(t \in [0, T^*],\)

\[ \nabla \Gamma(t) = K(t) \star \nabla \Gamma_0 - \int_0^t \nabla K(t-\tau) \star (b(\tau) \cdot \nabla \Gamma(\tau) + 2\omega_z(\tau)) \, d\tau. \]  

(34)

Then, from (34), we deduce that for all \((s, t) \in [0, T^*] \times [0, T^*],\)

\[ \nabla \Gamma(t) - \nabla \Gamma(s) = K(t) \star \nabla \Gamma_0 - K(s) \star \nabla \Gamma_0 - \int_s^t \nabla K(t-\tau) \star (b(\tau) \cdot \nabla \Gamma(\tau) + 2\omega_z(\tau)) \, d\tau. \]  

(35)

For all \(\tau \in [0, T^*], \| b(\tau) \cdot \nabla \Gamma(\tau) \|_{L^1} \leq \| u(\tau) \|_{L^2} \| \nabla \Gamma(\tau) \|_{L^2},\) then taking the \(L^1\)-norm in (35), we deduce that there exists a constant \(C > 0\) such that,

\[ \| \nabla \Gamma(t) - \nabla \Gamma(s) \|_{L^1} \leq \| K(t) \star \nabla \Gamma_0 - K(s) \star \nabla \Gamma_0 \|_{L^1} + C \int_s^t \frac{1}{\sqrt{t-\tau}} \left( \| u(\tau) \|_{L^2} \| \nabla \Gamma(\tau) \|_{L^2} + 2\| \omega_z(\tau) \|_{L^1} \right) \, d\tau. \]  

(36)

Then, thanks to Lemma 4.2, Energy equality (4) and (8), from (36), we obtain,

\[ \| \nabla \Gamma(t) - \nabla \Gamma(s) \|_{L^1} \leq \| K(t) \star \nabla \Gamma_0 - K(s) \star \nabla \Gamma_0 \|_{L^1} + C \sqrt{t-s} \left( \| u_0 \|_{L^2} \| \nabla \Gamma_0 \|_{L^2} + 2\Omega_1 \right), \]

which conclude the proof.

\[ \square \]

**Lemma 4.4** For any \(2 \leq p < \infty\) and \(1 \leq q < 2,\) we have \(\Gamma \in C([0, T^*]; L^p(\mathbb{R}^3))\) and \(\nabla \Gamma \in C([0, T^*]; L^q(\mathbb{R}^3)).\)
Proof. Thanks to Lemma 4.1 and thanks to (16), for any \(2 \leq p < \infty\), we deduce that the function \(h : t \mapsto \Gamma(t)\) defined from \([0, T^*]\) to \(L^p(\mathbb{R}^3)\) is uniformly continuous and since \(L^p(\mathbb{R}^3)\) is a Banach space therefore the function \(h\) can be prolong continuously at \(t = T^*\) and we denote this value for \(t = T^*\) by \(\Gamma(T^*)\). Then, we get for all \(2 \leq p < \infty\), \(\Gamma \in C([0, T^*]; L^p(\mathbb{R}^3))\). Thanks to Lemmata 4.2 and 4.3, we get a similar result for \(\nabla \Gamma\) that means that for all \(1 \leq q < 2\), \(\nabla \Gamma \in C([0, T^*]; L^q(\mathbb{R}^3))\), which conclude the proof.

Since \(u \in C([0, T^*]; H^2(\mathbb{R}^3))\), then we get for all \(T \in [0, T^[, \sup_{\tau \in [0, T]} \|\omega_z(\tau)\|_{L^2} < \infty\) and thanks to the Sobolev embedding \(H^2(\mathbb{R}^3) \hookrightarrow BC(\mathbb{R}^3)\), we have also \(\sup_{\tau \in [0, T]} \|u(\tau)\|_{L^\infty} < \infty\), hence, thanks to Lemma 4.2, from (34), we can prove that \(\nabla \Gamma \in C([0, T^*]; L^2(\mathbb{R}^3))\). Since we do not have the uniform continuity of the function \(g : t \mapsto \nabla \Gamma(t)\) defined from \([0, T^*]\) to \(L^2(\mathbb{R}^3)\), we can not deduce the continuity at \(t = T^*\) of \(g\) proceeding as in Lemma 4.4. Then, in the following Lemma, we give the proof of the continuity of \(g\) until the time \(T^*\) included.

Lemma 4.5 We have \(\nabla \Gamma \in C([0, T^*]; L^2(\mathbb{R}^3))\).

Proof. Thanks to Lemma 4.4, we have \(\Gamma \in C([0, T^*]; L^2(\mathbb{R}^3))\) and thanks to Lemma 4.2, we deduce that \(\nabla \Gamma \in L^\infty([0, T^*], L^2(\mathbb{R}^3))\) and for all \(t \in [0, T^*]\),

\[
\|\nabla \Gamma(t)\|_{L^2} \leq \|\nabla \Gamma_0\|_{L^2}.
\]

(37)

Let \(1 < q < 2\). We introduce the one-variable function \(\phi_q\) defined on \(\mathbb{R}_+\) by \(\phi_q(\sigma) = \frac{2}{q}(\sigma + 1)\frac{2}{q} - 1\) for all \(\sigma \geq 0\). Then, for all \(\sigma \geq 0\), we have \(\phi_q'(\sigma) = (\sigma + 1)\frac{2}{q}\). Since for all \(0 < a < b\) and \(0 < \alpha < 1\), \(|b^\alpha - a^\alpha| \leq (b - a)a^{\alpha - 1}\), then we have for all \(\sigma \geq 0\),

\[
0 \leq \phi_q(\sigma) \leq \sigma.
\]

(38)

Therefore thanks to (37), we have for all \(t \in [0, T^*]\), \(0 \leq \int \phi_q(|\nabla \Gamma(t)|^2) \leq \|\nabla \Gamma_0\|_{L^2}^2 < \infty\). From Equation (15), we deduce that \(\nabla \Gamma\) satisfies the Equation (39),

\[
(\partial_t + \nabla b + b \cdot \nabla - \Delta) \nabla \Gamma + 2\nabla \omega_z = 0,
\]

(39)

where, we have used \(\omega_z = \frac{1}{p}\partial_t \Gamma\). Since \(\nabla \cdot b = 0\), we have for all \(t \in [0, T^*]\), \(\int b(t) \cdot \nabla \phi_q(|\nabla \Gamma(t)|^2) = 0\), then we multiply Equation (39) by \(\nabla \Gamma(t) \phi_q'(|\nabla \Gamma(t)|^2)\), integrate it on \(\mathbb{R}^3\) and we obtain for all \(t \in [0, T^*]\),

\[
\frac{1}{2} \frac{d}{dt} \int \phi_q(|\nabla \Gamma(t)|^2) = \int \Delta \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi_q'(|\nabla \Gamma(t)|^2) - 2 \int \nabla b(t) \cdot \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi_q'(|\nabla \Gamma(t)|^2) - 2 \int \nabla \omega_z(t) \cdot \nabla \Gamma(t) \phi_q'(|\nabla \Gamma(t)|^2).
\]

(40)

Let us estimate each term at the right-hand side of Equation (40).

Using integrations by parts, we get,

\[
\int \Delta \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi_q'(|\nabla \Gamma(t)|^2) = - \int \Delta \Gamma(t) \div(\nabla \Gamma(t) \phi_q'(|\nabla \Gamma(t)|^2))
\]

\[
= - \int |\Delta \Gamma(t)|^2 \phi_q'(|\nabla \Gamma(t)|^2) - 2 \int \Delta \Gamma(t) \nabla \Gamma(t) \cdot \div \Gamma(t) \phi_q''(|\nabla \Gamma(t)|^2).
\]

We get also,

\[
\int \nabla \omega_z(t) \cdot \nabla \Gamma(t) \phi_q'(|\nabla \Gamma|^2) = - \int \omega_z(t) \Delta \Gamma(t) \phi_q'(|\nabla \Gamma|^2) - 2 \int \omega_z(t) \Delta \Gamma(t) \cdot \div \Gamma(t) \phi_q''(|\nabla \Gamma|^2).
\]
There exists a constant $C > 0$ such that $|\nabla \text{div} \Gamma| \leq C|\nabla^2 \Gamma|$, furthermore $|\Delta \Gamma| \leq |\nabla^2 \Gamma|$, then we infer,
\[
\left| \int \Delta \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right| \leq \int |\nabla^2 \Gamma(t)|^2 \phi'_q(|\nabla \Gamma|^2) + 2C \int |\nabla^2 \Gamma(t)|^2 |\nabla \Gamma(t)| \phi''_q(|\nabla \Gamma|^2).
\]

