Spectral analysis and stabilization of a chain of serially connected Euler-Bernoulli beams and strings

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Abstract. We consider N Euler-Bernoulli beams and N strings alternatively connected to one another and forming a particular network which is a chain beginning with a string. We consider two stabilization problems on the same network. The spectrum of the conservative system is studied: the characteristic equation as well as its asymptotic behavior are given. We prove that the energy of the solutions of the first dissipative system tends to zero when the time tends to infinity under some irrationality assumptions of the length of the strings and beams. On another hand we prove a polynomial decay result of the energy of the second system, independently of the length of the strings and beams, for all regular initial data. Our technique is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

2010 Mathematics Subject Classification. 35L05, 35M10, 35R02, 47A10, 93D15, 93D20.
Key words and phrases. Network, wave equation, Euler-Bernoulli beam equation, spectrum, resolvent method, feedback stabilization.

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1 Introduction

We consider the evolution problems (P_1) and (P_2) described by the following systems of $2N$ equations :

$$(P_1)\begin{cases}(\partial_t^2 u_{2j-1}-\partial_x^2 u_{2j-1})(t,x)=0,\ x\in(0,l_{2j-1}),\ t\in(0,\infty),\ j=1,...,N,\\ (\partial_t^2 u_{2j}+\partial_x^4 u_{2j})(t,x)=0,\ x\in(0,l_{2j}),\ t\in(0,\infty),\ j=1,...,N,\\ u_1(t,0)=0,\ \ u_{2N}(t,l_{2N})=0,\ t\in(0,\infty),\\ \partial_x^2 u_{2j}(t,0)=\partial_x^2 u_{2j}(t,l_{2j})=0,\ t\in(0,\infty),\ j=1,...,N,\\ u_j(t,l_j)=u_{j+1}(t,0),\ t\in(0,\infty),\ j=1,...,2N-1,\\ \partial_x^3 u_{2j}(t,0)+\partial_x u_{2j-1}(t,l_{2j-1})=\ -\boldsymbol{\partial_t u_{2j-1}(t,l_{2j-1})},\ t\in(0,\infty),\ j=1,...,N,\\ \partial_x^3 u_{2j}(t,l_{2j})+\partial_x u_{2j+1}(t,0)=\boldsymbol{\partial_t u_{2j}(t,l_{2j})},\ t\in(0,\infty),\ j=1,...,N,\\ u_j(0,x)=u_j^0(x),\ \ \partial_t u_j(0,x)=u_j^1(x),\ x\in(0,l_j),\ j=1,...,2N,\end{cases}$$

and

$$(P_2)\begin{cases}(\partial_t^2 u_{2j-1}-\partial_x^2 u_{2j-1})(t,x)=0,\ x\in(0,l_{2j-1}),\ t\in(0,\infty),\ j=1,...,N,\\ (\partial_t^2 u_{2j}+\partial_x^4 u_{2j})(t,x)=0,\ x\in(0,l_{2j}),\ t\in(0,\infty),\ j=1,...,N,\\ u_1(t,0)=0,\ \ u_{2N}(t,l_{2N})=0,\ \ \partial_x^2 u_{2N}(t,l_{2N})=0,\ t\in(0,\infty),\\ \partial_x^2 u_{2j}(t,0)=\boldsymbol{\partial_{tx}^2 u_{2j}(t,0)},\ t\in(0,\infty),\ j=1,...,N,\\ \partial_x^2 u_{2j}(t,l_{2j})=-\boldsymbol{\partial_{tx}^2 u_{2j}(t,l_{2j})},\ t\in(0,\infty),\ j=1,...,N-1,\\ u_j(t,l_j)=u_{j+1}(t,0),\ t\in(0,\infty),\ j=1,...,2N-1,\\ \partial_x^3 u_{2j}(t,0)+\partial_x u_{2j-1}(t,l_{2j-1})=-\ \boldsymbol{\partial_t u_{2j-1}(t,l_{2j-1})},\ t\in(0,\infty),\ j=1,...,N,\\ \partial_x^3 u_{2j}(t,l_{2j})+\partial_x u_{2j+1}(t,0)=\boldsymbol{\partial_t u_{2j+1}(t,0)},\ \ t\in(0,\infty),\ j=1,...,N-1,\\ u_j(0,x)=u_j^0(x),\ \ \partial_t u_j(0,x)=u_j^1(x),\ x\in(0,l_j),\ j=1,...,2N,\end{cases}$$

where $l_j>0,\ \forall\, j=1,...,2N$.

Models of the transient behavior of some or all of the state variables describing the motion of flexible structures have been of great interest in recent years, for details about physical motivation for the models, see [11], [14], [16] and the references therein. Mathematical analysis of transmission partial differential equations is detailed in [16].

Let us first introduce some notation and definitions which will be used throughout the rest of the paper, in particular some which are linked to the notion of C^ν- networks, $\nu\in\mathbb{N}$ (as introduced in [13] and recalled in [19]).
Let Γ be a connected topological graph embedded in $\mathbb{R}^2$, with $2N$ edges ($N\in\mathbb{N}^*$). Let $K=\{k_j\ :\ 1\le j\le 2N\}$ be the set of the edges of Γ. Each edge k_j is a Jordan curve in

$\mathbb{R}^2$ and is assumed to be parametrized by its arc length x_j such that the parametrization $\pi_j : [0, l_j] \to k_j : x_j \mapsto \pi_j(x_j)$ is ν-times differentiable, i.e. $\pi_j \in C^\nu([0, l_j], \mathbb{R}^2)$ for all $1 \le j \le 2N$. The length of the edge k_j is $l_j > 0$. The C^ν- network G associated with Γ is then defined as the union

$$G = \bigcup_{j=1}^{2N} k_j.$$

We study two feedback stabilization problems for a string-beam network, see [1]-[8], [16] and [27]-[28]. In the following, only chains will be considered as mathematically described in Section 5 of [20]. See also [21] and Figure 1.

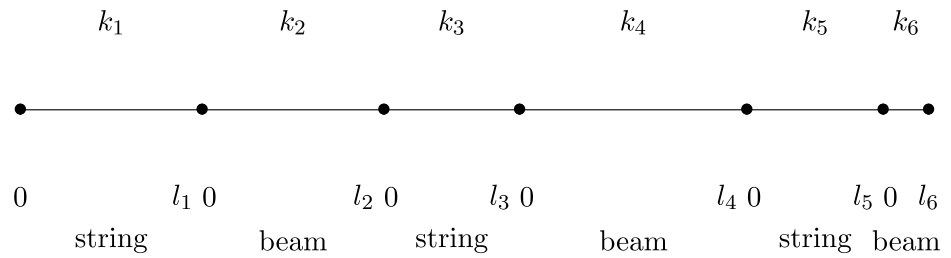

Figure 1: A chain with $2N = 6$ edges

Following Ammari/Jellouli/Mehrenberger ([9]), we study a linear system modelling the vibrations of a chain of alternated Euler-Bernoulli beams and strings but with N beams and N strings (instead of one string-one beam). For each edge k_j (representing a string if j is odd and a beam if j is even), the scalar function $u_j(x, t)$ for $x \in G$ and $t > 0$ contains the information on the vertical displacement of the string if j is odd and of the beam if j is even $(1 \le j \le 2N)$.

Our aim is to study the spectrum of the conservative spatial operator which is defined in Section 3 and to obtain stability results for (P_1) and (P_2).

We define the natural energy $E(t)$ of a solution $\underline{u} = (u_1, ..., u_{2N})$ of (P_1) or (P_2) by

$$E(t) = \frac{1}{2} \sum_{j=1}^{N} \left(\int_0^{l_{2j-1}} \left(|\partial_t u_{2j-1}(t,x)|^2 + |\partial_x u_{2j-1}(t,x)|^2 \right) \mathrm{d}x \right.$$
$$\left. + \int_0^{l_{2j}} \left(|\partial_t u_{2j}(t,x)|^2 + |\partial_x^2 u_{2j}(t,x)|^2 \right) \mathrm{d}x \right). \quad (1.1)$$

We can easily check that every sufficiently smooth solution of (P_1) satisfies the following dissipation law

$$E'(t) = - \sum_{j=1}^{2N-1} \left| \partial_t u_j(t, l_j) \right|^2 \leq 0, \quad (1.2)$$

and therefore, the energy is a nonincreasing function of the time variable t.

The first result concerns the well-posedness of the solutions of (P_1) and the decay of the energy $E(t)$ of the solutions of (P_1). We also study the spectrum of the corresponding conservative system. We give, in particular, the characteristic equation and the asymptotic behavior of the eigenvalues of the corresponding conservative system. We deduce that the generalized gap condition holds: if we denote by $(\lambda_n)_{n\in\mathbb{N}^*}$ the sequence of eigenvalues counted with their multiplicities, then

$$\exists \gamma > 0, \, \forall n \geq 1, \, \lambda_{n+2N} - \lambda_n \geq \gamma. \quad (1.3)$$

Contrary to [9], it seems that the (simple) gap condition fails in general (for any $N \geq 2$). Therefore we do not succeed to obtain an observability inequality (and then to deduce stability results for (P_1)) directly by the study of the spectrum and the eigenvectors (see, for instance, [22]). In fact, the difficulties are to locate precisely the type of eigenvalues in the packets.

However, we prove that the energy $E(t)$ of the solutions of (P_1) tends to zero when $t \to +\infty$ in an appropriate energy space (described later), under some assumptions about the irrationality properties of the length of the strings and beams. For that, we use a result from [10].

As we do not succeed to obtain the explicit decay rate to zero of the energy of the solutions of (P_1), we change a little the system, by considering more dissipation conditions. That is why we introduce in problem (P_2), in addition, the following dissipation conditions

$$\partial_x^2 u_{2j}(t,0) = \partial_{\mathbf{tx}}^{\mathbf{2}} \mathbf{u_{2j}(t,0)}, \quad t \in (0,\infty), \quad j = 1, ..., N,$$
$$\partial_x^2 u_{2j}(t,l_{2j}) = -\partial_{\mathbf{tx}}^{\mathbf{2}} \mathbf{u_{2j}(t,l_{2j})}, \quad t \in (0,\infty), \quad j = 1, ..., N-1.$$

In this case, we are able to prove more interesting stability results for system (P_2) and to give the explicit decay rate of the energy of the solutions of (P_2) in an appropriate space.

