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Singularity Analysis of the H4 Robot using Grassmann-Cayley Algebra

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Abstract- This paper extends a recently proposed singularity analysis method to lower-mobility parallel manipulators having an articulated nacelle. Using screw theory, a twist graph is introduced in order to simplify the constraint analysis of such manipulators. Then, a wrench graph is obtained in order to represent some points at infinity on the Plücker lines of the Jacobian matrix. Using Grassmann-Cayley algebra, the rank deficiency of the Jacobian matrix amounts to the vanishing condition of the superbracket. Accordingly, the parallel singularities are expressed in three different forms involving superbrackets, meet and join operators, and vector cross and dot products, respectively. The approach is explained through the singularity analysis of the H4 robot. All the singularity conditions of this robot are enumerated and the motions associated with these singularities are characterized.

Keywords: Grassmann-Cayley algebra, parallel singularity, screw theory, articulated nacelle, projective space, superbracket.

1 Introduction

Parallel singularities are critical configurations in which a Parallel Manipulator (PM) loses its inherent stiffness and the motion of its end-effector becomes uncontrollable. Generally, such configurations are related to the degeneracy of a wrench system expressed in a matrix form, namely, a $6 \times 6$ Jacobian matrix $J$. Accordingly, the parallel singularities of a PM correspond to the vanishing conditions of the determinant of $J$ that can be found using either geometrical [1], symbolic [2] or numerical methods [3].

Lower-mobility PMs are suitable for a wide range of applications that require fewer than six degrees of freedom ($dof$). The legs of such manipulators apply some constraint wrenches, also known as structural constraints [4], on the end-effector. The wrench system resulting from all the constraints of the legs is called constraint wrench system of the PM, also known as platform constraint system [5]. In turn, the actuators apply some actuation wrenches on the end-effector. The wrench system resulting from all the actuation wrenches of the legs is called actuation wrench system of the PM. In a general configuration, by locking actuators, the end-effector of a PM becomes fully constrained. In screw theory, a zero-pitch wrench corresponds to the Plücker coordinate vector of a finite line in the 3-dimensional projective space $\mathbb{P}^3$ whereas an infinite-pitch wrench corresponds to the Plücker coordinate vector of a line at infinity in $\mathbb{P}^3$.

For a 6-$dof$ PM, $J$ is a $6 \times 6$ matrix that transforms the velocities of the actuators to the linear and angular velocities of the end-effector. Zlatanov et al. [6] proposed an approach, using an input-output velocity relationship, to formulate a $6 \times 6$ Jacobian matrix for 6-$dof$ PMs and also for lower-mobility PMs whose legs and end-effector have the same $dof$. Nevertheless, their approach cannot provide a $6 \times 6$ Jacobian matrix for a more general lower-mobility PM and thus it does not allow the examination of all singular configurations of such a PM.

Based on the theory of reciprocal screws [7–11], Joshi and Tsai [12] developed a general methodology to derive a $6 \times 6$ Jacobian matrix $J$ providing information about both constraint and actuation wrench systems of a lower-mobility PM. Accordingly, the rows of $J$ for a ($n < 6$)-$dof$ PM can be composed of $n$ linearly independent actuation wrenches plus $(6 - n)$ linearly independent constraint wrenches. As a result, a lower-mobility PM can exhibit two different types of parallel singularities: (i) constraint singularities [13], also known as platform singularities [5] and (ii) actuation singularities, also called architecture singularities [12]. Constraint singularities correspond to the degeneracy of the constraint wrench system. In such configurations, the end-effector loses some constraints and gains some extra $dof$. On the other hand, actuation singularities occur if $J$ is rank deficient while the constraint wrench does not degenerate.

The Grassmann-Cayley Algebra (GCA) was developed by H. Grassmann as a calculus for linear varieties. The superbracket of GCA is a determinant of six 2-extensors that are associated with six Plücker lines. Accordingly, by selecting two points on each Plücker line of $J$, a superbracket expression can be formulated and amounts to the determinant.

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of $\mathbf{J}$, up to scale. By exploring the foregoing superbracket thanks to the GCA operators, the singularity conditions of a PM can be obtained. In turn, Grassmann geometry provides a classification of linear varieties.

For Gough-Stewart PMs, the rows of $\mathbf{J}$ are Plücker coordinate vectors of six finite lines that are six actuation forces applied by the actuators on the end-effector. Parallel singularities of such manipulators occur when those lines become linearly dependent, i.e., when the linear variety spanned by these lines degenerates. Merlet [1] analyzed these singularities using Grassmann geometry by using a set of geometric rules for which a linear variety of dimension $n \leq 6$ degenerates. On the other hand, Ben-Horin et al. [14–16] used rules for which a linear variety of dimension $n \leq 6$ degenerates. Merlet [1] analyzed these singularities of such manipulators occur when those lines become coordinate vectors of six finite lines that are six actuation forces a classification of linear varieties.