Since $q < 2$, then for all $\sigma \geq 0$, we have $0 \leq \phi'_q(\sigma) \leq 1$ and $|\phi''_q(\sigma)| \leq \frac{2 - q}{2} \frac{1}{\sqrt{\sigma + 1}}$. Then, we deduce for all $t \in [0,T^*]$, \[
\left| \int \Delta \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right| \leq (1 + (2 - q)C) \int |\nabla^2 \Gamma(t)|^2. \tag{41}
\]
We deduce also, \[
\left| \int \nabla \omega_s(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right| \leq (1 + (2 - q)C) \int |\omega_s(t)||\nabla^2 \Gamma(t)|. \tag{42}
\]
We get also, \[
\left| \int \nabla b(t) \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right| \leq \int |\nabla b(t)||\nabla \Gamma(t)|^2. \tag{43}
\]
We integrate Equation (40) over $[s, \tau] \ s \in [0,T^*], \ \tau \in [0,T^*]$, then we get,
\[
\frac{1}{2} \left( \int \phi_q(|\nabla \Gamma(s)|^2) - \phi_q(|\nabla \Gamma(s)|^2) \right) = \int_s^\tau \left( \left| \int \Delta \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right| dt - \int_s^\tau \left( \int \nabla b(t) \nabla \Gamma(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right) dt \right. \tag{44}
- 2 \int_s^\tau \left( \int \nabla \omega_s(t) \cdot \nabla \Gamma(t) \phi'_q(|\nabla \Gamma|^2) \right) dt.
\]
Thanks to (41)-(43), we have for all $0 \leq s \leq \tau < T^*$,
\[
\frac{1}{2} \left| \int \phi_q(|\nabla \Gamma(s)|^2) - \phi_q(|\nabla \Gamma(s)|^2) \right| \leq (1 + (2 - q)C) \int_s^\tau \left( \int |\nabla^2 \Gamma(t)|^2 \right) dt + (1 + (2 - q)C) \int_s^\tau \left( \int |\omega_s(t)||\nabla^2 \Gamma(t)| \right) dt + \int_s^\tau \left( \int |\nabla b(t)||\nabla \Gamma(t)|^2 \right) dt. \tag{45}
\]
Thanks to Cauchy-Schwarz inequality and the Gagliardo-Nirenberg inequality $\|\nabla f\|_{L^2} \lesssim \|f\|_{L^\infty} \|\nabla^2 f\|_{L^2}$, we deduce that there exists a constant $C_1 > 0$ such that for all $t \in [0,T^*]$, \[
\int |\nabla b(t)||\nabla \Gamma(t)|^2 \leq C_1 \|\nabla b(t)\|_{L^2} \|\Gamma(t)\|_{L^\infty} \|\nabla^2 \Gamma(t)\|_{L^2}.
\]
Thanks to Cauchy-Schwarz inequality, we get, \[
\int |\omega_s(t)||\nabla^2 \Gamma(t)| \leq \|\omega_s(t)\|_{L^2} \|\nabla^2 \Gamma(t)\|_{L^2}.
\]
Since also for all $f \in H^2(\mathbb{R}^3)$, $\|\nabla^2 f\|_{L^2} = \|\Delta f\|_{L^2}$, then from (45), we deduce for all $0 \leq s \leq \tau < T^*$,
\[
\int \phi_q(|\nabla \Gamma(s)|^2) - \phi_q(|\nabla \Gamma(s)|^2) \right| \leq 2(1 + (2 - q)C) \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt + 2(1 + (2 - q)C) \int_s^\tau |\omega_s(t)||\Delta \Gamma(t)||_{L^2} dt + 2C_1 \int_s^\tau |\nabla b(t)||\Gamma(t)||_{L^\infty} \|\Delta \Gamma(t)\|_{L^2} dt. \tag{46}
\]
Since $|\nabla b| \leq |\nabla u|$, $|\omega_2| \leq |\omega| \leq |\nabla u|$ and thanks to (16), Cauchy-Schwarz inequality, we deduce for all $0 \leq s \leq \tau < T^*$,

$$\left| \int \phi_q(|\nabla \Gamma(\tau)|^2) - \phi_q(|\nabla \Gamma(s)|^2) \right| \leq 2(1 + (2 - q)C) \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt$$

$$+ 2((1 + (2 - q)C) + C_1 \|\Gamma_0\|_{L^\infty}) \left( \int_s^\tau \|\nabla u(t)\|_{L^2}^2 dt \right)^\frac{1}{2} \left( \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt \right)^\frac{1}{2}. \quad (47)$$

Now, we show that inequality (47) is valid for $\tau = T^*$. Let $(\tau_n)_{n \geq 0}$ be a sequence of real of $[0, T^*]$ such that $\tau_n$ tends to $T^*$ as $n \to \infty$. For all $a > 0, b > 0$ and $0 < \alpha < 1$, we have,

$$|b^\alpha - a^\alpha| \leq |b - a|^\alpha. \quad (48)$$

Thanks to (48), we have,

$$\int \phi_q(|\nabla \Gamma(T^*)|^2) - \phi_q(|\nabla \Gamma(\tau_n)|^2) \leq \frac{2}{q} \int |\nabla \Gamma(T^*)|^2 - |\nabla \Gamma(\tau_n)|^2 |^\frac{2}{q}$$

$$= \frac{2}{q} \int |\nabla \Gamma(T^*)|^2 - |\nabla \Gamma(\tau_n)|^2 |^\frac{2}{q} |\nabla \Gamma(T^*)| + |\nabla \Gamma(\tau_n)| |^\frac{2}{q}$$

$$\leq \frac{2}{q} \int |\nabla \Gamma(T^*) - \nabla \Gamma(\tau_n)|^\frac{2}{q} \|\nabla \Gamma(T^*) + \nabla \Gamma(\tau_n)\|_{L^q}^\frac{2}{q},$$

by using Cauchy-Schwarz inequality. Furthermore, thanks to Lemma 4.4, we have $\|\nabla \Gamma(T^*) + \nabla \Gamma(\tau_n)\|_{L^q} \leq \|\nabla \Gamma(T^*)\|_{L^q} + \|\nabla \Gamma(\tau_n)\|_{L^q} \leq 2\|\nabla \Gamma\|_{L^\infty([0, T^*]; L^q)}$, then we get,

$$\int \phi_q(|\nabla \Gamma(T^*)|^2) - \phi_q(|\nabla \Gamma(\tau_n)|^2) \leq \frac{2}{q} (2\|\nabla \Gamma\|_{L^\infty([0, T^*]; L^q)})^\frac{2}{q} \|\nabla \Gamma(T^*) - \nabla \Gamma(\tau_n)\|_{L^q}^\frac{2}{q}.$$}

Thanks again to Lemma 4.4, $\|\nabla \Gamma(T^*) - \nabla \Gamma(\tau_n)\|_{L^q} \to 0$ as $n \to \infty$ and hence,

$$\int \phi_q(|\nabla \Gamma(T^*)|^2) - \phi_q(|\nabla \Gamma(\tau_n)|^2) \to 0 \text{ as } n \to \infty. \quad (49)$$

Setting $\tau = \tau_n$ and taking the limit as $n \to \infty$ in (47), we deduce, thanks to (49), Lemma 4.2 and Energy equality (4),

$$\left| \int \phi_q(|\nabla \Gamma(T^*)|^2) - \phi_q(|\nabla \Gamma(s)|^2) \right| \leq 2(1 + (2 - q)C) \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt$$

$$+ 2((1 + (2 - q)C) + C_1 \|\Gamma_0\|_{L^\infty}) \left( \int_s^\tau \|\nabla u(t)\|_{L^2}^2 dt \right)^\frac{1}{2} \left( \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt \right)^\frac{1}{2}, \quad (50)$$

which show that Inequality (47) is valid also for $\tau = T^*$. Therefore, thanks to (47) and (50), we get that for all $0 \leq s \leq \tau \leq T^*$,

$$\left| \int \phi_q(|\nabla \Gamma(\tau)|^2) - \phi_q(|\nabla \Gamma(s)|^2) \right| \leq 2(1 + (2 - q)C) \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt$$

$$+ 2((1 + (2 - q)C) + C_1 \|\Gamma_0\|_{L^\infty}) \left( \int_s^\tau \|\nabla u(t)\|_{L^2}^2 dt \right)^\frac{1}{2} \left( \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 dt \right)^\frac{1}{2}. \quad (51)$$
Since for all $1 < q < 2$, for all $t \in [0, T^*]$, $0 \leq \int \phi_q(|\nabla \Gamma(t)|^2) \leq \int |\nabla \Gamma_0|^2$. By monotone and dominated convergence theorems, we deduce that for all $t \in [0, T^*]$, $\int \phi_q(|\nabla \Gamma(t)|^2) \to \int |\nabla \Gamma(t)|^2$ as $q \to 2$. Therefore, taking the limit as $q \to 2$ in Inequality (51), we infer that for all $0 \leq s \leq \tau \leq T^*$,

$$\left| \int |\nabla \Gamma(\tau)|^2 - |\nabla \Gamma(s)|^2 \right| \leq 2 \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 \, dt + 2(1 + C_1\|\Gamma_0\|_{L^\infty}) \left( \int_s^\tau \|\nabla u(t)\|_{L^2}^2 \, dt \right)^\frac{1}{2} \left( \int_s^\tau \|\Delta \Gamma(t)\|_{L^2}^2 \, dt \right)^\frac{1}{2}. \quad (52)$$

Let $\tau \in [0, T^*]$ and $(\tau_n)_{n \geq 0}$ a sequence of real of $[0, T^*]$ such that $\tau_n$ tends to $\tau$ as $n \to \infty$. Since $\Gamma \in C([0, T^*]; L^2(R^3))$, then $\Gamma(\tau_n)$ converges strongly to $\Gamma(\tau)$ in $L^2(R^3)$ and therefore $\nabla \Gamma(\tau_n)$ converges weakly to $\nabla \Gamma(\tau)$ in $L^2(R^3)$. Thanks to (52), Lemma 4.2 and Energy equality (4), we have $\int |\nabla \Gamma(\tau_n)|^2 \to \int |\nabla \Gamma(\tau)|^2$ as $n \to \infty$. Since $L^2(R^3)$ is a Hilbert space, we deduce that $\|\nabla \Gamma(\tau_n) - \nabla \Gamma(\tau)\|_{L^2} \to 0$ as $n \to \infty$, which allows us to conclude the proof.

□

**Lemma 4.6** We have $\Gamma \in C([0, T^*]; H^1(R^3))$, moreover for all $t \in [0, T^*]$, the trace of $\Gamma(t)$ on $r = 0$ vanishes.