In the same manner as previously and with the same energy $E(t)$ (defined by (1.1)), every sufficiently smooth solution of (P_2) satisfies the following dissipation law

$$E'(t) = -\sum_{j=1}^{2N-1} \left|\partial_t u_j(t, l_j)\right|^2 - \sum_{j=1}^{N-1} \left|\partial_{tx}^2 u_{2j}(t, l_{2j})\right|^2 - \sum_{j=1}^{N} \left|\partial_{tx}^2 u_{2j}(t, 0)\right|^2 \leq 0, \qquad (1.4)$$

and therefore, the energy is a nonincreasing function of the time variable t.

The main result of this paper then concerns the precise asymptotic behavior of the solutions of (P_2). As it was shown in [9] in the case of one string and one beam connected together (i.e. $N = 1$), we can not except to obtain an exponential decay rate of the solutions of (P_2). However we are able to prove that the decay rate to zero of the energy is $\ln^4(t)/t^2$, independently of the length of the strings and beams and by taking more regular initial data in an appropriate space. Our technique is based on a frequency domain method from [17] and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

This paper is organized as follows: In Section 2, we give the proper functional setting for systems (P_1) and (P_2) and prove that these two systems are well-posed. In Section 3, we study the spectrum of the corresponding conservative system and we give the asymptotic behavior of the eigenvalues. We then show that the energies of systems (P_1) and (P_2) tend to zero. Finally, in Section 4, we study the stabilization result for (P_2) by the frequency domain technique and give the explicit decay rate of the energy of the solutions of (P_2).

2 Well-posedness of the systems

In order to study systems (P_1) and (P_2) we need a proper functional setting. We define the following space

$$V = \left\{ \underline{u} = (u_1, ..., u_{2N}) \in \prod_{j=1}^{N} \left(H^1(0, l_{2j-1}) \times H^2(0, l_{2j})\right), \right.$$
$$\left. u_j(l_j) = u_{j+1}(0), j = 1, \ldots, 2N-1,\ u_1(0) = 0,\ u_{2N}(l_{2N}) = 0 \right\},$$

equipped with the sesquilinear form

$$< \underline{u},\, \underline{\tilde{u}} >_V = \sum_{j=1}^{N} \left(\int_0^{l_{2j-1}} \partial_x u_{2j-1}(x) \partial_x \overline{\tilde{u}_{2j-1}(x)} dx + \int_0^{l_{2j}} \partial_x^2 u_{2j}(x) \partial_x^2 \overline{\tilde{u}_{2j}(x)} dx \right). \quad (2.5)$$

Note the following lemma:

Lemma 2.1. *We have that* 0 *is an eigenvalue associated to* (P_1) *and* (P_2) *of multiplicity* $N-1$*, i.e. there exists a subspace of* V *of dimension* $N-1$ *such that any* $\underline{\phi}$ *in this subspace satisfies*

$$(EP_0) \begin{cases} \partial_x^2 \phi_{2j-1}(x) = 0,\ x \in (0, l_{2j-1}),\ j = 1, ..., N, \\ \partial_x^4 \phi_{2j}(x) = 0,\ x \in (0, l_{2j}),\ j = 1, ..., N, \\ \phi_1(0) = 0,\ \ \phi_{2N}(l_{2N}) = 0, \\ \partial_x^2 \phi_{2j}(0) = \partial_x^2 \phi_{2j}(l_{2j}) = 0,\ j = 1, ..., N, \\ \phi_j(l_j) = \phi_{j+1}(0),\ j = 1, ..., 2N-1, \\ \partial_x^3 \phi_{2j}(0) + \partial_x \phi_{2j-1}(l_{2j-1}) = 0,\ j = 1, ..., N, \\ \partial_x^3 \phi_{2j}(l_{2j}) + \partial_x \phi_{2j+1}(0) = 0,\ j = 1, ..., N. \end{cases}$$

Proof. Let $\underline{\phi}$ be a non-trivial solution of (EP_0). By the two first equations of (EP_0), for $j \in \{1, \cdots, N\}$, ϕ_{2j-1} is a first order polynomial and ϕ_{2j} is a third order polynomial. Moreover, with the fourth equation of (EP_0), ϕ_{2j} also is a first order polynomial. The two last equations of (EP_0) become

$$\partial_x \phi_{2j-1}(0) = \partial_x \phi_{2j-1}(l_{2j-1}) = 0, \quad j = 1, \cdots, N.$$

Consequently there exists $b_{2j-1} \in \mathbb{C}$ such that $\phi_{2j-1} = b_{2j-1}$ for $j \in \{1, \cdots, N\}$. The third equation of (EP_0) implies $b_1 = 0$. Moreover we find, by the fifth equation of (EP_0), that

$$\phi_{2j}(x) = \frac{b_{2j+1} - b_{2j-1}}{l_{2j}} x + b_{2j-1}, \quad x \in (0, l_{2j}),\ j = 1, \cdots, N,$$

where we set $b_{2N+1} = 0$.

The function $\underline{\phi}$ defined above with $(b_3, b_5, \cdots, b_{2N-1}) \in \mathbb{C}^{N-1}$ then satisfies (EP_0), which finishes the proof. $\square$

It is well-known that system (P_1) may be rewritten as the first order evolution equation

$$\begin{cases} U' = \mathcal{A}_1 U, \\ U(0) = (\underline{u}^0,\, \underline{u}^1) = U_0, \end{cases} \quad (2.6)$$

where U is the vector $U = (\underline{\mathrm{u}},\, \partial_t \underline{\mathrm{u}})^t$ and the operator $\mathcal{A}_1 : Y_1 \to V \times \prod_{j=1}^{2N} L^2(0, l_j)$ is defined by

$$\mathcal{A}_1(\underline{u}, \underline{v})^t := (\underline{v}, (\partial_x^2 u_{2j-1}, -\partial_x^4 u_{2j})_{1 \le j \le N})^t,$$

with

$$Y_1 := \left\{ (\underline{u},\, \underline{v}) \in \prod_{j=1}^{N} \left(H^2(0, l_{2j-1}) \times H^4(0, l_{2j})\right) \times V \ : \right.$$

$$\left. \text{satisfies (2.7) to (2.10) hereafter} \right\},$$

$$\partial_x^2 u_{2N}(l_{2N}) = 0 \tag{2.7}$$

$$\partial_x^2 u_{2j}(0) = 0 \quad j = 1, ..., N \quad \text{and} \quad \partial_x^2 u_{2j}(l_{2j}) = 0, \quad j = 1, ..., N-1 \tag{2.8}$$

$$\partial_x^3 u_{2j}(0) + \partial_x u_{2j-1}(l_{2j-1}) = -v_{2j-1}(l_{2j-1}), \quad j = 1, ..., N \tag{2.9}$$

$$\partial_x^3 u_{2j}(t, l_{2j}) + \partial_x u_{2j+1}(0) = v_{2j}(l_{2j}), \quad j = 1, ..., N-1. \tag{2.10}$$

It is clear that $< .\,, . >_V$ does not define a norm for V but only a semi-norm since, for all $\underline{u} \in V$, we have $< \underline{u},\, \underline{u} >_V = 0$ if and only if $\underline{u}$ satisfies (EP_0). In order to get a Hilbert space we define by E_0, the eigenspace of $\mathcal{A}_1$ associated to the eigenvalue 0, i.e.

$$E_0 = \left\{ (\underline{\phi}, 0) \in V \times \prod_{j=1}^{2N} L^2(0, l_j) \ : \ \underline{\phi} \text{ satisfies } (EP_0) \right\},$$

and $P_{0,1} : V \times \prod_{j=1}^{2N} L^2(0, l_j) \to E_0$ the projection onto E_0 defined by

$$P_{0,1} = \frac{1}{2\pi i} \oint_\gamma (\lambda I - \mathcal{A}_1)^{-1} d\lambda,$$

where γ is a simple closed curve enclosing the eigenvalue 0 (see Theorem 6.17 of [15]). Now let $\mathcal{H}_1$ the Hilbert space defined by

$$V \times \prod_{j=1}^{2N} L^2(0, l_j) = E_0 \oplus \mathcal{H}_1, \tag{2.11}$$

where $\mathcal{H}_1 = (I - P_{0,1})(V \times \prod_{j=1}^{2N} L^2(0, l_j))$ and $E_0 = P_{0,1}(V \times \prod_{j=1}^{2N} L^2(0, l_j))$. Then $P_{0,1}$ is the projection onto E_0 parallel to $\mathcal{H}_1$. Note that, if $N = 1$, $\mathcal{H}_1 = V \times \prod_{j=1}^{2N} L^2(0, l_j)$. Then $\mathcal{H}_1$ is a Hilbert space, equipped with the usual inner product

$$\left\langle \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix}, \begin{pmatrix} \underline{\tilde{u}} \\ \underline{\tilde{v}} \end{pmatrix} \right\rangle_{\mathcal{H}_1} = \sum_{j=1}^{N} \left(\int_0^{l_{2j-1}} \left(v_{2j-1}(x)\overline{\tilde{v}_{2j-1}(x)} + \partial_x u_{2j-1}(x) \partial_x \overline{\tilde{u}_{2j-1}(x)} \right) \mathrm{d}x \right.$$

$$\left. + \int_0^{l_{2j}} \left(v_{2j}(x)\overline{\tilde{v}_{2j}(x)} + \partial_x^2 u_{2j}(x) \partial_x^2 \overline{\tilde{u}_{2j}(x)} \right) \mathrm{d}x \right).$$

The domain $\mathcal{D}(\mathcal{A}_1)$ of the operator $\mathcal{A}_1$ is defined by

$$\mathcal{D}(\mathcal{A}_1) := \mathcal{H}_1 \cap Y_1.$$

Therefore

$$\mathcal{A}_1 : \mathcal{D}(\mathcal{A}_1) \to \mathcal{H}_1,$$

since $\mathcal{A}_1$ commutes with $P_{0,1}$.

Moreover the norm on $\mathcal{D}(\mathcal{A}_1)$ is defined by

$$||(\underline{u}, \underline{v})||^2_{\mathcal{D}(\mathcal{A}_1)} = ||\mathcal{A}_1(\underline{u}, \underline{v})||^2_{\mathcal{H}_1} + ||(\underline{u}, \underline{v})||^2_{\mathcal{H}_1} . \tag{2.12}$$

Note that, with all these notation, problem (P_1) is rewritten in an abstract way as: find $(\underline{u}, \underline{v})^t \in \mathcal{D}(\mathcal{A}_1)$ such that $(\underline{u}, \underline{v})^t_t = \mathcal{A}_1(\underline{u}, \underline{v})^t$.

Now we can prove the well-posedness of system (P_1) and that the solution of (P_1) satisfies the dissipation law (1.2).

