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To cite this version:
Emmanuel Gobet, Plamen Turkedjiev. Complementary proofs to ”Approximation of discrete BSDE using least-squares regression”. 2011. hal-00642656

HAL Id: hal-00642656
https://hal.archives-ouvertes.fr/hal-00642656
Submitted on 18 Nov 2011

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Complementary proofs to "Approximation of discrete BSDE using least-squares regression"

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Abstract

This short note gives complementary proofs to our work [GT11], of which we follow the notations and assumptions.

1. Assumption (AF-iii) for non-uniform grids

The time grid \( (t_k = T - T(1 - k/N)^{1/\theta_p})_{0 \leq k \leq N} \) with \( \theta_p \in (0, 1] \) satisfies

\[
C_\pi := \sup_{k < N} \frac{\Delta_k}{(T - t_k)^{1 - \theta_L}} \leq \frac{T^{\theta_L}}{\theta_p \pi^{1 + \theta P}} N^{1 - \theta P},
\]

\[
R_\pi := \sup_{0 \leq k \leq N - 2} \frac{\Delta_k}{\Delta_{k+1}} \leq \frac{1}{\theta_p} \left( 1 + \left( \frac{1}{2\theta_p} \right)^{1 - \theta P} \right),
\]

where \( \Delta_k = t_{k+1} - t_k \) and \( \theta_L \in (0, 1] \).

Proof. Set \( 1/\theta_p = \mu \geq 1 \) and \( g(x) = 1 - (1 - x)^\mu \); we have \( t_k = Tg(k/N) \). Note that \( g \) is increasing and concave; thus we have

\[
\frac{\Delta_k}{(T - t_k)^{1 - \theta_L}} \leq \frac{\mu T^{\mu - 1}}{T^{1 - \theta_L} (1 - k/N)^{\mu(1 - \theta_L)}} = \frac{T^{\theta_L}}{\theta_p N} (1 - k/N)^{\theta_L/\theta_p - 1} \]

and the bound on \( C_\pi \) follows by considering either \( \theta_L \geq \theta_p \) or \( \theta_L < \theta_p \).

Now, we study \( R_\pi \). Since \( g \) is concave, we have \( \Delta_{k-1} \geq \Delta_k \geq \cdots \geq \Delta_{N-1} = TN^{-\mu} \) and \( \Delta_{k-1} \leq \frac{\mu T}{N} (1 - k/N)^{\mu - 1} \). This gives a first upper bound for the
We set \((\Delta_{k-1}/\Delta_k) \leq \Delta_{k-1}/\Delta_{k-N+1} \leq \mu T N^{-1} \leq \mu (N-k)^{-1} \leq \mu n_{0}^{-1} \). \hspace{1cm} (1)

We are now in a position to complete the upper bound on \(R_{\pi}\).

- \(\mu \in [1, 2]: \) we prove \(\Delta_{k-1}/\Delta_k \leq \mu\). For \(k = N - 1\), the inequality is true owing to (1). Now take \(k < N - 1\). Since \(g''\) is non-increasing \((\mu \in [1, 2])\), we have \(\Delta_k \geq \mu T N^{-1} (1 - k/N)^{\mu - 1} + T 2N^{2} g''((k + 1)/N)\), and we easily deduce

\[
\frac{\Delta_{k-1}}{\Delta_k} \leq \frac{\mu T N^{-1} (1 - k/N)^{\mu - 1}}{\mu T N^{-1} (1 - k/N)^{\mu - 1} + T 2N^{2} \mu (\mu - 1)(1 - (k + 1)/N)^{\mu - 2}} \leq \frac{1}{1 - (\mu - 2)^{-1}} \leq \mu.
\]

- \(\mu \geq 2: \) we prove \(\Delta_{k-1}/\Delta_k \leq \mu (k/\Delta_k)\). Set \(n_0 = \lfloor \frac{\mu}{2} \rfloor\): \(n_0 \leq \frac{\mu}{2} < n_0 + 1\). For \(k \geq N - n_0\), the announced upper bound directly follows from (1). Now take \(k \leq N - n_0 - 1\) (which implies \(N - k > \frac{\mu}{2}\)): \(g''\) being non-decreasing for \(\mu \geq 2\), we have

\[
\Delta_k \geq \mu T N^{-1} (1 - k/N)^{\mu - 1} + \frac{T 2N^{2}}{\mu (\mu - 1)} (1 - (k + 1)/N)^{\mu - 2} \geq \mu T N^{-1} \frac{1}{\mu} \geq \Delta_{k-1}/\mu.
\]

\[\square\]

