# Stochastic dominance with respect to a capacity and risk measures

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**Abstract**

Pursuing our previous work in which the classical notion of *increasing convex* stochastic dominance relation with respect to a probability has been extended to the case of a normalised monotone (but not necessarily additive) set function also called a capacity, the present paper gives a generalization to the case of a capacity of the classical notion of *increasing* stochastic dominance relation. This relation is characterized by using the notions of distribution function and quantile function with respect to the given capacity. Characterizations, involving Choquet integrals with respect to a distorted capacity, are established for the classes of monetary risk measures (defined on the space of bounded real-valued measurable functions) satisfying the properties of comonotonic additivity and consistency with respect to a given generalized stochastic dominance relation. Moreover, under suitable assumptions, a "Kusuoka-type" characterization is proved for the class of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the generalized increasing convex stochastic dominance relation. Generalizations to the case of a capacity of some well-known risk measures (such as the Value at Risk or the Tail Value at Risk) are provided as examples. It is also established that some

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well-known results about Choquet integrals with respect to a distorted probability do not necessarily hold true in the more general case of a distorted capacity.

**Keywords:** Choquet integral, stochastic orderings with respect to a capacity, distortion risk measure, quantile function with respect to a capacity, distorted capacity, Choquet expected utility, ambiguity, non-additive probability, Value at Risk, Rank-dependent expected utility, behavioural finance, maximal correlation risk measure, quantile-based risk measure, Kusuoka's characterization theorem

# 1 Introduction

Capacities (which are normalised monotone set functions) and integration with respect to capacities were introduced by Choquet and were afterwards applied in different areas such as economics and finance among many others (cf. for instance Wang and Yan 2007 for an overview of applications). In economics and finance, capacities and Choquet integrals have been used, in particular, to build alternative theories to the "classical" setting of expected utility of Von Neumann and Morgenstern. Indeed, the classical expected utility paradigm has been challenged by various empirical experiments and "paradoxes" (such as Allais's and Ellsberg's) thus leading to the development of new theories. One of the proposed alternative theories is the Choquet expected utility, abridged as CEU, where agent's preferences are represented by a capacity $\mu$ and a non-decreasing real-valued function $u$. The agent's "satisfaction" with a claim $X$ is then assessed by the Choquet integral of $u(X)$ with respect to the capacity $\mu$. Choquet expected utility intervenes in situations where an objective probability measure is not given and where the agents are not able to derive a subjective probability over the set of different scenarios. Other alternative theories, such as the rank-dependent expected utility theory (Quiggin 1982) and Yarii's dual theory (Yarii 1997), can be seen as particular cases of the CEU-theory.

On the other hand, stochastic orders have also been extensively used in the decision theory. They represent partial order relations on the space of random variables on some probability space $(\Omega, \mathcal{F}, P)$ (more precisely, stochastic orders are partial order relations on the set of the corresponding distribution functions). Different kinds of stochastic orders, such as the *increasing* stochastic dominance (also known as first-order stochastic dominance) and the *increasing convex* stochastic dominance, have been studied and applied and links to the expected utility theory have been explored. The reader is referred to Müller and Stoyan (2002) and Shaked and Shanthikumar (2006) for a general presentation of the subject. Hereafter, the term "classical" will be used to designate the results in the case where the initial space $(\Omega, \mathcal{F}, P)$ is a *probability* space. We recall that a random variable $X$ is said to be dominated by a random variable $Y$ in the "classical" *increasing* (resp. ***inceasing convex***) stochastic dominance with respect to a given probability $P$ if $\mathbb{E}_P(u(X)) \leq \mathbb{E}_P(u(Y))$ for all $u : \mathbb{R} \to \mathbb{R}$ *non-decreasing* (resp. ***non-decreasing and convex***) provided the expectations (taken in the Lebesgue sense) exist in $\mathbb{R}$. The definition of the "classical" stop-loss order, well-known in the insurance literature (cf., for instance, Denuit et al. 2006), is also recalled: $X$ is said to be dominated by $Y$ in the "classical" stop-loss order with respect to a given probability $P$ if $\mathbb{E}_P((X-b)^+) \leq \mathbb{E}_P((Y-b)^+)$ for all $b \in \mathbb{R}$ provided the expectations (taken in the Lebesgue sense) exist in $\mathbb{R}$. We also recall that in the "classical" case of a probability the notions of increasing convex stochastic dominance and stop-loss order coincide.

In Grigorova (2010), motivated by the CEU-theory, we have generalized the "classical" notion of *inceasing convex* stochastic dominance to the case where the measurable space $(\Omega, \mathcal{F})$ is endowed with a given capacity $\mu$ which is not necessarily a probability measure. It has been established in particular (cf. prop. 3.2 in Grigorova 2010) that the "classical" equivalence between the notions of increasing convex ordering and stop-loss ordering extends to the case where the capacity $\mu$ is assumed to be continuous from below and from above (see section 2 below for more details).

A closely related notion to the concepts mentioned above is the notion of risk measures having the properties of comonotonic additivity and consistency with respect to a given stochastic dominance relation. Risk measures having the property of consistency with respect to a given "classical" stochastic dominance relation have been extensively studied in the literature - cf. Dana (2005), Denuit et al. (2006), Song and Yan (2009) and the references given by these authors. It is argued in Denuit et al. (2006) that "it seems reasonable to require that risk measures agree with some appropriate stochastic orders". On the other hand, risk measures having the property of comonotonic additivity have been introduced and links to the Choquet integrals have been explored (see, for instance, Schmeidler's representation theorem recalled in section 2 below). For the economic interpretation of the property of comonotonic additivity and further references the reader is referred to Föllmer and Schied (2004). Monetary risk measures having the properties of comonotonic additivity and consistency with respect to a given "classical" stochastic dominance relation have been linked to the so-called distortion risk measures, introduced in the insurance literature by Wang (1996) (cf. also Wang et al. 1997, as well as Dhaene et al. 2006 and references therein). Let us denote by $\chi$ the space of bounded real-valued measurable functions on $(\Omega, \mathcal{F})$ where $(\Omega, \mathcal{F})$ is a given measurable space. It is well-known (cf. the overview by Song and Yan 2009) that the set of monetary risk measures defined on $\chi$ having the properties of comonotonic additivity and consistency with respect to the "classical" *increasing* stochastic dominance with respect to a given probability $P$ can be characterized by means of Choquet integrals with respect to a capacity of the form $\psi \circ P$ where $\psi$ is a distortion function (i.e. $\psi$ is a non-decreasing function on $[0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$). We recall that a capacity of the form $\psi \circ P$ where $P$ is a probability and $\psi$ is a distortion function is called a distorted probability (see the end of subsection 2.1 below for more details). Under a non-atomicity assumption on the initial probability space $(\Omega, \mathcal{F}, P)$, the set of monetary risk measures defined on $\chi$ having the properties of comonotonic additivity and consistency with respect to the "classical" *stop-loss* stochastic dominance with respect to the probability $P$ is known to be characterized by means of Choquet integrals with respect to a capacity of the form

$\psi \circ P$ where $\psi$ is a *concave* distortion function.

Moreover, some frequently used risk measures, such as the Value at Risk or the Tail Value at Risk among others, can be represented by means of Choquet integrals with respect to a distorted probability (cf., for instance, Dhaene et al. 2006).

The notion of risk measures which are consistent with respect to a given "classical" stochastic dominance relation is also linked to the notion of law-invariance of risk measures introduced by Kusuoka (2001). Kusuoka (2001) has provided a characterization of the class of convex law-invariant comonotonic additive monetary risk measures on the space $L^{\infty}(\Omega, \mathcal{F}, P)$ in the case where the probability space $(\Omega, \mathcal{F}, P)$ is atomless (cf. theorem 7 in Kusuoka 2001, as well as theorem 1.4 in Ekeland and Schachermayer 2011).

In the present paper we pursue our previous work from Grigorova (2010) by generalizing the "classical" notion of *increasing* stochastic dominance to the case where the measurable space $(\Omega, \mathcal{F})$ is endowed with a capacity $\mu$ which is not necessarily a probability measure. We characterize this "generalized" relation by using the notions of distribution function with respect to the capacity $\mu$ and quantile function with respect to the capacity $\mu$. Next, we study the set of monetary risk measures defined on $\chi$ having the properties of comonotonic additivity and consistency with respect to the *"generalized" increasing stochastic dominance* with respect to the capacity $\mu$, as well as the set of monetary risk measures defined on $\chi$ having the properties of comonotonic additivity and consistency with respect to the *"generalized" stop-loss stochastic dominance* with respect to the capacity $\mu$. Under suitable assumptions on the space $(\Omega, \mathcal{F}, \mu)$ we provide characterizations analogous to the classical ones. More precisely, in the case where the initial capacity $\mu$ is assumed to be continuous from below and from above, the former class of risk measures is characterized in terms of Choquet integrals with respect to a capacity of the form $\psi \circ \mu$ (which we call a distorted capacity) where $\psi$ is a distortion function. Under suitable assumptions on the space $(\Omega, \mathcal{F}, \mu)$ the latter class of risk measures is characterized by means of Choquet integrals with respect to a distorted capacity of the form $\psi \circ \mu$ whose distortion function $\psi$ is concave. We also establish that some well-known results concerning Choquet integrals with respect to a distorted probability do not necessarily hold true in the more general case of a distorted capacity (cf. subsection 3.4, as well as remarks 4.1 and 4.2). After reformulating Kusuoka's theorem in a form which is suitable for the needs of the present paper, we establish a "Kusuoka-type" characterization of the class of monetary risk measures defined on $\chi$ having the properties of comonotonic additivity and consistency with respect to the *"generalized" stop-loss stochastic dominance* with respect to the capacity $\mu$. According to this characterization (cf. theorem 4.3 below) the risk measures $\rho_\infty$ and $\rho^Y$ defined by $\rho_\infty(X) := \sup_{t<1} r^+_{X,\mu}(t)$ for all $X \in \chi$ and $\rho^Y(X) := \int_0^1 r^+_{Y,\mu}(t) r^+_{X,\mu}(t)dt$ for all $X \in \chi$, where $Y$ is a non-negative measurable function on $(\Omega, \mathcal{F})$ such that $\int_0^1 r^+_{Y,\mu}(t)dt = 1$, can be viewed as the "building blocks" of the latter class of risk measures [1]. Under additional assumptions on the initial capacity $\mu$ (namely continuity from below and from above, and concavity) a characterization involving the value function of an optimization problem studied in Grigorova (2010) is given in theorem 4.4. We end this paper by giving some examples generalizing the "classical" Value at Risk and the "classical" Tail Value at Risk to the case of a capacity which is not necessarily a probability measure. In the case of the "generalized" Value at Risk some particular subcases are studied and an economic interpretation is provided.

The remainder of the paper is organised in the following manner. Section 2 is divided in three subsections. In subsection 2.1 some definitions and results about capacities and Choquet integrals, which are used in the sequel, are recalled - all the results in this subsection but three (namely lemmas 2.3 and 2.4, and proposition 2.5) are not new. Subsection 2.2 recalls the definitions and characterizations of the "generalized" increasing

[1] The symbol $r^+_{X,\mu}$ (resp. $r^+_{Y,\mu}$) denotes the upper quantile function of $X$ (resp. of $Y$) with respect to the capacity $\mu$; the reader is referred to section 2 for more details.

convex ordering and the "generalized" stop-loss ordering with respect to a capacity in a form which is suitable for the needs of the present paper; the proofs of the results of this subsection can be found in Grigorova (2010). The terminology about risk measures is recalled in subsection 2.3.

Section 3 is divided in four subsections. In subsection 3.1 we define the "generalized" increasing stochastic ordering with respect to a capacity and provide characterizations analogous to those in the classical case of a probability measure. In subsection 3.2 we characterize the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the "generalized" increasing stochastic ordering (with respect to a given capacity $\mu$). Subsection 3.3 is devoted to the characterization of the set of monetary risk measures which are comonotonic additive and consistent with respect to the "generalized" stop-loss stochastic ordering (with respect to a given capacity $\mu$). Subsection 3.4 deals with the property of convexity of a Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$.

In section 4 (theorem 4.3 and theorem 4.4) we provide "Kusuoka-type" characterizations of the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the "generalized" stop-loss stochastic ordering (with respect to a given capacity $\mu$).

Section 5 is devoted to the examples.

# 2 Some basic definitions and results

## 2.1 Capacities and Choquet integrals

Most of the definitions and results of this subsection can be found in the book by Föllmer and Schied (2004) (cf. section 4.7 of this reference) and/or in the one by Denneberg (1994).

Let $(\Omega, \mathcal{F})$ be a measurable space. We denote by $\chi$ the space of measurable, real-valued and bounded functions on $(\Omega, \mathcal{F})$.

**Definition 2.1** *Let $(\Omega, \mathcal{F})$ be a measurable space. A set function $\mu : \mathcal{F} \longrightarrow [0,1]$ is called a capacity if it satisfies $\mu(\varnothing) = 0$, $\mu(\Omega) = 1$ (normalisation) and the following monotonicity property: $A, B \in \mathcal{F}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$.*

We recall the definition of the Choquet integral with respect to a capacity $\mu$ (cf. Denneberg 1994).

**Definition 2.2** *For a measurable real-valued function $X$ on $(\Omega, \mathcal{F})$, the Choquet integral with respect to a capacity $\mu$ is defined as follows*

$$\mathbb{E}_\mu(X) := \int_0^{+\infty} \mu(X > x)dx + \int_{-\infty}^0 (\mu(X > x) - 1)dx.$$

Note that the Choquet integral in the preceding definition may not exist (namely if one of the two (Riemann) integrals on the right side is equal to $+\infty$ and the other to $-\infty$), may be in $\mathbb{R}$ or may be equal to $+\infty$ or $-\infty$. The Choquet integral always exists if the function $X$ is bounded from below or from above. The Choquet integral exists and is finite if $X$ is in $\chi$.

Thus we come to the notion of the (non-decreasing) distribution function of $X$ with respect to a capacity $\mu$.

**Definition 2.3** *Let $X$ be a measurable function defined on $(\Omega, \mathcal{F})$. We call a distribution function of $X$ with respect to $\mu$ the non-decreasing function $G_X$ defined by*

$$G_X(x) := 1 - \mu(X > x), \quad \forall x \in \mathbb{R}.$$

**Remark 2.1** The non-decreasingness of $G_X$ is due to the monotonicity of $\mu$.

In the case where $\mu$ is a probability measure, the distribution function $G_X$ coincides with the usual distribution function $F_X$ of $X$ defined by $F_X(x) := \mu(X \leq x)$, $\forall x \in \mathbb{R}$.