**Proof.** Thanks to Lemmata 4.4 and 4.5, we have $\Gamma \in C([0, T^*]; H^1(R^3))$. For all $t \in [0, T^*]$, the function $t \mapsto \int_R \int_{R^+} \Gamma(r, z, t) \partial_r \Gamma(r, z, t) \, dr \, dz$ is well-defined, indeed, we have,

$$\left| \int_R \int_{R^+} \Gamma(r, z, t) \partial_r \Gamma(r, z, t) \, dr \, dz \right| = \left| \int_R \int_{R^+} \left( \frac{\Gamma(r, z, t)}{r} \right) \partial_r \Gamma(r, z, t) \, dr \, dz \right|$$

$$= \left| \int_R \int_{R^+} u_0(r, z, t) \partial_r \Gamma(r, z, t) \, dr \, dz \right| = \frac{1}{2\pi} \left| \int u_0(r, z, t) \partial_r \Gamma(r, z, t) \, dr \right| \leq \frac{1}{2\pi} \|u_0(t)\|_{L^2} \|\nabla \Gamma(t)\|_{L^2} \leq \frac{1}{2\pi} \|u_0(t)\|_{L^2} \|\nabla \Gamma(t)\|_{L^2} \quad (53)$$

where we have used Cauchy-Schwarz inequality and Energy equality (4). Due to the regularity of $u$ on $[0, T^*]$, we have for all $t \in [0, T^*]$, for all $z \in \mathbb{R}$,

$$\Gamma(0, z, t) = 0. \quad (54)$$

Furthermore, thanks to (54), we have for all $t \in [0, T^*]$,

$$\int_R |\Gamma(0, z, T^*)|^2 \, dz = \int_R |\Gamma(0, z, T^*) - \Gamma(0, z, t)|^2 \, dz$$

$$= - \int_0^\infty \int_0^\infty \partial_r (\Gamma(r, z, t) - \Gamma(r, z, T^*))^2 \, dr \, dz$$

$$= -2 \int_R \int_{R^+} (\Gamma(r, z, t) - \Gamma(r, z, T^*)) \partial_r (\Gamma(r, z, t) - \Gamma(r, z, T^*)) \, dr \, dz. \quad (55)$$

Using the same arguments as (53), we deduce that for all $t \in [0, T^*]$,

$$\left| \int_R \int_{R^+} (\Gamma(r, z, t) - \Gamma(r, z, T^*)) \partial_r (\Gamma(r, z, t) - \Gamma(r, z, T^*)) \, dr \, dz \right| \leq \frac{1}{\pi} \|u_0\|_{L^2} \|\nabla \Gamma(t) - \nabla \Gamma(T^*)\|_{L^2}. \quad (56)$$
Thanks to (55) and (56), we get for all \( t \in [0, T^*] \),
\[
\int_{\mathbb{R}} |\Gamma(0, z, T^*)|^2 dz \leq \frac{2}{\pi} \|u_0\|_{L^2} \|\nabla \Gamma(t) - \nabla \Gamma(T^*)\|_{L^2}.
\] (57)

Since \( \nabla \Gamma \in C([0, T^*]; L^2(\mathbb{R}^4)) \), from (57), we deduce that,
\[
\int_{\mathbb{R}} |\Gamma(0, z, T^*)|^2 dz = 0,
\]
which allow us to conclude the proof.

□

Thanks to the celebrated results obtained on partial regularity of suitable weak solutions to the three dimensional Navier-Stokes equations both in [CKN] and [LS], the singularity set as defined in [CKN] and [LS] of any suitable weak solution of the three-dimensional Navier-Stokes equations has one-dimensional Hausdorff-measure 0. In the case of axisymmetric three-dimensional Navier-Stokes equations with swirl, if there is any singularity, it must be along the symmetry axis which is the \( z \)-axis. Therefore, we get \( u \in BC(\{ \mathbb{R}^4 \setminus \mathcal{S} \} \times [0, T^*]) \), where \( \mathcal{S} \) is the \( z \)-axis and we recall that we have already \( u \in BC(\mathbb{R}^4 \times [0, T^*]) \).

Then, we infer, thanks also to Lemma 4.6, that \( \Gamma = ru \theta \) cannot develop singularities at time \( t = T^* \) and hence,
\[
\Gamma \in BC(\mathbb{R}^4 \times [0, T^*]).
\] (58)

Before to turn to the proof of Lemma 4.8, we need the Lemma 4.7 which states,

**Lemma 4.7** For all \( \varepsilon > 0 \) and \( 0 < t_* < T^* \), there exists \( R > 0 \) such that,
\[
\sup_{t \in [t_*, T^*]} \|\Gamma(t)\chi_{B(0, R)}\|_{L^\infty} \leq \varepsilon.
\]

**Proof.** For the proof, we borrow the arguments used in section Energy estimates of [CSYT]. Let \( 0 < t_1 < T^* \), \( R > 0 \), \( 0 < \sigma < 1 \) and \( k > 0 \).

Consider \( \zeta_1 \) a radial smooth function of \( \mathbb{R}^3 \) such that \( 0 \leq \zeta_1 \leq 1 \), \( \zeta_1 = 0 \) on \( B(0, \sigma) \) and \( \zeta_1 = 1 \) on \( B(0, 1)^c \).

We consider also \( \zeta_2 \) a smooth real function defined on \( (0, \frac{t_1}{t_1 - t_1}) \) such that \( \zeta_2(0) = 0 \), \( 0 \leq \zeta_2 \leq 1 \) and \( \zeta_2 = 1 \) on \( (1 - \sigma, \frac{t_1}{t_1 - t_1}) \). We introduce the function \( \zeta \) defined on \( \mathbb{R}^3 \times [0, T^*] \) by \( \zeta(x, t) = \zeta_1(\frac{t}{t_1})\zeta_2(\frac{|x|}{x^2}) \).

Define \( u_\pm = \max(\pm u, 0) \) for a scalar function. We multiply Equation (15) by \( 2(\Gamma - k)_{\pm} \zeta^2 \) and since \( \zeta(\sigma t_1) = 0 \), as in [CSYT], we obtain for all \( t \in [\sigma t_1, T^*] \),
\[
\int_{\mathbb{R}^3} \zeta^2(t)(\Gamma(t) - k)^2_{\pm} + 2\int_{\sigma t_1}^t \int_{\mathbb{R}^3} |\nabla((\Gamma(s) - k)_{\pm})|^2
\]
\[
= 2\int_{\sigma t_1}^t ds \int_{\mathbb{R}^3} (\Gamma(s) - k)^2_{\pm} \left( \zeta(s) \frac{\partial \zeta(s)}{\partial t} + |\nabla \zeta(s)|^2 + 2\zeta(s) \frac{\partial \zeta(s)}{r} + b(s)\zeta(s) \cdot \nabla \zeta(s) \right).
\] (59)

Further, we choose \( \zeta_1 \) such that \( \zeta_1 \) decay like \( \left( \frac{|x| - \sigma}{1 - \sigma} \right)^n \) near the boundary \( B(0, \sigma) \) with \( n \geq 3 \) a given integer. Let \( v_{\pm} = (\Gamma - k)_{\pm} \), then thanks to Young inequality used with exponent \( (\frac{4}{3}, 4) \), we get,
\[
\int_{\mathbb{R}^3} v_{\pm}^2 b \cdot \nabla \zeta \leq \int_{\mathbb{R}^3} (R^{-4} v_{\pm}^2 b \cdot \zeta^2) \cdot (R^4 v_{\pm}^4 \zeta^4)
\]
\[
\leq 3 R^{-\frac{4}{3}} \int_{\mathbb{R}^3} v_{\pm}^2 \zeta^2 b + R^4 \int_{\mathbb{R}^3} v_{\pm}^4 |\nabla \zeta|^4 \zeta^2.
\] (60)

Thanks to properties of \( \zeta_1 \) and \( \zeta_2 \), we deduce that there exists a constant \( C > 0 \) depending on \( n \) such that,
\[
\frac{|\nabla \zeta|^4 \zeta^2}{\zeta^2} \leq \frac{C}{R^4(1 - \sigma)^4} \chi_{B(0, \sigma R)}.
\] (61)
Then, we get,
\[
\frac{R^2}{4} \int_{\mathbb{R}^3} v_\pm^2 \frac{|\nabla \zeta|^4}{\zeta^2} \leq \frac{C}{4R^2(1 - \sigma)^4} \int_{B(0,\sigma R)^c} v_\pm^2.
\] (62)

Thanks to Hölder, Sobolev inequalities and energy equality (4), we deduce that there exists a constant \(C_2 > 0\) such that,
\[
\frac{3R^{-\frac{4}{3}}}{4} \int_{\mathbb{R}^3} v_\pm^2 \zeta^2 b_\pm^4 \leq \frac{3R^{-\frac{2}{3}}}{4} \|v_\pm \zeta\|_{L^2}^2 \|b\|_{L^2}^2 \\
\leq C_2 R^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \int_{\mathbb{R}^3} |\nabla v_\pm \zeta|^2 \\
\leq C_2 R^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \int_{\mathbb{R}^3} |\nabla v_\pm \zeta|^2.
\] (63)

From (60), using (62) and (63), we obtain,
\[
\int_{\mathbb{R}^3} v_\pm^2 b_\pm \zeta \cdot \nabla \zeta \leq \frac{C}{4R^2(1 - \sigma)^4} \int_{B(0,\sigma R)^c} v_\pm^2 + C_2 R^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \int_{\mathbb{R}^3} |\nabla v_\pm \zeta|^2.
\] (64)

Thanks to (64), from (59), we deduce that there exists a constant \(C_3 > 0\) such that,
\[
\int_{\mathbb{R}^3} \zeta^2(t)(\Gamma(t) - k)_{\pm}^2 + \left(2 - C_2 R^{-\frac{2}{3}} \|u_0\|_{L^2}^2\right) \int_{\sigma_{t_1}}^t \int_{\mathbb{R}^3} |\nabla ((\Gamma(s) - k)_{\pm})|^2 \\
\leq \frac{C_3}{(1 - \sigma)^2} \left(\frac{1}{R^2} + \frac{1}{t_1}\right) \int_{\sigma_{t_1}}^t ds \int_{B(0,\sigma R)^c} (\Gamma(s) - k)_{\pm}^2.
\]

We require that \(R\) be such that,
\[
R \geq \sqrt{t_1} \\
1 - C_2 R^{-\frac{2}{3}} \|u_0\|_{L^2}^2 \geq 0 \text{ that means } R \geq C_2^\frac{3}{2} \|u_0\|_{L^2}^2.
\] (65)