2. Proof of Proposition 3.3.

**Proposition 3.3.** Assume \((A'_L)\) and \((A_{P})\). For any \(R \in [0, +\infty]\) and for any \(\pi\) with \(N\) large enough (such that \(C_{\pi} L_{L}^{2} \leq \frac{1}{2\theta_{L}}\)), the following almost sure error bounds on \(Y_i - Y_i^{R}\) and \(Z_i - Z_i^{R}\) hold for any \(0 < i < N\):

\[
|Y_i - Y_i^{R}| \leq C_{\psi} \exp \left( \frac{T}{8} + \frac{12qL_{L}^{2}}{\theta_{L}} T^{\theta_{L}} \right) \exp \left( - \frac{1}{4} R^{2} \right) \sqrt{N},
\]

\[
\left( \sum_{k=i}^{N-1} \mathbb{E} |Z_k - Z_k^{R}|^{2} \Delta_k \right)^{\frac{1}{2}} \leq C_{\psi} \exp \left( \frac{12qL_{L}^{2}}{\theta_{L}} T^{\theta_{L}} \right) (8q + T \exp(\frac{T}{4})) \frac{1}{2} \exp \left( - \frac{1}{4} R^{2} \right) \sqrt{N}.
\]

**Proof.** We set \(T_{R} := \mathbb{E}[(N - (-R) \vee N \wedge R)^{2}]\) where \(N\) is a Gaussian random variable with mean 0 and variance 1. An explicit computation gives

\[
T_{R} = 2 \mathbb{P}(N > R)(R^{2} + 1) - R \frac{e^{-\frac{1}{2} R^{2}}}{\sqrt{2\pi}} \leq 2 \mathbb{P}(N > R)(R^{2} + 1 - R^{2}) \leq 2e^{-\frac{1}{2} R^{2}},
\]
where the two last inequalities are derived from the Mill inequality and the Markov exponential inequality.

Now, we follow the arguments of Lemma 3.1 and we consider \( \gamma \in (0, +\infty)^N \) such that \( 8q(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} \leq 1 \) for \( 0 \leq k < N \). Define \( \Delta Y_k := Y_k - Y_k^R \) and \( \Delta Z_k := Z_k - Z_k^R \).

**Preliminary bound on \( \Delta Z \).** Applying the Cauchy-Schwartz inequality and the almost sure bounds on \( Y \) and \( Y^R \) (Proposition 3.2), we obtain:

\[
\Delta_k |\Delta Z_k|^2 = \Delta_k^{-1} |E_k[Y_{k+1} \Delta W_k - Y_{k+1}^R \Delta W_k]|^2
\leq 2 \Delta_k^{-1} |E_k[Y_{k+1} (\Delta W_k - |\Delta W_k|_w)]|^2 + 2 \Delta_k^{-1} |E_k[\Delta Y_{k+1} \Delta W_k]|^2
\leq 2q C_y^2 T_R + 2q (E_k[|\Delta Y_{k+1}^2|] - (E_k[|\Delta Y_{k+1}|])^2).
\]

(2)

**Bound on \( \Delta Y \).** Using Young’s inequality \((a+b)^2 \leq (1+\Delta_k \gamma_k) a^2 + (1+\frac{1}{\Delta_k \gamma_k}) b^2\), the Lipschitz property of \((y,z) \rightarrow f_k(y,z)\), and using (2), we obtain

\[
\Delta Y_k^2 \leq (1 + \Delta_k \gamma_k)(E_k[\Delta Y_{k+1}^2])^2
+ 2(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} \Delta_k (E_k[\Delta Y_{k+1}^2] + |\Delta Z_k|^2)
\leq (1 + \Delta_k \gamma_k - 4q(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}}) (E_k[\Delta Y_{k+1}^2])^2
+ 2(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} \Delta_k \Delta Y_{k+1}^2
+ 4q C_y^2 (\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} T_R.
\]

(3)