For some results (such as lemma 2.3 for instance) we will need to extend the definition of $G_X$ to the extended real line $\bar{\mathbb{R}}$ which will be done by setting $G_X(+\infty) := 1 - \mu(X >$

$+\infty)$ and $G_X(-\infty) := 1 - \mu(X > -\infty)$.

Let us now define the generalized inverse of the function $G_X$.

**Definition 2.4** *For a measurable real-valued function $X$ defined on $(\Omega, \mathcal{F})$ and for a capacity $\mu$, let $G_X$ denote the distribution function of $X$ with respect to $\mu$. We call a quantile function of $X$ with respect to $\mu$ every function $r_X : (0,1) \longrightarrow \bar{\mathbb{R}}$ verifying*

$$\sup\{x \in \mathbb{R} \mid G_X(x) < t\} \leq r_X(t) \leq \sup\{x \in \mathbb{R} \mid G_X(x) \leq t\}, \ \forall t \in (0,1),$$

*where the convention* $\sup\{\varnothing\} = -\infty$ *is used.*

*The functions $r_X^-$ and $r_X^+$ defined by*

$$r_X^-(t) := \sup\{x \in \mathbb{R} \mid G_X(x) < t\}, \forall t \in (0,1) \quad \textit{and} \quad r_X^+(t) := \sup\{x \in \mathbb{R} \mid G_X(x) \leq t\}, \forall t \in (0,1)$$

*are called the lower and upper quantile functions of $X$ with respect to $\mu$.*

For notational convenience, we omit the dependence on $\mu$ in the notation $G_X$ and $r_X$ when there is no ambiguity. The following observation can be found in Föllmer and Schied (2004).

**Remark 2.2** The lower and upper quantile functions of $X$ with respect to $\mu$ can be expressed in the following manner as well:

$$r_X^-(t) := \inf\{x \in \mathbb{R} \mid G_X(x) \geq t\}, \forall t \in (0,1) \quad \text{and} \quad r_X^+(t) := \inf\{x \in \mathbb{R} \mid G_X(x) > t\}, \forall t \in (0,1)$$

**Remark 2.3** Let $\mu$ be a capacity and let $X$ be a measurable real-valued function such that

$$\lim_{x \to -\infty} G_X(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} G_X(x) = 1. \tag{2.1}$$

We denote by $G_X(x-)$ and $G_X(x+)$ the left-hand and right-hand limits of $G_X$ at $x$. A function $r_X$ is a quantile function of $X$ (with respect to $\mu$) if and only if

$$(G_X(r_X(t)-) \leq t \leq G_X(r_X(t)+), \quad \forall t \in (0,1).$$

In this case $r_X$ is real-valued. Note that the condition (2.1) is satisfied if $X \in \chi$ and $\mu$ is arbitrary. The condition (2.1) is satisfied for an arbitrary $X$ if $\mu$ is continuous from below and from above (see definition 2.5).

We have the following well-known result (cf. for instance Föllmer and Schied 2004 for the bounded case, or Denneberg 1994 - pages 61-62 in chapter 5) where we make the convention that the assertion is valid provided the expressions make sense.

**Proposition 2.1** *Let $X$ be a real-valued measurable function and let $r_X$ be a quantile function of $X$ with respect to a capacity $\mu$, then*

$$\mathbb{E}_\mu(X) = \int_0^1 r_X(t)dt.$$

The following lemma is the analogue of lemma A.23. in Föllmer and Schied (2004) and can be found in Denneberg (1994) (cf. also proposition 3.2 in Yan 2009).

**Lemma 2.1** *Let $X = f(Y)$ where $f$ is a non-decreasing function and let $r_Y$ be a quantile function of $Y$ with respect to a capacity $\mu$. Suppose that $f$ and $G_Y$ have no common discontinuities, then $f \circ r_Y$ is a quantile function of $X$ with respect to $\mu$. In particular,*

$$r_X(t) = r_{f(Y)}(t) = f(r_Y(t)) \text{ for almost every } t \in (0,1),$$

*where $r_X$ denotes a quantile function of $X$ with respect to $\mu$.*

**Remark 2.4** If the capacity $\mu$ satisfies the additional properties of continuity from below and from above, the assumption of no common discontinuities of the functions $f$ and $G_Y$ can be dropped in the previous lemma. The proof is then analogous to the proof in the classical case of a probability measure (cf. lemma A.23. in Föllmer and Schied 2004 for a proof in the classical case) and is left to the reader.

We recall some well-known definitions about capacities which will be needed later on.

**Definition 2.5** *A capacity $\mu$ is called convex (or equivalently, supermodular) if*

$$A, B \in \mathcal{F} \Rightarrow \mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

*A capacity $\mu$ is called concave (or submodular, or 2-alternating) if*

$$A, B \in \mathcal{F} \Rightarrow \mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

*A capacity $\mu$ is called continuous from below if*

$$(A_n) \subset \mathcal{F} \text{ such that } A_n \subset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n \to \infty} \mu(A_n) = \mu(\cup_{n=1}^{\infty} A_n).$$

*A capacity $\mu$ is called continuous from above if*

$$(A_n) \subset \mathcal{F} \text{ such that } A_n \supset A_{n+1}, \forall n \in \mathbb{N} \Rightarrow \lim_{n \to \infty} \mu(A_n) = \mu(\cap_{n=1}^{\infty} A_n).$$

We recall the notion of comonotonic functions (cf. Föllmer and Schied 2004).

**Definition 2.6** *Two measurable functions $X$ and $Y$ on $(\Omega, \mathcal{F})$ are called comonotonic if*

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0, \ \forall (\omega, \omega') \in \Omega \times \Omega.$$

We have the following characterization of comonotonic functions which corresponds to proposition 4.5 in Denneberg (1994) (see also Föllmer and Schied 2004)

**Proposition 2.2** *For two real-valued measurable functions $X$, $Y$ on $(\Omega, \mathcal{F})$ the following conditions are equivalent:*

**(i)** *$X$ and $Y$ are comonotonic.*

**(ii)** *There exists a measurable function $Z$ on $(\Omega, \mathcal{F})$ and two non-decreasing functions $f$ and $g$ on $\mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$.*

**(iii)** *There exist two continuous, non-decreasing functions $u$ and $v$ on $\mathbb{R}$ such that $u(z) + v(z) = z$, $z \in \mathbb{R}$, and $X = u(X+Y)$ , $Y = v(X+Y)$.*

The notion of comonotonic functions proves to be very useful while dealing with Choquet integrals thanks to the following result (cf. lemma 4.84 in Föllmer and Schied 2004, as well as corollary 4.6 in Denneberg 1994).

**Lemma 2.2** *If $X, Y : \Omega \to \mathbb{R}$ is a pair of comonotonic measurable functions and if $r_X, r_Y, r_{X+Y}$ are quantile functions (with respect to a capacity $\mu$) of $X, Y, X+Y$ respectively, then*

$$r_{X+Y}(t) = r_X(t) + r_Y(t), \text{ for almost every } t.$$

In the following propositions we summarize some of the main properties of Choquet integrals for reader's convenience (cf. proposition 5.1 in Denneberg 1994) and we make the convention that the properties are valid provided the expressions make sense (which is always the case when we restrain ourselves to elements in $\chi$).

**Proposition 2.3** *Let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ and $X$ and $Y$ be measurable real-valued functions on $(\Omega, \mathcal{F})$ , then we have the properties:*

- *(positive homogeneity)* $\mathbb{E}_\mu(\lambda X) = \lambda \mathbb{E}_\mu(X), \forall \lambda \in \mathbb{R}_+$
- *(monotonicity)* $X \leq Y \Rightarrow \mathbb{E}_\mu(X) \leq \mathbb{E}_\mu(Y)$
- *(translation invariance)* $\mathbb{E}_\mu(X + b) = \mathbb{E}_\mu(X) + b, \forall b \in \mathbb{R}$
- *(asymmetry)* $\mathbb{E}_\mu(-X) = -\mathbb{E}_{\bar{\mu}}(X)$, *where $\bar{\mu}$ is the dual capacity of $\mu$* ($\bar{\mu}(A)$ *is defined by* $\bar{\mu}(A) = 1 - \mu(A^c), \forall A \in \mathcal{F}$)
- *(comonotonic additivity) If $X$ and $Y$ are comonotonic, then* $\mathbb{E}_\mu(X + Y) = \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y)$.

Finally, we recall the subadditivity property of the Choquet integral with respect to a concave capacity.

**Proposition 2.4** *Let $\mu$ be a concave capacity on $(\Omega, \mathcal{F})$ and $X$ and $Y$ be measurable real-valued functions on $(\Omega, \mathcal{F})$ such that $\mathbb{E}_\mu(X) > -\infty$ and $\mathbb{E}_\mu(Y) > -\infty$, then we have the following property*

$$\textit{(subadditivity)} \qquad \mathbb{E}_\mu(X+Y) \leq \mathbb{E}_\mu(X) + \mathbb{E}_\mu(Y).$$

**Remark 2.5** The reader is referred to Denneberg (1994) for a slightly weaker assumption than the one given in the previous proposition.

The next theorem is known as Schmeidler's representation theorem (cf. theorem 11.2 in Denneberg 1994; cf. also theorem 4.82 in Föllmer and Schied 2004).

**Theorem 2.1 (Schmeidler's representation theorem)** *Let $\rho : \chi \longrightarrow \mathbb{R}$ be a given functional satisfying the properties of:*

**(i)** *(monotonicity)* $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$

**(ii)** *(comonotonic additivity)* $X, Y$ *comonotonic* $\Rightarrow \rho(X+Y) = \rho(X) + \rho(Y)$

**(iii)** *(normalisation)* $\rho(\mathbb{I}) = 1$.

*Then, there exists a capacity $\nu$ on $(\Omega, \mathcal{F})$ such that*

$$\rho(X) = \mathbb{E}_\nu(X), \ \forall X \in \chi.$$

**Remark 2.6** We note that the normalisation property $(iii)$ of the previous theorem is satisfied by any functional $\rho : \chi \longrightarrow \mathbb{R}$ which is assumed to have the properties of comonotonic additivity (property $(ii)$) and translation invariance. Indeed, the comonotonic additivity of $\rho$ implies that $\rho(0+0) = 2\rho(0)$ which gives $\rho(0) = 0$. This property combined with the translation invariance of $\rho$ implies the normalisation property $(iii)$. In particular, the normalisation property $(iii)$ is satisfied by any monetary risk measure $\rho : \chi \longrightarrow \mathbb{R}$ (in the sense of definition 2.9) having the property of comonotonic additivity.

Finally, we state a useful result about monotonic transformations of measurable functions and the corresponding upper quantile functions.

**Lemma 2.3** *Let $Z$ be a real-valued measurable function on $(\Omega, \mathcal{F})$, let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ and let $f$ be a non-decreasing right-continuous function. Denote by $r_Z^+$ and by $r_{f(Z)}^+$ the upper quantile functions of $Z$ and $f(Z)$ (with respect to $\mu$). Suppose that $f$ and $G_Z$ have no common discontinuities, then*

$$r_{f(Z)}^+(t) = f(r_Z^+(t)),\ \forall t \in (0, 1).$$

**Proof:** The proof of the lemma uses arguments similar to those used in the proof of proposition 3.2 in Yan (2009) and is given in the appendix.

□

An analogous result to that of lemma 2.3 holds true in the case of lower quantile functions with respect to a capacity. The result can be shown by using similar arguments to the ones used in the proof of the previous lemma 2.3 - its proof is therefore omitted.

**Lemma 2.4** *Let $Z$ be a real-valued measurable function on $(\Omega, \mathcal{F})$, let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ and let $f$ be a non-decreasing* left-continuous *function. Denote by $r_Z^-$ and by $r_{f(Z)}^-$ the* lower *quantile functions of $Z$ and $f(Z)$ (with respect to $\mu$). Suppose that $f$ and $G_Z$ have no common discontinuities, then*

$$r_{f(Z)}^-(t) = f(r_Z^-(t)),\ \forall t \in (0, 1).$$

Using the previous two lemmas 2.3 and 2.4, we state a proposition representing a generalization to the case of a capacity of a well-known "classical" result about the upper and lower quantile functions of comonotonic random variables - cf. for instance theorem 4.2.1 in Dhaene et al. (2006) for the classical case.

**Proposition 2.5** *If $X$ and $Y$ are two comonotonic real-valued measurable functions, then*

$$r^+_{X+Y}(t) = r^+_X(t) + r^+_Y(t), \ \forall t \in (0,1) \qquad \textit{and} \tag{2.2}$$

$$r^-_{X+Y}(t) = r^-_X(t) + r^-_Y(t), \ \forall t \in (0,1). \tag{2.3}$$

**Proof:** The arguments of the proof of proposition 2.5 are similar to those used in the proof of corollary 4.6 in Denneberg (1994). The proof is placed in the appendix.

□

**Remark 2.7** The previous proposition 2.5 is to be compared with lemma 2.2. In fact, lemma 2.2 can be viewed as a consequence of proposition 2.5 after recalling that a quantile function (with respect to a given capacity) of a given real-valued measurable function is unique except on an at most countable set.

We end this subsection by two examples of a capacity. The first example is well-known in the decision theory (think for instance of the rank-dependent expected utility theory - Quiggin 1982 or of Yarii's distorted utility theory in Yarii 1997); the second is a slight generalization of the first - it can be found in Denneberg (1994).

1. Let $\mu$ be a probability measure on $(\Omega, \mathcal{F})$ and let $\psi : [0,1] \to [0,1]$ be a non-decreasing function on $[0,1]$ such that $\psi(0) = 0$ and $\psi(1) = 1$. Then the set function $\psi \circ \mu$ defined by $\psi \circ \mu(A) := \psi(\mu(A)), \forall A \in \mathcal{F}$ is a capacity in the sense of definition 2.1. The function $\psi$ is called a distortion function and the capacity $\psi \circ \mu$ is called a *distorted probability*. If the distortion function $\psi$ is concave, the capacity $\psi \circ \mu$ is a concave capacity in the sense of definition 2.5.

2. Let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ and let $\psi$ be a distortion function. Then the set function $\psi \circ \mu$ is a capacity which, by analogy with the previous example, will be

called a *distorted capacity*. Moreover, we have the following property : if $\mu$ is a concave capacity and $\psi$ is concave, then $\psi \circ \mu$ is concave. The proof uses the same arguments as the proof of proposition 4.69 in Föllmer and Schied (2004) and is left to the reader (see also exercice 2.10 in Denneberg 1994).

## 2.2 Stochastic orderings with respect to a capacity

Most of the definitions and results in this subsection can be found in Grigorova (2010).