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2.2 Theory of Reciprocal Screws

Screw theory is suitable for the type synthesis and the study of the instantaneous motion of PMs. The principle of reciprocal screws was studied in [7–9] and then developed in [10,13,18].

A unit screw is given by $\mathbf{S} = [s; (s_0 \times s + \lambda s)]$ where $s$ is a unit vector along the screw axis, $s_0$ is the position vector of a point on the screw axis with respect to a reference frame and $\lambda$ is the pitch of the screw. A screw of intensity $\rho$ could be written as: $\mathbf{S} = \rho \mathbf{S}$. A zero-pitch screw $\mathbf{S}_0 = [s; \rho] = \mathbf{s}$ corresponds to the Plücker coordinate vector of a finite line in $\mathbb{P}^3$. An infinite-pitch screw $\mathbf{S}_\infty = \mathbf{T}$ corresponds to the Plücker coordinate vector of a line at infinity in $\mathbb{P}^3$.

A screw system of order $n$ $(0 \leq n \leq 6)$, also called $n$-system, comprises all screws that are linearly dependent on given linearly independent screws. The reciprocity conditions of finite- and/or infinite-pitch screws are:

(a) Two zero-pitch screws are reciprocal to each other if and only if their axes are coplanar;

(b) A zero-pitch screw $\mathbf{S}_0$ is reciprocal to an infinite-pitch screw $\mathbf{S}_\infty$ if their directions are orthogonal to each other;

(c) Two infinite-pitch screws are always reciprocal to each other.

All screws that are reciprocal to a $n$-system of screws $(n < 6)$ form a $(6 - n)$-system.

\[\mathbf{S} = [s; \mathbf{r} \times s] \]

1In the scope of this paper, $(s; \mathbf{r} \times s)$ denotes a Plücker coordinate vector given by $[s; \mathbf{r} \times s]$. 

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All screws that are reciprocal to a $n$-system of screws $(n < 6)$ form a $(6 - n)$-system.
A twist is a screw representing the instantaneous motion of a rigid body, a wrench is a screw representing a system of forces and moments acting on a rigid body. A zero-pitch twist \( \bar{\tau}_0 \) is associated with a pure rotation while an infinite-pitch twist \( \bar{\tau}_\infty \) is associated with a pure translation. A zero-pitch wrench \( \bar{F} = (s; r \times s) \) represents a pure force whereas an infinite-pitch wrench \( \bar{M} = (\mathbf{0}_3 \times \mathbf{n}) \) represents a pure moment.

In this paper, screw theory is used to determine the Jacobian matrix \( J \) of the manipulator under study. Accordingly, the reciprocity condition is applied in order to characterize, for each leg-chain of the manipulator, the constraint wrench system reciprocal to its twist system as well as the actuation wrench system obtained by locking actuators.

### 2.3 Grassmann-Cayley Algebra

The GCA was developed by H. Grassmann (1809-1877) as a calculus for linear varieties operating on extensors with the join \( "\lor" \) and meet \( "\land" \) operators. The latter are associated with the span and intersection of vector spaces of extensors characterized with their step. In the \((d = 4)\)-dimensional vector space \( V \) associated with \( \mathbb{P}^3 \), extensors of step 1, 2 and 3 represent points, lines and planes, respectively. They are also associated with subspaces of \( V \) of dimension 1, 2 and 3, respectively. The GCA makes it possible to work at the symbolic level, and therefore, to produce coordinate-free algebraic expressions for the geometric singularity conditions of spatial PMs. In this section, some properties of GCA are recalled. For further details on GCA, the reader is referred to [15, 19–21].

#### 2.3.1 The superbracket decomposition

Generally, the rows of \( J \) of a PM are Plücker coordinate vectors of six lines in \( \mathbb{P}^3 \). The superjoin of these six vectors in \( \mathbb{P}^3 \) corresponds to the determinant of \( J \) up to a scalar multiple, which is the superbracket in GCA \( \Lambda(V^{(2)}) \) [21]. Thus, a singularity occurs when these six projective lines become linearly dependent, which amounts to a superbracket equal to zero. The superbracket is an expression involving 12 points selected on these lines and can be developed into a linear combination of 24 bracket monomials \( y_i \) \((i = 1, \ldots, 24) \) [15, 22], each one being the product of three brackets of four projective points:

\[
[ab, cd, ef, gh, i, j, kl] = \sum_{i=1}^{24} y_i \quad (1)
\]

These 24 bracket monomials are given in Appendix A. A bracket \([abcd]\) is null if and only if the projective points \( a, b, c \) and \( d \) are coplanar. The bracket of four projective points is defined as the determinant of the matrix whose columns are the homogeneous coordinates of these points.

#### 2.3.2 Geometric incidences in \( \mathbb{P}^3 \)

The join and meet operators were used in [16, 17] to describe many geometric incidences between elements in the projective space \( \mathbb{P}^3 \). For instance, the following geometric incidence developed in Appendix B is used in Sec. 3.2.

\[
gh \land (abc \land def) = [gabc][hdef] - [habc][gdef] = [gabc][hdef] \quad (2)
\]

where dotted letters stand for parametric elements as mentioned in [21]. Equation (2) is equal to zero if line (gh) intersects with the intersection line of planes (abc) and (def).