The condition \( 8q(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} \leq 1 \) ensures that \( 1 + \Delta_k \gamma_k - 4q(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} \geq 0 \); this given, we may use Jensen’s inequality on the term (4) to obtain:

\[
\Delta Y_k^2 \leq (1 + \Delta_k \gamma_k + 2(\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} \Delta_k) E_k[\Delta Y_{k+1}^2]
+ 4q C_y^2 (\Delta_k + \frac{1}{\gamma_k}) \frac{L_f^2}{(T-t_k)^{1-\theta_L}} T_R
\leq (1 + \Delta_k \gamma_k + \frac{\Delta_k}{4}) E_k[\Delta Y_{k+1}^2] + \frac{1}{2} C_y^2 T_R
\]

using again the relation between \( \Delta_k \) and \( 1/\gamma_k \). Multiplying by \( \lambda_k := \prod_{j=0}^{k-1} (1 + \Delta_j \gamma_j + \frac{\Delta_k}{4}) \), taking conditional expectation \( E_k \), and summing over \( k = i, \ldots, N-1 \), we obtain a pointwise uniform bound for \( \Delta Y_{i}^2 \):

\[
\Delta Y_{i}^2 \Gamma_i \leq \Delta Y_{i}^2 \lambda_i \leq \frac{1}{2} C_y^2 e^{T/4} \Gamma_N N T_R.
\]

(5)
such that $G \perp \perp H$. 

Proof. We start with a standard result. Let $\mathcal{G} \perp \perp \mathcal{H}$. 

3. Proof of Lemma 4.2

Lemma 4.2. With the current notation and assumptions, for all $m$ we have

$$
\bar{y}_k^{R,M}(X_k^m) = E_N^M \left[ \Phi(\hat{X}_N^{k,m}) + \sum_{i=k}^{N-1} f_i(\hat{X}_i^{k,m}, y_{i+1}^{R,M}(\hat{X}_{i+1}^{k,m}), z_i^{R,M}(\hat{X}_i^{k,m})) \Delta_i \right],
$$

$$
\Delta_k \hat{z}_{l,k}^{R,M}(X_k^m) = E_N^M \left[ [\Delta \hat{W}_{l,k}^m]_{l,k} \Phi(\hat{X}_N^{k,m}) + \sum_{i=k+1}^{N-1} f_i(\hat{X}_i^{k,m}, y_{i+1}^{R,M}(\hat{X}_{i+1}^{k,m}), z_i^{R,M}(\hat{X}_i^{k,m})) \Delta_i \right].
$$

Proof. We start with a standard result. Let $\mathcal{G}$ and $\mathcal{H}$ be sub-$\sigma$-algebras of $\mathcal{F}$, such that $\mathcal{G} \perp \perp \mathcal{H}$. Let $F : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ be bounded and $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, and $U : \Omega \rightarrow \mathbb{R}^d$ be $\mathcal{H}$-measurable. Then, by the Monotone Class Theorem for
functions, $\mathbb{E}[F(U)|\mathcal{H}] = j(U)$ where $j(h) = \mathbb{E}[F(h)]$ for all $h \in \mathbb{R}^d$.

In order to apply the above result, we require some standard results about the ghost path $(X, \Delta W)$. Let $k$ be fixed. Since $\bar{X}$ is a Markov chain, then for all $i > k$ there is a mapping $V_i : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ measurable with respect to $\mathcal{G}_i \otimes \mathcal{B}(\mathbb{R}^d)$ such that $\bar{X}_i = V_i(x)$, where the filtration $(\mathcal{G}_i)_{k \leq i} \leq N$ is independent of $\mathcal{F}_N$ and $\Delta \bar{W}_k$ is $\mathbb{G}_{k+1}$-measurable. 

Now, by defining

\[
\begin{align*}
F_1(x) &:= \Psi^{R,M}_k(x, V_{k+1}(x), \ldots, V_N(x)), \\
F_2(x) &:= [\Delta \bar{W}^{k, m}_k]_u \Psi^{R,M}_{k+1}(V_{k+1}(x), V_{k+2}(x), \ldots, V_N(x)),
\end{align*}
\]

the result of the previous paragraph can be applied, because $F_1$ and $F_2$ are $\mathcal{G}_N \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, hence the representations for $\bar{g}^{R,M}_k(X^m_k)$ and $\bar{e}^{R,M}_k(X^m_k)$. □