**Definition 2.7** *Let $X$ and $Y$ be two measurable functions on $(\Omega, \mathcal{F})$ and let $\mu$ be a capacity on $(\Omega, \mathcal{F})$. We say that $X$ is smaller than $Y$ in the* increasing convex *ordering (with respect to the capacity $\mu$) denoted by $X \leq_{icx} Y$ if*

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

*for all functions $u : \mathbb{R} \to \mathbb{R}$ which are non-decreasing and convex,*
*provided the Choquet integrals exist in $\mathbb{R}$.*

The previous definition coincides with the usual definition of the increasing convex order when the capacity $\mu$ is a probability measure on $(\Omega, \mathcal{F})$ (cf. Shaked and Shanthikumar 2006 for details in the classical case).
As in the previous section, the dependence on the capacity $\mu$ in the notation for the stochastic dominance relation $\leq_{icx}$ is intentionally omitted. Nevertheless, we shall note $\leq_{icx,\mu}$ when an explicit mention of the capacity to which we refer is needed.

**Remark 2.8** The economic interpretation of the increasing convex ordering with respect to a capacity $\mu$ is the following: $X \leq_{icx,\mu} Y$ if all the CEU-maximizers whose preferences are described by the (common) capacity $\mu$ and a non-decreasing convex utility function prefer the claim $Y$ to the claim $X$.
If the measurable functions $X$ and $Y$ are interpreted as *losses* (which will be the case in the sequel of the paper), the increasing convex stochastic ordering with respect to a

capacity $\mu$ can be interpreted as follows: $X \leq_{icx,\mu} Y$ if all the CEU-*minimizers* whose preferences are described by the (common) capacity $\mu$ and a non-decreasing convex *"pain"* function (see Denuit et al. 1999 for the terminology) prefer losing $X$ to losing $Y$.

We define the notion of stop-loss ordering (or stop-loss dominance relation) below.

**Definition 2.8** *Let $X$ and $Y$ be two measurable functions on $(\Omega, \mathcal{F})$ and let $\mu$ be a capacity on $(\Omega, \mathcal{F})$. We say that $X$ is smaller than $Y$ in the* stop-loss *ordering with respect to the capacity $\mu$ denoted by $X \leq_{\mathrm{sl}} Y$ if*

$$\mathbb{E}_\mu((X-b)^+) \leq \mathbb{E}_\mu((Y-b)^+), \ \forall b \in \mathbb{R},$$

*provided the Choquet integrals exist in $\mathbb{R}$.*

In the classical case where the capacity $\mu$ is a probability measure the previous definition is reduced to the usual definition of stop-loss order well-known in the insurance literature (see for instance Dhaene et al. 2006). The interpretation of the stop-loss dominance relation in the classical case is the following: $X \leq_{sl} Y$ if and only if $X$ has lower stop-loss premia than $Y$. A similar interpretation could be given in our more general setting if we see the number $\mathbb{E}_\mu((X-b)^+)$ for a given $b \in \mathbb{R}$ as a "generalized" stop-loss premium of $X$.

We have the following characterization of the stop-loss ordering relation with respect to a capacity which is due to propositions 3.3 and 3.4 in Grigorova (2010).

**Proposition 2.6** *Let $\mu$ be a capacity and let $X$ and $Y$ be two real-valued measurable functions such that $\int_0^1 |r_X(t)|dt < +\infty$ and $\int_0^1 |r_Y(t)|dt < +\infty$ where $r_X$ and $r_Y$ denote (the) quantile functions of $X$ and $Y$ with respect to $\mu$. The following three statements are equivalent:*

**(i)** $X \leq_{\mathrm{sl},\mu} Y$.

**(ii)** $\int_x^{+\infty} \mu(X > u)du \leq \int_x^{+\infty} \mu(Y > u)du$, $\forall x \in \mathbb{R}$.

**(iii)** $\int_y^1 r_X(t)dt \leq \int_y^1 r_Y(t)dt, \forall y \in (0, 1)$.

Another useful characterization of the relation $\leq_{sl,\mu}$ is given in the following proposition. Its analogue in the classical case of a probability measure is due to Dana (2005) (see also thm. 5.2.1 in Dhaene et al. 2006 for a related result). The version presented here can be found in Grigorova (2010).

**Proposition 2.7** *Let $X \in \chi$ and $Y \in \chi$ be given. Then the following statements are equivalent:*

**(i)** $X \leq_{\mathrm{sl},\mu} Y$

**(ii)** $\int_0^1 g(t)r_X(t)dt \leq \int_0^1 g(t)r_Y(t)dt$, $\forall g : [0, 1] \to \bar{\mathbb{R}}_+$, *integrable, non-decreasing.*

We have the following proposition establishing the equivalence between the increasing convex stochastic dominance and the stop-loss stochastic dominance in the case of a capacity which is continuous from below and from above (cf. proposition 3.2 in Grigorova 2010).

**Proposition 2.8** *Let $\mu$ be a capacity which is continuous from below and from above and let $X$ and $Y$ be two real-valued measurable functions. Then the following two statements are equivalent:*

**(i)** $X \leq_{\mathrm{sl},\mu} Y$.

**(ii)** $X \leq_{\mathrm{icx},\mu} Y$.

**Remark 2.9** As observed in Grigorova (2010), it can be easily seen from the definition of the increasing convex ordering that the assumption on the continuity of the capacity $\mu$ is not needed in the proof of the implication $(ii) \Rightarrow (i)$ in proposition 2.8.

## 2.3 Monetary risk measures

We will use the following definitions:

**Definition 2.9** *1. A mapping $\rho : \chi \to \mathbb{R}$ is called a monetary measure of risk if it satisfies the following properties for all $X, Y \in \chi$:*

**(i)** *(monotonicity)* $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$

**(ii)** *(translation invariance)* $\rho(X + b) = \rho(X) + b, \forall b \in \mathbb{R}$

*2. A monetary measure of risk $\rho$ is called convex if it satisfies the additional property of*

**(iii)** *(convexity)* $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$*,* $\forall \lambda \in [0, 1]$*,* $\forall X, Y \in \chi$*.*

*3. A convex monetary measure of risk $\rho$ is called coherent if it satisfies the additional property of*

**(iv)** *(positive homogeneity)* $\rho(\lambda X) = \lambda\rho(X), \forall \lambda \in \mathbb{R}_+$.

Let us remark that the above definiton of a coherent monetary risk measure coincides, up to a minus sign, with the definiton given by Artzner et al. (1999). The "sign convention" used in the present paper is frequently adopted in the insurance literature when the measurable functions are interpreted as potential losses or payments that have to be made (see, for instance, Dhaene et al. 2006 for explanations in the context of insurance; for the same "sign convention" as the one used in the present paper, the reader is also referred to Wang and Yan 2007, or Ekeland et al. 2009).

# 3 Stochastic orderings with respect to a capacity and generalized distortion risk measures

## 3.1 The increasing stochastic dominance with respect to a capacity

In this subsection we define the notion of increasing stochastic dominance with respect to a capacity and provide characterizations analogous to those existing in the "classical" case of a probability measure. The reader is referred to Shaked and Shanthikumar (2006) for details in the classical case.

**Definition 3.1** *Let $X$ and $Y$ be two measurable functions on $(\Omega, \mathcal{F})$ and let $\mu$ be a capacity on $(\Omega, \mathcal{F})$. We say that $X$ is dominated by $Y$ in the* increasing *stochastic dominance (with respect to the capacity $\mu$) denoted by $X \leq_{\mathrm{mon},\mu} Y$ if*

$$\mathbb{E}_\mu(u(X)) \leq \mathbb{E}_\mu(u(Y))$$

*for all non-decreasing functions $u : \mathbb{R} \to \mathbb{R}$ provided the Choquet integrals exist in $\mathbb{R}$.*

In the case where $\mu$ is a probability measure the preceding definition is reduced to the usual definition of increasing stochastic dominance (also known as first-order stochastic dominance).

**Remark 3.1** The economic interpretation of the increasing stochastic dominance with respect to a capacity $\mu$ is the following: $X \leq_{mon,\mu} Y$ if all CEU-maximizers whose preferences are described by the (common) capacity $\mu$ and a non-decreasing utility function prefer the claim $Y$ to the claim $X$.

We have the following characterization of the increasing stochastic dominance with respect to $\mu$.

**Proposition 3.1** *Let $\mu$ be a capacity which is continuous from below and from above. Let $X$ and $Y$ be two real-valued measurable functions. The following three statements are equivalent:*

**(i)** $X \leq_{\text{mon},\mu} Y$.

**(ii)** $G_X(x) \geq G_Y(x)$, $\forall x \in \mathbb{R}$.

**(iii)** $r_X^+(t) \leq r_Y^+(t), \forall t \in (0,1)$.

**Proof:** Let us first prove the implication $(i) \Rightarrow (ii)$. We fix $x \in \mathbb{R}$ and we remark that $G_X(x) = 1 - \mathbb{E}_\mu(u(X))$ where $u(y) := \mathbb{I}_{(x,+\infty)}(y)$ which proves the desired implication as the function $u$ is non-decreasing.
The implication $(ii) \Rightarrow (iii)$ is a consequence of the definition of the upper quantile functions $r_X^+$ and $r_Y^+$.
To conclude, we prove the implication $(iii) \Rightarrow (i)$. Suppose that $r_X^+(t) \leq r_Y^+(t), \forall t \in (0,1)$ and let $u : \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. Thanks to proposition 2.1 and to remark 2.4 (where the assumption of continuity from below and from above of $\mu$ is used) we have $\mathbb{E}_\mu(u(X)) = \int_0^1 u(r_X^+(t))dt$; the same type of representation holds for $\mathbb{E}_\mu(u(Y))$. Thus we obtain $\mathbb{E}_\mu(u(X)) = \int_0^1 u(r_X^+(t))dt \leq \int_0^1 u(r_Y^+(t))dt = \mathbb{E}_\mu(u(Y))$ which concludes the proof.

□

**Remark 3.2** We note that the implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ in the proof of proposition 3.1 have been established without using the assumption of continuity from below and from above of $\mu$.

We end this subsection by giving some vocabulary which will be useful in the sequel while dealing with risk measures.

**Definition 3.2** *Let $\rho : \chi \to \mathbb{R}$ be a given functional and let $\mu$ be a capacity. We say that $\rho$ satisfies the property of:*

1. *(*consistency with respect to $\leq_{\text{mon},\mu}$*) if $X \leq_{\text{mon},\mu} Y$ implies $\rho(X) \leq \rho(Y)$.*

2. *(*consistency with respect to $\leq_{\text{sl},\mu}$*) if $X \leq_{\text{sl},\mu} Y$ implies $\rho(X) \leq \rho(Y)$.*

3. *(*consistency with respect to $\leq_{\mathrm{icx},\mu}$*) if* $X \leq_{\mathrm{icx},\mu} Y$ *implies* $\rho(X) \leq \rho(Y)$.

The following result which is easy to establish provides a link between the notions introduced in definition 3.2.

**Proposition 3.2** *Let* $\rho : \chi \to \mathbb{R}$ *be a given functional and let* $\mu$ *be a capacity. The following statements hold:*

1. $\rho$ *is consistent with respect to* $\leq_{\mathrm{sl},\mu} \Rightarrow \rho$ *is consistent with respect to* $\leq_{\mathrm{icx},\mu} \Rightarrow \rho$ *is consistent with respect to* $\leq_{\mathrm{mon},\mu}$.

2. *If the capacity* $\mu$ *is continuous from below and from above, the consistency with respect to the relation* $\leq_{\mathrm{icx},\mu}$ *is equivalent to the consistency with respect to the relation* $\leq_{\mathrm{sl},\mu}$.

**Proof:** The first statement is due to the definitions of the relations $\leq_{icx,\mu}$, $\leq_{sl,\mu}$ and $\leq_{mon,\mu}$. The second statement is a consequence of proposition 2.8.

$\square$

## 3.2 Generalized distortion risk measures

In this subsection we are interested in risk measures which can be represented as Choquet integrals with respect to a distorted capacity. Such risk measures will be called generalized distortion risk measures.

**Definition 3.3** *Let* $\mu$ *be a capacity and let* $\psi$ *be a distortion function. A monetary risk measure* $\rho : \chi \longrightarrow \mathbb{R}$ *of the form*

$$\rho(X) = \mathbb{E}_{\psi \circ \mu}(X), \ \forall X \in \chi$$

*is called a generalized distortion risk measure.*

In the case where $\mu$ is a probability measure the previous definition is reduced to the definition of a distortion risk measure (or a distortion premium principle) well-known in finance and insurance - see, for instance, Dhaene et al. (2006) for a survey and examples. The generalization considered in definition 3.3 is suggested at the end of an article by Denneberg (1990). In Grigorova (2010) an example of a generalized distortion risk measure is obtained as the value function of the following financial optimization problem:

$$\text{(D)} \qquad \begin{gathered} \text{Maximize } \mathbb{E}_\mu(ZC) \\ \text{under the constraints } C \in \chi_+ \text{ s.t. } C \leq_{icx,\mu} X \end{gathered}$$

where $\chi_+$ denotes the set of non-negative bounded measurable functions, $\mu$ is a given (concave and continuous from below) capacity, $Z$ is a given non-negative measurable function such that $\int_0^1 r_Z(t)dt < \infty$ and $X$ is a given function in $\chi_+$.

**Remark 3.3** Any generalized distortion risk measure in the sense of definition 3.3 is a monetary risk measure satisfying the properties of positive homogeneity and comonotonic additivity. A generalized distortion risk measure is convex if and only if the distorted capacity $\psi \circ \mu$ appearing in definition 3.3 is a concave capacity. The "if part" in the previous statement has already been recalled in proposition 2.4; the "only if part" is easy to establish by using, for instance, exercise 5.1 in Denneberg (1994).

A well-known representation result in the classical case of a probability measure is generalized to the case of a capacity in the following lemma. For the statement and the proof of this result in the "classical" case we refer to Song and Yan (2009) as well as to exercise 11.3 in Denneberg (1994); the "classical" result is related to the work of Wang et al. (1997) and to the work of Kusuoka (2001) as well.

**Lemma 3.1** *Let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ and let $\rho : \chi \to \mathbb{R}$ be a functional satisfying the following properties*

**(i)** $G_{X,\mu}(x) \geq G_{Y,\mu}(x), \forall x \in \mathbb{R} \Rightarrow \rho(X) \leq \rho(Y)$

**(ii)** *(comonotonic additivity)* $X, Y$ *comonotonic* $\Rightarrow \rho(X+Y) = \rho(X) + \rho(Y)$

**(iii)** *(normalisation)* $\rho(\mathbb{I}) = 1$.

*Then, there exists a distortion function* $\psi : [0,1] \longrightarrow [0,1]$ *such that*

$$\rho(X) = \mathbb{E}_{\psi \circ \mu}(X),\ \forall X \in \chi.$$

This lemma is based on Schmeidler's representation theorem (theorem 2.1). Before we prove the lemma, let us make a remark which will be used in the proof.