### 3 Singularity Analysis of The H4 Robot

The H4 robot belongs to a recent family of 4-dof parallel robots designed for high-speed pick and place operations. There exist two varieties of architectures of the H4 family with either revolute or prismatic actuators [23, 24]. The H4 robot shown in Fig. 1 is composed of four identical legs \( l_i = \mathbb{R}^i-(4S)^i \) \((i = 1, \ldots, 4) \) attached to a common base \( B \) and linked to the end-effector \( E \) by means of an articulated nacelle. The actuated joints are the \( \mathbb{R}^i \)-joints of legs \( l_i \), \( i = 1, \ldots, 4 \). The axes of \( \mathbb{R}^1 \) and \( \mathbb{R}^2 \) (respectively \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \)) are directed along \( y \) (respectively \( x \)). The nacelle is composed of three bodies: (i) link \( b_I \) connecting \( l_1 \) and \( l_2 \) in parallel and making a resulting chain \( l_{12} \); (ii) link \( b_{II} \) connecting \( l_3 \) and \( l_4 \) in parallel and making a resulting chain \( l_{34} \); (iii) the end-effector \( E \) linked to \( b_I \) and \( b_{II} \) with two revolute joints \( R_I \) and \( R_{II} \) having two parallel vertical axes. Finally, the H4 robot has two compound legs: \( L_I = l_{12} - R_I \) and \( L_{II} = l_{34} - R_{II} \).

A 4S joint is composed of four links \( mn, mp, nq \) and \( pq \) connected with four spherical joints centered at \( m, n, p \) and \( q \), respectively. According to [23, 25], the H4 robot is designed in such a way that points \( m, n, p \) and \( q \) of a \( (4S)^i \) joint, represented in Fig. 2, form a parallelogram in any robot configuration. In a leg \( l_i \) shown in Fig. 2, let \( \mathbf{n}_i \) denote the unit vector normal to this parallelogram and let \( \mathbf{s}_1 \) and \( \mathbf{l}_1 \) denote the unit vectors along lines \( mn \) and \( mp \), respectively. Note that lines \( mn \) and \( pq \) and the axis of the actuated joint \( \mathbb{R}^i \) are parallel. Therefore, \( \mathbf{s}_1 \parallel \mathbf{s}_3 \parallel y \) and \( \mathbf{s}_2 \parallel \mathbf{s}_4 \parallel x \).

\( ^2R, \mathbf{S} \) and \( (4S) \) denote a revolute joint, a spherical joint and a spatial parallelogram joint, respectively.
3.1 Constraint analysis

Based on the results obtained in [11, 26], the twist system of a (4S)$^i$ joint is equivalent to the twist system of a virtual serial chain composed of two virtual revolute joints $R_1^i$ and $R_2^i$, a virtual prismatic joint $P^i$ and a virtual $\Pi^i$-joint$^3$ whose associated twists are: $\hat{\varepsilon}_{02}^i = (s; r_x^i \times s_i)$, $\hat{\varepsilon}_{03}^i = (l; r_y^i \times l_i)$, $\hat{\varepsilon}_{04}^i = (0_{3 \times 1}; n_i)$ and $\hat{\varepsilon}_{05}^i = (0_{3 \times 1}; l_i \times n_i)$, respectively. Accordingly, the twist graph of the $H4$ robot is shown in Fig. 3. It represents the independent joint twists of the legs. In this graph, lines and circles represent joints and links, respectively whereas dashed lines and dashed circles represent virtual joints and virtual links, respectively. Let $\hat{\varepsilon}_{01}^i$ be the twist associated with $R^i$. The constraint wrench system $W_i$ of leg $l_i = R^i-(4S)^i$ ($i = 1, \ldots, 4$) is reciprocal to twists $\hat{\varepsilon}_{01}^i, \hat{\varepsilon}_{02}^i, \hat{\varepsilon}_{03}^i, \hat{\varepsilon}_{04}^i$ and $\hat{\varepsilon}_{05}^i$. Thus, it is a 1-system spanned by $\hat{M}_i$ such that:

$$\hat{M}_1 = (0_{3 \times 1}; y \times l_1) = (0_{3 \times 1}; n_1) \tag{3}$$
$$\hat{M}_2 = (0_{3 \times 1}; x \times l_2) = (0_{3 \times 1}; n_2) \tag{4}$$
$$\hat{M}_3 = (0_{3 \times 1}; y \times l_3) = (0_{3 \times 1}; n_3) \tag{5}$$
$$\hat{M}_4 = (0_{3 \times 1}; x \times l_4) = (0_{3 \times 1}; n_4) \tag{6}$$