Proposition 2.2. *(i) For an initial datum $U_0 \in \mathcal{H}_1$, there exists a unique solution $U \in C([0, +\infty), \mathcal{H}_1)$ to problem (2.6). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}_1)$, then*

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}_1)) \cap C^1([0, +\infty), \mathcal{H}_1).$$

(ii) The solution $\underline{u}$ of (P_1) with initial datum in $\mathcal{D}(\mathcal{A}_1)$ satisfies (1.2). Therefore the energy is decreasing.

Proof. (i) By Lumer-Phillips' theorem (see [24, 26]), it suffices to show that $\mathcal{A}_1$ is dissipative and maximal.

We first prove that $\mathcal{A}_1$ is dissipative. Take $U = (\underline{u}, \underline{v})^t \in \mathcal{D}(\mathcal{A}_1)$. Then

$$\begin{aligned}\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1} = \sum_{j=1}^N \Bigg(\int_0^{l_{2j-1}} \left(\partial_x^2 u_{2j-1}(x)\overline{v_{2j-1}(x)} + \partial_x v_{2j-1}(x)\partial_x \overline{u_{2j-1}(x)}\right) \mathrm{d}x \\ + \int_0^{l_{2j}} \left(-\partial_x^4 u_{2j}(x)\overline{v_{2j}(x)} + \partial_x^2 v_{2j}(x)\partial_x^2 \overline{u_{2j}(x)}\right) \mathrm{d}x \Bigg).\end{aligned}$$

By integration by parts, we have

$$\Re\left(\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1}\right) = \Re\left(\sum_{j=1}^N \left[\partial_x u_{2j-1}\overline{v_{2j-1}}\right]_0^{l_{2j-1}} + \sum_{j=1}^N \left[-\partial_x^3 u_{2j}\overline{v_{2j}}\right]_0^{l_{2j}} + \sum_{j=1}^N \left[\partial_x^2 u_{2j}\partial_x \overline{v_{2j}}\right]_0^{l_{2j}} \right).$$

Moreover, we have

$$\sum_{j=1}^{N} \left[\partial_x^2 u_{2j} \partial_x \overline{v_{2j}}\right]_0^{l_{2j}} = 0,$$

by (2.7) and (2.8), and by the continuity of $\underline{v}$ at the interior nodes, we obtain

$$\begin{aligned}
&\sum_{j=1}^{N} \left[\partial_x u_{2j-1} \overline{v_{2j-1}}\right]_0^{l_{2j-1}} + \sum_{j=1}^{N} \left[-\partial_x^3 u_{2j} \overline{v_{2j}}\right]_0^{l_{2j}} \\
&= \sum_{j=1}^{N} \left(\partial_x u_{2j-1}(l_{2j-1}) + \partial_x^3 u_{2j}(0)\right) \overline{v_{2j-1}}(l_{2j-1}) - \sum_{j=1}^{N-1} \left(\partial_x u_{2j+1}(0) + \partial_x^3 u_{2j}(l_{2j})\right) \overline{v_{2j}}(l_{2j}) \\
&\quad -\partial_x u_1(0)\overline{v_1}(0) - \partial_x^3 u_{2N}(l_{2N})\overline{v_{2N}}(l_{2N}) \\
&= -\sum_{j=1}^{N} |v_{2j-1}(l_{2j-1})|^2 - \sum_{j=1}^{N-1} |v_{2j}(l_{2j})|^2
\end{aligned}$$

by (2.9), (2.10) and since $\underline{v} \in V$. Therefore

$$\Re\left(\langle \mathcal{A}_1 U, U\rangle_{\mathcal{H}_1}\right) = -\sum_{j=1}^{2N-1} |v_j(l_j)|^2 \leq 0. \tag{2.13}$$

This shows the dissipativeness of $\mathcal{A}_1$.

Let us now prove that $\mathcal{A}_1$ is maximal, i.e. that $\lambda I - \mathcal{A}_1$ is surjective for some $\lambda > 0$.

Let $(\underline{f}, \underline{g})^t \in \mathcal{H}_1$. We look for $U = (\underline{u}, \underline{v})^t \in \mathcal{D}(\mathcal{A}_1)$ solution of

$$(\lambda I - \mathcal{A}_1) \begin{pmatrix} \underline{u} \\ \underline{v} \end{pmatrix} = \begin{pmatrix} \underline{f} \\ \underline{g} \end{pmatrix}, \tag{2.14}$$

or equivalently

$$\begin{cases} \lambda u_j - v_j = f_j & \forall j \in \{1, ..., 2N\}, \\ \lambda v_{2j-1} - \partial_x^2 u_{2j-1} = g_{2j-1} & \forall j \in \{1, ..., N\}, \\ \lambda v_{2j} + \partial_x^4 u_{2j} = g_{2j} & \forall j \in \{1, ..., N\}. \end{cases} \tag{2.15}$$

Suppose that we have found $\underline{u}$ with the appropriate regularity. In the following, due to (2.11), we set V_1 the Hilbert space defined by

$$\mathcal{H}_1 = V_1 \times \prod_{j=1}^{2N} L^2(0, l_j),$$

equipped with the inner product (2.5). Then for all $j \in \{1, ..., 2N\}$, we have

$$v_j := \lambda u_j - f_j \in V_1 \subset V. \tag{2.16}$$

It remains to find $\underline{u}$. By (2.15) and (2.16), u_j must satisfy, for all $j = 1, ..., N$,

$$\lambda^2 u_{2j-1} - \partial_x^2 u_{2j-1} = g_{2j-1} + \lambda f_{2j-1},$$

and

$$\lambda^2 u_{2j} + \partial_x^4 u_{2j} = g_{2j} + \lambda f_{2j}.$$

Multiplying these identities by a test function $\underline{\phi}$, integrating in space and using integration by parts, we obtain

$$\sum_{j=1}^{N} \int_0^{l_{2j-1}} \left(\lambda^2 u_{2j-1}\overline{\phi_{2j-1}} + \partial_x u_{2j-1}\partial_x\overline{\phi_{2j-1}}\right) dx - \sum_{j=1}^{N} \left[\partial_x u_{2j-1}\overline{\phi_{2j-1}}\right]_0^{l_{2j-1}}$$
$$+ \sum_{j=1}^{N} \int_0^{l_{2j}} \left(\lambda^2 u_{2j}\overline{\phi_{2j}} + \partial_x^2 u_{2j}\partial_x^2\overline{\phi_{2j}}\right) dx + \sum_{j=1}^{N} \left(\left[\partial_x^3 u_{2j}\overline{\phi_{2j}}\right]_0^{l_{2j}} - \left[\partial_x^2 u_{2j}\partial_x\overline{\phi_{2j}}\right]_0^{l_{2j}}\right)$$
$$= \sum_{j=1}^{2N} \int_0^{l_j} \left(g_j + \lambda f_j\right) \overline{\phi_j} dx.$$

Since $\underline{u} \in \mathcal{D}(\mathcal{A}_1)$ and $(\underline{u}, \underline{v})$ satisfies (2.16), we then have

$$\sum_{j=1}^{N} \int_0^{l_{2j-1}} \left(\lambda^2 u_{2j-1}\overline{\phi_{2j-1}} + \partial_x u_{2j-1}\partial_x\overline{\phi_{2j-1}}\right) dx + \sum_{j=1}^{N} \int_0^{l_{2j}} \left(\lambda^2 u_{2j}\overline{\phi_{2j}} + \partial_x^2 u_{2j}\partial_x^2\overline{\phi_{2j}}\right) dx$$
$$+ \sum_{j=1}^{2N-1} \lambda u_j(l_j)\overline{\phi_j}(l_j) = \sum_{j=1}^{2N} \int_0^{l_j} \left(g_j + \lambda f_j\right) \overline{\phi_j} dx + \sum_{j=1}^{2N-1} f_j(l_j)\overline{\phi_j}(l_j). \quad (2.17)$$

This problem has a unique solution $\underline{u} \in V_1$ by Lax-Milgram's lemma, because the left-hand side of (2.17) is coercive on V_1. If we consider $\underline{\phi} \in \prod_{j=1}^{2N} \mathcal{D}(0,\, l_j) \subset V_1$, then $\underline{u}$ satisfies

$$\lambda^2 u_{2j-1} - \partial_x^2 u_{2j-1} = g_{2j-1} + \lambda f_{2j-1} \quad \text{in } \mathcal{D}'(0,\, l_{2j-1}), \quad j = 1, \cdots, N,$$
$$\lambda^2 u_{2j} + \partial_x^4 u_{2j} = g_{2j} + \lambda f_{2j} \quad \text{in } \mathcal{D}'(0,\, l_{2j}), \quad j = 1, \cdots, N.$$

This directly implies that $\underline{u} \in \prod_{j=1}^{N} \left(H^2(0,\, l_{2j-1}) \times H^4(0,\, l_{2j})\right)$ and then $\underline{u} \in V_1 \cap \prod_{j=1}^{N} \left(H^2(0,\, l_{2j-1}) \times H^4(0,\, l_{2j})\right)$. Coming back to (2.17) and by integrating by parts, we find

$$\sum_{j=1}^{N} \left(\partial_x^2 u_{2j}(l_{2j})\partial_x\overline{\phi_{2j}}(l_{2j}) - \partial_x^2 u_{2j}(0)\partial_x\overline{\phi_{2j}}(0)\right)$$
$$+ \sum_{j=1}^{N} \left(\partial_x u_{2j-1}(l_{2j-1}) + \partial_x^3 u_{2j}(0)\right) \overline{\phi_{2j-1}}(l_{2j-1})$$
$$- \sum_{j=1}^{N-1} \left(\partial_x u_{2j+1}(0) + \partial_x^3 u_{2j}(l_{2j})\right) \overline{\phi_{2j}}(l_{2j}) + \sum_{j=1}^{2N-1} \lambda u_j(l_j)\overline{\phi_j}(l_j) = \sum_{j=1}^{2N-1} f_j(l_j)\overline{\phi_j}(l_j).$$

Consequently, by taking particular test functions $\underline{\phi}$, we obtain

$$\partial_x^2 u_{2j}(l_{2j}) = 0 \quad \text{and} \quad \partial_x^2 u_{2j}(0) = 0,\, j = 1, \cdots, N,$$

$$\begin{aligned} \partial_x u_{2j-1}(l_{2j-1}) + \partial_x^3 u_{2j}(0) &= -\lambda u_{2j-1}(l_{2j-1}) + f_{2j-1}(l_{2j-1}) \\ &= -v_{2j-1}(l_{2j-1}),\, j = 1, \cdots, N, \end{aligned}$$

$$\partial_x u_{2j+1}(0) + \partial_x^3 u_{2j}(l_{2j}) = \lambda u_{2j}(l_{2j}) - f_{2j}(l_{2j}) = v_{2j}(l_{2j}),\, j = 1, \cdots, N-1.$$

In summary we have found $(\underline{u}, \underline{v})^t \in \mathcal{D}(\mathcal{A}_1)$ satisfying (2.14), which finishes the proof of (i).