**Remark 3.4** The property (i) in the previous lemma implies the property of monotonicity of $\rho$ (i.e. $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$), as well as the following property which, for the easing of the presentation, will be called *ditrsibution invariance of $\rho$ with respect to $\mu$*:

$$G_{X,\mu}(x) = G_{Y,\mu}(x),\ \forall x \in \mathbb{R} \Rightarrow \rho(X) = \rho(Y).$$

**Proof of lemma 3.1:** The functional $\rho$ being monotonic, comonotonic additive and normalised, Schmeidler's representation theorem (theorem 2.1) can be applied in order to obtain the existence of a capacity $\nu$ on $(\Omega, \mathcal{F})$ such that

$$\rho(X) = \mathbb{E}_{\nu}(X),\ \forall X \in \chi. \tag{3.1}$$

We will now prove that there exists a distortion function $\psi$ such that $\nu(A) = \psi \circ \mu(A),\ \forall A \in \mathcal{F}$. The arguments are similar to those in the "classical" case and follow the proof of proposition 2.1 in Song and Yan (2009).
Let us first note that for $A, B \in \mathcal{F}$, the distribution functions (with respect to $\mu$) $G_{\mathbb{I}_A,\mu}$ and $G_{\mathbb{I}_B,\mu}$ of the measurable functions $\mathbb{I}_A$ and $\mathbb{I}_B$ coincide if and only if $\mu(A) = \mu(B)$. Thus, the functional $\rho$ being distribution invariant with respect to $\mu$, we have that $\mu(A) = \mu(B)$ implies $\rho(\mathbb{I}_A) = \rho(\mathbb{I}_B)$ which in turn implies that $\nu(A) = \nu(B)$. Therefore, we can define a function $\psi$ on the set $S := \{\mu(A), A \in \mathcal{F}\}$ as follows:

$$\psi : \{\mu(A), A \in \mathcal{F}\} \longrightarrow [0,1]$$
$$\psi(x) := \nu(A) \text{ if } x = \mu(A).$$

The function $\psi$ is such that $\nu(A) = \psi \circ \mu(A),\ \forall A \in \mathcal{F}$. Moreover, $\psi(0) = 0$ and $\psi(1) = 1$ and $\psi$ is a non-decreasing function on $S$. The non-decreasingness of $\psi$ is a consequence of property (i). Indeed, let $A, B \in \mathcal{F}$ be such that $\mu(A) \leq \mu(B)$. Then, for all $x \in \mathbb{R}$, $G_{\mathbb{I}_A,\mu}(x) = 1 - \mu(\mathbb{I}_A > x) \geq 1 - \mu(\mathbb{I}_B > x) = G_{\mathbb{I}_B,\mu}(x)$. The inequality $\nu(A) \leq \nu(B)$ follows thanks to property (i) and to the representation (3.1). We conclude the proof as in Song and Yan (2009) by arguing that the function $\psi$ can be extended to a non-decreasing function on the closure of the set $S$ and then to a non-decreasing function on $[0, 1]$.

$\square$

**Remark 3.5** The converse statement in lemma 3.1 also holds true. More precisely, let $\mu$ be a capacity and let $\rho : \chi \to \mathbb{R}$ be a functional of the form $\rho(.) = \mathbb{E}_{\psi\circ\mu}(.)$ where $\psi$ is a distortion function. As a Choquet integral with respect to a capacity, the functional $\rho$ obviously satisfies properties (ii) and (iii) in lemma 3.1. Property (i) in lemma 3.1 is also satisfied as the functional $\rho$ can be written in the following manner: $\rho(X) = \mathbb{E}_{\psi\circ\mu}(X) = \int_0^{+\infty} \psi(1 - G_{X,\mu}(x))dx + \int_{-\infty}^0 \psi(1 - G_{X,\mu}(x)) - 1dx,\ \forall X \in \chi$.

The following theorem is a "generalization" to the case of a capacity of a well-known representation result for monetary risk measures satisfying the properties of comonotonic additivity and consistency with respect to the "classical" increasing stochastic dominance (see for instance Song and Yan (2009) for the classical case).

**Theorem 3.1** *Let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ which is continuous from below and from above and let $\rho : \chi \to \mathbb{R}$ be a monetary risk measure satisfying the properties of*

**(i)** *(consistency with respect to $\leq_{\mathrm{mon},\mu}$) $X \leq_{\mathrm{mon},\mu} Y \Rightarrow \rho(X) \leq \rho(Y)$*

**(ii)** *(comonotonic additivity) $X, Y$ comonotonic $\Rightarrow \rho(X + Y) = \rho(X) + \rho(Y)$.*

*Then, there exists a distortion function $\psi$ such that*

$$\rho(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi.$$

**Proof:** The result follows directly from lemma 3.1 and proposition 3.1.

□

**Remark 3.6** Note that properties (i) and (ii) in the previous theorem are satisfied by any monetary risk measure on $\chi$ of the form $\mathbb{E}_{\psi\circ\mu}(.)$ where $\psi$ is a given distortion function and where $\mu$ is a given capacity. The statement is due to remark 3.5, to proposition 3.1 and to remark 3.2.

**Remark 3.7** We also note that the distortion function $\psi$ in the representation formula of the previous theorem is unique on the set $S := \{\mu(A), A \in \mathcal{F}\}$.

We conclude from the previous theorem 3.1 combined with remark 3.6 that in the case where the initial capacity $\mu$ is continuous from below and from above the class of generalized distortion risk measures with respect to $\mu$ (in the sense of definition 3.3) coincides with the class of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation.

As already mentioned, risk measures satisfying the property of comonotonic additivity (property (ii) in the previous theorem) have been extensively studied in the literature and the financial interpretation of this property has been aknowledged (see for instance Föllmer and Schied 2004).

Nevertheless, the notion of consistency with respect to the $\leq_{mon,\mu}$ −relation for a risk measure (as well as the notion of consistency with respect to the $\leq_{sl,\mu}$ −relation, or with respect to the $\leq_{icx,\mu}$ −relation) being introduced in the present paper, an interpretation is given hereafter. The interpretation provided in this paper is from the point of view of an insurance company. Consider an insurance company which is willing to compare measurable functions (interpreted in this context as random losses) according to the CEU-theory. The use of a stochastic dominance relation deriving from the CEU-theory

(such as the $\leq_{mon,\mu}$ - stochastic dominance relation, the $\leq_{sl,\mu}$ $-$relation or the $\leq_{icx,\mu}$ $-$relation) is suitable as it gives a way of comparing random losses according to the desired economic theory. The CEU-theory and the stochastic dominance relations to which it gives rise may intervene, for instance, in situations where the insurance company is facing ambiguity. However, as it is the case of the "classical" stochastic dominance relations with respect to a probability, the stochastic dominance relations with respect to a capacity have the following "drawback": the relations are not "total" which means that for some measurable functions $X$ and $Y$ it is possible to have neither $X \leq_{mon,\mu} Y$ nor $Y \leq_{mon,\mu} X$ (if the $\leq_{mon,\mu}$ $-$relation is taken as an example).
In the present paper, risk measures having the property of consistency with respect to the given stochastic dominance relation with respect to a capacity are used as a way of circumventing the previous "drawback". This approach is analogous to the one used in the "classical" case of a probability where risk measures consistent with respect to the "classical" stochastic dominance relations are studied.

**Remark 3.8** The $\leq_{mon,\mu}$ $-$relation and the property of consistency with respect to the $\leq_{mon,\mu}$ $-$relation could be interpreted in terms of ambiguity. The interpretation is based on the characterization of the $\leq_{mon,\mu}$ $-$relation established in proposition 3.1 in the case of a capacity $\mu$ which is continuous from below and from above. Let us recall that the measurable functions on $(\Omega, \mathcal{F})$ are interpreted as losses in the present paper and let $X$ and $Y$ be two measurable functions in $\chi$ such that

$$G_{X,\mu}(t) \leq G_{Y,\mu}(t) \text{ for all } t \in \mathbb{R} \tag{3.2}$$

which is equivalent to $\mu(X > t) \geq \mu(Y > t)$ for all $t \in \mathbb{R}$.
Let us first consider the inequality $\mu(X > t) \geq \mu(Y > t)$ where $t \in \mathbb{R}$ is fixed. Bearing in mind that the capacity $\mu$ models the agent's perception of "uncertain" (or "ambiguous") events, the reader may interpret the previous inequality as having the following meaning: the event $\{X > t\}$ is perceived by the agent as being less uncertain than (or

equally uncertain to) the event $\{Y > t\}$.

Thus, the relation (3.2) (which, thanks to proposition 3.1, is equivalent to $Y \leq_{mon,\mu} X$ in the case of a capacity $\mu$ assumed to be continuous from below and from above) can be loosely read as follows: the agent "feels less (or equally) uncertain about the loss $X$'s taking great values than about the loss $Y$'s ".

Thus, if a loss $X \in \chi$ is perceived (through a capacity $\mu$ which is continuous from below and from above) as being more or equally certain to take great values (in the previous sense) than a loss $Y \in \chi$, the "risk" [2] $\rho(X)$ associated to the loss $X$ by a risk measure $\rho : \chi \longrightarrow \mathbb{R}$ having the property of *consistency with respect to the* $\leq_{\text{mon},\mu}$ *−relation* is greater than or equal to the "risk" $\rho(Y)$ associated to the loss $Y$.

Thanks to proposition 2.6, an analogous interpretation could be given of the $\leq_{sl,\mu}$ −relation and of the property of consistency with respect to the $\leq_{sl,\mu}$ −relation.

## 3.3 Characterizing risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation

We have seen that, for a given capacity $\mu$, the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation is included in the set of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation. Besides, in the case where the initial capacity $\mu$ is continuous from below and from above, a characterization of the latter set in terms of Choquet integrals with respect to a distorted capacity has been established in theorem 3.1 combined with remark 3.6. This subsection is devoted to a characterization of the former set of risk measures in terms of Choquet integrals with respect to a distorted capacity where the distortion function is concave. Two separate theorems, corresponding to the two implications of which the characterization consists,

[2] The expression *"the risk" of a loss* $X \in \chi$ designates here the number $\rho(X)$ associated to $X$ by a risk measure $\rho$.

are presented.

The following theorem is a representation result for monetary risk measures satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation.

**Theorem 3.2** *Let $\mu$ be a capacity. Assume that there exists a real-valued measurable function $Z$ such that the distribution function $G_Z$ of $Z$ is continuous and satisfies the following property:* $\lim_{x\to-\infty} G_Z(x) = 0$ *and* $\lim_{x\to+\infty} G_Z(x) = 1$.

*If $\rho : \chi \to \mathbb{R}$ is a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ −relation, then there exists a* concave *distortion function $\psi$ such that*

$$\rho(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi.$$

The proof of this theorem is based on the representation result of lemma 3.1, on proposition 2.6, and on lemma 3.2 below. The lemma 3.2 is well-known in the classical case of a probability measure as a way of constructing a random variable with a uniform distribution on $[0, 1]$.

**Lemma 3.2** *Let $\mu$ be a capacity. Assume that there exists a real-valued measurable function $Z$ such that the distribution function $G_Z$ of $Z$ is continuous and satisfies*

$$\lim_{x\to-\infty} G_Z(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} G_Z(x) = 1. \tag{3.3}$$

*Set $U := G_Z(Z)$. The distribution function $G_U$ of $U$ is given by:* $G_U(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } x \in [0,1] \\ 1, & \text{if } x > 1. \end{cases}$

**Proof of lemma 3.2:** The measurable function $U$ can be written in the following manner: $U = f(Z)$, where, for the easing of the presentation, we have set $f := G_Z$. As in the proof of lemma 2.3, we define the upper generalized inverse $\check{f}$ of the non-decreasing function $f$ by $\check{f}(x) := \inf\{y \in \mathbb{R} : f(y) > x\}$, $\forall x \in \mathbb{R}$. The function $f$ being

non-decreasing and continuous, we know from the proof of proposition 3.2 in Yan (2009) that for all $x \in \mathbb{R}$, $G_U(x) = G_{f(Z)}(x) = G_Z \circ \check{f}(x)$.
Therefore, for all $x \in \mathbb{R}$, $G_U(x) = G_Z \circ \check{G}_Z(x)$.
Now, according to the definitions of $\check{G}_Z$ and of the upper quantile function $r_Z^+$, we have $\check{G}_Z(x) = r_Z^+(x), \forall x \in (0,1)$. Moreover, thanks to the assumption (3.3), $r_Z^+(x)$ belongs to $\mathbb{R}$, $\forall x \in (0,1)$.
Thus, if $x \in (0,1)$, then $G_Z \circ \check{G}_Z(x) = G_Z \circ r_Z^+(x) = x$. The last equality in the previous computation is due to the continuity of $G_Z$ on $\mathbb{R}$.
If $x \geq 1$, then $\check{G}_Z(x) = +\infty$ and $G_Z \circ \check{G}_Z(x) = 1$.
If $x < 0$, then $\check{G}_Z(x) = -\infty$ and $G_Z \circ \check{G}_Z(x) = G_Z(-\infty) = 1 - \mu(Z > -\infty) = 0$.
Finally, if $x = 0$, then either $\check{G}_Z(0) = -\infty$ or $\check{G}_Z(0) \in \mathbb{R}$. In both of the situations, $G_Z \circ \check{G}_Z(0) = 0$.
The expression for $G_U$ is thus proved.

□

The following two remarks concern the assumptions of the previous lemma.

**Remark 3.9** The existence of a measurable function $Z$ on $(\Omega, \mathcal{F})$ with a continuous distribution function with respect to the capacity $\mu$ has been assumed in the previous lemma 3.2. In the "classical" case where $\mu$ is a probability measure this assumption is equivalent to the usual assumption of non-atomicity of the measure space $(\Omega, \mathcal{F}, \mu)$ (cf. Föllmer and Schied 2004).