The wrench system $W_{12}$ and the twist system $T_{12}$ of chain $l_{12}$ are$^4$:

$$W_{12} = W_1 + W_2 = \text{span}(\hat{M}_1, \hat{M}_2) \tag{7}$$
$$T_{12} = W_{12}^T = \text{span}(\hat{\varepsilon}_{01}^1, \hat{\varepsilon}_{04}^1, \hat{\varepsilon}_{012}) \tag{8}$$

where $\hat{\varepsilon}_{01}^1 = (0_{3 \times 1}; x)$, $\hat{\varepsilon}_{04}^1 = (0_{3 \times 1}; y)$ and $\hat{\varepsilon}_{012} = (0_{3 \times 1}; z)$ and the axis of $\hat{\varepsilon}_{012}$ is directed along $n_{12} = n_1 \times n_2$. Thus, this chain can be assimilated to a parallel manipulator with three translations and one rotation about an axis directed along $n_{12}$. Similarly for chain $l_{34}$,

$$T_{34} = W_{34}^T = \text{span}(\hat{\varepsilon}_{01}^3, \hat{\varepsilon}_{04}^3, \hat{\varepsilon}_{013}) \tag{9}$$

where the axis of $\hat{\varepsilon}_{013}$ is directed along $n_{34} = n_3 \times n_4$.

The twist system of leg $l_I = l_{12} - R_y$ is $T_I = T_{12} + T_{R_y}$. $R_y$ is a revolute joint of axis $Z_I$ (Fig. 4), represented with the twist $\hat{\varepsilon}_{0Z_I} = (z; r_z \times z)$. Thus,

$$T_I = \text{span}(\hat{\varepsilon}_{01}, \hat{\varepsilon}_{04}, \hat{\varepsilon}_{013}, \hat{\varepsilon}_{012}, \hat{\varepsilon}_{0Z_I}) \tag{10}$$

Therefore,

$$W_I = T_I^T = \text{span}(\hat{M}_I) : \hat{M}_I = (0_{3 \times 1}; n_{12} \times z) \tag{11}$$

Likewise,

$$W_{II} = \text{span}(\hat{M}_{II}) : \hat{M}_{II} = (0_{3 \times 1}; n_{34} \times z) \tag{12}$$

Finally, the constraint wrench system of the $H4$ robot is a 2-system spanned by two infinite-pitch wrenches of directions.

---

$^3$A $\Pi^i$-joint couples two links while allowing a relative translation along a circular trajectory [27].

$^4$The notation “$\perp$” denotes the reciprocity of two screw systems.
Correspond to two lines at infinity and four finite lines expressed as the six unit screws \( \hat{M}_1, \hat{M}_ll, \hat{F}_1, \hat{F}_2, \hat{F}_3 \) and \( \hat{F}_4 \) that correspond to two lines at infinity and four finite lines orthogonal to \( z \), as follows:

\[
\mathcal{W}_{H4} = \mathcal{W}_I + \mathcal{W}_II = \text{span}(\hat{M}_1, \hat{M}_III) \tag{13}
\]

Accordingly, the end effector \( E \) of the robot provides three independent translations and one rotation about an axis of fixed direction along \( z \), namely, a Schönflies motion \([28]\).

Legs \( l_1 \) and \( l_2 \) (respectively \( l_3 \) and \( l_4 \)) are connected with \( R_I \) (respectively \( R_II \)). The actuated joints of the \( H4 \) robot are the \( R'_I \) joints of legs \( l_i \) (\( i = 1, \ldots, 4 \)). The actuation wrench \( \hat{F}_i \) of leg \( l_i \), \( i = 1, 2 \) (respectively \( i = 3, 4 \)), is a zero-pitch wrench of axis parallel to \( I_i \), intersecting axis \( Z_i \) (respectively \( Z_{II} \)) and lying in the plane \( \Omega_i \) of the \((4S)^I \) joint. Consequently, \( \hat{F}_i = (I_i; r'_i \times 1_i) \) (\( i = 1, \ldots, 4 \)) and the actuation wrench system of the \( H4 \) robot is:

\[
\mathcal{W}_{H4} = \text{span}(\hat{F}_1, \hat{F}_2, \hat{F}_3, \hat{F}_4) \tag{14}
\]

### 3.2 Superbracket of the \( H4 \) Robot

The rows of the Jacobian matrix of the \( H4 \) robot can be expressed as the six unit screws \( \hat{M}_1, \hat{M}_II, \hat{F}_1, \hat{F}_2, \hat{F}_3 \) and \( \hat{F}_4 \) that correspond to two lines at infinity and four finite lines in \( \mathbb{P}^3 \), respectively. To obtain the twelve points of the superbracket of the robot, we must select two points on each of these six lines. The choice of the latter points must lead to the simplest expression of the superbracket decomposition, i.e., to a reduced number of non-zero monomials. A monomial in Eq. (1) is null if one of its three brackets vanishes. In turn, a bracket is null if its four projective points are coplanar. For example, a bracket taking the following form: \([abca] \) is null.