4. Proof of Lemma 1 in Appendix B

**Lemma 1.** Let $\mathcal{G}$ be a countable set of functions $g : \mathbb{R}^d \mapsto [0, B]$ with $B > 0$. Let $X, X^1, \ldots, X^M (M \geq 1)$ be i.i.d. $\mathbb{R}^d$ valued random variables. For any $\alpha > 0$ and $\varepsilon \in (0, 1)$ one has

$$
\mathbb{P}\left( \sup_{g \in \mathcal{G}} \frac{1}{M} \sum_{m=1}^M g(X^m) - \mathbb{E}[g(X)] \geq \varepsilon \right) \leq 4\mathbb{E}(\mathcal{N}_1(\frac{\alpha \varepsilon}{5}, \mathcal{G}, X^1:M)) \exp\left( -\frac{3\varepsilon^2 \alpha M}{40B} \right),
$$

$$
\mathbb{P}\left( \sup_{g \in \mathcal{G}} \mathbb{E}[g(X)] - \frac{1}{M} \sum_{m=1}^M g(X^m) \geq \varepsilon \right) \leq 4\mathbb{E}(\mathcal{N}_1(\frac{\alpha \varepsilon}{8}, \mathcal{G}, X^1:M)) \exp\left( -\frac{6\varepsilon^2 \alpha M}{169B} \right).
$$

**Proof.** The first inequality is stated in [GKKW02, Theorem 11.6] for $B \geq 1$. For $B \in (0, 1)$, we rescale the class of functions $\{g/B : g \in \mathcal{G}\}$ (now bounded by 1), replace $\alpha$ by $\alpha/B$ and apply the previous case: this gives the announced upper bound.

To establish the second inequality, we adapt the proof of the first inequality from the proof of [GKKW02, Theorem 11.6]. The first step consists in taking a ghost sample $\bar{X}^{1:M}$ and observing that for a given $g \in \mathcal{G}$, $\mathbb{E}[g(\bar{X})] - \frac{1}{M} \sum_{m=1}^M g(\bar{X}^m) > \varepsilon (\alpha + \frac{1}{M} \sum_{m=1}^M g(\bar{X}^m) + \mathbb{E}[g(X)])$ and $\mathbb{E}[g(X)] - \frac{1}{M} \sum_{m=1}^M g(X^m) \leq \frac{\varepsilon}{4}(\alpha + \frac{1}{M} \sum_{m=1}^M g(\bar{X}^m) + \mathbb{E}[g(X)])$ imply

\[
(1 + \frac{5\varepsilon}{8})(\frac{1}{M} \sum_{m=1}^M g(\bar{X}^m) - \frac{1}{M} \sum_{m=1}^M g(X^m)) > \frac{3\varepsilon}{8}(2\alpha + \frac{1}{M} \sum_{m=1}^M g(X^m) + \frac{1}{M} \sum_{m=1}^M g(\bar{X}^m)) + \frac{3\varepsilon}{4}\mathbb{E}[g(X)].
\]
Since the r.h.s. positive, the l.h.s. is also positive; using $\frac{13}{8} \geq 1 + \frac{5\varepsilon}{8}$ implies

$$\frac{1}{M} \sum_{m=1}^{M} g(\tilde{X}^m) - \frac{1}{M} \sum_{m=1}^{M} g(X^m) > \frac{3\varepsilon}{13} (2\alpha + \frac{1}{M} \sum_{m=1}^{M} g(X^m) + \frac{1}{M} \sum_{m=1}^{M} g(\tilde{X}^m)).$$

Then we proceed as in [GKKW02, pp. 205-207] to show that the probability to estimate is bounded by

$$2\mathbb{P} \left( \exists g \in \mathcal{G} : \frac{1}{M} \sum_{m=1}^{M} g(\tilde{X}^m) - \frac{1}{M} \sum_{m=1}^{M} g(X^m) > \frac{3\varepsilon}{13} (2\alpha + \frac{1}{M} \sum_{m=1}^{M} g(X^m) + \frac{1}{M} \sum_{m=1}^{M} g(\tilde{X}^m)) \right)$$

for $M > \frac{8B}{\varepsilon^2\alpha}$ (however for $M \leq \frac{8B}{\varepsilon^2\alpha}$ the upper bound in Lemma 1 is obviously true). The rest of the proof is identical to [GKKW02, pp. 208-210], except that one should take a $L_1$ $\delta$-cover of $\mathcal{G}$ w.r.t. $X^{1:M}$ with $\delta = \frac{\alpha\varepsilon}{8}$ (instead of $\delta = \frac{\alpha\varepsilon}{5}$).

It leads to a new upper bound, $4\mathbb{E} \left( \mathcal{N}_1 \left( \frac{\alpha\varepsilon}{8}, \mathcal{G}, X^{1:M} \right) \right) \exp \left( - \frac{6\varepsilon^2\alpha M}{169B} \right)$. \qedsymbol

**References**