(ii) To prove (ii), it suffices to derivate the energy (1.1) for regular solutions and to use system (P_1). The calculations are analogous to those of the proof of the dissipativeness of $\mathcal{A}_1$ in (i), and then, are left to the reader. $\square$

We see, in the same manner, that problem (P_2) can be rewritten in an abstract way as: find $(\underline{u}, \underline{v})^t \in \mathcal{D}(\mathcal{A}_2)$ such that $(\underline{u}, \underline{v})^t_t = \mathcal{A}_2(\underline{u}, \underline{v})^t$, where $\mathcal{A}_2 : Y_2 \to V \times \prod_{j=1}^{2N} L^2(0, l_j)$ for

$$Y_2 := \left\{ (\underline{u},\, \underline{v}) \in \prod_{j=1}^{N} \left(H^2(0, l_{2j-1}) \times H^4(0, l_{2j})\right) \times V \;: \right.$$

$$\left. \text{satisfies (2.7), (2.9), (2.10) and (2.18) hereafter} \right\},$$

$$\partial_x^2 u_{2j}(0) = \partial_x v_{2j}(0),\, j = 1, ..., N \text{ and } \partial_x^2 u_{2j}(l_{2j}) = -\partial_x v_{2j}(l_{2j}),\, j = 1, ..., N-1, \quad (2.18)$$

$$\mathcal{A}_2(\underline{u}, \underline{v})^t := (\underline{v}, (\partial_x^2 u_{2j-1}, -\partial_x^4 u_{2j})_{1 \leq j \leq N})^t.$$

Then we define the Hilbert space $\mathcal{H}_2$ by

$$V \times \prod_{j=1}^{2N} L^2(0, l_j) = E_0 \oplus \mathcal{H}_2, \quad \mathcal{H}_2 = (I - P_{0,2})(V \times \prod_{j=1}^{2N} L^2(0, l_j))$$

with $P_{0,2} : V \times \prod_{j=1}^{2N} L^2(0, l_j) \to E_0$ the projection onto E_0 defined by

$$P_{0,2} = \frac{1}{2i\pi} \oint_\gamma (\lambda I - \mathcal{A}_2)^{-1} d\lambda$$

(with γ is a simple closed curve enclosing the eigenvalue 0), and

$$\mathcal{D}(\mathcal{A}_2) := \mathcal{H}_2 \cap Y_2.$$

Then

$$\mathcal{A}_2 : \mathcal{D}(\mathcal{A}_2) \to \mathcal{H}_2.$$

The following proposition holds:

Proposition 2.3. *(i) For an initial datum $U_0 \in \mathcal{H}_2$, there exists a unique solution $U \in C([0, +\infty), \mathcal{H}_2)$ to*

$$\begin{cases} U' = \mathcal{A}_2 U, \\ U(0) = (\underline{u}^0, \underline{u}^1) = U_0. \end{cases}$$

Moreover, if $U_0 \in \mathcal{D}(\mathcal{A}_2)$, then

$$U \in C([0, +\infty), \mathcal{D}(\mathcal{A}_2)) \cap C^1([0, +\infty), \mathcal{H}_2).$$

(ii) The solution $\underline{u}$ of (P_2) with initial datum in $\mathcal{D}(\mathcal{A}_2)$ satisfies (1.4). *Therefore the energy is decreasing.*

Proof. The proof of (i) and (ii) is the same as the proof of Proposition 2.2, and therefore is left to the reader. □

3 Spectral analysis of a chain of serially connected Euler-Bernoulli beams and strings

In this section, we study the spectral analysis of the corresponding conservative system. Let $\underline{\Phi}$ be the solution of the conservative system derived from problems (P_1) and (P_2) given in the introduction, i.e. $\underline{\Phi}$ is the solution of the following system

$$(P_c) \begin{cases} (\partial_t^2 \Phi_{2j-1} - \partial_x^2 \Phi_{2j-1})(t,x) = 0,\ x \in (0, l_{2j-1}),\ t \in (0,\infty),\ j = 1, ..., N, \\ (\partial_t^2 \Phi_{2j} + \partial_x^4 \Phi_{2j})(t,x) = 0,\ x \in (0, l_{2j}),\ t \in (0,\infty),\ j = 1, ..., N, \\ \Phi_1(t,0) = 0,\ \ \Phi_{2N}(t, l_{2N}) = 0,\ t \in (0,\infty) \\ \partial_x^2 \Phi_{2j}(t,0) = \partial_x^2 \Phi_{2j}(t, l_{2j}) = 0,\ t \in (0,\infty),\ j = 1, ..., N, \\ \Phi_j(t, l_j) = \Phi_{j+1}(t,0), t \in (0,\infty),\ j = 1, ..., 2N-1, \\ \partial_x^3 \Phi_{2j}(t,0) + \partial_x \Phi_{2j-1}(t, l_{2j-1}) = 0,\ t \in (0,\infty),\ j = 1, ..., N, \\ \partial_x^3 \Phi_{2j}(t, l_{2j}) + \partial_x \Phi_{2j+1}(t,0) = 0,\ t \in (0,\infty),\ j = 1, ..., N, \\ \Phi_j(0,x) = u_j^0(x),\ \ \partial_t \Phi_j(0,x) = u_j^1(x),\ x \in (0, l_j),\ j = 1, ..., 2N, \end{cases}$$

where we have replaced the dissipative conditions (in bold in systems (P_1) and (P_2)) by the conservative ones.

We can rewrite system (P_c) in an abstract way as: find $(\underline{\Phi}, \underline{\Psi})^t \in \mathcal{D}(\mathcal{A}_c)$ such that

$(\underline{\Phi}, \underline{\Psi})_t^t = \mathcal{A}_c(\underline{\Phi}, \underline{\Psi})^t$, where $\mathcal{A}_c : Y_c \to V \times \prod_{j=1}^{2N} L^2(0, l_j)$, for

$$Y_c := \left\{ (\underline{\Phi},\ \underline{\Psi}) \in \prod_{j=1}^{N} \left(H^2(0, l_{2j-1}) \times H^4(0, l_{2j})\right) \times V\ : \right.$$

$$\left. \text{satisfies (2.7), (2.8), and (3.19), (3.20) hereafter} \right\},$$

$$\partial_x^3 u_{2j}(0) + \partial_x u_{2j-1}(l_{2j-1}) = 0, \quad j = 1, ..., N \tag{3.19}$$

$$\partial_x^3 u_{2j}(t, l_{2j}) + \partial_x u_{2j+1}(0) = 0, \quad j = 1, ..., N-1, \tag{3.20}$$

and

$$\mathcal{A}_c(\underline{\Phi}, \underline{\Psi})^t := (\underline{\Psi}, (\partial_x^2 \Phi_{2j-1}, -\partial_x^4 \Phi_{2j})_{1 \le j \le N})^t.$$

Then we define the Hilbert space $\mathcal{H}_c$ by

$$V \times \prod_{j=1}^{2N} L^2(0, l_j) = E_0 \oplus \mathcal{H}_c, \quad \mathcal{H}_c = (I - P_{0,c})(V \times \prod_{j=1}^{2N} L^2(0, l_j)) = V_c \times \prod_{j=1}^{2N} L^2(0, l_j)$$

with $P_{0,c} : V \times \prod_{j=1}^{2N} L^2(0, l_j) \to E_0$ the projection onto E_0 defined by with

$$P_{0,c} = \frac{1}{2i\pi} \oint_\gamma (\lambda I - \mathcal{A}_c)^{-1} d\lambda$$

(with γ is a simple closed curve enclosing the eigenvalue 0), and

$$\mathcal{D}(\mathcal{A}_c) := \mathcal{H}_c \cap Y_c.$$

Following Section 2, it is clear that system (P_c) is well-posed in the natural energy space. If we suppose that $(\underline{u}^0, \underline{u}^1) \in \mathcal{H}_c = V_c \times \prod_{j=1}^{2N} L^2(0, l_j)$, then problem (P_c) admits a unique solution

$$\underline{\Phi} \in C([0, T], V_c) \cap C^1([0, T], \prod_{j=1}^{2N} L^2(0, l_j)).$$

This system is obviously conservative, i.e. its energy is constant.

3.1 The characteristic equation

Let $\underline{\phi}$ be a non-trivial solution of the eigenvalue problem (EP) associated to the conservative problem (P_c) and λ^2 be the corresponding eigenvalue. That is to say, $\underline{\phi} \in V_c$ satisfies the transmission and boundary conditions (3.21)-(3.25) hereafter as well as

$$(EP) \begin{cases} \partial_x^2 \phi_{2j-1} = \lambda^2 \phi_{2j-1} & \text{on } (0, \mathrm{l}_{2j-1}),\ \ \forall\, \mathrm{j} \in \{1, ..., \mathrm{N}\}, \\ -\partial_x^4 \phi_{2j} = \lambda^2 \phi_{2j} & \text{on } (0, \mathrm{l}_{2j}),\ \ \forall\, \mathrm{j} \in \{1, ..., \mathrm{N}\}, \\ \phi_{2j-1} \in H^2(0, l_{2j-1}), & \forall j \in \{1, ..., N\},\ \phi_{2j} \in H^4(0, l_{2j}), \forall j \in \{1, ..., N\}, \end{cases}$$

$$\phi_1(0) = 0,\ \phi_{2N}(l_{2N}) = 0, \tag{3.21}$$

$$\partial_x^2 \phi_{2j}(0) = \partial_x^2 \phi_{2j}(l_{2j}) = 0, \quad j = 1, ..., N \tag{3.22}$$

$$\phi_j(l_j) = \phi_{j+1}(0), \quad j = 1, ..., 2N-1 \tag{3.23}$$

$$\partial_x^3 \phi_{2j}(0) + \partial_x \phi_{2j-1}(l_{2j-1}) = 0, \quad j = 1, ..., N \tag{3.24}$$

$$\partial_x^3 \phi_{2j}(l_{2j}) + \partial_x \phi_{2j+1}(0) = 0, \quad j = 1, ..., N-1. \tag{3.25}$$