In order to represent the lines at infinity in the superbracket, it is necessary to use points at infinity. A finite line has one unique point at infinity. Since the set of all points at infinity is a plane \( \Omega_\infty \), the bracket of four points at infinity is null. Accordingly, in case some lines at infinity are among the six Plücker lines of the robot\(^5\), it is convenient to include as many points at infinity as possible in the superbracket. Therefore, a finite line is represented with one finite point and its unique point at infinity.

Since points at infinity cannot be shown in Fig. 4, a representation in \( \mathbb{P}^3 \) of the wrenches of the \( H4 \) robot is given in Fig. 5. This representation is based on the properties of the projective space that are given in Sec. 2.

Let \( a \) (respectively \( c \)) denote the intersection point of \( \hat{F}_1 \) (respectively \( \hat{F}_2 \)) and \( Z_I \) and let \( e \) (respectively \( g \)) denote the intersection point of \( \hat{F}_3 \) (respectively \( \hat{F}_4 \)) and \( Z_{II} \) (Fig. 4). Besides, \( Z_I \) and \( Z_{II} \) are directed along \( z \), i.e., \( ac \) and \( eg \) are parallel finite lines and they intersect in \( \Omega_\infty \) at \( \hat{a} = (z; 0) \), the latter being the point at infinity corresponding to \( z \).

Let \( \hat{b} = (l_1; 0) \), \( \hat{d} = (l_2; 0) \), \( \hat{f} = (l_3; 0) \) and \( \hat{h} = (l_4; 0) \) be the points at infinity corresponding to \( l_1, l_2, l_3 \) and \( l_4 \), respectively. Therefore,

\[
\hat{F}_1 = ab \; ; \; \hat{F}_2 = cd \; ; \; \hat{F}_3 = ef \; ; \; \hat{F}_4 = gh \tag{15}
\]

\(^5\)In general, this is the case of lower mobility PMs with at least one limited rotational dof.
The finite line $\mathcal{M}_1 = (0_3 \times 1; y \times l_1)$ passes through the points at infinity $y = (y; 0)$ and $h = (l_1; 0)$:

$$\mathcal{M}_1 = (0_3 \times 1; y \times l_1) = (0_3 \times 1; n_1) = yh$$

Likewise,

$$\mathcal{M}_2 = (0_3 \times 1; x \times l_2) = (0_3 \times 1; n_2) = xd$$

Furthermore, $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect at the point at infinity $\mathcal{h} = (n_{12}; 0)$, where $n_{12} = n_1 \times n_2$.

In the same vein,

$$\mathcal{M}_3 = (0_3 \times 1; y \times l_3) = (0_3 \times 1; n_3) = yf$$

$$\mathcal{M}_4 = (0_3 \times 1; x \times l_4) = (0_3 \times 1; n_4) = xh$$

Lines $\mathcal{M}_3$ and $\mathcal{M}_4$ intersect at point $\mathcal{h} = (n_{34}; 0)$ where $n_{34} = n_3 \times n_4$. Finally,

$$\mathcal{M}_I = (0_3 \times 1; n_{12} \times z) = \frac{h}{j}$$

$$\mathcal{M}_II = (0_3 \times 1; n_{34} \times z) = \frac{k}{j}$$

From Eqs. (15), (20) and (21), the superbracket of the $H4$ robot takes the following expression:

$$S_{H4} = [a b c d e f g h i j k]$$

3.3 Superbracket decomposition

From Appendix A, $S_{H4}$ is decomposed into a linear combination of 24 bracket monomials. Since the bracket of four coplanar projective points is null, the superbracket decomposition leads to 5 non-zero monomials as follows:

$$S_{H4} = \left[ a b c d e f h j g i k j \right] - \left[ a b c f d e h j g i k j \right] + \left[ a b d f c g h j e i k j \right] + \left[ a b c h d e f j g i k j \right] - \left[ a b d h c e f j g i k j \right]$$

We know that

$$[g \, i \, k \, j] = \begin{vmatrix} g_1 & i_1 & k_1 & j_1 \\ g_2 & i_2 & k_2 & j_2 \\ g_3 & i_3 & k_3 & j_3 \\ 1 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} i_1 & k_1 & j_1 \\ i_2 & k_2 & j_2 \\ i_3 & k_3 & j_3 \end{vmatrix} = [e \, i \, k \, j]$$

(24)

Hence, $[g \, i \, k \, j]$ is a common factor of the five monomials of Eq. (23) as follows:

$$S_{H4} = [g \, i \, k \, j] \left( -[a \, b \, c \, d \, e \, f \, h \, j] + [a \, b \, c \, f \, d \, e \, h \, j] \right) + [a \, b \, d \, f \, c \, g \, h \, j] + [a \, b \, c \, h \, d \, e \, f \, j] - [a \, b \, d \, h \, c \, e \, f \, j]$$

(25)