Thanks to (65), we get,
\[
\int_{\mathbb{R}^3} \zeta^2(t)(\Gamma(t) - k)_{\pm}^2 + \int_{\sigma_{t_1}}^t \int_{\mathbb{R}^3} |\nabla ((\Gamma(s) - k)_{\pm})|^2 \\
\leq \frac{2C_3}{(1 - \sigma)^4 t_1} \int_{\sigma_{t_1}}^t ds \int_{B(0,\sigma R)^c} (\Gamma(s) - k)_{\pm}^2,
\]

which implies that for all \(t \in [t_1, T^*]\),
\[
\int_{B(0,R)^c} (\Gamma(t) - k)_{\pm}^2 + \int_{\sigma_{t_1}}^t \int_{B(0,\sigma R)^c} |\nabla ((\Gamma(s) - k)_{\pm})|^2 \\
\leq \frac{2C_3}{(1 - \sigma)^4 t_1} \int_{\sigma_{t_1}}^t ds \int_{B(0,\sigma R)^c} (\Gamma(s) - k)_{\pm}^2.
\]

Therefore, we deduce that,
\[
\sup_{t_1 \leq t \leq T^*} \int_{B(0,R)^c} (\Gamma(t) - k)_{\pm}^2 + \int_{t_1}^{T^*} \int_{B(0,\sigma R)^c} |\nabla ((\Gamma(s) - k)_{\pm})|^2 \\
\leq \frac{4C_3}{(1 - \sigma)^4 t_1} \int_{\sigma_{t_1}}^{T^*} ds \int_{B(0,\sigma R)^c} (\Gamma(s) - k)_{\pm}^2.
\] (66)

We introduce the parabolic cylinder, \(Q_{R,t_1} = B(0,R)^c \times [t_1, T^*]\). Then we re-write (66) as follows,
\[
\sup_{t_1 \leq t \leq T^*} \int_{B(0,R)^c} (\Gamma(t) - k)_{\pm}^2 + \int_{Q_{R,t_1}} |\nabla ((\Gamma(s) - k)_{\pm})|^2 \\
\leq \frac{4C_3}{(1 - \sigma)^4 t_1} \int_{Q_{R,t_1}} (\Gamma(s) - k)_{\pm}^2.
\] (67)
Let $R_\ast \geq 0$, $0 < t_\ast < T^\ast$ and $k_\ast > 0$ such that inequalities (65) holds for $\left( \frac{R_\ast}{2}, t_\ast \right)$ instead of $(R, t_1)$ and $N \in \mathbb{N}$, we set,

$$
\sigma = \frac{1 - 2^{-N+1}}{1 - 2^{-N+2}} \quad R = R_{N+1} \quad t_1 = t_{N+1} \quad k = k_{N+1}
$$

(68)

where $(R_m)_{m \in \mathbb{N}}$, $(t_m)_{m \in \mathbb{N}}$ and $(k_m)_{m \in \mathbb{N}}$ are sequences of real defined by $R_m = (1 - 2^{-(m+1)})R_\ast$, $t_m = (1 - 2^{-(m+1)})t_\ast$ and $k_m = k_m^\pm = (1 + 2^{-m})k_\ast$ for all $m \in \mathbb{N}$. We notice that for all $m \in \mathbb{N}$, $R_m \geq \frac{R_\ast}{2}$, then (65) holds for all $m \in \mathbb{N}$ for $(R_m, t_\ast)$. We notice also that, $\frac{R_N}{R_{N+1}} = \sigma$, $\frac{t_N}{t_{N+1}} = \sigma$, $1 - \sigma = \frac{2^{-N+1}}{2}$ and $t_{N+1} \geq \frac{t_\ast}{2}$. Then using (68), from (67), we deduce that for any $N \in \mathbb{N}$,

$$
\sup_{t_{N+1} \leq t \leq T^\ast} \int_{B(0,R_{N+1})} (\Gamma(t) - k_{N+1})^2 + \int_{Q_{N+1}} |\nabla((\Gamma(s) - k_{N+1})_\pm)|^2 \leq \frac{128C_\ast 2^{4(N+1)}}{t_\ast} \int_{Q_N} (\Gamma(s) - k_{N+1})^2_\pm.
$$

(69)

where $Q_N = Q_{R_N,t_N}$. Using (69) and the same arguments as the proof of Lemma 2.2 in [CSYT], we deduce that there exists a constant $C_4 > 0$,

$$
\sup_{Q_{R_\ast,t_\ast}} |\Gamma| \leq C_4 \left( t_\ast^{\frac{1}{2}} \int_{Q_{R_{N+1}}} |\Gamma|^2 \right)^{\frac{1}{2}},
$$

which implies,

$$
\sup_{Q_{R_\ast,t_\ast}} |\Gamma| \leq C_4 \left( t_\ast^{\frac{1}{2}} \int_{Q_{R_N}} |\Gamma|^2 \right)^{\frac{1}{2}},
$$

(70)

where $R_\ast$ and $t_\ast$ are real such that $0 < t_\ast < T^\ast$, $R_\ast \geq 2 \max(\sqrt{\tau}, C_2^3 \|u_0\|_{L^2})$ and $Q_{R_\ast,t_\ast} = B(0,R_\ast)^c \times [t_\ast,T^\ast]$. Thanks to (21), we get $\int_{0}^{T^\ast} \int_{\mathbb{R}^3} |\Gamma|^2 < +\infty$, therefore using Lebesgue dominated convergence Theorem, we deduce,

$$
\int_{Q_{R_N}} |\Gamma|^2 \rightarrow 0 \text{ as } R_\ast \rightarrow \infty.
$$

(71)

Then, thanks to (71) and (70), we deduce that for any $\varepsilon > 0$, there exists $R_\ast \geq 2 \max(\sqrt{\tau}, C_2^3 \|u_0\|_{L^2})$ such that,

$$
\sup_{Q_{R_\ast,t_\ast}} |\Gamma| \leq \varepsilon,
$$

which conclude the proof.

□

Now, we can proceed to the proof of Lemma 4.8.

**Lemma 4.8** For all $\varepsilon > 0$, there exists $r_1 > 0$, such that for all $t \in [0,T^\ast]$, $\|\Gamma(t)\chi_{(t \leq r_1)}\|_{L^\infty} \leq \varepsilon$.

**Proof.** Let $\varepsilon > 0$. Since $m \geq 2$, then $u \in C([0,T^\ast[, H^2(\mathbb{R}^3))^3$ and thanks to Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow BC(\mathbb{R}^3)$, we get that $u \in BC([0,T^\ast[ \times \mathbb{R}^3)$. We can assume that for all $t \in [0,T^\ast[, \|u(t)\|_{L^\infty} > 0$ indeed if there exists $t_0 \in [0,T^\ast]$ such that $\|u(t_0)\|_{L^\infty} = 0$, from Navier-Stokes equations, we can show
that \(u(t) = 0\) for all \(t \in [t_0, T^*]\) leading a contradiction with (5). We take \(0 < t_* < T^*\), for all \(t \in [0, t_*]\) with \(r_0 = \|u(t)\|_{L^\infty([0, t_*], \mathbb{R}^3)}\), we get,
\[
\|\Gamma(t)\psi_{\{0 \leq r \leq r_0\}}\|_{L^\infty} \leq \varepsilon
\]
(72)

Thanks to Lemma 4.7, there exists \(L > 0\) such that
\[
\|\Gamma\|_{L^\infty([t_*, T^*) \times [\mathbb{R}^3, B(0, L)])} \leq \frac{\varepsilon}{3}
\]
(73)

Let \((R_n)_{n \geq 0}\) be a non-increasing sequence such that \(R_n \to 0\) as \(n \to \infty\), we introduce \((\zeta_n)_{n \geq 0}\) the sequence of functions defined on \(\mathbb{R}_+\) as follows :
\[
\zeta_n(r) = 1 - \frac{r}{2R_n} \text{ for all } 0 \leq r \leq 2R_n,
\]
\[
\zeta_n(r) = 0 \text{ for all } r > 2R_n.
\]
(74)

We introduce also the sequence of functions \((h_n)_{n \geq 0}\) defined on \([t_*, T^*] \times \mathbb{R}^3\) by,
\[
h_n(x, t) = [\Gamma(x, t)\zeta_n(r), \text{ for all } x = (x', z) \in \mathbb{R}^3 \text{ with } r = |x'|.\]
(75)

Since \(\Gamma \in C(\mathbb{R}^3 \times [t_*, T^*])\) (see (58)), then we get that for all \(n \geq 0\), \(h_n\) is continuous on \([t_*, T^*] \times \mathbb{R}^3\).

For all \(r > 0\), we have \(\zeta_n(r) \to 0\) as \(n \to \infty\) and since for all \(t \in [t_*, T^*]\), \(\Gamma(t)\) vanishes on \(r = 0\), then we deduce that for all \(t \in [t_*, T^*]\), for all \(x \in \mathbb{R}^3\), \(h_n(x, t) \to 0\) as \(n \to \infty\). Since \((R_n)_{n \geq 0}\) is a non-increasing sequence then \((\zeta_n)_{n \geq 0}\) is a non-increasing sequence of real functions and therefore \((h_n)_{n \geq 0}\) is a non-increasing sequence of functions. Then, thanks to Dini’s Theorem, we deduce,
\[
\|h_n\|_{L^\infty([t_*, T^*] \times B(0, L))} \to 0 \text{ as } n \to \infty,
\]
(76)

this implies that there exists \(N \in \mathbb{N}\) such that,
\[
|h_N|_{L^\infty([t_*, T^*] \times B(0, L))} \leq \frac{\varepsilon}{3}.
\]
(77)

Since \(\chi_{\{0 \leq r \leq R_N\}} \leq 2\zeta_n\), then we deduce that,
\[
\|\Gamma \chi_{\{0 \leq r \leq R_N\}}\|_{L^\infty([t_*, T^*] \times B(0, L))} \leq 2\|h_N\|_{L^\infty([t_*, T^*] \times B(0, L))}
\]
(78)

Using (77) and (78), we get,
\[
\|\Gamma \chi_{\{0 \leq r \leq R_N\}}\|_{L^\infty([t_*, T^*] \times B(0, L))} \leq \frac{2\varepsilon}{3}.
\]
(79)

Thanks to (79) and (73), we deduce,
\[
\|\Gamma \chi_{\{0 \leq r \leq R_N\}}\|_{L^\infty([t_*, T^*] \times \mathbb{R}^3)} \leq \varepsilon,
\]
(80)

Taking \(r_1 = \min(r_0, R_N)\) and using (72), (80), we conclude the proof.
\[\square\]

5 Global regularity

In this section, we prove our Theorem 5.1 which states that for any \(u_0 \in H^m(\mathbb{R}^3)\) axisymmetric solenoidal vector field, with \(m \geq 2\), \(\Gamma_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\) and \(\omega_0 \in L^1(\mathbb{R}^3)\), there exists an unique strong solution \(u \in C([0, +\infty[; H^m(\mathbb{R}^3)) \cap L^2([0, T^*]; H^{m+2}(\mathbb{R}^3))^3\) to the Navier-Stokes equations (2) for the initial data \(u_0\). The proof of our Theorem is obtained in three steps :

- First, thanks to the convection term, we eliminate an annoying term in (11), \(\frac{u_0}{r} \omega_0\), by using the change of unknowns from \((u_0, \psi_0, \omega_0)\) to \((u_1, \psi_1, \omega_1)\) (see (81)).
• Second, Thanks to Lemma 5.2 combined with Lemma 4.8, we establish in Lemma 5.3 a dynamic control of \( ||u_1(t)||^3 + ||\omega_1(t)||^2 \) which reveals a dynamic interaction between the angular velocity and the angular vorticity fields.