Note that this also means that $(\underline{\phi}, \lambda \underline{\phi}) \in \mathcal{D}(\mathcal{A}_c)$ is an eigenvector of $\mathcal{A}_c$ associated to the eigenvalue λ. By the definition of $\mathcal{A}_c$ and of its domain, 0 is not an eigenvalue of $\mathcal{A}_c$. Moreover 0 is not an eigenvalue of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Define z by $\lambda = iz^2$ where z lies in $\mathbb{R}^{+*}$ with $i^2 = -1$.
Following Paulsen ([23]) and Mercier ([18]), we will rewrite this eigenvalue problem on a chain of $2N$ beams and strings using only square matrices of order 2 in the following way: we define, for each $j \in \{1, ..., N\}$, the vector functions V_{2j-1} and V_{2j} by

$$V_{2j-1}(x) = \left(\phi_{2j-1}(x),\ \frac{1}{z^2} \partial_x \phi_{2j-1}(x) \right)^t,\ \forall x \in [0, l_{2j-1}],$$

$$V_{2j}(x) = \left(\phi_{2j}(x),\ \frac{1}{z^3} \partial_x^3 \phi_{2j}(x) \right)^t,\ \forall x \in [0, l_{2j}].$$

Define the matrices A_j by

$$A_{2j-1}(z, l_{2j-1}) := \begin{pmatrix} c_{2j-1} & s_{2j-1} \\ -s_{2j-1} & c_{2j-1} \end{pmatrix},$$

$$A_{2j}(z, l_{2j}) := \frac{1}{e^{2l_{2j}z} - 2e^{l_{2j}z} s_{2j} - 1} \cdot \begin{pmatrix} e^{2l_{2j}z}(c_{2j} - s_{2j}) - c_{2j} - s_{2j} & 2s_{2j}\left(1 - e^{2l_{2j}z}\right) \\ e^{2l_{2j}z} c_{2j} - 2e^{l_{2j}z} + c_{2j} & e^{2l_{2j}z}(c_{2j} - s_{2j}) - c_{2j} - s_{2j} \end{pmatrix},$$

with $j \in \{1, \cdots, N\}$ and with the notation

$$\begin{cases} c_{2j-1} = \cos(l_{2j-1} \cdot z^2), & s_{2j-1} = \sin(l_{2j-1} \cdot z^2) \\ c_{2j} = \cos(l_{2j} \cdot z), & s_{2j} = \sin(l_{2j} \cdot z). \end{cases} \tag{3.26}$$

The matrix T is defined by:

$$T(z) := \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{z} \end{pmatrix}.$$

To finish with, the matrix $M(z)$ is the square matrix of order 2 given by

$$M(z) = A_{2N} T A_{2N-1} ... T^{-1} A_2 T A_1. \tag{3.27}$$

Lemma 3.1. *(A few trivial but useful properties)*
With the notation introduced above, we have:

$$\begin{aligned}
&V_j(l_j) = A_j V_j(0),\ \forall j \in \{1, ..., 2N\},\\
&V_{2j}(0) = T V_{2j-1}(l_{2j-1}),\ \forall j \in \{1, ..., N\},\\
&V_{2j+1}(0) = T^{-1} V_{2j}(l_{2j}),\ \forall j \in \{1, ..., N-1\},\\
&V_{2N}(l_{2N}) = M(z) V_1(0).
\end{aligned}$$

Proof. First, for j odd and $j \in \{1, \ldots, 2N\}$, since u_j satisfies the first equation of the eigenvalue problem (EP), u_j is a linear combination of the vectors of the fundamental basis

$$\left(\cos(z^2\, .),\, \sin(z^2\, .)\right).$$

The first equation of the lemma follows from that property after some calculations.
Now, for j even and $j \in \{1, \ldots, 2N\}$, since u_j satisfies the second equation of the eigenvalue problem (EP), u_j is a linear combination of the vectors of the fundamental basis

$$\left(\cos(z.),\, \sin(z\, .),\, e^{z\, \cdot},\, e^{-z\, \cdot}\right).$$

In this basis, if we consider the two following functions d_1, d_2 with coordinates

$$d_1 := (-e^{l_j z} \sin(l_j z), e^{l_j z} \cos(l_j z) - 1, 0, -e^{l_j z} \sin(l_j z))$$

$$d_2 := (e^{l_j z} - e^{-l_j z}, 0, \cos(l_j z) - e^{-l_j z}, e^{l_j z} - \cos(l_j z),$$

we can see that they are independent and satisfy (3.21). Consequently u_j can be expressed as a linear combination of these two functions. Now, to find A_j, we proceed as follows: let $(\alpha, \beta)^t$ the coordinates of u_j in the basis (d_1, d_2). There exist two matrices M_0, M_1 such that $V_j(0) = M_0(\alpha, \beta)^t$ and $V_j(l_j) = M_1(\alpha, \beta)^t$, then A_j is the matrix $M_1 M_0^{-1}$.

Moreover the transmission conditions (3.23), (3.24) and (3.25) imply the second and third equations.
The fourth one is the logical consequence of the first three applied successively for $j = 1$, $j = 2$, etc... □

Theorem 3.2. *(The characteristic equation for the eigenvalue problem corresponding to a chain of alternated beams and strings)*
The complex number $\lambda = iz^2$ $(z \in \mathbb{R}^{+*})$ is an eigenvalue of $\mathcal{A}_c$ if and only if z satisfies the characteristic equation

$$f(z) = m_{12}(z) = 0, \tag{3.28}$$

where $m_{12}(z)$ is the term on the first line and second column of the matrix $M(z)$.

Proof. Let $\underline{\phi}$ be a non-trivial solution of the eigenvalue problem (EP) and λ^2 be the corresponding eigenvalue, where $\lambda = iz^2$ $(z \in \mathbb{R}^{+*})$.

Using the boundary conditions as well as $V_{2N}(l_{2N}) = M(z)V_1(0)$, it follows:

$$\begin{pmatrix} 0 \\ \dfrac{1}{z^3}\partial_x^3\phi_{2N}(l_{2N}) \end{pmatrix} = M(z) \begin{pmatrix} 0 \\ \dfrac{1}{z^2}\partial_x\phi_1(0) \end{pmatrix}.$$

It is clear that the vector of the second part of the previous equality is non-trivial since $\underline{\phi}$ is a non-trivial solution of problem (EP). Hence the result. $\square$

Proposition 3.3. *(Asymptotic behavior of the characteristic equation)*
Assume that the characteristic equation is given by Theorem 3.2. Then

$$f(z) = z\left(f_\infty(z) + g(z)\right)$$

where

$$f_\infty(z) = s_1(z) \cdot c_2(z) \cdot s_3(z) \cdots s_{2N-1}(z) \cdot (c_{2N}(z) - s_{2N}(z)) \tag{3.29}$$

(with c_j, s_j defined by (3.26)) and g satisfies $\lim_{z\to+\infty} g(z) = 0$. Thus, the asymptotic behavior of the spectrum $\sigma(\mathcal{A}_c)$ corresponds to the roots of the asymptotic characteristic equation

$$f_\infty(z) = 0. \tag{3.30}$$

Proof. In the following, the notation $o(h(\lambda))$ is used for a square matrix of order 2 such that all its terms are dominated by the function $\lambda \mapsto h(\lambda)$ asymptotically. For any $j \in \{1, \ldots, N\}$,

$$A_{2j}(z, l_{2j}) = \frac{1}{e^{2l_{2j}z} - 2e^{l_{2j}z}s_{2j} - 1}\left[e^{2l_{2j}z}\begin{pmatrix} c_{2j} - s_{2j} & 2s_{2j} \\ c_{2j} & c_{2j} - s_{2j} \end{pmatrix} + o(2l_{2j}z)\right].$$

Thus

$$A_{2j}(z,l_{2j}) = \begin{pmatrix} c_{2j}-s_{2j} & 2s_{2j} \\ c_{2j} & c_{2j}-s_{2j} \end{pmatrix} + o(1),$$

which leads, after some calculations, to:

$$T^{-1}A_{2j}TA_{2j-1} = \begin{pmatrix} (c_{2j}-s_{2j})c_{2j-1} & (c_{2j}-s_{2j})s_{2j-1} \\ -zc_{2j}c_{2j-1} & -zc_{2j}s_{2j-1} \end{pmatrix} + o(1).$$

Likewise

$$T^{-1}A_{2j+2}TA_{2j+1} = \begin{pmatrix} (c_{2j+2}-s_{2j+2})c_{2j+1} & (c_{2j+2}-s_{2j+2})s_{2j+1} \\ -zc_{2j+2}c_{2j+1} & -zc_{2j+2}s_{2j+1} \end{pmatrix} + o(1).$$

Thus

$$\begin{aligned} &T^{-1}A_{2j+2}TA_{2j+1}T^{-1}A_{2j}TA_{2j-1} \\ &= \begin{pmatrix} -z(c_{2j+2}-s_{2j+2})s_{2j+1}c_{2j}c_{2j-1} & -z(c_{2j+2}-s_{2j+2})s_{2j+1}c_{2j}s_{2j-1} \\ z^2c_{2j+2}s_{2j+1}c_{2j}c_{2j-1} & z^2c_{2j+2}s_{2j+1}c_{2j}s_{2j-1} \end{pmatrix} + o(1). \end{aligned}$$

The result follows by induction. □

Remark 3.4. *We can note that the eigenvalues $\lambda = iz^2$ of (EP) have $2N$ families of asymptotic behavior:*

$$\left(i\,\frac{k\pi}{l_{2j-1}}\right)_{k\in\mathbb{N}^*}, \quad j=1,\cdots,N, \quad \left(i\left(\frac{\pi+2k\pi}{2l_{2j}}\right)^2\right)_{k\in\mathbb{N}^*}, \quad j=1,\cdots,N-1,$$
$$\text{and} \quad \left(i\left(\frac{\pi/4+k\pi}{l_{2N}}\right)^2\right)_{k\in\mathbb{N}^*}.$$

It follows that the generalized gap condition (1.3) holds.

Proposition 3.5. *(Geometric multiplicity of the eigenvalues)*
If $\lambda \neq 0$ is an eigenvalue of the operator $\mathcal{A}_c$ and E_λ is the associated eigenspace, then the dimension of E_λ is one.