From the syzygies or Grassmann-Plücker relations [15, 21], it follows that:

$$[a b c e d f h j] = [a b c d e f h j] + [a b c f d e h j] + [a b c h d e f j] + [a b c h d e f j]$$

(26)

We know that $[d f h j] = 0$, $[d e h j] = [d g h j]$ and $[d e f j] = -[d e f j]$. Furthermore, points $a$, $c$ and $j$ being on the same projective line, $[a b c j] = 0$. Thus,

$$-[a b c d e f h j] + [a b c f d e h j] + [a b c h d e f j] = 0$$

Therefore,

$$S_{H4} = [g \, i \, k \, j] \left( [a b d f c g h j] - [a b d h c e f j] \right)$$

(27)

We know that $j = g - e$, i.e., $e = g - j$. As a result,

$$[c e f j] = [c (g - j) f j] = [c g f j]$$

(28)

Finally,

$$S_{H4} = [g \, i \, k \, j] \left( [a b d f c g h j] - [a b d h c g f j] \right) = [g \, i \, k \, j] \left( [a b d f \, c g h j] - [a b d h \, c g f j] \right)$$

(29)

where dotted letters stand for the permuted elements, as mentioned in [21].

3.4 Geometric singularity conditions of the $H4$ robot

From Eq. (29), a parallel singularity occurs when:

$$[g \, i \, k \, j] = 0 \iff [i \, k \, j] = 0 \iff \frac{h}{j} \text{ and } \frac{k}{j} \text{ belong to the same projective line. According to Eq. (24) and since } \frac{h}{j} = (n_{12}; 0), \frac{k}{j} = (n_{34}; 0) \text{ and } \frac{j}{z} = (z; 0), \text{ this condition is expressed in a vector form as follows:}$$

$$\begin{vmatrix} n_{12} \\ n_{34} \end{vmatrix} \cdot z = 0$$

(30)

where $n_{12} = n_1 \times n_2$ and $n_{34} = n_3 \times n_4$. In turn, $n_1 = y \times l_1$, $n_2 = x \times l_2$, $n_3 = y \times l_3$ and $n_4 = x \times l_4$. Thus, Eq. (30) is equivalent to:

$$\left( (y \times l_1) \times (x \times l_2) \right) \cdot z = 0$$

(31)

Equation (30) expresses the dependency of the two constraint moments $\mathcal{M}_I = (0_3 \times 1; n_{12} \times z) = \frac{h}{j}$ and $\mathcal{M}_II = (0_3 \times 1; n_{34} \times z) = \frac{k}{j}$. Therefore, it corresponds to the constraint singularities in which the robot loses one or two constraints and gains one or two extra dof.
2. \[ \begin{bmatrix} a & b & d & f \\ c & g & h & j \end{bmatrix} = 0 \Leftrightarrow (a, b, d, f) \cap (c, g, h, j) \neq \emptyset \] projective line \((f, l)\) intersects with the intersection line of planes \((a, b, d, f)\) and \((c, g, h, j)\) (see Eq. (2)). This condition is expressed in a vector form as follows:

\[
(\mathbf{l}_1 \times \mathbf{l}_2) \times ((\mathbf{r}_g - \mathbf{r}_e) \times \mathbf{z}) \cdot (\mathbf{l}_3 \times \mathbf{l}_4) = 0 \quad (32)
\]

Equation (32) corresponds to the actuation singularities, namely, configurations in which the actuators cannot control the end-effector’s motion.

4 Motions associated with Parallel Singularities

In a constraint singularity, the motion pattern of a PM may change and its end-effector might gain some limited motion(s). On the other hand, in an actuation singularity, the actuators of a PM cannot control its end-effector’s motion, namely, the end-effector might generate some infinitesimal motion(s) even when the actuators are locked. In this section, the different cases of both constraint and actuation singularities are enumerated and the gained and/or the uncontrollable motions are interpreted.

4.1 Constraint singularities and gained motions

Constraint singularities correspond to the degeneracy of the constraint wrench system, which produces a change in the limited motion of the end-effector. By solving Eq. (30), the conditions for constraint singularities turn out to be:

(a) \( \mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}_{3 \times 1} \), i.e., planes \( \Omega_1 \) and \( \Omega_2 \) are parallel and, as a consequence, \( \mathbf{n}_{12} = \mathbf{0}_{3 \times 1} \). Since \( \mathbf{n}_1 = y \times \mathbf{l}_1 \) and \( \mathbf{n}_2 = x \times \mathbf{l}_2 \), this condition can occur if and only if \( \mathbf{l}_1 \) and \( \mathbf{l}_2 \) are parallel to plane \((\mathbf{x}, \mathbf{O}_y)\), i.e., if \( \mathbf{n}_1 \parallel \mathbf{n}_2 \parallel \mathbf{z} \). It means that points \( b \) and \( d \) belong to the line at infinity \((\mathbf{x}, \mathbf{y})\), which coincides with lines \( \hat{M}_1 \) and \( \hat{M}_2 \). Accordingly, point \( \mathbf{i} \) cannot be defined. In that case, the constraint moment \( \hat{M}_1 \) is null and the robot gains one extra dof. Indeed, the only limited dof is the rotation about an axis directed along \( \mathbf{z} \times \mathbf{n}_{34} \);