**Remark 3.10** We note that assumption (3.3) of the previous lemma is not redundant in the case of a capacity $\mu$ which is not a probability measure. We also note that if $\mu$ and $Z$ do not satisfy the assumption (3.3), the result on the distribution function $G_U$ of $U$ of the lemma may not hold true. Indeed, let us consider the following counter-example. Let $(\Omega, \mathcal{F}, P)$ be a probability space such that there exists a random variable $Z$ whose distribution function $F_Z$ (with respect to $P$) is continuous and satisfies $0 < F_Z(x) <$

$1, \forall x \in \mathbb{R}$. Let $\mu$ be a capacity of the form $\mu := \psi \circ P$ where $\psi$ is a distortion function which is continuous on $(0,1)$ and such that $b := \sup_{x<1} \psi(x) < 1$. Then, the distribution function $G_{Z,\mu}$ of $Z$ (with respect to $\mu$) is continuous but fails to satisfy the assumption (3.3) in lemma 3.2 as

$$\lim_{x \to -\infty} G_{Z,\mu}(x) = \lim_{x \to -\infty} \Big(1 - \psi(1 - F_Z(x))\Big) = 1 - \sup_{x \in \mathbb{R}} \psi(1 - F_Z(x))$$
$$= 1 - \sup_{y<1} \psi(y) = 1 - b > 0.$$

Let us compute $G_{U,\mu}(x)$ for $x \in (0, 1-b)$. For $x \in (0, 1-b)$, $\check{G}_{Z,\mu}(x) = r^+_{Z,\mu}(x) = -\infty$. Therefore, for $x \in (0, 1-b)$, $G_{U,\mu}(x) = G_{Z,\mu} \circ \check{G}_{Z,\mu}(x) = G_{Z,\mu}(-\infty) = 0 \neq x$ which provides the desired counter-example.

Let us now prove theorem 3.2.

**Proof of theorem 3.2:** It is easy to check that the monetary risk measure $\rho$ satisfies the properties (i), (ii) and (iii) in lemma 3.1. Therefore, there exists a distortion function $\psi$ such that $\rho(X) = \mathbb{E}_{\psi \circ \mu}(X)$, $\forall X \in \chi$. It remains to show that the distortion function $\psi$ is concave.

Let $x \in [0,1]$ and $y \in [0,1]$ be such that $x < y$. There exist measurable sets $A$ and $B$ satisfying the following properties: $A \subset B$, $\mu(A) = x$ and $\mu(B) = y$. Indeed, if we set $A := \{U > 1 - x\}$ and $B := \{U > 1 - y\}$ where $U := G_Z(Z)$, we have that $A \subset B$. Moreover, according to lemma 3.2, $\mu(A) = \mu(U > 1-x) = 1 - G_U(1-x) = 1-(1-x) = x$. Similarly, we compute $\mu(B) = y$. Therefore, the sets $A$ and $B$ are as desired.

Furthermore, there exists a measurable set $C$ such that $\mu(C) = \frac{x+y}{2}$ (the set $C$ can be constructed by setting $C := \{U > 1 - \frac{x+y}{2}\}$).

We now set $X := \frac{1}{2}\mathbb{I}_A + \frac{1}{2}\mathbb{I}_B$ and $Y := \mathbb{I}_C$ and we note that the measurable functions $\frac{1}{2}\mathbb{I}_A$ and $\frac{1}{2}\mathbb{I}_B$ are comonotonic as $A \subset B$.

Let us show that $X \leq_{sl,\mu} Y$. According to proposition 2.6, it suffices to prove that $\forall t \in (0,1)$,

$$\int_t^1 r^+_X(s)ds \leq \int_t^1 r^+_Y(s)ds. \tag{3.4}$$

Now, $r_Y^+(t) = \mathbb{I}_{[1-\mu(C),1)}(t)$ and $r_X^+(t) = \frac{1}{2}\mathbb{I}_{[1-\mu(A),1)}(t) + \frac{1}{2}\mathbb{I}_{[1-\mu(B),1)}(t)$ for almost every $t$ where lemma 2.2 and lemma 2.3 have been used to compute $r_X^+(t)$. Therefore, equation (3.4) is equivalent to $\frac{1}{2}\Big(1-\max\{t, 1-\mu(A)\}\Big)+\frac{1}{2}\Big(1-\max\{t, 1-\mu(B)\}\Big) \leq 1-\max\{t, 1-\mu(C)\}$ which is equivalent to $\frac{1}{2}\min\{1-t, \mu(A)\}+\frac{1}{2}\min\{1-t, \mu(B)\} \leq \min\{1-t, \mu(C)\}$. The observation that, for a fixed $t \in (0,1)$, the mapping $z \longrightarrow \min\{1-t, z\}$ is concave allows us to conclude that equation (3.4) holds true.

The consistency of $\rho$ with respect to the $\leq_{sl,\mu}$ −relation implies that $\rho(X) \leq \rho(Y)$ which is equivalent to $\mathbb{E}_{\psi\circ\mu}(\frac{1}{2}\mathbb{I}_A + \frac{1}{2}\mathbb{I}_B) \leq \mathbb{E}_{\psi\circ\mu}(\mathbb{I}_C)$. The positive homogeneity and the comonotonic additivity of the Choquet integral then give $\frac{1}{2}\psi\circ\mu(A)+\frac{1}{2}\psi\circ\mu(B) \leq \psi\circ\mu(C)$. The concavity of $\psi$ follows as $\mu(A) = x$, $\mu(B) = y$, $\mu(C) = \frac{x+y}{2}$ and as $x$ and $y$ are arbitrary.

$\square$

**Remark 3.11** The distortion function in the representation result of the previous theorem (theorem 3.2) is unique. Indeed, suppose that there exists a distortion function $\tilde{\psi}$ such that $\rho(X) = \mathbb{E}_{\tilde{\psi}\circ\mu}(X), \forall X \in \chi$. Let $x \in [0,1]$. Under the assumptions of theorem 3.2 there exists a measurable set $A$ such that $\mu(A) = x$ (see the proof of theorem 3.2 for the construction of the set $A$). On the other hand, $\rho(\mathbb{I}_A) = \psi \circ \mu(A) = \tilde{\psi} \circ \mu(A)$, which implies the desired equality, namely $\psi(x) = \tilde{\psi}(x)$.

**Remark 3.12** One may wonder if the Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$ (as the one which appears in the representation formula of theorem 3.2) can be compared with the Choquet integral with respect to the initial capacity $\mu$. In the case where the distortion function $\psi$ is concave (which is the case in the representation formula of theorem 3.2), the following inequality holds: $\psi \circ \mu(A) \geq \mu(A), \forall A \in \mathcal{F}$. Therefore, $\mathbb{E}_{\psi\circ\mu}(X) \geq \mathbb{E}_{\mu}(X), \forall X \in \chi$. We conclude that, under the assumptions of theorem 3.2, a monetary risk measure $\rho$ having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation satisfies the property:

$\rho(X) \geq \mathbb{E}_\mu(X), \forall X \in \chi.$

In the particular case where, along with the assumptions made in theorem 3.2, the additional assumption of concavity of the capacity $\mu$ is made, a monetary risk measure $\rho$ satisfying the properties of theorem 3.2, namely comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation, is necessarily a convex monetary risk measure. The result is formulated in the following corollary. The convexity of $\rho$ in this case is due to the concavity of the distorted capacity $\psi \circ \mu$ in the representation of $\rho$ and to the sub-additivity of the Choquet integral with respect to a concave capacity. For the corresponding result in the "classical" case of a probability the reader is referred to Song and Yan (2009), as well as to Föllmer and Schied (2004).

**Corollary 3.1** *Let $\mu$ be a concave capacity and assume that there exists a real-valued measurable function $Z$ such that the distribution function $G_Z$ of $Z$ is continuous and satisfies the following property:* $\lim_{x\to-\infty} G_Z(x) = 0$ *and* $\lim_{x\to+\infty} G_Z(x) = 1$.
*Let $\rho : \chi \to \mathbb{R}$ be a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ −relation. Then $\rho$ is a* convex *monetary risk measure on $\chi$.*

**Remark 3.13** We note that if, along with the assumptions on the space $(\Omega, \mathcal{F}, \mu)$ in the previous theorem 3.2 (respectively in corollary 3.1), the additional assumption of continuity from below and from above on the capacity $\mu$ is made, then the property of consistency with respect to the $\leq_{sl,\mu}$ −relation in theorem 3.2 (resp. corollary 3.1) can be replaced by the property of consistency with respect to the $\leq_{icx,\mu}$ −relation. The statement is due to the second assertion in proposition 3.2. We note, furthermore, that the assumption on the limits of the distribution function $G_Z$ of $Z$ in theorem 3.2 (resp. corollary 3.1) is made redundant by this additional continuity assumption on the capacity $\mu$ (cf. remark 2.3).

It has been established in the previous theorem 3.2 that, under suitable assumptions on the initial space $(\Omega, \mathcal{F}, \mu)$, a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation can be represented as a Choquet integral with respect to a distorted capacity of the form $\psi \circ \mu$ where the distortion function $\psi$ is concave. In order to complete the desired characterization it remains to show that the converse statement holds true which is the purpose of the following theorem.

**Theorem 3.3** *Let $\mu$ be a capacity and let $\psi$ be a concave distortion function. The functional $\rho$ defined by $\rho(X) := \mathbb{E}_{\psi\circ\mu}(X), \forall X \in \chi$ is a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ −relation.*

The following lemma will be used in the proof of theorem 3.3. The lemma is a generalization of a well-known "classical" expression for Choquet integrals with respect to a distorted probability whose distortion function is concave (see, for instance, Föllmer and Schied 2004 or Carlier and Dana 2006 for the classical case). Our proof follows the proof given by Föllmer and Schied (2004) and is included for reader's convenience.

**Lemma 3.3** *Let $\mu$ be a capacity and let $\psi$ be a concave distortion function. For all $X \in \chi$,*

$$\mathbb{E}_{\psi\circ\mu}(X) = \psi(0+) \sup_{t<1} r_X^+(t) + \int_0^1 \psi'(1-t) r_X^+(t) dt \tag{3.5}$$

**Proof of the lemma:** It suffices to prove equation (3.5) for non-negative elements of $\chi$, the terms on both sides of the equality being translation invariant. Let $X$ be in $\chi_+$. The following expression is similar to the "classical" one; the proof is due to the non-decreasingness of $G_X$ and to the definition of $r_X^+$ and is left to the reader:

$$r_X^+(t) = \int_0^{+\infty} \mathbb{I}_{\{G_X(s)\leq t\}} ds, \ \forall t \in (0,1). \tag{3.6}$$

Thanks to (3.6) we compute

$$\begin{aligned}\int_0^1 \psi'(1-t) r_X^+(t) dt &= \int_0^1 \psi'(1-t) \int_0^{+\infty} \mathbb{I}_{\{G_X(s)\le t\}} ds\, dt \\ &= \int_0^{+\infty} \int_0^{1-G_X(s)} \psi'(y) dy\, ds \\ &= \int_0^{+\infty} \Big(\psi(1-G_X(s)) - \psi(0+)\Big) \mathbb{I}_{\{G_X(s)<1\}} ds\end{aligned}$$

where the equation $\int_0^y \psi'(s)ds = \big(\psi(y) - \psi(0+)\big)\mathbb{I}_{y>0}$ has been used to obtain the last line.

Using the definition of the Choquet integral and the fact that

$$\sup_{t<1} r_X^+(t) = \int_0^{+\infty} \mathbb{I}_{\{G_X(s)<1\}} ds$$

whose proof is left to the reader, we obtain

$$\begin{aligned}\int_0^1 \psi'(1-t) r_X^+(t) dt &= \int_0^{+\infty} \psi(1-G_X(s)) ds - \psi(0+) \int_0^{+\infty} \mathbb{I}_{\{G_X(s)<1\}} ds \\ &= \mathbb{E}_{\psi\circ\mu}(X) - \psi(0+) \sup_{t<1} r_X^+(t).\end{aligned}$$

The lemma is thus proved.

□

**Proof of theorem 3.3:** As recalled in proposition 2.3, the Choquet integral satisfies the properties of monotonicity, translation invariance and comonotonic additivity. Therefore, the only property of the functional $\rho$ which has to be proved is the property of consistency with respect to the $\le_{sl,\mu}$ −relation.

Let $X, Y \in \chi$ be such that $X \le_{sl,\mu} Y$. Let us prove that $\mathbb{E}_{\psi\circ\mu}(X) \le \mathbb{E}_{\psi\circ\mu}(Y)$ which, thanks to lemma 3.3, is equivalent to

$$\psi(0+) \sup_{t<1} r_X^+(t) + \int_0^1 \psi'(1-t) r_X^+(t) dt \le \psi(0+) \sup_{t<1} r_Y^+(t) + \int_0^1 \psi'(1-t) r_Y^+(t) dt.$$

Proposition 2.7 implies that $\int_0^1 \psi'(1-t) r_X^+(t) dt \le \int_0^1 \psi'(1-t) r_Y^+(t) dt$. The number $\psi(0+)$ being non-negative, it remains to show that $\sup_{t<1} r_X^+(t) \le \sup_{t<1} r_Y^+(t)$.

Suppose, by way of contradiction, that $\sup_{t<1} r_X^+(t) > \sup_{t<1} r_Y^+(t)$. Then, there exists $t_0 \in [0,1)$ such that $r_X^+(s) \geq r_X^+(t_0) > \sup_{t<1} r_Y^+(t)$, $\forall s \geq t_0$. This implies that $r_X^+(s) > r_Y^+(s)$, $\forall s \geq t_0$ leading to $\int_{t_0}^1 \big(r_X^+(s) - r_Y^+(s)\big)ds > 0$. The last inequality contradicts the relation $X \leq_{sl,\mu} Y$ (cf. the characterization of the $\leq_{sl,\mu}$ −relation in proposition 2.6). The previous reasoning leads to the desired implication, namely $X \leq_{sl,\mu} Y \Rightarrow \sup_{t<1} r_X^+(t) \leq \sup_{t<1} r_Y^+(t)$, and concludes the proof.

□

## 3.4 Convex generalized distortion risk measures: a counter-example

As recalled in remark 3.3, a generalized distortion risk measure of the form $\mathbb{E}_{\psi\circ\mu}(.)$ is convex if and only if the distorted capacity $\psi \circ \mu$ is concave in the sense of definition 2.5. The purpose of this section is to investigate the question whether the concavity of a distorted capacity $\psi \circ \mu$ (and therefore, the convexity of $\mathbb{E}_{\psi\circ\mu}(.)$) can be characterized by means of the concavity of the distortion function $\psi$.

It has been seen in example 2. of subsection 2.1 that, in the case where $\mu$ is a concave capacity, a distorted capacity of the form $\psi \circ \mu$ is concave if the distortion function $\psi$ is concave. On the other hand, it is well-known that in the "classical" case where $\mu$ is a probability measure, under a non-atomicity assumption on the measure space $(\Omega, \mathcal{F}, \mu)$, the converse statement also holds true, namely the concavity of a distorted probability of the form $\psi \circ \mu$ implies the concavity of the distortion function $\psi$ (cf. proposition 4.69 in Föllmer and Schied 2004).