Proof. The eigenvectors $\underline{\phi} \in V_c$ associated to the eigenvalue λ^2 (cf. problem (EP)) are entirely determined by their values at the nodes of the network (i.e. where the beams and strings are connected to one another). Due to Lemma 3.1, they are also determined by $V_1(0) = \left(\phi_1(0), \frac{1}{z^2}\partial_x\phi_1(0)\right)^t$. Now $\phi_1(0) = 0$ (cf. condition (3.21)) and $\partial_x\phi_1(0)$ may take any value in $\mathbb{R}^*$. Hence the result. □

3.2 Strong stability of (P_1) and (P_2)

We first prove the following lemma:

Lemma 3.6. *If there exist $i, j \in \{1, \cdots, N\}$ such that*

$$\frac{l_{2i-1}}{l_{2j-1}} \notin \mathbb{Q} \quad or \quad \frac{l_{2i}}{l_{2j}} \notin \mathbb{Q}, \tag{3.31}$$

or if there exist $i, j \in \{1, \cdots, N\}$ such that

$$\frac{(l_{2i})^2}{l_{2j-1}} \neq \frac{p^2}{q}\pi, \quad where \quad p, q \in \mathbb{Z}, \tag{3.32}$$

then

$$\sum_{j=1}^{2N-1} |\phi_j(l_j)|^2 \neq 0, \tag{3.33}$$

for all eigenvectors $\underline{\phi} \in V_c$ of (EP).

Proof. Let $\underline{\phi} \in V_c$ be an eigenvector of (EP) associated to the eigenvalue λ^2, where $\lambda = iz^2$ $(z \in \mathbb{R}^{+*})$. Assume that (3.33) is false, i.e. that we have

$$\sum_{j=2}^{2N} |\phi_j(0)|^2 = 0. \tag{3.34}$$

We use in the following the basis introduced in the proof of Lemma 3.1.

First, since $\phi_{2j-1}(0) = 0$ for $j = 1, \cdots, N$, it is easy to see that there exists a_{2j-1} such that

$$\phi_{2j-1} = a_{2j-1} \sin(z^2 \cdot), \quad \forall j = 1, \cdots, N.$$

Then, by the continuity at the interior nodes (3.23), we get

$$a_{2j-1} \sin(z^2 l_{2j-1}) = 0, \quad \forall j = 1, \cdots, N.$$

Second, there exist a_{2j}, b_{2j}, $\tilde{a}_{2j}$ and $\tilde{b}_{2j}$ such that

$$\phi_{2j} = a_{2j} \sin(z \cdot) + b_{2j} \cos(z \cdot) + \tilde{a}_{2j} \sinh(z \cdot) + \tilde{b}_{2j} \cosh(z \cdot).$$

By (3.22) and (3.34), we obtain

$$b_{2j} = \tilde{b}_{2j} = \tilde{a}_{2j} = 0 \quad \text{and} \quad a_{2j} \sin(z l_{2j}) = 0, \quad \forall j = 1, \cdots, N,$$

since $z \neq 0$. Then, we have, with the notation introduced in (3.26),

$$a_j s_j = 0, \quad j = 1, \cdots, 2N. \tag{3.35}$$

Moreover (3.24) gives

$$a_{2j} = \frac{1}{z} a_{2j-1} c_{2j-1},$$

and (3.25) yields

$$a_{2j+1} = z a_{2j} c_{2j}.$$

By induction, we obtain, for all $j \geq 2$,

$$a_j = z^{\epsilon_j} a_1 c_{j-1} c_{j-2} \cdots c_1, \tag{3.36}$$

with $\epsilon_{2j} = -1$ and $\epsilon_{2j-1} = 0$. Therefore $a_1 \neq 0$ (otherwise $a_j = 0$ for all j, and then $\phi = 0$, which is impossible). Now, by (3.35), we have $s_1 = 0$ and $c_1 = \pm 1$. Then, since (3.36) holds, $a_2 \neq 0$ and $s_2 = 0$, again with (3.35). Then $c_2 = \pm 1$... We see, by induction, that $s_j = 0$ for all $j \in \{1, \cdots, 2N\}$. Therefore, it suffices to have one $s_j \neq 0$ for some $j \in \{1, \cdots, 2N\}$ to obtain (3.33). It is the case if there exist $i, j \in \{1, \cdots, N\}$ such that (3.31) or (3.32) hold. $\square$

As a consequence of the previous lemma, we can prove the following proposition.

Proposition 3.7. *We have*

$$\lim_{t \to +\infty} E(t) = 0 \tag{3.37}$$

for all solution $\underline{u}$ of (P_1) with $(\underline{u}^0, \underline{u}^1)$ in $\mathcal{H}_1$ if and only if (3.33) holds for all eigenvectors $\underline{\phi} \in V_c$ of (EP). Consequently, if there exist $i, j \in \{1, \cdots, N\}$ such that (3.31) or (3.32) hold, then (3.37) holds.

Proof. $\boxed{\Leftarrow}$ Let us show that (3.33) implies (3.37). For that purpose we closely follow [25].

First, we show that $\mathcal{A}_1$ has no eigenvalue on the imaginary axis. If it is not the case, let $i\omega$ be an eigenvalue of $\mathcal{A}_1$ where $\omega \in \mathbb{R}^*$. Let $Z \in \mathcal{D}(\mathcal{A}_1)$ be an eigenvector associated with $i\omega$. Then Z is of the form

$$Z = \begin{pmatrix} \underline{\phi} \\ i\omega\underline{\phi} \end{pmatrix},$$

with

$$\begin{aligned} \partial_x^2 \phi_{2j-1} &= -\omega^2 \phi_{2j-1}, \quad j = 1, \cdots, N, \\ \partial_x^4 \phi_{2j} &= \omega^2 \phi_{2j}, \quad j = 1, \cdots, N. \end{aligned} \tag{3.38}$$

It is an immediate consequence of the identity $(i\omega I - \mathcal{A}_1)Z = 0$.

We now take the inner product $\langle ., . \rangle_{\mathcal{H}_1}$ between $\mathcal{A}_1 Z$ and Z. By (2.13), we have

$$\Re \left(\langle \mathcal{A}_1 Z, Z \rangle_{\mathcal{H}_1} \right) = -\omega^2 \sum_{j=1}^{2N-1} |\phi_j(l_j)|^2 .$$

Since Z is an eigenvector of $\mathcal{A}_1$ associated with $i\omega$ and $\omega \neq 0$, we obtain

$$\sum_{j=1}^{2N-1} |\phi_j(l_j)|^2 = 0.$$

Note that Z satisfies the eigenvalue problem (EP) and Z belongs to $\mathcal{D}(\mathcal{A}_c)$, since

$$P_{0,c} Z = \frac{1}{2i\pi} \oint_\gamma (\lambda I - \mathcal{A}_c)^{-1} Z d\lambda = \frac{1}{2i\pi} \oint_\gamma \frac{1}{\lambda - i\omega} Z d\lambda = 0,$$

(where we use $(\lambda I - \mathcal{A}_c)(\frac{1}{\lambda - i\omega} Z) = Z$ and where γ is a simple closed curve enclosing 0), and thus $Z = Z - P_{0,c}Z \in (I - P_{0,c})(V \times \prod_{j=1}^{2N} L^2(0, l_j)) = \mathcal{H}_c$. Then this contradicts (3.33). Therefore $\mathcal{A}_1$ has no eigenvalue on the imaginary axis.

Now, we can apply the main theorem of Arendt and Batty [10]: Since $\sigma(\mathcal{A}_1) \cap i\mathbb{R}$ is empty, we obtain (3.37).

$\boxed{\Rightarrow}$ Let us show that (3.37) implies (3.33). For that purpose we use a contradiction argument. Suppose that there exists an eigenvector $\underline{\phi} \in V_c$ of (EP) of associated eigenvalue λ^2 (where $\lambda = iz^2$, $z \in \mathbb{R}^{+*}$) such that

$$\sum_{j=1}^{2N-1} |\phi_j(l_j)|^2 = 0.$$

Let us set

$$u(., t) = \phi \cos(z^2 t).$$

Then u is solution of (P_1) and satisfies

$$E(t) = E(0),$$

because

$$\phi_j(l_j) = 0, \quad \forall j = 1, \cdots, 2N.$$

This contradicts (3.37).

It suffices to use Lemma 3.6 to finish the proof. $\square$

Moreover, with the same method as previously, we are able to prove the decay to zero of the energy of solutions without restriction about the irrational properties of the lengths.

Proposition 3.8. *We have* $\lim_{t \to +\infty} E(t) = 0$ *for any solution of* (P_2) *with* $(\underline{u}^0, \underline{u}^1)$ *in* $\mathcal{H}_2$.

Proof. As in the proof of Proposition 3.7, we can show that the energy of solutions of (P_2) tends to zero if and only if

$$\sum_{j=1}^{2N-1} |\phi_j(l_j)|^2 + \sum_{j=1}^{N-1} \left(|\partial_x \phi_{2j}(l_{2j})|^2 + |\partial_x \phi_{2j}(0)|^2 \right) \neq 0, \tag{3.39}$$

for all eigenvectors $\underline{\phi}$ of (EP). Let $\underline{\phi}$ be an eigenvector of (EP) such that (3.39) is false. By the same proof as Lemma 3.6, this implies that $\underline{\phi} = 0$, which is impossible. Then (3.39) holds and therefore the energy decays to 0. □

Remark 3.9. *If we take the initial data in* $V \times \prod_{j=1}^{2N} L^2(0, l_j)$, *the energy of the solutions of* (P_1) *and* (P_2) *do not decay to* 0, *since* $u = \phi$, *where* $(\phi, 0)^t$ *is an eigenvector of* $\mathcal{A}_i$ $(i = 1, 2)$ *associated to the eigenvalue* 0, *is solution of* (P_1) *and* (P_2) *with constant energy.*

4 Stabilization result for (P_2)

We prove a decay result of the energy of system (P_2), independently of the length of the strings and beams, for all regular initial data. In [9], the authors prove that the system described by (P_2) is not exponentially stable in $\mathcal{H}_2$ with $N = 1$ (i.e. with one string and one beam). Therefore, in the general case (for $N \in \mathbb{N}^*$), we can not except to obtain an exponential decay for the energy of the solutions of (P_2), but only a weaker decay rate, and in this general case, we prove a polynomial decay rate. To obtain this, our technique is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

The following theorem is a direct generalisation of the result in [9], which we note, due to a mistake in the choice of θ, the decay rate in the following $\frac{ln^4(t)}{t^2}$ has been written $\frac{ln^6(t)}{t^4}$ (corresponding to a choice of $\theta = 1$ and not to $\theta = 1/2$).