• Third, using this dynamic control, we obtain the proof of our Theorem 5.1.

In this section, we assume that \( u_0 \in H^m \) is a axisymmetric solenoidal vector field, with \( m \geq 2 \), then Theorem 3.1 holds. We rewrite \( u_0 \) and \( \psi_0 \) as follows:

\[
\begin{align*}
\psi_0(r, z, t) &= ru_1(r, z, t), \\
\omega_0(r, z, t) &= \omega_1(r, z, t), \\
\psi_0(r, z, t) &= \psi_1(r, z, t).
\end{align*}
\] (81)

Since \( m \geq 2 \), then \( u \in C([0, T^*]; H^2(\mathbb{R}^3)) \cap L^2([0, T^*]; H^4(\mathbb{R}^3)) \) (see Section 2) and thanks to Lemmata 3-6 in [NP], we deduce that,

\[
\begin{align*}
\psi &\in C([0, T^*]; H^1(\mathbb{R}^3)) \\
\omega &\in C([0, T^*]; L^2(\mathbb{R}^3)) \cap L^2([0, T^*]; H^4(\mathbb{R}^3)).
\end{align*}
\] (82)

As in [HII], from (11), we derive the following equivalent system for \((u_1, \omega_1, \psi_1)\):

\[
\begin{align*}
\partial_t u_1 + u_r \partial_r u_1 + u_z \partial_z u_1 &= 2u_1 \partial_z \psi_1(t) + \left( \partial_r^2 u_1 + \frac{3}{r} \partial_r u_1 \right), \\
\partial_t \omega_1 + u_r \partial_r \omega_1 + u_z \partial_z \omega_1 &= \partial_z u_1^2 + \left( \partial_r^2 \omega_1 + \frac{3}{r} \partial_r \omega_1 \right), \\
- \left( \partial_r^2 \psi_1 + \frac{3}{r} \partial_r \psi_1 \right) &= \omega_1
\end{align*}
\] (83)

where,

\[
u_r = -r \frac{\partial \psi_1}{\partial z}, \quad u_z = \frac{1}{r} \frac{\partial (r^2 \psi_1)}{\partial r}.
\] (84)

To obtain the proof of the crucial Lemma 5.3, we use Lemma 5.1 and Lemma 5.2. Lemma 5.1 is an immediate consequence of CKN-type inequalities proved in [CKN2].

**Lemma 5.1** There exists a constant \( C > 0 \) such that for all \( v \in C_0^\infty([0, +\infty[ \backslash \{0\}) \) and \( \alpha > \frac{1}{2} \), we have,

\[
\int_0^\infty |v(r)|^2 r^{2(\alpha - 1)} \, dr \leq C \int_0^\infty |v'(r)|^2 r^{2\alpha} \, dr.
\]

Here, we give the proof of Lemma 5.2.

**Lemma 5.2** There exists a constant \( C > 0 \) such that for all \( R > 0 \) and \( t \in [0, T^*] \), we have,

\[
\int |u_1(t)|^4 \leq C ||f(t)||_{L^\infty(R)} ||L^\infty \int |\nabla|u_1|^2|^{\frac{1}{2}} + \frac{C ||\Gamma_0||_{L^\infty}}{R^2} \int |u_1(t)|^3
\]

**Proof.** Let \( R > 0 \) and \( \varepsilon > 0 \).

Consider the cut-off function \( \zeta \) defined on \( \mathbb{R}_+ \) for which \( 0 \leq \zeta \leq 1 \), \( \zeta = 1 \) on \([0, \frac{R}{2}]\), \( \text{supp} \zeta \subset [0, 1] \). Now, we consider for any \( \eta > 0 \), the rescaled cut-off function \( \zeta_\eta \) defined on \( \mathbb{R}_+ \) by \( \zeta_\eta(r) = \zeta \left( \frac{r}{\eta} \right) \). For any \( x \in \mathbb{R}^3 \), we write \( x \) under the form \( x = (x', z) \) where \( x' \in \mathbb{R}^2 \). Then, we have,

\[
\int |u_1(t)|^4 = \int_{\mathbb{R}^3} \big( u_1(x,t)(\zeta_R(|x'|) + |u_1(x,t)||1 - \zeta_R(|x'|)) \big)^4 \, dx
\]

\[
\leq 4 \int_{\mathbb{R}^3} \big( |u_1(x,t)||\zeta_R(|x'|) \big)^4 \, dx + 4 \int_{\mathbb{R}^3} \big( |u_1(x,t)||1 - \zeta_R(|x'|) \big)^4 \, dx
\]

\[
\leq 4 \int_{\mathbb{R}^3} \big( |u_1(x,t)||\zeta_R(|x'|) \big)^4 \, dx + 4 \int_{\mathbb{R}^3} |u_1(x,t)|^4 \chi_{\{|x'| \geq \frac{4}{\eta} \}} \, dx.
\]
With \( r = |x'| \), we recall that \( \Gamma = ru_\theta \) and \( \Gamma_0 = ru_\theta(0) \), we notice that \( r^2u_1(t) \equiv \Gamma(t) \), then \( |u_1(t)|^4\chi_{(r \geq \frac{R}{2})} = \frac{\Gamma(t)|\chi_{(r \geq \frac{R}{2})}|}{r^2}|u_1(t)|^3 \leq \frac{4\|\Gamma(t)\|_{L^\infty}}{R^2}|u_1(t)|^3 \) and thanks to (16), we obtain,

\[
|u_1(t)|^4\chi_{(r \geq \frac{R}{2})} \leq \frac{4\|\Gamma_0\|_{L^\infty}}{R^2}|u_1(t)|^3.
\]

Therefore, we deduce for all \( t \in [0,T^*] \),

\[
\int |u_1(t)|^4 \leq 4 \int_{\mathbb{R}^3} (|u_1(x,t)|\zeta_R(|x'|))^4 \ dx + \frac{16\|\Gamma_0\|_{L^\infty}}{R^2} \int_{\mathbb{R}^3} |u_1(t)|^3. \tag{85}
\]

It remains to prove that there exists a constant \( C > 0 \) such that,

\[
\int_{\mathbb{R}^3} (|u_1(x,t)|\zeta_R(|x'|))^4 \ dx \leq C\|\Gamma(t)\chi_{(r \leq R)}\|_{L^\infty} \int |\nabla |u_1|^2|^2. \]

For this, we introduce the function \( v \) defined on \( \mathbb{R}^3 \times [0,T^*] \) by

\[
v(x,t) = |u_1(x,t)|\zeta_R(|x'|). \tag{86}
\]

We have,

\[
\int_{\mathbb{R}^3} v(x,t)^4 \ dx = \int_{\mathbb{R}^3} v(x,t)\zeta(|x'|) + v(x,t)(1 - \zeta(|x'|)))^4 \ dx \\
\leq 4 \int_{\mathbb{R}^3} v(x,t)\zeta(|x'|) \ dx + 4 \int_{\mathbb{R}^3} v(x,t)(1 - \zeta(|x'|)))^4 \ dx \\
\leq 4 \int_{\mathbb{R}^3} v(x,t)^4\chi_{\{|x'| \leq \varepsilon\}} \ dx + 4 \int_{\mathbb{R}^3} v(x,t)(1 - \zeta(|x'|)))^4 \ dx. \tag{87}
\]

Notice that \( v \) depend only on \( r = |x'| \) and \( z \), then we write \( v(x,t) \) under the form \( v(r,z,t) \) and we have,

\[
\int_{\mathbb{R}^3} v(x,t)(1 - \zeta(|x'|)))^4 \ dx = 2\pi \int_{-\infty}^\infty \int_0^\infty (v(r,z,t)(1 - \zeta(r))^4 \ r \ dr \ dz. \tag{88}
\]

Thanks to Lemma 5.1, we deduce that there exists a real \( C_1 > 0 \) such that for a.e \( z \in \mathbb{R} \),

\[
\int_0^\infty (v(r,z,t)(1 - \zeta(r))^4 \ r \ dr \ = \int_0^\infty h(r,z,t)^2 \ r \ dr \\
\leq C_1 \int_0^\infty (\partial_r h(r,z,t))^2 \ r^3 \ dr, \tag{89}
\]

where \( h \) is the function defined on \( \mathbb{R}^3 \times \mathbb{R} \times [0,T^*] \) by \( h(r,z,t) = (v(r,z,t)(1 - \zeta(r)))^2 \). Since, we have,

\[
(\partial_r h(r,z,t))^2 = 4(v(r,z,t)(1 - \zeta(r))) \partial_r (v(r,z,t)(1 - \zeta(r)))^2 \\
= 4(v(r,z,t)(1 - \zeta(r))^2 \partial_r v(r,z,t) - v(r,z,t)^2(1 - \zeta(r))^2)^2 \\
\leq 8v(r,z,t)^2(\partial_r v(r,z,t))^2 + 8v(r,z,t)^4(\zeta'(r))^2.
\]