Nevertheless, in the more general case where $\mu$ is a concave capacity which is not necessarily a probability measure, this converse statement may not be true even if the existence of a measurable function $Z$ with a continuous distribution function $G_Z := G_{Z,\mu}$ is assumed. Let us consider the following counter-example.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space. Let $\phi$ be a distortion function which is concave and continuous and set $\mu := \phi \circ \mathbb{P}$. Then, the capacity $\mu$ is a concave capacity, the distortion function $\phi$ being concave. Furthermore, $\mu$ is continuous from below and from above, the function $\phi$ being continuous. Moreover, there exists a measurable function $Z$ on $(\Omega, \mathcal{F})$ such that the distribution function (with respect to $\mu$) $G_Z := G_{Z,\mu}$ of $Z$ is continuous (in fact, one can easily verify that any random variable $Z$ whose distribution function with respect to $\mathbb{P}$ is continuous satisfies this property; the existence of such a random variable is guaranteed by the non-atomicity assumption on $(\Omega, \mathcal{F}, \mathbb{P})$).

To be more concrete, let us specify the definition of $\phi$: $\phi(x) := x^\beta, \forall x \in [0,1]$ where $\beta \in (0,1)$. Let us further define a distortion function $\psi : [0,1] \to [0,1]$ by $\psi(x) := x^{\frac{\alpha}{\beta}}, \forall x \in [0,1]$ where $\alpha \in (0,1)$ is such that $\alpha > \beta$. Let us consider the distorted capacity $\psi \circ \mu$ where $\mu := \phi \circ \mathbb{P}$ as above.

The distortion function $\psi$ is not concave; in fact, $\psi$ is a strictly convex function. Nevertheless, the distorted capacity $\psi \circ \mu$ is a concave capacity. The latter property is easily obtained by observing that $\psi \circ \mu = (\psi \circ \phi) \circ \mathbb{P}$ and that $\psi \circ \phi$ is a concave distortion function as $\psi \circ \phi(x) = x^\alpha, \forall x \in [0,1]$. Thus, the capacity $\psi \circ \mu$ is concave as it can be represented as a distorted probability with respect to a concave distortion function.

To summarize, we have given an example of a measurable space $(\Omega, \mathcal{F})$ endowed with a capacity $\mu$ which is concave, continuous from below and from above (but not necessarily additive) and such that there exists a measurable function whose distribution function with respect to $\mu$ is continuous. We have then shown that it is possible to contruct a distorted capacity of the form $\psi \circ \mu$ which is concave (in the sense of definition 2.5) but whose distortion function $\psi$ is not concave thus providing the desired counter-example.

# 4 "Kusuoka-type" characterization of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$-relation

The purpose of this section is to provide a "Kusuoka-type" characterization of the class of monetary risk measures having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$-relation under suitable assumptions on the space $(\Omega, \mathcal{F}, \mu)$ where $\mu$ is a capacity. We recall, for reader's convenience, the classical Kusuoka's result (cf. theorem 7 in Kusuoka 2001) in a form which is given in Ekeland and Schachermayer (2011) (theorem 1.4):

**Theorem 4.1 (Kusuoka's theorem)** *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\rho : L^{\infty}(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ be a given functional. Then, the following two statements are equivalent:*

**(i)** *The functional $\rho$ is a convex monetary risk measure having the properties of comonotonic additivity and law-invariance.*

**(ii)** *There exists $\alpha \in [0,1]$ and a random variable $Y \in L^1_+(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}_P(Y) = 1$ such that*

$$\rho(X) = \alpha \; ess \; sup(X) + (1-\alpha)\, \rho_Y(X), \quad \forall X \in L^{\infty}(\Omega, \mathcal{F}, P),$$

*where $\rho_Y(X) := \sup_{\tilde{X} \in L^{\infty}(\Omega, \mathcal{F}, P) : \tilde{X} \sim X} \mathbb{E}_P(Y\tilde{X})$ and the notation $\tilde{X} \sim X$ means that $\tilde{X}$ and $X$ have the same law (with respect to $P$).*

Let us further remark that the law-invariance property in statement (i) of the previous theorem can be replaced by the property of consistency with respect to the "classical" stop-loss order relation $\leq_{sl,\, P}$ (with respect to the probability $P$). More precisely, in the case where the probability space $(\Omega, \mathcal{F}, P)$ is atomless, the following well-known result holds true; the result is recalled for reader's convenience.

**Proposition 4.1** *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ be a given functional. Then, the following statements are equivalent:*

**(i)** *The functional $\rho$ is a convex monetary risk measure having the properties of comonotonic additivity and law-invariance.*

**(ii)** *The functional $\rho$ is a convex monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{mon, P}}$-relation.*

**(iii)** *The functional $\rho$ is a convex monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl, P}}$-relation.*

**(iv)** *The functional $\rho$ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\text{sl, P}}$-relation.*

**Proof:** The equivalence between assertions $(iii)$ and $(iv)$ is a consequence of corollary 3.1 applied to the particular case of an atomless probability space. The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are obvious. The implication $(i) \Rightarrow (iii)$ can be found in Cherny and Grigoriev (2007) (page 294).

□

Thus, theorem 4.1 can be viewed as a way of characterizing (convex) monetary risk measures having the properties of comonotonic additivity and consistency with respect to the "classical" $\leq_{sl,\ P} -$ relation in the case where the probability space $(\Omega, \mathcal{F}, P)$ is atomless.

We note as well that, thanks to lemma 4.5.5. in Föllmer and Schied (2004), statement (ii) in theorem 4.1 can be reformulated in the following manner:

**(ii bis)** There exists $\alpha \in [0,1]$ and a random variable $Y \in L^1_+(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}_P(Y) = 1$ such that

$$\rho(X) = \alpha \text{ ess sup}(X) + (1-\alpha) \int_0^1 q_Y(t) q_X(t) dt, \quad \forall X \in L^\infty(\Omega, \mathcal{F}, P),$$

where $q_X$ (resp. $q_Y$) denotes (the) quantile function of $X$ (resp. $Y$) with respect to the probability $P$.

Thanks to the previous considerations, theorem 4.1 can be reformulated as follows:

**Theorem 4.2 (Kusuoka's theorem - equivalent formulation)** *Let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\rho : L^\infty(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ be a given functional. Then the following two statements are equivalent:*

**(i)** *The functional $\rho$ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl,\ P}}$-relation.*

**(ii)** *There exists $\alpha \in [0,1]$ and a random variable $Y \in L^1_+(\Omega, \mathcal{F}, P)$ satisfying $\mathbb{E}_P(Y) = 1$ such that*

$$\rho(X) = \alpha \text{ } ess \text{ } sup(X) + (1-\alpha) \int_0^1 q_Y(t) q_X(t) dt, \quad \forall X \in L^\infty(\Omega, \mathcal{F}, P),$$

*where $q_X$ (resp. $q_Y$) denotes (the) quantile function of $X$ (resp. $Y$) with respect to $P$.*

A "generalization" of theorem 4.2 to the setting of a capacity (which is not necessarily a probability measure) is established in the following theorem.

**Theorem 4.3 (Kusuoka-type characterization in the case of a capacity)** *Let $\mu$ be a capacity. Assume that there exists a real-valued measurable function $Z$ such that the distribution function $G_Z$ of $Z$ is continuous and satisfies the following property: $\lim_{x \to -\infty} G_Z(x) = 0$ and $\lim_{x \to +\infty} G_Z(x) = 1$.*
*Let $\rho : \chi \longrightarrow \mathbb{R}$ be a given functional. Then the following two statements are equivalent:*

**(i)** *The functional $\rho$ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$-relation.*

**(ii)** *There exists* $\alpha \in [0,1]$ *and a non-negative measurable function* $Y$ *satisfying* $\int_0^1 r_{Y,\mu}(t)dt = 1$ *such that*

$$\rho(X) = \alpha \sup_{t<1} r^+_{X,\mu}(t) + (1-\alpha)\int_0^1 r_{Y,\mu}(t) r_{X,\mu}(t)dt, \quad \forall X \in \chi.$$

The following lemma summarizes some of the main properties of the functional $X \mapsto \sup_{t<1} r^+_{X,\mu}(t)$ and will be used in the proof of theorem 4.3.

**Lemma 4.1** *Let* $\mu$ *be a capacity. The functional* $\rho_\infty : \chi \longrightarrow \mathbb{R}$ *defined by* $\rho_\infty(X) := \sup_{t<1} r^+_X(t), \forall X \in \chi$ *is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the* $\leq_{\mathrm{sl},\mu}$ *−relation.*
*Moreover, the functional* $\rho_\infty$ *can be represented in the following manner:*

$$\rho_\infty(X) = \mathbb{E}_{\psi\circ\mu}(X), \forall X \in \chi$$

*where* $\psi$ *is a concave distortion function given by* $\psi(x) = \begin{cases} 1, \ \text{if } x > 0 \\ 0, \ \text{if } x = 0. \end{cases}$

**Proof of the lemma:** The translation invariance of the functional $\rho_\infty$ follows from lemma 2.3. The monotonicity of $\rho_\infty$ is due to the definition of the upper quantile function and to the monotonicity of the capacity $\mu$.
Let us prove the comonotonic additivity of $\rho_\infty$. Let $X$ and $Y$ be two comonotonic functions in $\chi$. According to proposition 2.2, there exists $Z \in \chi$ and two non-decreasing continuous functions $f$ and $g$ on $\mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$. Therefore,

$$\rho_\infty(X+Y) = \sup_{t<1} r^+_{X+Y}(t) = \sup_{t<1} r^+_{(f+g)(Z)}(t) = \sup_{t<1}(f+g)(r^+_Z(t))$$

where lemma 2.3 has been used to obtain the last equality.
As the function $f+g$ is non-decreasing and continuous on $\mathbb{R}$ and as $\sup_{t<1} r^+_Z(t) \in \mathbb{R}$, we have $\sup_{t<1}(f+g)(r^+_Z(t)) = (f+g)(\sup_{t<1} r^+_Z(t))$. The same argument is used to show

that $f\Big(\sup_{t<1} r^+_Z(t)\Big) = \sup_{t<1} f\big(r^+_Z(t)\big)$ and $g\Big(\sup_{t<1} r^+_Z(t)\Big) = \sup_{t<1} g\big(r^+_Z(t)\big)$. Thus,

$$\sup_{t<1}(f+g)(r^+_Z(t)) = (f+g)\big(\sup_{t<1} r^+_Z(t)\big) = \sup_{t<1} f\big(r^+_Z(t)\big) + \sup_{t<1} g\big(r^+_Z(t)\big) =$$

$$= \sup_{t<1} r^+_{f(Z)}(t) + \sup_{t<1} r^+_{g(Z)}(t) = \sup_{t<1} r^+_X(t) + \sup_{t<1} r^+_Y(t)$$

where lemma 2.3 has been used again to obtain the last but one equality. The comonotonic additivity of $\rho_\infty$ is thus proved.

The property of consistency with respect to the $\leq_{sl,\mu} -$ relation has already been shown at the end of the proof of theorem 3.3.

Finally, an application of Schmeidler's representation theorem (theorem 2.1) gives the existence of a capacity $\nu$ such that $\rho_\infty(X) = \mathbb{E}_\nu(X)$, $\forall X \in \chi$. The capacity $\nu$ is given by

$$\nu(A) = \rho_\infty(\mathbb{I}_A) = \sup_{t<1} r^+_{\mathbb{I}_A}(t) = \sup_{t<1} \mathbb{I}_{[1-\mu(A),1)}(t) = \begin{cases} 1, \text{ if } \mu(A) > 0 \\ 0, \text{ if } \mu(A) = 0. \end{cases}$$

Thus, $\nu(A) = \psi(\mu(A))$ which concludes the proof.

□

Some of the main properties of the functional $X \longmapsto \int_0^1 r_Y(t) r_X(t)dt$ (for a given $Y \geq 0$ such that $\int_0^1 r_Y(t)dt = 1$) have already been studied in section 5.1 of Grigorova (2010) and are summarized in the following lemma for reader's convenience.

**Lemma 4.2** *Let $\mu$ be a capacity. Let $Y$ be a non-negative measurable function such that $\int_0^1 r_Y(t)dt = 1$. The functional $\rho^Y : \chi \longrightarrow \mathbb{R}$ defined by $\rho^Y(X) := \int_0^1 r_Y(t) r_X(t)dt, \forall X \in \chi$ is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$ $-$relation.*

*Moreover, the functional $\rho^Y$ can be represented in the following manner:*

$$\rho^Y(X) = \mathbb{E}_{\psi^Y \circ \mu}(X), \forall X \in \chi$$

*where $\psi^Y$ is a concave distortion function given by $\psi^Y(x) = \int_{1-x}^1 r_Y(t)dt, \forall x \in [0,1]$.*

Let us now prove theorem 4.3.

**Proof of theorem 4.3:** The implication $(ii) \Rightarrow (i)$ is a consequence of lemma 4.1 and lemma 4.2.

To prove the converse implication, let $\rho$ be a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation. Thanks to theorem 3.2 and to lemma 3.3, there exists a concave distortion function $\psi$ such that $\forall X \in \chi$,

$$\rho(X) = \psi(0+) \sup_{t<1} r_X^+(t) + \int_0^1 \psi'(1-t) r_X^+(t) dt.$$

- If $\psi(0+) = 1$, then $\rho(X) = \sup_{t<1} r_X^+(t)$, $\forall X \in \chi$ which proves the desired result with $\alpha = 1$.

- Otherwise, by setting $\alpha := \psi(0+)$, we have

$$\rho(X) = \alpha \sup_{t<1} r_X^+(t) + (1-\alpha) \int_0^1 \frac{\psi'(1-t)}{1-\psi(0+)} r_X^+(t) dt, \; \forall X \in \chi.$$

  Let us remark that $\int_0^1 \frac{\psi'(1-t)}{1-\psi(0+)} dt = 1$. Therefore, in order to prove statement $(ii)$, it suffices to prove that there exists a non-negative measurable function $Y$ such that $r_Y(t) = \frac{\psi'(1-t)}{1-\psi(0+)}$ for almost every $t \in (0,1)$.
  Set $U := G_Z(Z)$ and define a function $g$ by setting $g(t) := \frac{\psi'_+(1-t)}{1-\psi(0+)}$, $\forall t \in (0,1)$ where $\psi'_+$ denotes the right-hand derivative of the concave function $\psi$. Let $Y$ be defined by $Y := g(U)$ (where, in order to assure that $Y$ is well-defined on $\Omega$, the definition of $g$ has been extended to $[0,1]$ by setting $g(0) := \lim_{t\downarrow 0} \frac{\psi'_+(1-t)}{1-\psi(0+)}$ and $g(1) := \lim_{t\uparrow 1} \frac{\psi'_+(1-t)}{1-\psi(0+)}$).
  Then, the measurable function $Y$ is as wanted. Indeed, $Y \geq 0$. Moreover, the distribution function $G_U$ of $U$ being continuous (according to lemma 3.2) and the function $g$ being non-decreasing, we can apply lemma 2.1 to obtain:

  $$r_Y(t) = r_{g(U)}(t) = g(r_U(t)) \text{ for almost every } t \in (0,1). \tag{4.1}$$

  Now, it can be deduced from lemma 3.2 that $r_U(t) = t$ for all $t \in (0,1)$. This observation combined with equality (4.1) allows to conclude that $r_Y(t) = g(t)$ for almost every $t \in (0, 1)$.