Theorem 4.1. *There exists a constant $C > 0$ such that, for all $(\underline{u}^0, \underline{u}^1) \in \mathcal{D}(\mathcal{A}_2)$, the solution of system (P_2) satisfies the following estimate*

$$E(t) \leq C \, \frac{\ln^4(t)}{t^2} \, \left\| (\underline{u}^0, \underline{u}^1) \right\|^2_{\mathcal{D}(\mathcal{A}_2)}, \qquad \forall \, t > 0. \tag{4.40}$$

Proof. We will employ the following frequency domain theorem for polynomial stability (see Liu-Rao [17]) of a C_0 semigroup of contractions on a Hilbert space:

Lemma 4.2. *A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space satisfies*

$$||e^{t\mathcal{L}} U_0|| \leq C \, \frac{\ln^{1+\frac{1}{\theta}}(t)}{t^{\frac{1}{\theta}}} ||U_0||_{\mathcal{D}(\mathcal{L})}$$

for some constant $C > 0$ and for $\theta > 0$ if

$$\rho(\mathcal{L}) \supset \left\{ i\beta \mid \beta \in \mathbb{R} \right\} \equiv i\mathbb{R}, \tag{4.41}$$

and

$$\limsup_{|\beta| \to \infty} \frac{1}{\beta^\theta} \, \|(i\beta - \mathcal{L})^{-1}\| < \infty, \tag{4.42}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Then the proof of Theorem 4.1 is based on the following two lemmas.

Lemma 4.3. *The spectrum of $\mathcal{A}_2$ contains no point on the imaginary axis.*

Proof. Since $\mathcal{A}_2$ has compact resolvent, its spectrum $\sigma(\mathcal{A}_2)$ only consists of eigenvalues of $\mathcal{A}_2$. We will show that the equation

$$\mathcal{A}_2 Z = i\beta Z \tag{4.43}$$

with $Z = (\underline{y}, \, \underline{v})^t \in \mathcal{D}(\mathcal{A}_2)$ and $\beta \neq 0$ has only the trivial solution.
By taking the inner product of (4.43) with Z and using

$$\begin{aligned} \Re \left(< \mathcal{A}_2 Z, Z >_{\mathcal{H}_2} \right) = - \sum_{j=1}^{N} \left(|v_{2j}(0)|^2 + \left| \frac{dv_{2j}}{dx}(0) \right|^2 \right) \\ - \sum_{j=1}^{N-1} \left(|v_{2j}(l_{2j})|^2 + \left| \frac{dv_{2j}}{dx}(l_{2j}) \right|^2 \right), \end{aligned} \tag{4.44}$$

we obtain that

$$v_{2j}(0) = 0, \; \frac{dv_{2j}}{dx}(0) = 0, \; j = 1, ..., N \text{ and } v_{2j}(l_{2j}) = 0, \; \frac{dv_{2j}}{dx}(l_{2j}) = 0, \; j = 1, ..., N-1.$$

Next, we eliminate $\underline{v}$ in (4.43) to get an ordinary differential equation:

$$\begin{cases} (\beta^2 y_{2j-1} + \partial_x^2 y_{2j-1})(x) = 0, \ x \in (0, l_{2j-1}), j = 1, ..., N, \\ (\beta^2 y_{2j} - \partial_x^4 y_{2j})(x) = 0, \ x \in (0, l_{2j}), j = 1, ..., N, \\ y_1(0) = 0, \ y_{2N}(l_{2N}) = 0, \ \partial_x^2 y_{2N}(l_{2N}) = 0, \\ \partial_x^2 y_{2j}(0) = 0, \ j = 1, ..., N, \\ \partial_x^2 y_{2j}(l_{2j}) = 0, \ j = 1, ..., N-1, \\ y_j(l_j) = y_{j+1}(0), \ j = 1, ..., 2N-1, \\ \partial_x^3 y_{2j}(0) + \partial_x y_{2j-1}(l_{2j-1}) = 0, \ j = 1, ..., N, \\ \partial_x^3 y_{2j}(l_{2j}) + \partial_x y_{2j+1}(0) = 0, \ j = 1, ..., N-1. \end{cases} \tag{4.45}$$

Then, we can easily see that the only solution of the above system is the trivial one. $\square$

The second lemma shows that (4.42) holds with $\mathcal{L} = \mathcal{A}_2$ and $\theta = 1$.

Lemma 4.4. *The resolvent operator of $\mathcal{A}_2$ satisfies condition* (4.42) *for $\theta = 1$.*

Proof. Suppose that condition (4.42) is false with $\theta = 1$. By the Banach-Steinhaus Theorem (see [12]), there exists a sequence of real numbers $\beta_n \to +\infty$ and a sequence of vectors $Z_n = (\underline{y}_n, \underline{v}_n)^t \in \mathcal{D}(\mathcal{A}_2)$ with $\|Z_n\|_{\mathcal{H}_2} = 1$ such that

$$||\beta_n(i\beta_n I - \mathcal{A}_2)Z_n||_{\mathcal{H}_2} \to 0 \quad \text{as} \quad n \to \infty, \tag{4.46}$$

i.e.,

$$\beta_n^{1/2}\,(i\beta_n y_n - v_n) \equiv f_n \to 0 \quad \text{in} \ \ V, \tag{4.47}$$

$$\beta_n^{1/2}\left(i\beta_n v_{n,2j-1} - \frac{d^2 y_{n,2j-1}}{dx^2}\right) \equiv g_{n,2j-1} \to 0 \quad \text{in} \ \ L^2(0, l_{2j-1}), \tag{4.48}$$

$$\beta_n^{1/2}\left(i\beta_n v_{n,2j} + \frac{d^4 y_{n,2j}}{dx^4}\right) \equiv k_{n,2j} \to 0 \quad \text{in} \ \ L^2(0, l_{2j}), \tag{4.49}$$

since $\beta_n^{1/2} \le \beta_n$.

Our goal is to derive from (4.46) that $||Z_n||_{\mathcal{H}_2}$ converges to zero, thus there is a contradiction. The proof is divided into four steps:

First step. We first notice that we have

$$||\beta_n(i\beta_n I - \mathcal{A}_2)Z_n||_{\mathcal{H}_2} \ge |\Re\left(\langle \beta_n(i\beta_n I - \mathcal{A}_2)Z_n, Z_n\rangle_{\mathcal{H}_2}\right)|. \tag{4.50}$$

Then, by (4.44) and (4.46),

$$\beta_n^{\frac{1}{2}}\, v_{n,2j}(0) \to 0, \quad \beta_n^{\frac{1}{2}}\, \frac{dv_{n,2j}}{dx}(0) \to 0, \quad j = 1, ..., N \tag{4.51}$$

and

$$\beta_n^{\frac{1}{2}}\, v_{n,2j}(l_{2j}) \to 0, \quad \beta_n^{\frac{1}{2}}\, \frac{dv_{n,2j}}{dx}(l_{2j}) \to 0, \quad j = 1, ..., N-1. \tag{4.52}$$

This further leads, by (4.47) and the trace theorem, to

$$|\beta_n|^{\frac{3}{2}}\ |y_{n,2j}(0)| \to 0, \quad |\beta_n|^{3/2} \left|\frac{dy_{n,2j}}{dx}(0)\right| \to 0, \quad j = 1, ..., N, \tag{4.53}$$

and

$$|\beta_n|^{\frac{3}{2}}\ |y_{n,2j}(l_{2j})| \to 0, \quad |\beta_n|^{3/2} \left|\frac{dy_{n,2j}}{dx}(l_{2j})\right| \to 0, \quad j = 1, ..., N-1. \tag{4.54}$$

Moreover, since $Z_n \in \mathcal{D}(\mathcal{A}_2)$ and thus satisfies (2.18), we have, by (4.51) and (4.52),

$$|\beta_n|^{\frac{1}{2}} \left|\frac{d^2 y_{n,2j}}{dx^2}(0)\right| \to 0,\ j = 1, ..., N, \quad |\beta_n|^{\frac{1}{2}} \left|\frac{d^2 y_{n,2j}}{dx^2}(l_{2j})\right| \to 0,\ j = 1, ..., N-1. \tag{4.55}$$

Then, note that, by continuity at the interior nodes and by (4.53) and (4.54), we have

$$|\beta_n|^{\frac{3}{2}}\ |y_{n,2j-1}(0)| \to 0,\ j = 2, ..., N, \quad |\beta_n|^{\frac{3}{2}}\ |y_{n,2j-1}(l_{2j-1})| \to 0,\ j = 1, ..., N. \tag{4.56}$$

Second step. We now express $\underline{v}_n$ as a function of $\underline{y}_n$ from (4.47) and substitute it into (4.48)-(4.49) to get

$$\beta_n^{1/2}\left(-\beta_n^2 y_{n,2j-1} - \frac{d^2 y_{n,2j-1}}{dx^2}\right) = g_{n,2j-1} + i\beta_n f_{n,2j-1}, \quad j = 1, ..., N, \tag{4.57}$$

$$\beta_n^{1/2}\left(-\beta_n^2 y_{n,2j} + \frac{d^4 y_{n,2j}}{dx^4}\right) = k_{n,2j} + i\beta_n f_{n,2j}, \quad j = 1, ..., N. \tag{4.58}$$

Next, we take the inner product of (4.57) with $q_{2j-1}(\cdot)\dfrac{dy_{n,2j-1}}{dx}$ in $L^2(0, l_{2j-1})$ where $q_{2j-1} \in C^1([0, l_{2j-1}])$ and $q_{2j-1}(0) = 0$. We obtain that

$$\begin{aligned}
&\int_0^{l_{2j-1}} \beta_n^{1/2}\Big(-\beta_n^2 y_{n,2j-1} - \frac{d^2 y_{n,2j-1}}{dx^2}\Big) q_{2j-1}(x)\frac{d\bar{y}_{n,2j-1}}{dx}\, dx \\
&= \int_0^{l_{2j-1}} \Big(g_{n,2j-1} + i\beta_n\, f_{n,2j-1}\Big)\ q_{2j-1}(x)\frac{d\bar{y}_{n,2j-1}}{dx}\, dx \\
&= \int_0^{l_{2j-1}} g_{n,2j-1}\ q_{2j-1}(x)\frac{d\bar{y}_{n,2j-1}}{dx}\, dx - i\int_0^{l_{2j-1}} q_{2j-1}\frac{df_{n,2j-1}}{dx}\ \beta_n \bar{y}_{n,2j-1}\, dx \\
&-i\int_0^{l_{2j-1}} f_{n,2j-1}\frac{dq_{2j-1}}{dx}\ \beta_n \bar{y}_{n,2j-1}\, dx + i f_{n,2j-1}(l_{2j-1}) q_{2j-1}(l_{2j-1})\beta_n \bar{y}_{n,2j-1}(l_{2j-1}).
\end{aligned} \tag{4.59}$$

It is clear that the right-hand side of (4.59) converges to zero. Indeed, $f_{n,2j-1}$ and $g_{n,2j-1}$ converge to zero in H^1 and L^2 respectively, $\|Z_n\|_{\mathcal{H}_2} = 1$ and (4.56) holds, and, finally, $|\beta_n y_{n,2j-1}| = \left|\frac{f_{n,2j-1}}{\beta_n^{1/2}} + v_{n,2j-1}\right|$ is bounded in $L^2(0, l_{2j-1})$.