Since, for all \( r \geq 0 \), we have \( |\zeta'(r)| \leq \frac{C_2}{\varepsilon} \chi_{(r \leq \varepsilon)} \), where \( C_2 > 0 \) is the real given by \( C_2 = \|\zeta'\|_{L^\infty(\mathbb{R}^3)} \), then we get,

\[
(\partial_r h(r,z,t))^2 \leq 8v(r,z,t)^2(\partial_r v(r,z,t))^2 + 8\frac{C_2^2}{\varepsilon} v(r,z,t)^4 \chi_{(r \leq \varepsilon)}.
\]

Therefore, from (89), we deduce,

\[
\int_0^\infty (v(r,z,t)(1 - \zeta(r)))^4 \ r \ dr \ \leq 8C_1 \int_0^\infty v(r,z,t)^2(\partial_r v(r,z,t))^2 \ r^3 \ dr \\
+ 8\frac{C_2^2}{\varepsilon} C_1 \int_0^\infty v(r,z,t)^4 \chi_{(r \leq \varepsilon)} r^3 \ dr. \tag{90}
\]
We notice that,

\[ \frac{1}{\varepsilon^2} \int_0^\infty v(r, z, t)^4 \chi_{[r \leq \varepsilon]} r^3 \, dr \leq \int_0^\infty v(r, z, t)^4 \chi_{[r \leq \varepsilon]} r \, dr \quad (91) \]

Using (88), (90) and (91), we deduce that there exists a constant \( C_3 > 0 \) such that for all \( t \in [0, T^*] \),

\[
\int_{\mathbb{R}^3} (v(x, t)(1 - \zeta_{\varepsilon}(|x'|)))^4 \, dx \leq 2\pi C_3 \int_0^\infty \int_{\mathbb{R}^3} (v(r, z, t))^2 (\partial_r v(r, z, t))^2 \, r^3 \, dr \, dz \\
+ 2\pi C_3 \int_{-\infty}^0 \int_{\mathbb{R}^3} (v(r, z, t))^4 \chi_{r \leq \varepsilon} \, r \, dr \, dz. \quad (92)
\]

Owing to (86), we get,

\[
v(r, z, t)^2 (\partial_r v(r, z, t))^2 r^3 = r^2 |u_1(r, z, t)| \zeta_R(r) |(v(r, z, t))(\partial_r v(r, z, t))^2 r| \\
= |\Gamma(r, z, t) \zeta_R(r)(v(r, z, t))(\partial_r v(r, z, t))^2 r| \leq |\Gamma(t) \chi_{r \leq R}| L^\infty |v(r, z, t)(\partial_r v(r, z, t))^2 r|,
\]

and therefore, from (92), we deduce,

\[
\int_{\mathbb{R}^3} (v(x, t)(1 - \zeta_{\varepsilon}(|x'|)))^4 \, dx \leq C_3 |\Gamma(t) \chi_{r \leq R}| L^\infty \int_{\mathbb{R}^3} |v(x, t)||\nabla v(x, t)|^2 \, dx \\
+ C_3 \int_{\mathbb{R}^3} v(x, t)^4 \chi_{|x'| \leq \varepsilon} \, dx. \quad (93)
\]

Thanks to (93), from (87), we deduce that there exists a constant \( C_4 > 0 \) such that for all \( t \in [0, T^*] \),

\[
\int_{\mathbb{R}^3} v(x, t)^4 \, dx \leq C_4 |\Gamma(t) \chi_{r \leq R}| L^\infty \int_{\mathbb{R}^3} |v(x, t)||\nabla v(x, t)|^2 \, dx + C_4 \int_{\mathbb{R}^3} v(x, t)^4 \chi_{|x'| \leq \varepsilon} \, dx. \quad (94)
\]

Thanks again to (86), we get,

\[
|\nabla v(x, t)|^2 = |\zeta_R(|x'|)||\nabla u_1(x, t)| + |u_1(x, t)||\nabla \zeta_R(|x'|)|^2 \\
\leq 2 \zeta_R(|x'|)^2 |\nabla u_1(x, t)|^2 + 2 |u_1(x, t)|^2 |\nabla \zeta_R(|x'|)|^2 |u_1(x, t)|^2 \\
\leq 2 |\nabla u_1(x, t)|^2 + 2 |u_1(x, t)|^2 |\zeta_R|^2 L^\infty \\
\leq 2 |\nabla u_1(x, t)|^2 + \frac{2C_2}{R} |u_1(x, t)|^2.
\]

Therefore, replacing \( v \) by its value in Inequality (94), we deduce,

\[
\int_{\mathbb{R}^3} |u_1(x, t)||\zeta_R(|x'|)|^4 \, dx \leq 2C_4 |\Gamma(t) \chi_{r \leq R}| L^\infty \int_{\mathbb{R}^3} |u_1(x, t)||\nabla u_1(x, t)|^2 \, dx \\
+ \frac{4C_2C_4 |\Gamma(t) \chi_{r \leq R}| L^\infty}{R} \int_{\mathbb{R}^3} |u_1(x, t)|^3 \\
+ C_4 \int_{\mathbb{R}^3} |u_1(x, t)|^4 \chi_{|x'| \leq \varepsilon} \, dx.
\]

Thanks to (82) and Sobolev inequalities, we have \( u_1 \in C([0, T^*]; L^4(\mathbb{R}^3)) \), then thanks to Lebesgue Dominated Convergence Theorem, we have \( \int_{\mathbb{R}^3} |u_1(x, t)|^4 \chi_{|x'| \leq \varepsilon} \, dx \to 0 \) as \( \varepsilon \to 0 \) and therefore,

\[
\int_{\mathbb{R}^3} |u_1(x, t)||\zeta_R(|x'|)|^4 \, dx \leq 2C_4 |\Gamma(t) \chi_{r \leq R}| L^\infty \int_{\mathbb{R}^3} |u_1(x, t)||\nabla u_1(x, t)|^2 \, dx \\
+ \frac{4C_2C_4 |\Gamma(t) \chi_{r \leq R}| L^\infty}{R} \int_{\mathbb{R}^3} |u_1(x, t)|^3. \quad (95)
\]
Thanks to (95) and (16), from (85), we deduce that there exists a constant $C_5 > 0$ such that for all $t \in [0, T^*]$, 
\[
\int |u_1(t)|^4 \leq C_5 \|\Gamma(t)\chi_{(r \leq R)}\|_{L^\infty} \int |u_1(t)| \|\nabla u_1(t)\|^2 + \frac{C_5 \|\Gamma_0\|_{L^\infty}}{R^2} \int_{\mathbb{R}^3} |u_1(t)|^3,
\]
which allow us to conclude the proof.

\[\Box\]

**Lemma 5.3** There exists a real $\rho > 0$ depending on $u_0$ and $T^*$ such that for all $t \in [0, T^*]$, 
\[
\frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_3^2 \leq \left( \frac{1}{3} \|u_1(0)\|_3^3 + \frac{1}{2} \|\omega_1(0)\|_3^2 \right) \exp \left( \rho \left( 1 + \|\Gamma_0\|_{L^\infty} \right) \right).
\]

**Proof.** Let us introduce $\delta > 0$ and $R > 0$ two non-negative real.
We multiply the first equation of (83) by $|u_1(t)|$, integrate it over $\mathbb{R}^3$, use the incompressibility condition (13), then we obtain for all $t \in [0, T^*]$, 
\[
\frac{1}{3} \frac{d}{dt} \|u_1(t)\|_3^3 + \frac{8}{9} \int |\nabla u_1(t)|^2 = 2 \int |u_1(t)|^3 \partial_z \psi_1(t). \tag{97}
\]

Note that, in order to treat the convective term, we have integrated by parts and the boundary integrals have vanished at $r = 0$ due to the fact that $u_0(0, z, t) = 0$, while near $r = \infty$ due to the standard density argument.

We observe that for all $t \in [0, T^*]$, $r \in \mathbb{R}_+$ and $z \in \mathbb{R}$, 
\[
r \cdot |u_1(r, z, t)|^3 = \partial_r^2 \left( \int_0^r \int_0^{r_1} r_2 |u_1(r_2, z, t)| \, dr_2 \, dr_1 \right).
\]

Then, we can write the term at the right hand side of Equation (97), as follows, 
\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) = 2 \pi \int_{-\infty}^\infty \int_0^\infty \frac{1}{r} \left( \int_0^r \int_0^{r_1} r_2 |u_1(r_2, z, t)| \, dr_2 \, dr_1 \right)^2 \, dr d z.
\]

By using integration by parts. From (98), thanks to Cauchy-Schwarz inequality, we obtain, 
\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq 4 \pi \left( \int_{-\infty}^\infty \int_0^\infty \frac{1}{r} \left( \int_0^r \int_0^{r_1} r_2 |u_1(r_2, z, t)| \, dr_2 \, dr_1 \right)^2 \, dr d z \right)^\frac{1}{2} \times \left( \int_{-\infty}^\infty \int_0^\infty (\partial_r^2 \partial_z \psi_1(r, z, t))^2 r \, dr d z \right)^\frac{1}{2}.
\]

Since, we have $2 \pi \int_{-\infty}^\infty \int_0^\infty (\partial_r^2 \partial_z \psi(r, z, t))^2 r \, dr d z \leq \int |\nabla^2 \partial_z \psi_1(t)|^2$, then, we get, 
\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq 2 \left( 2 \pi \int_{-\infty}^\infty \int_0^\infty \frac{1}{r} \left( \int_0^r \int_0^{r_1} r_2 |u_1(r_2, z, t)| \, dr_2 \, dr_1 \right)^2 \, dr d z \right)^\frac{1}{2} \times \left( \int |\nabla^2 \partial_z \psi_1(t)|^2 \right)^\frac{1}{2}.
\]
Thanks to Lemma 1 in [HLL] used with $u = \partial_z \psi_1(t)$, $f = \partial_z \omega_1(t)$ and using the same choice of the weight $w$ as in Lemma 2 ([HLL]), we deduce that there exists a constant $C_1 > 0$ such that for all $t \in [0, T^*[$,
\[
\int |\nabla^2 \partial_z \psi_1(t)|^2 \leq C_1 \int |\partial_z \omega_1(t)|^2,
\] (101)