□

**Remark 4.1** Let us remark that, unlike the classical case of proposition 4.1, under the more general assumptions on $(\Omega, \mathcal{F}, \mu)$ of theorem 4.3 a monetary risk measure $\rho$ satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation (as the one of statement $(i)$ in theorem 4.3) is not necessarily convex. A counter-example similar to the one constructed in subsection 3.4 is given in the appendix.
Let us recall, nevertheless, that if, along with the assumptions made in theorem 4.3, the assumption of concavity of the capacity $\mu$ is made, a monetary risk measure $\rho$ satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation is convex (cf. corollary 3.1).

**Remark 4.2** Let us remark also that, unlike the classical case of proposition 4.1, under the more general assumptions on $(\Omega, \mathcal{F}, \mu)$ of theorem 4.3 a convex monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation is not necessarily consistent with respect to the $\leq_{sl,\mu}$ −relation even if the additional assumption of concavity of the capacity $\mu$ is made. A counter-example, based on the one of subsection 3.4, is given in the appendix.

One may wonder if, in our setting of a capacity (which is not necessarily a probability measure), statement $(ii)$ in theorem 4.3 could be linked to the value function of an optimization problem analogous to the one appearing in statement $(ii)$ of the "classical" Kusuoka's theorem (theorem 4.1). The following result has been established in Grigorova (2010) - the formulation given hereafter is suitable for the needs of the present paper and is due to theorem 5.1 combined with remark 5.1, remark 5.3 and proposition 3.2 of the above-mentioned work.

**Proposition 4.2** *Let $\mu$ be a capacity which is assumed to be concave and continuous from below and from above. Let $Y$ be a given non-negative measurable function such that $\int_0^1 r_{Y,\mu}(t)dt = 1$. Then the funcional $\rho_Y : \chi_+ \longrightarrow \mathbb{R}$ defined by*

$$\rho_Y(X) := \sup_{\tilde{X}\in\chi_+ : \tilde{X}\leq_{\mathrm{sl},\mu} X} \mathbb{E}_\mu(Y\tilde{X}),\ \forall X \in \chi_+$$

*can be expressed in the following manner: $\rho_Y(X) = \int_0^1 r_{Y,\mu}(t) r_{X,\mu}(t)dt$.*

The previous proposition 4.2 combined with theorem 4.3 and remark 2.3 leads to the following

**Theorem 4.4** *Let $\mu$ be a capacity which is assumed to be* concave *and continuous from below and from above and assume that there exists a real-valued measurable function $Z$ on $(\Omega, \mathcal{F})$ such that the distribution function $G_Z$ of $Z$ (with respect to $\mu$) is continuous. Let $\rho : \chi_+ \longrightarrow \mathbb{R}$ be a given functional. Then the following two statements are equivalent:*

**(i)** *The functional $\rho$ is a (convex) monetary risk measure on $\chi_+$ having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{sl},\mu}$-relation.*

**(ii)** *There exists $\alpha \in [0,1]$ and a non-negative measurable function $Y$ satisfying $\int_0^1 r_{Y,\mu}(t)dt = 1$ such that*

$$\rho(X) = \alpha \sup_{t<1} r^+_{X,\mu}(t) + (1-\alpha)\,\rho_Y(X), \quad \forall X \in \chi_+,$$

*where $\rho_Y(X) := \sup_{\tilde{X}\in\chi_+ : \tilde{X}\leq_{\mathrm{sl},\mu} X} \mathbb{E}_\mu(Y\tilde{X}),\ \forall X \in \chi_+$.*

The previous theorem may be seen as an analogue of theorem 4.1 in the setting of a capacity which is assumed to be concave and continuous from below and from above.

# 5 Some examples of generalized distortion risk measures

In this section some generalizations to the case of a capacity of some well-known "classical" risk measures are given.

## 5.1 A "generalized" Value at Risk

Let us recall, for reader's convenience, the well-known "classical" definition of the Value at Risk at level $\lambda \in (0,1)$ with respect to a given probability $P$ of a given "potential loss" $X \in \chi$ (denoted by $VaR_\lambda(X)$ or $VaR_\lambda^P(X)$):

$$VaR_\lambda(X) := q_X^-(\lambda),$$

where, as before, the symbol $q_X^-$ stands for the lower quantile function of $X$ with respect to the probability $P$. The same sign convention in the definition of the $VaR_\lambda(X)$ as the one used in the present paper is used, for instance, by Dhaene et al. (2006) or Song and Yan (2009).

We now consider a generalization of the previous definition to the case of a capacity which is not necessarily a probability measure. The definition and some properties of the "gerneralized" Value at Risk are given in the following

**Definition/Proposition 5.1** *Let $\mu$ be a capacity on $(\Omega, \mathcal{F})$ and $\lambda$ be in $(0,1)$. The functional $GVaR_\lambda^\mu : \chi \to \mathbb{R}$ defined by*

$$GVaR_\lambda^\mu(X) := r_{X,\mu}^-(\lambda),\ \forall X \in \chi$$

*is a monetary risk measure having the properties of comonotonic additivity and consistency with respect to the $\leq_{\mathrm{mon},\mu}$ $-$relation. Moreover, the functional $GVaR_\lambda^\mu$ has the following representation*

$$GVaR_\lambda^\mu(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi \tag{5.1}$$

*where $\psi(x) := \psi_\lambda(x) := \mathbb{I}_{(1-\lambda,1]}(x),\ \forall x \in [0,1]$.*

**Proof:** The monotonicity and the translation invariance of $GVaR_\lambda^\mu(.)$ are a consequence of the definition of the lower quantile $r_{.,\mu}^-(\lambda)$. The comonotonic additivity is due to proposition 2.5.

Let us now prove the representation formula (5.1). Schmeidler's representation theorem (theorem 2.1) and remark 2.6 give the existence of a capacity $\nu$ on $(\Omega, \mathcal{F})$ such that $GVaR^{\mu}_{\lambda}(X) = \mathbb{E}_{\nu}(X)$, $\forall X \in \chi$. For all $A \in \mathcal{F}$, we have

$$\nu(A) = GVaR^{\mu}_{\lambda}(\mathbb{I}_A) = r^{-}_{\mathbb{I}_A,\mu}(\lambda) = \mathbb{I}_{(1-\mu(A),1)}(\lambda).$$

Therefore, the capacity $\nu$ is of the form $\nu(A) = \psi(\mu(A)), \forall A \in \mathcal{F}$. The representation formula (5.1) is thus proved.
The representation result (5.1) being established, the property of consistency with respect to the $\leq_{mon,\mu}$ −relation follows from remark 3.6.

□

In general, the risk measure $GVaR^{\mu}_{\lambda}(.)$ is not consistent with respect to the $\leq_{sl,\mu}$ −relation, the distortion function $\psi$ in the representation formula (5.1) not being concave (cf. theorem 3.2 and remark 3.11).

**Remark 5.1** In the previous definition/proposition the lower quantile $r^{-}_{.,\mu}(\lambda)$ with respect to a given capacity $\mu$ at a given point $\lambda$ is perceived as a "generalized" distortion risk measure (with respect to the capacity $\mu$). An analogous result holds true for the upper quantile $r^{+}_{.,\mu}(\lambda)$ thus providing another example of a "generalized" distortion risk measure. In the latter case, the distortion function $\psi$ in the representation (5.1) has to be replaced by the function $x \longmapsto \mathbb{I}_{[1-\lambda,1]}(x)$.
We note that the risk measure $r^{+}_{.,\mu}(\lambda)$ can be viewed as a generalization (to the case of a capacity) of the risk measure $Q^{+}_{\lambda}(.)$ introduced in Dhaene et al. (2006).

Two particular cases are considered below - the case where the capacity $\mu$ is a distorted probability and the case where the capacity $\mu$ is an "upper envelope" of a given set of prior probability measures.

### 5.1.1 The case of a distorted probability

Let $P$ be a given probability measure and $\phi$ be a given continuous distortion function. The first particular case to be considered is the case where the initial capacity $\mu$ is of the form $\mu = \phi \circ P$. The following result establishes a link, in this case, between the lower quantile function $r^-_{X,\mu}$ with respect to the capacity $\mu$ of a given measurable function $X$ and the corresponding lower quantile function $q^-_X$ with respect to the probability $P$.

**Proposition 5.1** *Let $P$ be a probability measure and $\phi$ be a given continuous distortion function. Let $\mu$ be a capacity of the form $\mu = \phi \circ P$. Let $X$ be a given real-valued measurable function. Then, the following equality holds true for all $t \in (0,1)$:*

$$r^-_{X,\mu}(t) = q^-_X\left(1 - \check{\phi}(1-t)\right),$$

*where $\check{\phi}$ denotes the upper generalized inverse of the non-decreasing function $\phi$ defined by $\check{\phi}(y) := \sup\{z : \phi(z) \le y\}$, $\forall y \in [0,1]$.*

**Proof:** The proof of the previous proposition is placed in the appendix.

$\square$

**Remark 5.2** Under the assumptions of proposition 5.1, the following link between the *upper* quantile functions $r^+_{X,\mu}$ and $q^+_X$ can be established:

$$r^+_{X,\mu}(t) = q^+_X(1 - \check{\phi}^-(1-t)),\ \forall t \in (0,1), \tag{5.2}$$

where $\check{\phi}^-$ denotes the *lower* generalized inverse of the distortion function $\phi$. The proof is based on arguments similar to those used in the proof of proposition 5.1 and is omitted. Let us note, however, that in the proof of the equality (5.2) we use the following equivalence, which is due to the assumption of continuity of the distortion function $\phi$:

$$\phi(a) \ge t \text{ if and only if } a \ge \check{\phi}^-(t).$$

According to proposition 5.1, in the case where $\mu = \phi \circ P$ (and where the distortion function $\phi$ is continuous), the "generalized" Value at Risk with respect to $\mu$ at level $\lambda \in (0,1)$ is equal to the "classical" Value at Risk with respect to $P$ at level $\tilde{\lambda}$ where $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda)$.

One may wonder if the above relation between the risk measure $GVaR^{\mu}_{\lambda}$ and the risk measure $VaR_{\tilde{\lambda}}$ has an economic interpretation. Can the CEU-theory (upon which the motivation of the present paper is based) explain the behaviour of an economic agent who, instead of assessing the risk of a given loss $X$ by the Value at Risk of $X$ at a level $\lambda$, assesses the risk of $X$ by the Value at Risk of $X$ at the (possibly different) level $\tilde{\lambda}$ ?

The measurable functions on $(\Omega, \mathcal{F})$ in the present paper being interpreted as *losses*, we will consider an economic agent (an insurer, for instance) who is a CEU-*minimizer*. The agent's preferences are described by a "pain" function $u$ and a capacity $\mu$ which, in the particular case that we consider, is of the form $\mu = \phi \circ P$.[3] The agent's "dissatisfaction" of a loss $X$ is then assessed by the Choquet integral of $u(X)$ with respect to the capacity $\mu$.

When interpreting proposition 5.1 we will focus on three particular sub-cases: the case where there is "no distortion", the case of a concave (continuous) distortion $\phi$, and the case of a convex (continuous) distortion $\phi$. Let us remark that when an agent who is a CEU-*minimizer* is considered, the concavity (resp. the convexity) of the distortion function $\phi$ is interpreted in terms of the agent's being a pessimist (resp. an optimist)[4].

1. The sub-case of a distortion function $\phi$ of the form $\phi(x) := x, \ \forall x \in [0,1]$

   In this sub-case we have $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda) = \lambda$ where $\lambda \in (0,1)$ is a given level. This

[3]In the case where the capacity $\mu$ is a distorted probability, the CEU-theory coincides with the so-called Rank-Dependent Expected Utility theory - see, for instance, Wang and Yan (2007) for a review.

[4]The situation considered more frequently in the literature (cf. Wang and Yan 2007, or Carlier and Dana 2003) is that of CEU-maximizers (the measurable functions on $(\Omega, \mathcal{F})$ being often interpreted as gains, instead of losses), in which case the interpretation of the concavity (resp. convexity) of the distortion function $\phi$ in terms of pessimism (resp. optimism) is reversed.

equality and proposition 5.1 lead to $GVaR^{\mu}_{\lambda}(X) = VaR_{\tilde{\lambda}}(X) = VaR_{\lambda}(X), \forall X \in \chi$. Hence, in the sub-case where the probability of events is perceived objectively (i.e. $\phi = id$), the risk measure $GVaR^{\mu}_{\lambda}$ at level $\lambda \in (0,1)$ is equal to the "usual" $VaR_{\lambda}$ at the same level $\lambda$. We thus recover, by means of proposition 5.1, an observation which can be derived from the definitions of the two risk measures.

2. The sub-case of a concave (continuous) distortion function $\phi$

   Let $\lambda \in (0,1)$ be a given level. The concavity of $\phi$ implies that $\tilde{\lambda} := 1-\check{\phi}(1-\lambda) \geq \lambda$. Therefore, $VaR_{\tilde{\lambda}}(X) \geq VaR_{\lambda}(X), \forall X \in \chi$. By combining this inequality with proposition 5.1 we obtain that $GVaR^{\mu}_{\lambda}(X) = VaR_{\tilde{\lambda}}(X) \geq VaR_{\lambda}(X), \forall X \in \chi$. Thus, in the case where the agent is pessimistic (the distortion function $\phi$ being concave), the risk attributed to a loss $X$ by means of the $GVaR^{\mu}_{\lambda}(X)$ is higher than (or equal to) the risk, equal to $VaR_{\lambda}(X)$, the agent would have attributed if he/she had perceived events objectively without distorting them.

3. The sub-case of a convex (continuous) distortion function $\phi$

   The convexity of $\phi$ implies that $\tilde{\lambda} := 1 - \check{\phi}(1-\lambda) \leq \lambda$. Therefore, in this sub-case, the inequality $GVaR^{\mu}_{\lambda}(X) \leq VaR_{\lambda}(X)$ holds for all $X \in \chi$.

   The risk $GVaR^{\mu}_{\lambda}(X)$ attributed by an optimistic agent (the distortion function $\phi$ in this sub-case being convex) to a given loss $X$ is lower than (or equal to) the risk $VaR_{\lambda}(X)$ attributed to $X$ by an agent who is objective.