By a straight-forward calculation,

$$\Re\left\{\int_0^{l_{2j-1}} -\beta_n^2 y_{n,2j-1}\, q_{2j-1}\frac{d\bar{y}_{n,2j-1}}{dx}\,dx\right\} = -\frac{1}{2}q_{2j-1}(l_{2j-1})|\beta_n y_{n,2j-1}(l_{2j-1})|^2 + \frac{1}{2}\int_0^{l_{2j-1}} \frac{dq_{2j-1}}{dx}|\beta_n y_{n,2j-1}|^2 dx$$

and

$$\Re\left\{\int_0^{l_{2j-1}} -\frac{d^2 y_{n,2j-1}}{dx^2}\, q_{2j-1}\frac{d\bar{y}_{n,2j-1}}{dx}\,dx\right\} = -\frac{1}{2}\, q_{2j-1}(l_{2j-1})\left|\frac{dy_{n,2j-1}}{dx}(l_{2j-1})\right|^2 + \frac{1}{2}\int_0^{l_{2j-1}}\left|\frac{dy_{n,2j-1}}{dx}\right|^2 \frac{dq_{2j-1}}{dx}dx.$$

We then take the real part of (4.59), and (4.56) leads to

$$\int_0^{l_{2j-1}} \frac{dq_{2j-1}}{dx}\,|\beta_n y_{n,2j-1}|^2\,dx + \int_0^{l_{2j-1}} \frac{dq_{2j-1}}{dx}\left|\frac{dy_{n,2j-1}}{dx}\right|^2 dx - q_{2j-1}(l_{2j-1})\left|\frac{dy_{n,2j-1}}{dx}(l_{2j-1})\right|^2 \to 0. \quad (4.60)$$

Similarly, we take the inner product of (4.58) with $q_{2j}(\cdot)\dfrac{dy_{n,2j}}{dx}$ in $L^2(0,l_{2j})$ with $q_{2j}\in C^3([0,l_{2j}])$ and $q_{2j}(l_{2j})=0$. We then repeat the above procedure. Since

$$\int_0^{l_{2j}}\left|\frac{dy_{n,2j}}{dx}\right|^2 dx = -\frac{1}{i\beta_n}\int_0^{l_{2j}} v_{n,2j}\frac{d^2\bar{y}_{n,2j}}{dx^2} - \frac{1}{i\beta_n}\int_0^{l_{2j}}(i\beta_n\, y_{n,2j} - v_{n,2j})\frac{d^2\bar{y}_{n,2j}}{dx^2}dx - \frac{d\bar{y}_{n,2j}}{dx}(0)\, y_{n,2j}(0) + \frac{d\bar{y}_{n,2j}}{dx}(l_{2j})\, y_{n,2j}(l_{2j}),$$

then, from the boundedness of $v_{n,2j}$, $i\beta_n y_{n,2j} - v_{n,2j}$, $\frac{d^2 y_{n,2j}}{dx^2}$ in $L^2(0,l_{2j})$ and (4.53)-(4.54), $\frac{dy_{n,2j}}{dx}$ converges to zero in $L^2(0,l_{2j})$. This will give, after some calculations,

$$\int_0^{l_{2j}} \frac{dq_{2j}}{dx}|\beta_n y_{n,2j}|^2 dx + \int_0^{l_{2j}} 3\,\frac{dq_{2j}}{dx}\left|\frac{d^2 y_{n,2j}}{dx^2}\right|^2 dx - 2\Re\left(\frac{d^3 y_{n,2j}}{dx^3}(0)q_{2j}(0)\frac{d\bar{y}_{n,2j}}{dx}(0)\right) \to 0. \quad (4.61)$$

Third step. Next, we show that $\dfrac{dy_{n,2j-1}}{dx}(l_{2j-1})$ and $\dfrac{d^3 y_{n,2j}}{dx^3}(0)$ converge to zero. We take the inner product of (4.58) with $\frac{1}{\beta_n^{1/2}}e^{-\beta_n^{1/2}x}$ in $L^2(0,l_{2j})$. We have, with (4.58),

$$\int_0^{l_{2j}}\left(-\beta_n^2 y_{n,2j} + \frac{d^4 u_{n,2j}}{dx^4}\right)e^{-\beta_n^{1/2}x}dx = \int_0^{l_{2j}} \frac{1}{\beta_n^{1/2}}k_{n,2j}e^{-\beta_n^{1/2}x}dx + i\int_0^{l_{2j}} \beta_n^{1/2} f_{n,2j}e^{-\beta_n^{1/2}x}dx. \quad (4.62)$$

It is clear that the first term of the right hand side of (4.62) tends to zero by (4.49). Moreover, by integration by parts,

$$\int_0^{l_{2j}} \beta_n^{1/2} f_{n,2j} e^{-\beta_n^{1/2} x} dx = \int_0^{l_{2j}} \frac{df_{n,2j}}{dx} e^{-\beta_n^{1/2} x} dx - f_{n,2j}(l_{2j}) e^{-\beta_n^{1/2} l_{2j}} + f_{n,2j}(0),$$

which tends to zero since $f_{n,2j}$ tends to zero in H^2 and by the trace theorem.

This leads to

$$\int_0^{l_{2j}} \left(\beta_n^2 e^{-\beta_n^{1/2} x} \, y_{n,2j} - e^{-\beta_n^{1/2} x} \, \frac{d^4 y_{n,2j}}{dx^4} \right) dx \to 0. \tag{4.63}$$

Performing four integrations by parts in the second term on the left-hand side of (4.63), we obtain

$$\begin{aligned} \int_0^{l_{2j}} \left(\beta_n^2 e^{-\beta_n^{1/2} x} \, y_{n,2j} - e^{-\beta_n^{1/2} x} \, \frac{d^4 y_{n,2j}}{dx^4} \right) dx &= \frac{d^3 y_{n,2j}}{dx^3}(0) + \beta_n^{1/2} \, \frac{d^2 y_{n,2j}}{dx^2}(0) \\ &\quad + \beta_n \, \frac{d y_{n,2j}}{dx}(0) + \beta_n^{3/2} y_{n,2j}(0) + o(1), \end{aligned} \tag{4.64}$$

with (4.54)-(4.55) and since

$$\begin{aligned} \left| \frac{d^3 y_{n,2j}}{dx^3}(l_{2j}) e^{-\beta_n^{1/2} l_{2j}} \right|^2 &\leq e^{-2\beta_n^{1/2} l_{2j}} \int_0^{l_{2j}} \left| \frac{d^4 y_{n,2j}}{dx^4}(x) \right|^2 dx \\ &\leq e^{-2\beta_n^{1/2} l_{2j}} \int_0^{l_{2j}} \left| \frac{k_{n,2j}}{\beta_n^{1/2}} - i\beta_n v_{n,2j} \right|^2 dx \\ &\leq \frac{2}{\beta_n} e^{-2\beta_n^{1/2} l_{2j}} \int_0^{l_{2j}} |k_{n,2j}|^2 \, dx \\ &\quad + 2\beta_n^2 e^{-2\beta_n^{1/2} l_{2j}} \int_0^{l_{2j}} |v_{n,2j}|^2 \, dx \to 0, \end{aligned}$$

because $\|Z_n\|_{\mathcal{H}_2} = 1$.

Thus, according to (4.53) and (4.55), we simplify (4.64) to

$$\frac{d^3 y_{n,2j}}{dx^3}(0) \to 0. \tag{4.65}$$

Consequently, since $Z_n \in \mathcal{D}(\mathcal{A}_2)$ and thus satisfies (2.9), we obtain

$$\frac{d y_{n,2j-1}}{dx}(l_{2j-1}) \to 0. \tag{4.66}$$

Then, (4.53) and (4.65) lead to

$$\frac{d\bar{y}_{n,2j}}{dx}(0) \, \frac{d^3 y_{n,2j}}{dx^3}(0) \to 0. \tag{4.67}$$

In view of (4.66)-(4.67), we simplify (4.60) and (4.61) to

$$\int_0^{l_{2j-1}} \frac{dq_{2j-1}}{dx}|\beta_n y_{n,2j-1}|^2 dx + \int_0^{l_{2j-1}} \frac{dq_{2j-1}}{dx}\left|\frac{dy_{n,2j-1}}{dx}\right|^2 dx \to 0, \tag{4.68}$$

$$\int_0^{l_{2j}} \frac{dq_{2j}}{dx}|\beta_n y_{n,2j}|^2 dx + \int_0^{l_{2j}} 3\,\frac{dq_{2j}}{dx}\left|\frac{d^2 y_{n,2j}}{dx^2}\right|^2 dx \to 0 \tag{4.69}$$

respectively.

Fourth step. Finally, we choose q_{2j-1} and q_{2j} such that $\dfrac{dq_{2j-1}}{dx}$ is strictly positive and $\dfrac{dq_{2j}}{dx}$ is strictly negative. This can be done by taking

$$q_{2j-1}(x) = e^x - 1, \qquad q_{2j}(x) = e^{(l_{2j}-x)} - 1.$$

Therefore, (4.68) and (4.69) imply

$$\|\beta_n y_{n,2j-1}\|_{L^2(0,l_{2j-1})} \to 0,\ \|\beta_n y_{n,2j}\|_{L^2(0,l_{2j})} \to 0,\ \|(y_{n,2j-1}, y_{n,2j})_{j\in\{1,\cdots,N\}}\|_V \to 0. \tag{4.70}$$

In view of (4.47), we also get

$$\|v_{n,2j-1}\|_{L^2(0,l_{2j-1})} \to 0, \quad \|v_{n,2j}\|_{L^2(0,l_{2j})} \to 0, \tag{4.71}$$

which clearly contradicts $\|Z_n\|_{\mathcal{H}_2} = 1$. □

The two hypothesis of Lemma 4.2 are proved by Lemma 4.3 and Lemma 4.4. Then (4.40) holds. The proof of Theorem 4.1 is then finished. □

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