(b) \( \mathbf{n}_3 \times \mathbf{n}_4 = \mathbf{0}_{3 \times 1} \); this condition can be explored similarly to the previous condition;

(c) \( \mathbf{n}_{12} \times \mathbf{n}_{34} = \mathbf{0}_{3 \times 1} \Leftrightarrow \mathbf{n}_{12} \parallel \mathbf{n}_{34} \); this happens when the intersection line of planes \( \Omega_1 \) and \( \Omega_2 \) becomes parallel to the intersection line of planes \( \Omega_3 \) and \( \Omega_4 \). This implies that \( \mathbf{i} \parallel \mathbf{k} \). Under this condition, the robot has only one constraint moment \( \hat{M}_1 \equiv \hat{M}_II \) and the end-effector has only one constraint, namely, the rotation about an axis directed along \( \mathbf{z} \times \mathbf{n}_{12} \parallel \mathbf{z} \times \mathbf{n}_{34} \). For example, when \( \mathbf{n}_1 \parallel \mathbf{n}_3 \) and \( \mathbf{n}_2 \parallel \mathbf{n}_4 \) such a configuration is illustrated in Fig. 6;

(d) \( \mathbf{n}_{12} \parallel \mathbf{z} \); this can occur if \( \mathbf{n}_{12} = \mathbf{n}_1 \times \mathbf{n}_2 \) is parallel to \( \mathbf{z} \), i.e., if \( \mathbf{n}_1 = y \times \mathbf{l}_1 \) and \( \mathbf{n}_2 = x \times \mathbf{l}_2 \) are parallel to plane \((\mathbf{x}, \mathbf{O}_y)\). This implies to have \( \mathbf{l}_1 \parallel \mathbf{z} \parallel \mathbf{l}_2 \), i.e.,

\[ \mathbf{h} \equiv \mathbf{j} \equiv \mathbf{d} \equiv \mathbf{k} \equiv \mathbf{i} \equiv \mathbf{j} \]

In that case, \( \mathbf{h} \equiv \mathbf{d} \equiv \mathbf{k} \equiv \mathbf{i} \). Under this condition, the constraint moment \( \hat{M}_1 \) is null and the robot gains one extra dof;

(e) \( \mathbf{n}_{34} \parallel \mathbf{z} \); this condition can be analyzed similarly to the previous condition;

(f) \( \mathbf{n}_{12} \parallel \mathbf{n}_{34} \parallel \mathbf{z} \Leftrightarrow \mathbf{i} \parallel \mathbf{k} \equiv \mathbf{j} \); in that case, \( \hat{M}_I \) and \( \hat{M}_{II} \) are null and the robot gains two rotational dofs;

(g) \( \mathbf{n}_{12}, \mathbf{n}_{34} \) and \( \mathbf{z} \) are orthogonal to a same direction but not parallel to each other. In that case: \( \hat{M}_I \equiv \hat{M}_{II} \) and the robot gains one rotational dof.

Table 1 illustrates the constraint singularity conditions of the H4 robot as well as the dimension of the corresponding constraint wrench system \( \mathcal{W}_{H4} \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Vector condition</th>
<th>Result</th>
<th>( \dim(\mathcal{W}_{H4}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>( \mathbf{n}_1 \parallel \mathbf{n}_2 )</td>
<td>( \hat{M}_I ) is null</td>
<td>1</td>
</tr>
<tr>
<td>(d)</td>
<td>( \mathbf{n}_{12} \parallel \mathbf{z} )</td>
<td>( \hat{M}_{II} ) is null</td>
<td>0</td>
</tr>
<tr>
<td>(b)</td>
<td>( \mathbf{n}_3 \parallel \mathbf{n}_4 )</td>
<td>( \hat{M}<em>I \equiv \hat{M}</em>{II} )</td>
<td>0</td>
</tr>
<tr>
<td>(c)</td>
<td>( \mathbf{n}<em>{12} \parallel \mathbf{n}</em>{34} \parallel \mathbf{z} )</td>
<td>( \hat{M}<em>I ) and ( \hat{M}</em>{II} ) are null</td>
<td>0</td>
</tr>
</tbody>
</table>

4.2 Actuation singularities and uncontrollable motions

Actuation singularities correspond to the rank deficiency of \( \mathbf{J} \) while the constraint wrench system does not degenerate. In that case, the wrenches \( \hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{M}_4 \) and \( \hat{M}_2 \) form a system for a given actuation singularity determine the uncontrollable mo-


\footnote{Note that all vectors are considered as unit vectors, i.e., \( \mathbf{n}_{12} \parallel \mathbf{n}_{34} \) implies that \( \mathbf{n}_{12} \equiv \mathbf{n}_{34} \).}
A translation along and A combined 1-