Thanks to (100) and (101), we infer that for all $t \in [0, T^*[$,
\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq 2 \sqrt{C_1} \left( \int |\partial_z \omega_1(t)|^2 \right)^{\frac{1}{2}} \sqrt{T},
\] (102)
where,
\[
I = 2\pi \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{r} \left( \int_{0}^{r} \int_{0}^{r_1} r_2 |u_1(r_2, z, t)|^3 dr_2 dr_1 \right) dr dz.
\] (103)
We observe that,
\[
I \leq 2\pi \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{r} \left( \int_{0}^{r} \int_{0}^{r_1} |u_1(r_2, z, t)|^3 dr_2 dr_1 \right) dr dz.
\] (104)

Thanks to Lemma 5.1 used twice, we deduce that there exists some real $C_2 > 0$ and $C_3 > 0$ such that for a.e $z \in \mathbb{R}$,
\[
\int_{0}^{\infty} r \left( \int_{0}^{r} \int_{0}^{r_1} |u_1(r_2, z, t)|^3 dr_2 dr_1 \right) dr \leq C_2 \int_{0}^{\infty} r^3 \left( \int_{0}^{r} |u_1(r_1, z, t)|^3 dr_1 \right) dr \leq C_3 \int_{0}^{\infty} r^5 |u_1(r, z, t)|^6 dr dz.
\] (105)

Thanks to (102)-(105), we deduce that there exists a real $C_4 > 0$ such that for all $t \in [0, T^*[$,
\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq C_4 \left( \int |\partial_z \omega_1(t)|^2 \right)^{\frac{1}{2}} \left( 2\pi \int_{-\infty}^{\infty} \int_{0}^{\infty} r^5 |u_1(r, z, t)|^6 dr dz \right)^{\frac{1}{2}}.
\] (106)

Recalling $\Gamma = ru_\theta$ and $\Gamma_0 = ru_\theta(0)$, we notice that $r^2 u_1(t) = \Gamma(t)$, then $r^5 |u_1(t)|^6 = |\Gamma(t)|^2 |u_1(t)|^4 r$ and thanks to (16), from (106), we obtain,
\[
2 \int |u_1(t)|^3 \partial_z \psi_1(t) \leq C_4 \|\Gamma_0\|_{L^\infty} \left( \int |\partial_z \omega_1(t)|^2 \right)^{\frac{1}{2}} \left( 2\pi \int_{-\infty}^{\infty} \int_{0}^{\infty} |u_1(r, z, t)|^4 r dr dz \right)^{\frac{1}{2}} \\
= C_4 \|\Gamma_0\|_{L^\infty} \|\partial_z \omega_1(t)\|_{L^2} \|u_1(t)\|_{L^4}^2 \\
\leq \frac{1}{4} \|\partial_z \omega_1(t)\|_{L^2}^2 + C_4^2 \|\Gamma_0\|_{L^\infty}^2 \int |u_1(t)|^4.
\] (107)

Using (107), from (97), we deduce that for all $t \in [0, T^*[$,
\[
\frac{1}{3} \frac{d}{dt} \|u_1(t)\|_{H^1}^3 + \frac{8}{3} \int |\nabla |u_1(t)||^2 \leq \frac{1}{4} \|\partial_z \omega_1(t)\|_{L^2}^2 + C_4^2 \|\Gamma_0\|_{L^\infty}^2 \int |u_1(t)|^4.
\] (108)

We multiply the first equation of (83) by $\omega_1(t)$, integrate it over $\mathbb{R}^3$, use the incompressibility condition (13), then we obtain for all $t \in [0, T^*[$,
\[
\frac{1}{2} \frac{d}{dt} \|\omega_1(t)\|_{H^1}^2 + \int |\nabla \omega_1(t)|^2 = \int \omega_1(t) \partial_z (u_1(t))^2).
\] (109)

Using an integration by parts, Cauchy-Schwarz inequality and Young inequality, we deduce that for all $t \in [0, T^*[$,
\[
\int \omega_1(t) \partial_z (u_1(t))^2 = - \int \partial_z \omega_1(t) u_1(t)^2 \\
\leq \|\partial_z \omega_1(t)\|_{L^2} \|u_1(t)\|_{L^4}^2 \\
\leq \frac{1}{4} \int |\partial_z \omega_1(t)|^2 + \int |u_1(t)|^4.
\] (110)
We sum Inequalities (108) and (111), then, we obtain for all $t \in [0, T^*]$, 
\[
\frac{1}{2} \frac{d}{dt} \|u_1(t)\|_3^2 + \frac{3}{4} \int |\nabla u_1(t)|^2 \leq \int |u_1(t)|^4. \tag{111}
\]

We sum Inequalities (108) and (111), then, we obtain for all $t \in [0, T^*]$, 
\[
\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) + \frac{8}{9} \int |\nabla u_1(t)|^2 + \frac{1}{2} \int |\nabla \omega_1(t)|^2 \leq (1 + C_6^2 \|\Gamma_0\|_{L^\infty}) \int |u_1(t)|^4 \tag{112}
\]

Thanks to Lemma 5.2, from (112), we deduce that there exists a constant $C_5 > 0$ such that for all $R > 0$ and $t \in [0, T^*]$, 
\[
\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) + \left( \frac{8}{9} - C_5(1 + \|\Gamma_0\|_{L^\infty}) \|\Gamma(t)\chi_{\{r \leq R\}}\|_{L^\infty} \right) \int |\nabla u_1(t)|^2 \leq C_5(1 + \|\Gamma_0\|_{L^\infty}) \frac{\|\Gamma_0\|_{L^\infty}}{R^2} \int |u_1(t)|^3. \tag{113}
\]

Thanks to Lemma 4.8, we fix $R > 0$ such that for all $s \in [0, T^*]$, 
\[
C_5(1 + \|\Gamma_0\|_{L^\infty}) \|\Gamma(s)\chi_{\{r \leq R\}}\|_{L^\infty} \leq \frac{7}{18}.
\]

Then, we deduce that for all $t \in [0, T^*]$, 
\[
\frac{d}{dt} \left( \frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right) \leq C_5(1 + \|\Gamma_0\|_{L^\infty}) \frac{\|\Gamma_0\|_{L^\infty}}{R^2} \int |u_1(t)|^3 
\leq 3C_5(1 + \|\Gamma_0\|_{L^\infty}) \frac{\|\Gamma_0\|_{L^\infty}}{R^2} \left( \frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \right). \tag{115}
\]

Thanks to Gronwall inequality, we deduce for all $t \in [0, T^*]$, 
\[
\frac{1}{3} \|u_1(t)\|_3^3 + \frac{1}{2} \|\omega_1(t)\|_2^2 \leq \left( \frac{1}{3} \|u_1(0)\|_3^3 + \frac{1}{2} \|\omega_1(0)\|_2^2 \right) \exp \left( 3C_5(1 + \|\Gamma_0\|_{L^\infty}) \frac{\|\Gamma_0\|_{L^\infty}}{R^2} t \right), \tag{116}
\]

which conclude the proof.

\[\square\]

**Theorem 5.1** For any $u_0 \in H^m(\mathbb{R}^3)$ axisymmetric solenoidal vector field, with $m \geq 2$ with $\Gamma_0 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $\omega_0 \in L^1(\mathbb{R}^3)$, there exists an unique strong solution $u \in C([0, +\infty], H^m(\mathbb{R}^3)) \cap L^2([0, +\infty]; H^{m+2}(\mathbb{R}^3))$ to the Navier-Stokes equations (2) for the initial data $u_0$.

**Proof.** We recall since $u_0 \in H^m(\mathbb{R}^3)$ axisymmetric solenoidal vector field, with $m \geq 2$, then Theorem 3.1 holds and the unique strong solution to the Navier-Stokes equations (2) belongs to $C([0, T^*], H^m(\mathbb{R}^3)) \cap L^2([0, T^*]; H^{m+2}(\mathbb{R}^3))$ (see Section 2).

To get the proof, we show that $T^* = \infty$.

We derive first an estimate of $\omega_0 \in L^\infty(\mathbb{R}^3)$. We multiply the first equation of (10) by $\omega_0$ and integrate it over $\mathbb{R}^3$. Then, we have for all $t \in [0, T^*]$, 
\[
\frac{1}{2} \frac{d}{dt} \|\omega_0(t)\|_{L^2}^2 + \int |\nabla \omega_0(t)|^2 + |\omega_0(t)|^2 \leq \int \frac{u_0(t)}{r} \omega_0(t) + 2 \int \frac{u_0(t)}{r} \omega_0(t) \omega_0(t). \tag{117}
\]
On one hand, we have,
\[
\int \frac{u_r(t)}{r} \omega_r(t) \leq \|u_r(t)\omega_r(t)\|_{L^2} \leq \|u_r(t)\|_{L^2} \|\omega_r(t)\|_{L^\infty} \leq C \|u_r(t)\|_{L^2} \|\nabla \omega_r(t)\|_{L^2} \\
\leq C \|u_r(t)\|_{L^2}^2 \int \frac{\omega_r(t)}{r} \leq \|\nabla \omega_r(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \omega_r(t)\|_{L^2}^2.
\]

(118)

where, we have used the Sobolev embedding \( \dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \) with \( C > 0 \) a constant and Young inequality. On the other hand, we have,
\[
\int \frac{u_\theta(t)}{r} \omega_\theta(t) \omega_r(t) \leq \|\omega_r(t)\|_{L^2}^2 \leq \|\omega_r(t)\|_{L^2} \leq \|\omega_r(t)\|_{L^2} \leq C \|\nabla \omega_r(t)\|_{L^2} \|\omega_r(t)\|_{L^2} \leq C^2 \|\omega_r(t)\|_{L^2}^2 \int \frac{\omega_r(t)}{r} \leq \|\nabla \omega_r(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \omega_r(t)\|_{L^2}^2.
\]

(119)

