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An analytical solution of Shallow Water system coupled to Exner equation

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Abstract

In this paper, an exact smooth solution for the equations modeling the bedload transport of sediment in Shallow Water is presented. This solution is valid for a large family of sedimentation laws which are widely used in erosion modeling such as the Grass model or those of Meyer-Peter & Müller. One of the main interest of this solution is the derivation of numerical benchmarks to validate the approximation methods. To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé


1. Introduction

Soil erosion is a consequence of the movements of sediments due to mechanical actions of flows. In the context of bedload transport, a mass conservation law, also called Exner equation [1], is used to update

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the bed elevation. This equation is coupled with the shallow water equations describing the overland flows (see [2] and references therein) as follows:

\[ \partial_t h + \partial_x (hu) = 0, \]  
\[ \partial_t (hu) + \partial_x (hu^2 + gh^2/2) + gh\partial_x z_b = 0, \]  
\[ \partial_t z_b + \partial_x q_b = 0, \]

where \( h \) is the water depth, \( u \) the flow velocity, \( z_b \) the thickness of sediment layer which can be modified by the fluid and \( q \) the acceleration due to gravity. The variable \( hu \) is also called water discharge and noted by \( q \). Finally, \( q_b \) is the volumetric bedload sediment transport rate. Its expressions are usually proposed for granular non-cohesive sediments and quantified empirically [3,4,5].

Many numerical schemes have been developed to solve system (1-3) (see [5] and references therein). The validation of such schemes by an analytical solution is a simple way to ensure their working. Nevertheless, analytical solutions are not proposed in the literature. Up to our knowledge, asymptotic solutions, derived by Hudson in [6], are in general adopted to perform some comparisons with approximated solutions. The solutions are derived for Grass model [3], i.e. \( q_b = A_g u^3 \), when the interaction constant \( A_g \) is smaller than \( 10^{-2} \). In this paper, we propose a non obvious analytical solution in the steady state condition of flow.

2. Solution of the equations

In order to obtain an analytic solution, we consider \( q_b \) as a function of the dimensionless bottom shear stress \( \tau_b^* \) (see [5]). By using the friction law of Darcy-Weisbach, \( \tau_b^* \) is given by

\[ \tau_b^* = \frac{fu^2}{8(s-1)gd_s}, \]

where \( f \) is the friction coefficient, \( s = \rho_s/\rho \) the relative density of sediment in water and \( d_s \) the diameter of sediment. The formula of \( q_b \) is usually expressed under the form

\[ q_b = \kappa(\tau_b^* - \tau_{cr}^*)^p \sqrt{(s-1)gd_s^3}, \]  

where \( \tau_{cr}^* \) is the threshold for erosion, \( \kappa \) an empirical coefficient and \( p \) an exponent which is often fixed to 3/2 in many applications. The expression (4) can be written in the simple form

\[ q_b = Au_{e}^{2p}, \]  

where the effective velocity \( u_e \) and the interaction coefficient \( A \) are defined by

\[ \begin{align*}
    u_e^2 &= u^2 - u_{cr}^2, \\
    u_{cr}^2 &= \tau_{cr}^* \left[ \frac{f}{8(s-1)gd} \right]^{-1}, \\
    A &= \kappa \left[ \frac{f}{8(s-1)gd} \right]^p \sqrt{(s-1)gd^3}. 
\end{align*} \]

Remark. The Grass model [3] is one of the simplest case by using \( p = 3/2, \tau_{cr}^* = 0 \) and an empirical coefficient \( A_g \) instead of \( A \). The Meyer–Peter & Müller model [4] is one of the most applied by using \( p = 3/2, \kappa = 8, \tau_{cr}^* = 0.047 \). The following result is valid for all models rewriting in form (5-6).

Proposition 2.1 Assume that \( q_b \) is defined by (4). For a given uniform discharge \( q \) such that \( \tau_b^* > \tau_{cr}^* \), system (1-3) has the following analytical unsteady solution
where $\alpha$, $\beta$, $C$ are constants and $A$, $u_{cr}$ are defined by (6).

**Proof.** We are here concerned by the smooth solution. In view of the assumption $hu = q = \text{cst}$, equations (1-3) reduce to

\[
\begin{align*}
\frac{\partial h}{\partial t} &= 0, \\
\frac{\partial}{\partial x} \left( \frac{q^2}{h} \right) + gh \frac{\partial}{\partial x} H &= 0, \\
\frac{\partial}{\partial t} H + \frac{\partial}{\partial x} q_b &= 0,
\end{align*}
\]

(8) (9)

where $H = h + z$ is the free surface elevation. Differentiating equation (8) with respect to $t$ and then equation (9) with respect to $x$, we obtain

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} H &= 0, \\
\frac{\partial^2}{\partial x^2} q_b &= 0.
\end{align*}
\]

(10)

Note that we can write $q_b = q_b(h, q)$ to have $\partial_t q_b = \partial_h q_b \partial_t h + \partial_q q_b \partial_t q = 0$, so $q_b$ is not time-depending. Thank to (10), the expression of $q_b$ is obtained under the form

\[q_b = \alpha x + \beta,
\]

(11)

where $\alpha$ and $\beta$ are constant. From (3), we obtain $\partial_t z_b = -\partial_x q_b = -\alpha$ to write

\[z_b = -\alpha t + z_b^0(x).
\]

(12)

Moreover, from (5) we deduce the effective velocity as follows:

\[u_e^2 = \left[ \frac{\alpha x + \beta}{A} \right]^{1/p}.
\]

Plugging (12) into the momentum equation (8) and using a direct calculation, we have

\[
\begin{align*}
\frac{\partial}{\partial x} z_b^0 &= \left( \frac{q}{u^2 - u} \right) \frac{\partial}{\partial x} u \Rightarrow z_b^0 = -\frac{u^3 + 2gq}{2gu} + C.
\end{align*}
\]

which concludes the proof.

**Remark.** As $h$ and $u$ are stationary, the initial condition of (7) is $(h, u, z_b^0)$. Moreover, the solution $(h, u)$ applied to the Grass model is also an analytical solution of the Shallow Water Equations with the variable topography $z_b^0$. Concerning the shallow-water model, other solutions can be found in [7].

**3. Numerical experiments**

In this section, we consider the analytical solution (7) applied to the Grass model with $q = 1$, $A_g = \alpha = \beta = 0.005$ and $C = 1$. A relaxation solver is applied to approximate the solution of the model. We will not give here the details of the relaxation solver (for details see [8]), but just the relaxation model for the equations (1-3). Thus, we solve the following relaxation system:
\[
\begin{align*}
\partial_t h + \partial_x (hu) &= 0, \\
\partial_t (hu) + \partial_x (hu^2 + \pi) + gh \partial_x z_b &= 0, \\
\partial_t \pi + u \partial_x \pi + \frac{a^2}{h} \partial_x u &= 0, \\
\partial_t z_b + \partial_x q_r &= 0, \\
\partial_t q_r + \left( \frac{b^2}{h^2} - u^2 \right) \partial_x z_b + 2u \partial_x q_r &= 0,
\end{align*}
\]

that is completed with \( \pi = gh^2/2 \) and \( q_r = q_b \) at the equilibrium. Figure 1 presents the numerical result with \( J = 500 \) space cells, a CFL fix condition of 1 and \( T = 7s \). We only notice little difference on the velocity, near the inflow boundary.

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