A translation along and A rotation about an axis directed along . Consequently, in such a configuration, one

degenerates. However, in that case, points and coincide. Thus, axis lies in plane . Consequently, the actuators cannot control the rotational dof of the end-effector about an axis directed along ;

(e) \((l_1 \times l_2) \parallel (l_1 \times l_1)\): lines \((b d)\) and \((f h)\) coincide and points \(b, d, f, h\) and \(c\) are aligned as shown in Fig. 8(c). In that case, the line at infinity \((b d) \equiv (f h)\) crosses all the wrenches of \(J\). In other words, one can find an infinite pitch twist \((0_{3 \times 1}; l_1 \times l_2)\) reciprocal to all the wrenches of \(J\). Consequently, the actuators cannot control the translational dof of the end-effector along \(l_1 \times l_2\);

(f) \((l_1 \times l_k)\) is parallel to \((r_{x} - r_{z}) \times z\). This case is similar to the previous one;

(g) The three vectors \((l_1 \times l_2), (l_1 \times l_k)\) and \((r_{x} - r_{z}) \times z\) are parallel to each other. In that case, points \(b, d, f, j\) and \(u = (u; 0)\) are aligned and \(l_1, l_2, l_j, z\) and \(u = (r_{x} - r_{z})\) are orthogonal to a given direction. Thus, the actuation forces \(F_1, F_2, F_j\) lie in plane \(c g j\). Consequently, in such a configuration, one can find two projective lines \((z; r_{x} \times z)\) and \((0_{3 \times 1}; l_1 \times l_2)\) that cross all the wrenches of \(J\). Accordingly, the end-effector has two uncontrolled dof: a translation along \(l_1 \times l_2\) and a rotation about an axis directed along .

Table 2. Actuation singularities of the \(H4\) robot.

<table>
<thead>
<tr>
<th>Case(s)</th>
<th>Uncontrolled motions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a), (b), (h)</td>
<td>A combined 1 dof motion</td>
</tr>
<tr>
<td>(c)</td>
<td>A translation along (l_1 \times l_2)</td>
</tr>
<tr>
<td>(d), (e), (f)</td>
<td>A rotation about an axis directed along (z)</td>
</tr>
<tr>
<td>(g)</td>
<td>A translation along (l_1 \times l_2) plus a rotation about an axis directed along (z)</td>
</tr>
</tbody>
</table>

Fig. 7. An actuation singularity: \(l_1 \parallel l_2\).

Fig. 8. Wrench graph associated with the actuation singularity conditions (c), (d), (e) and (g).
5 Discussion

The main contributions of this paper is to apply the GCA for the singularity analysis of lower-mobility PMs with complex kinematics as well as the determination of the motions associated with the parallel singular configurations of such PMs. In comparison with other singularity analysis methods, the GCA provides, through the superbracket decomposition, an enumeration of all the singularity conditions. The singularity analysis of the H4 robot was performed in [30, 31] where the condition for constraint singularities expressed in Eq. (30) was obtained. However, the condition for actuation singularities had not been obtained in a complete and general form before, such as the one given in Eq. (32). The results obtained with GCA for the singularity analysis of the H4 robot are consequently more general than the results obtained previously. Moreover, this paper shows that with GCA, the interpretation of singular configurations can be performed based on either a vector condition or an invariant algebraic expression between some projective points of the wrench graph.

On the other side, the constraint and singularity analysis of the H4 robot in [11, 32] focused more on screw theory. Here, more emphasis is given on GCA, the selection of the superbracket expression and the simplification of the superbracket decomposition. Indeed, the singularity analysis is completed and all the singularity conditions are enumerated. Furthermore, the motions associated with a given singular configuration of the H4 robot are determined for the first time in this paper.

6 Conclusions

This paper presented a systematic approach based on Grassmann-Cayley Algebra (GCA) to analyze the singularities of Lower-Mobility PMs with an articulated nacelle and to characterize the motion associated with a given singular configuration. The proposed approach can be considered as an extension of the method proposed in [17] to manipulators with an articulated nacelle. It was explained through the singularity analysis of the H4 robot. The results provided two geometric conditions that make it possible to enumerate and interpret all the singular configurations of the H4 robot.

The proposed approach consists of three main steps. First, the twist graph of the manipulator is obtained. This twist graph represents the independent twists between the base and the end-effector and is used in order to simplify the constraint analysis of the manipulator. Then, the wrench graph of the manipulator is obtained. This wrench graph is very useful for the singularity analysis of robots with an articulated nacelle. It represents the wrenches, obtained with the constraint analysis, in the projective space P^3. Moreover, it allows one to select and express the points at infinity of the superbracket. Finally, the superbracket is analyzed in order to determine and interpret the parallel singularity conditions of the manipulator. These three steps can be followed to perform an exhaustive determination of the singularity conditions of lower-mobility PMs, mainly those having an articulated nacelle like the Par4 robot [33].