Then, using (118) and (119), from (117), we deduce for all \( t \in [0,T^*]\),
\[
\frac{1}{2} \frac{d}{dt} \|\omega_r(t)\|_{L^2}^2 + \frac{1}{2} \int |\nabla \omega_r(t)|^2 + \int \left| \frac{\omega_r(t)}{r} \right|^2 \leq C^2 \|u_r(t)\|_{L^2}^2 \left( \|\omega_r(t)\|_{L^2} \right)^2 + C^2 \|\omega_r(t)\|_{L^2}^2 \left( \frac{u_r(t)}{r} \right)^2_{L^2},
\]

(120)

which implies that,
\[
\frac{1}{2} \frac{d}{dt} \|\omega_r(t)\|_{L^2}^2 \leq C^2 \|u_r(t)\|_{L^2}^2 \left( \|\omega_r(t)\|_{L^2} \right)^2 + C^2 \|\omega_r(t)\|_{L^2}^2 \left( \frac{u_r(t)}{r} \right)^2_{L^2}.
\]

(121)

After an integration over \([0,t]\) of Inequality (121), we deduce that for all \( t \in [0,T^*]\),
\[
\|\omega_r(t)\|_{L^2}^2 \leq \|\omega_{0,r}\|_{L^2}^2 + C \int_0^t \|u_r(s)\|_{L^2}^2 \left( \|\omega_r(s)\|_{L^2} \right)^2 ds + C \int_0^t \|\omega_r(s)\|_{L^2}^2 \left( \frac{u_r(s)}{r} \right)^2_{L^2} ds,
\]

(122)

where \( \omega_{0,r} \) is the angular component of the initial vorticity \( \omega_0 = \nabla \times u_0 \). Thanks to Lemma 5.3, there exists a real \( Q_0 > 0 \) depending only on \( u_0 \) and \( T^* \) such that for all \( s \in [0,T^*] \), we have,
\[
\|\omega_r(s)\|_{L^2}^2 + \left( \frac{u_r(s)}{r} \right)^2_{L^2} \leq Q_0.
\]

Then, from (122), we infer that for all \( t \in [0,T^*]\),
\[
\|\omega_r(t)\|_{L^2}^2 \leq \|\omega_{0,r}\|_{L^2}^2 + C \int_0^t \|u_r(s)\|_{L^2}^2 \left( \|\omega_r(s)\|_{L^2} \right)^2 ds + C \int_0^t \|\omega_r(s)\|_{L^2}^2 \left( \frac{u_r(s)}{r} \right)^2_{L^2} ds \leq \|\omega_{0,r}\|_{L^2}^2 + C \int_0^t \|u(s)\|_{L^2}^2 \left( \|\omega(s)\|_{L^2} \right)^2 ds.
\]

Thanks to Interpolation inequality, Cauchy-Schwarz inequality and (4), we deduce,
\[
\int_0^t \|u(s)\|_{L^2}^2 ds \leq \int_0^t \|u(s)\|_{L^2} \|\nabla u(s)\|_{L^2} ds \leq \left( \int_0^t \|u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \left( \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \right)^{\frac{1}{2}} \leq \|u_0\|_{L^2} \sqrt{\frac{t}{2}}.
\]
Moreover,
\[
\int_0^t \|\omega(s)\|_{L^2}^2 \, ds \leq \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \leq \frac{1}{2} |u_0|_{L^2}^2.
\]

Therefore, we obtain for all \( t \in [0, T^*] \),
\[
\|\omega(t)\|_{L^2}^2 \leq \|\omega_0\|_{L^2}^2 + C^2 Q_0 \left( \int_0^t \|u_r(s)\|_{L^2}^2 \, ds + \int_0^t \|\omega_r(s)\|_{L^2}^2 \, ds \right)
\leq \|\omega_0\|_{L^2}^2 + C^2 Q_0 \left( \frac{1}{2} + \sqrt{\frac{t}{2}} \right).
\]

Then, with \( \Omega_0 = \|\omega_0\|_{L^2}^2 + C^2 Q_0 \|u_0\|_{L^2}^2 \left( \frac{1}{2} + \sqrt{\frac{T^*}{2}} \right) \), we deduce that for all \( t \in [0, T^*] \),
\[
\|\omega(t)\|_{L^2}^2 \leq \Omega_0.
\] (123)

Now, we multiply the second equation of (10) by \( \omega_r \), the third equation of (10) by \( \omega_z \), integrate them over \( \mathbb{R}^3 \) and sum the equations obtained, then we get for all \( t \in [0, T^*] \),
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) + \int \left( |\nabla \omega_r(t)|^2 + |\nabla \omega_z(t)|^2 + \left\| \frac{\omega_r(t)}{r} \right\|^2 \right) = \int \left( \partial_r u_r(t) \omega_r^2(t) + (\partial_z u_r(t) + \partial_r u_z(t)) \omega_r(t) \omega_z(t) + \partial_z u_z(t) \omega_z^2(t) \right). \] (124)

Thanks to Lemma 2 in [CL] and Theorem 3.1.1 in [CHE], we deduce that there exists a constant \( C_1 > 0 \) such that for all \( t \in [0, T^*] \),
\[
\|\nabla u_r(t)\|_{L^2} \leq C_1 \|\omega_r(t)\|_{L^2},
\|\nabla u_z(t)\|_{L^2} \leq C_1 \|\omega_0(t)\|_{L^2}. \] (125)

Furthermore, thanks to Cauchy-Schwarz inequality and Young inequality, we have for all \( t \in [0, T^*] \),
\[
\|\omega_r(t) \omega_z(t)\|_{L^2} \leq \|\omega_r(t)\|_{L^4} \|\omega_z(t)\|_{L^4} \leq \frac{1}{4} \|\omega_r(t)\|_{L^8}^2 + \frac{1}{2} \|\omega_z(t)\|_{L^8}^2. \] (126)

From (124), using Cauchy-Schwarz inequality, (125) and (126), we deduce that there exists a constant \( C_2 > 0 \) such that for all \( t \in [0, T^*] \),
\[
\frac{1}{2} \frac{d}{dt}(\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) + \int \left( |\nabla \omega_r(t)|^2 + |\nabla \omega_z(t)|^2 + \left\| \frac{\omega_r(t)}{r} \right\|^2 \right) \leq C_2 \|\omega_0(t)\|_{L^2} (\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2). \] (127)

Thanks to Interpolation inequality, Sobolev embedding \( \dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \), we deduce that there exists a constant \( C_3 > 0 \) such for all \( t \in [0, T^*] \),
\[
\|\omega_r(t)\|_{L^6} \leq \|\omega_r(t)\|_{L^2}^{\frac{1}{6}} \|\omega_r(t)\|_{L^8}^{\frac{5}{6}} \leq C_3 \|\omega_r(t)\|_{L^2}^{\frac{1}{2}} \|\nabla \omega_r(t)\|_{L^2}^{\frac{5}{2}},
\]
and also,
\[
\|\omega_z(t)\|_{L^6} \leq C_3 \|\omega_z(t)\|_{L^2}^{\frac{1}{6}} \|\nabla \omega_z(t)\|_{L^2}^{\frac{5}{6}}.
\]

24
Then, from (127), we obtain for all \( t \in [0,T^*] \),
\[
\frac{1}{2} \frac{d}{dt} (\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) + \frac{1}{2} \int \left( |\nabla \omega_r(t)|^2 + |\nabla \omega_z(t)|^2 + \left| \frac{\omega_r(t)}{r} \right|^2 \right) \leq C_2 C_3 \|\omega_0(t)\|_{L^2} \|\omega_r(t)\|_{L^2} \|\nabla \omega_r(t)\|_{L^2}^\frac{3}{2} + \frac{1}{4} \|\nabla \omega_r(t)\|_{L^2}^2.
\] (128)

Thanks to Young inequality, there exists a constant \( C_4 > 0 \) such that for all \( t \in [0,T^*] \),
\[
C_2 C_3 \|\omega_0(t)\|_{L^2} \|\omega_r(t)\|_{L^2} \|\nabla \omega_r(t)\|_{L^2}^\frac{3}{2} \leq C_4 \|\omega_0(t)\|_{L^2}^2 \|\omega_r(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \omega_r(t)\|_{L^2}^2,
\]
and also,
\[
C_2 C_3 \|\omega_0(t)\|_{L^2} \|\omega_z(t)\|_{L^2} \|\nabla \omega_z(t)\|_{L^2}^\frac{3}{2} \leq C_4 \|\omega_0(t)\|_{L^2}^2 \|\omega_z(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla \omega_z(t)\|_{L^2}^2.
\]

Therefore, from (128), we deduce that for all \( t \in [0,T^*] \),
\[
\frac{1}{2} \frac{d}{dt} (\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) + \frac{1}{2} \int \left( |\nabla \omega_r(t)|^2 + |\nabla \omega_z(t)|^2 + \left| \frac{\omega_r(t)}{r} \right|^2 \right) \leq C_4 \|\omega_0(t)\|_{L^2}^4 (\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2),
\] (129)
which implies,
\[
\frac{1}{2} \frac{d}{dt} (\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) \leq C_4 \|\omega_0(t)\|_{L^2}^4 (\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2).
\] (130)

Then, using Gronwall inequality, we deduce that for all \( t \in [0,T^*] \),
\[
(\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) \leq (\|\omega_r(0)\|_{L^2}^2 + \|\omega_z(0)\|_{L^2}^2) \exp \left( 2C_4 \int_0^t \|\omega_0(s)\|_{L^2}^4 \, ds \right).
\] (131)

Then, thanks to (123) and (4), from (131), we deduce that for all \( t \in [0,T^*] \),
\[
(\|\omega_r(t)\|_{L^2}^2 + \|\omega_z(t)\|_{L^2}^2) \leq (\|\omega_r(0)\|_{L^2}^2 + \|\omega_z(0)\|_{L^2}^2) \exp \left( C_4 \Omega_0 \|\omega_0\|_{L^2}^2 \right).
\] (132)

Then, thanks to (123) and (132), we deduce that \( \lim_{t \to T^*} \|\omega(t)\|_{L^2} < +\infty \), which lead to a contradiction with (6) and therefore \( T^* = +\infty \), which conclude the proof.

\( \square \)

References


