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An analytical solution of Shallow Water system coupled to Exner equation

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Abstract

In this paper, an exact smooth solution for the equations modeling the bedload transport of sediment in Shallow Water is presented. This solution is valid for a large family of sedimentation laws which are widely used in erosion modeling such as the Grass model or those of Meyer-Peter & Müller. One of the main interest of this solution is the derivation of numerical benchmarks to validate the approximation methods. To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

1. Introduction

Soil erosion is a consequence of the movements of sediments due to mechanical actions of flows. In the context of bedload transport, a mass conservation law, also called Exner equation [1], is used to update

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the bed elevation. This equation is coupled with the shallow water equations describing the overland flows (see [2] and references therein) as follows:

\[
\begin{aligned}
\partial_t h + \partial_x (hu) &= 0, \\
\partial_t (hu) + \partial_x (hu^2 + gh^2/2) + gh \partial_x z_b &= 0, \\
\partial_t z_b + \partial_x q_b &= 0,
\end{aligned}
\]

where \( h \) is the water depth, \( u \) the flow velocity, \( z_b \) the thickness of sediment layer which can be modified by the fluid and \( g \) the acceleration due to gravity. The variable \( hu \) is also called water discharge and noted by \( q \). Finally, \( q_b \) is the volumetric bedload sediment transport rate. Its expressions are usually proposed for granular non-cohesive sediments and quantified empirically [3,4,5].

Many numerical schemes have been developed to solve system (1-3) (see [5] and references therein). The validation of such schemes by an analytical solution is a simple way to ensure their working. Nevertheless, analytical solutions are not proposed in the literature. Up to our knowledge, asymptotic solutions, derived by Hudson in [6], are in general adopted to perform some comparisons with approximated solutions. The solutions are derived for Grass model [3], i.e \( q_b = A_g u_e^2 \), when the interaction constant \( A_g \) is smaller than \( 10^{-2} \). In this paper, we propose a non obvious analytical solution in the steady state condition of flow.

2. Solution of the equations

In order to obtain an analytic solution, we consider \( q_b \) as a function of the dimensionless bottom shear stress \( \tau_b^* \) (see [5]). By using the friction law of Darcy-Weisbach, \( \tau_b^* \) is given by

\[
\tau_b^* = \frac{fu^2}{8(s-1)gd_s},
\]

where \( f \) is the friction coefficient, \( s = \rho_s/\rho \) the relative density of sediment in water and \( d_s \) the diameter of sediment. The formula of \( q_b \) is usually expressed under the form

\[
q_b = \kappa (\tau_b^* - \tau_{cr}^*)^p \sqrt{(s-1)gd_s^3},
\]

where \( \tau_{cr}^* \) is the threshold for erosion, \( \kappa \) an empirical coefficient and \( p \) an exponent which is often fixed to 3/2 in many applications. The expression (4) can be written in the simple form

\[
q_b = Au_e^{2p},
\]

where the effective velocity \( u_e \) and the interaction coefficient \( A \) are defined by

\[
\begin{aligned}
\left\{ \begin{array}{l}
u_e^2 = u^2 - u_{cr}^2, \\
u_{cr}^2 = \tau_{cr}^* \left[ \frac{f}{8(s-1)gd} \right]^{-1}, \\
A = \kappa \left[ \frac{f}{8(s-1)gd} \right]^p \sqrt{(s-1)gd_s^3}.
\end{array} \right.
\]

Remark. The Grass model [3] is one of the simplest case by using \( p = 3/2, \tau_{cr}^* = 0 \) and an empirical coefficient \( A_g \) instead of \( A \). The Meyer-Peter & Müller model [4] is one of the most applied by using \( p = 3/2, \kappa = 8, \tau_{cr}^* = 0.047 \). The following result is valid for all models rewriting in form (5-6).

Proposition 2.1 Assume that \( q_b \) is defined by (4). For a given uniform discharge \( q \) such that \( \tau_b^* > \tau_{cr}^* \), system (1-3) has the following analytical unsteady solution
\[ \begin{align*}
  u^2_e &= \left( \frac{\alpha x + \beta}{A} \right)^{1/p}, \\
  u &= \sqrt{u^2_e + u^2_{cr}}, \quad h = q/u, \\
  z^0_b &= -\frac{u^3 + 2qg}{2gu} + C, \\
  z_b &= -\alpha t + z^0_b.
\end{align*} \]

where \( \alpha, \beta, C \) are constants and \( A, u_{cr} \) are defined by (6).

**Proof.** We are here concerned by the smooth solution. In view of the assumption \( hu = q = \text{cst} \), equations (1-3) reduce to

\[ \begin{align*}
  \partial_t h &= 0, \\
  \partial_x q^2/h + gh\partial_x H &= 0, \\
  \partial_t H + \partial_x q_b &= 0,
\end{align*} \]

where \( H = h + z \) is the free surface elevation. Differentiating equation (8) with respect to \( t \) and then equation (9) with respect to \( x \), we obtain

\[ \begin{align*}
  \partial_{tt} H &= 0, \\
  \partial_{tt}^2 H &= 0.
\end{align*} \]

Note that we can write \( q_b = q_b(h, q) \) to have \( \partial_t q_b = \partial_h q_b \partial_t h + \partial_q q_b \partial_t q = 0 \), so \( q_b \) is not time-depending. Thank to (10), the expression of \( q_b \) is obtained under the form

\[ q_b = \alpha x + \beta, \]

where \( \alpha \) and \( \beta \) are constant. From (3), we obtain \( \partial_t z_b = -\partial_x q_b = -\alpha \) to write

\[ z_b = -\alpha t + z^0_b(x). \]

Moreover, from (5) we deduce the effective velocity as follows:

\[ u^2_e = \left( \frac{\alpha x + \beta}{A} \right)^{1/p}. \]

Plugging (12) into the momentum equation (8) and using a direct calculation, we have

\[ \partial_x z^0_b = \left[ \frac{q}{u^2} - \frac{u}{g} \right] \partial_x u \Rightarrow z^0_b = -\frac{u^3 + 2qg}{2gu} + C \]

which concludes the proof.

**Remark.** As \( h \) and \( u \) are stationary, the initial condition of (7) is \( (h, u, z^0_b) \). Moreover, the solution \( (h, u) \) applied to the Grass model is also an analytical solution of the Shallow Water Equations with the variable topography \( z^0_b \). Concerning the shallow-water model, other solutions can be found in [7].

### 3. Numerical experiments

In this section, we consider the analytical solution (7) applied to the Grass model with \( q = 1, A_g = \alpha = \beta = 0.005 \) and \( C = 1 \). A relaxation solver is applied to approximate the solution of the model. We will not give here the details of the relaxation solver (for details see [8]), but just the relaxation model for the equations (1-3). Thus, we solve the following relaxation system:
Figure 1. Comparison between the exact solution and the relaxation method for: the water height and the topography (left) and the velocity (right).

\[
\begin{align*}
\partial_t h + \partial_x (hu) &= 0, \\
\partial_t (hu) + \partial_x (hu^2 + \pi) + gh \partial_x z_b &= 0, \\
\partial_t \pi + u \partial_x \pi + \frac{a^2}{h} \partial_x u &= 0, \\
\partial_t z_b + \partial_x q_r &= 0, \\
\partial_t q_r + \left( \frac{b^2}{h^2} - u^2 \right) \partial_x z_b + 2u \partial_x q_r &= 0,
\end{align*}
\]

that is completed with \( \pi = gh^2/2 \) and \( q_r = q_b \) at the equilibrium. Figure 1 presents the numerical result with \( J = 500 \) space cells, a CFL fix condition of 1 and \( T = 7s \). We only notice little difference on the velocity, near the inflow boundary.

References