An analogous reasoning applies to the risk measure $r^{+}_{\cdot,\mu}(\lambda)$; remark 5.2 is in this case used in place of proposition 5.1.

#### 5.1.2 The case where $\mu$ is the upper envelope of a given set $\mathcal{P}$ of probability measures

We place oureselves in the context of model-uncertainty, expressed by a given non-empty set $\mathcal{P}$ of prior probability measures. The following result holds true.

**Proposition 5.2** *Let $\mathcal{P}$ be a given non-empty set of probability measures on $(\Omega, \mathcal{F})$. Let us define a capacity $\mu$ on $(\Omega, \mathcal{F})$ by $\mu(A) := \sup_{P \in \mathcal{P}} P(A)$ and let $X$ be a given real-valued measurable function on $(\Omega, \mathcal{F})$. Then, for all $t \in (0,1)$,*

$$r^{-}_{X,\mu}(t) = \sup_{P \in \mathcal{P}} q^{-}_{X,P}(t), \tag{5.3}$$

*where $q^{-}_{X,P}$ denotes the lower quantile function of $X$ with respect to the probability $P$.*

**Proof:** The proof of proposition 5.2 is given in the appendix.

□

If the capacity $\mu$ of the form $\mu(.) := \sup_{P \in \mathcal{P}} P(.)$ is interpreted as expressing a pessimistic attitude towards model-uncertainty [5], the relation (5.3) of the previous proposition can be loosely interpreted as follows:
the risk, equal to $GVaR^{\mu}_{\lambda}(X)$, attributed to a given loss $X$ by a pessimistic agent facing model-uncertainty, is equal to the supremum of the risks $VaR^{P}_{\lambda}(X)$ attributed to the loss $X$ in each of the prior models $P \in \mathcal{P}$.

**Remark 5.3** In the case where the capacity $\mu$ is the "lower envelope" of the set $\mathcal{P}$ of prior probability measures (i.e. $\mu(.) := \inf_{P \in \mathcal{P}} P(.)$) the following result about upper quantile functions can be shown:

$$r^{+}_{X,\mu}(t) = \inf_{P \in \mathcal{P}} q^{+}_{X,P}(t), \ \forall t \in (0,1).$$

An interpretation in terms of the agent's optimism could be given in this case.

[5]Our interpretation of the capacity $\mu$ of the form $\mu(.) := \sup_{P \in \mathcal{P}} P(.)$ as expressing a pessimistic attitude towards model-uncertainty is motivated by the following observation: $\mathbb{E}_{\mu}(u(X)) \geq \mathbb{E}_{P}(u(X)), \ \forall X \in \chi, \ \forall P \in \mathcal{P}$, where $u : \mathbb{R} \to \mathbb{R}$ is a given function. The inequality is due to proposition 5.2 (iii) in Denneberg (1994). A CEU-minimizer with a "pain" function $u$ and a capacity $\mu(.) := \sup_{P \in \mathcal{P}} P(.)$ assesses his/her "dissatisfaction" with a loss $X \in \chi$ by the number $\mathbb{E}_{\mu}(u(X))$ which, according to the previous observation, is greater than (or equal to) the "dissatisfaction" $\mathbb{E}_{P}(u(X))$ associated to the loss $X$ in any of the prior models $P \in \mathcal{P}$. Thus, in the context of model-uncertainty, a CEU-minimizer whose capacity $\mu$ is of the form $\mu(.) = \sup_{P \in \mathcal{P}} P(.)$ will be considered as being pessimistic.

**Remark 5.4** In the case where $\mu(.) := \sup_{P\in\mathcal{P}} P(.)$ the "generalized" Value at Risk defined in definition/proposition 5.1 of the present article can be linked to a risk measure introduced in definition III.15 of Kervarec (2008). More precisely, the "generalized" Value at Risk $GVaR^{\mu}_{\lambda}(X)$ at level $\lambda \in (0,1)$ of a given measurable function $X \in \chi$ is equal to Kervarec's "Value at Risk" at level $(1-\lambda)$ of the measurable function $(-X)$. Indeed, thanks to the above proposition 5.2 and to lemma 2.1 in Dhaene et al. (2006), we obtain $GVaR^{\mu}_{\lambda}(X) = \sup_{P\in\mathcal{P}} -q^{+}_{-X,P}(1-\lambda)$. The term on the right-hand side of the previous equality is equal to Kervarec's "Value at Risk" at level $(1-\lambda)$ of $(-X)$ by proposition III.17 in Kervarec (2008); the desired link between the two risk measures is thus established.
We note as well that the minus sign preceding $X$ in this relation is not surprising as the measurable functions on $(\Omega, \mathcal{F})$ in the present paper are viewed as losses, unlike the interpretation given in the work of Kervarec (2008) where they are perceived as gains.

## 5.2 A "generalized" Tail Value at Risk

The "classical" definition of the risk measure Tail Value at Risk is recalled hereafter for reader's convenience (cf., for instance, Dhaene et al. 2006). The "classical" Tail Value at Risk at level $\lambda \in (0,1)$ with respect to a given probability $P$ of a given "potential loss" $X \in \chi$ (denoted by $TVaR_{\lambda}(X)$ or by $TVaR^{P}_{\lambda}(X)$) is defined by:

$$TVaR_{\lambda}(X) := \frac{1}{1-\lambda}\int_{\lambda}^{1} q_X(t)dt$$

where the symbol $q_X$ denotes a (version of the) quantile function of $X$ with respect to the probability $P$. We note that the Tail Value at Risk of $X \in \chi$ at level $\lambda \in (0,1)$ (as defined above) is equal to the Average Value at Risk of $(-X)$ at level $(1-\lambda)$ appearing, for instance, in definition 4.43 of Föllmer and Schied (2004).
We consider hereafter a generalization of the previous definition to the case of a capacity which is not necessarily a probability measure. The definition and some properties of the "gerneralized" Tail Value at Risk are given in the following

**Definition/Proposition 5.2** *Let $\mu$ be a capacity and let $\lambda \in (0,1)$.*
*The functional $GTVaR^{\mu}_{\lambda} : \chi \to \mathbb{R}$ defined by $GTVaR^{\mu}_{\lambda}(X) := \frac{1}{1-\lambda}\int_{\lambda}^{1} r^{+}_{X}(t)dt,\ \forall X \in \chi$ can be represented in the form:*

$$GTVaR^{\mu}_{\lambda}(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi \tag{5.4}$$

*where $\psi$ is a concave distortion function given by $\psi(x) := \psi_{\lambda}(x) := \frac{1}{1-\lambda}\min\{1-\lambda; x\},\ \forall x \in [0,1]$.*
*In particular, if $\mu$ is a concave capacity, $GTVaR^{\mu}_{\lambda}$ is a sub-additive functional on $\chi$ i.e. $GTVaR^{\mu}_{\lambda}(X+Y) \le GTVaR^{\mu}_{\lambda}(X) + GTVaR^{\mu}_{\lambda}(Y), \forall X, Y \in \chi$.*

**Remark 5.5** The last statement in the previous definition/proposition 5.2 corresponds to exercise 6.7 in Denneberg (1994). The formulation given above is suitable for the needs of the present paper.

**Remark 5.6** The factor $\frac{1}{1-\lambda}$ in the definition of the functional $GTVaR^{\mu}_{\lambda}$ is necessary to obtain a normalised set function $\psi \circ \mu$ in the representation formula (5.4) in the sense that $\psi(\mu(\Omega)) = 1$.

Let us now prove the result; the proof is based on lemma 3.1.

**Proof:** It is easy to check that the functional $GTVaR^{\mu}_{\lambda}$ satisfies properties (i), (ii) and (iii) of lemma 3.1; it follows, in particular, that there exists a non-decreasing function $\psi$ defined on the set $S := \{\mu(A),\ A \in \mathcal{F}\}$ such that the representation (5.4) holds, namely $GTVaR^{\mu}_{\lambda}(X) = \mathbb{E}_{\psi\circ\mu}(X),\ \forall X \in \chi$. The expression of the function $\psi$ on the set $S$ can be computed from (5.4) as follows: for all $A \in \mathcal{F}$

$$\begin{aligned}\psi\circ\mu(A) = GTVaR^{\mu}_{\lambda}(\mathbb{I}_A) &= \frac{1}{1-\lambda}\int_{\lambda}^{1} \mathbb{I}_{[1-\mu(A),1)}(t)dt \\ &= \frac{1}{1-\lambda}\Big(1 - \max\{\lambda; 1-\mu(A)\}\Big) = \frac{1}{1-\lambda}\min\{1-\lambda; \mu(A)\}.\end{aligned}$$

Then, $\psi$ is extended to the whole interval $[0,1]$ by setting $\psi(x) := \frac{1}{1-\lambda}\min\{1-\lambda; x\},\ \forall x \in [0,1]$. The function $\psi$ is obviously a concave distortion function.

In the case where $\mu$ is a concave capacity, the distorted capacity $\psi \circ \mu$ in the representation (5.4) is concave as the distortion function $\psi$ is concave (see example 2. at the end of subsection 2.1). The representation (5.4) and the property of sub-additivity of the Choquet integral with respect to a concave capacity allow us to conclude that the functional $GTVaR_\lambda^\mu$ is sub-additive in this case.

□

Thanks to the representation formula (5.4) of the previous definition/proposition and to theorem 3.3 we conclude that the functional $GTVaR_\lambda^\mu$ is a monetary risk measure on $\chi$ having the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation.

**Remark 5.7** We note that the monetary risk measure $GTVaR_\lambda^\mu$ can be used to characterize the $\leq_{sl,\mu}$ −stochastic dominance relation with respect to a capacity $\mu$. More precisely, it follows from proposition 2.6 that:

$$X \leq_{sl,\mu} Y \text{ if and only if } GTVaR_\lambda^\mu(X) \leq GTVaR_\lambda^\mu(Y),\ \forall \lambda \in (0,1),$$

where $X$ and $Y$ are real-valued measurable functions such that $\int_0^1 |r_{X,\mu}(t)|dt < +\infty$ and $\int_0^1 |r_{Y,\mu}(t)|dt < +\infty$. The previous equivalence can be seen as a generalization to the case of a capacity of remark 4.44 in Föllmer and Schied (2004).

# A Appendix

**Proof of lemma 2.3:** The function $f$ being non-decreasing, we define the following (upper) inverse $\check{f}$ of $f$ by $\check{f}(y) := \sup\{z : f(z) \leq y\},\ \forall y \in \mathbb{R}$. Note that according to remark 2.2 the function $\check{f}$ can be expressed in the following manner $\check{f}(y) := \inf\{z : f(z) > y\},\ \forall y \in \mathbb{R}$. As the function $f$ is non-decreasing and as the functions $f$ and $G_Z$ have no common discontinuities, we know from Yan (2009) that

$$G_{f(Z)}(x) = G_Z \circ \check{f}(x),\ \forall x \in \mathbb{R}. \tag{A.1}$$

Thanks to (A.1) and to remark 2.2, the upper quantile function $r^+_{f(Z)}$ of $f(Z)$ can be expressed as follows

$$r^+_{f(Z)}(t) = \sup\{x : G_Z \circ \check{f}(x) \leq t\} = \inf\{x : G_Z \circ \check{f}(x) > t\}. \tag{A.2}$$

For a fixed $t \in (0,1)$, let us first prove that $r^+_{f(Z)}(t) \geq f(r^+_Z(t))$ which, thanks to the previous considerations, amounts to showing that $\inf\{x : G_Z \circ \check{f}(x) > t\} \geq f(r^+_Z(t))$. The case where the set $\{x : G_Z \circ \check{f}(x) > t\}$ is empty being trivial, let $x \in \mathbb{R}$ be such that

$$G_Z \circ \check{f}(x) > t. \tag{A.3}$$

Now, the inequality (A.3) and the fact that $r^+_Z(t) = \inf\{y : G_Z(y) > t\}$ imply that $\check{f}(x) \geq r^+_Z(t)$. We consider two cases

- 1st case: If $x$ is such that $\check{f}(x) > r^+_Z(t)$, then $f(r^+_Z(t)) \leq x$. This implication is due to the definition of $\check{f}(x)$.

- 2nd case: In the case where $x$ is such that $\check{f}(x) = r^+_Z(t)$, the inequality (A.3) gives $G_Z(r^+_Z(t)) > t$.

  In the sub-case where $\check{f}(x)$ and $r^+_Z(t)$ belong to $\mathbb{R}$, we conclude from the latter inequality that $r^+_Z(t)$ is a point of discontinuity of $G_Z$ which implies that $f$ is continuous at $r^+_Z(t)$. Thus we obtain that $f(r^+_Z(t)) = f(\check{f}(x)) = x$.

  In the sub-case where $\check{f}(x) = r^+_Z(t) = +\infty$, we have, thanks to the definition of $\check{f}(x)$, that $\sup_{y\in\mathbb{R}} f(y) \leq x$. Therefore, $f(r^+_Z(t)) = f(+\infty) \leq x$.

  The measurable function $Z$ being real-valued, the inequality (A.3) implies that $\check{f}(x) \neq -\infty$. Thus, only the two above-mentioned sub-cases are to be considered.

In both of the cases the inequality $x \geq f(r^+_Z(t))$ holds; the desired inequality $r^+_{f(Z)}(t) \geq f(r^+_Z(t))$ follows.

Let us prove the converse inequality namely $r^+_{f(Z)}(t) \leq f(r^+_Z(t))$ which is equivalent to $\sup\{x : G_Z \circ \check{f}(x) \leq t\} \leq f(r^+_Z(t))$. Let $x$ be such that $G_Z \circ \check{f}(x) \leq t$. This inequality implies that $\check{f}(x) \neq +\infty$ and that $\check{f}(x) \leq r^+_Z(t)$.

- If $\check{f}(x) \in \mathbb{R}$, then applying the non-decreasing function $f$ at both sides of the latter inequality gives $f(\check{f}(x)) \leq f(r_Z^+(t))$. Now, the function $f$ being right-continuous and the function $\check{f}$ being a generalized inverse of $f$ we have $f(\check{f}(x)) = f(\check{f}(x)+) \geq x$. Thus we obtain $x \leq f(r_Z^+(t))$.

- If $\check{f}(x) = -\infty$, then $x \leq \inf_{y \in \mathbb{R}} f(y)$ (due to the definition of $\check{f}(x)$). Therefore, $x \leq f(r_Z^+(t))$ which concludes the proof.

□

**Proof of proposition 2.5:** Let us prove the result concerning the upper quantile functions (equation (2.2)). The proof is based on lemma 2.3. The assertion concerning the lower quantile functions follows from lemma 2.4 by means of similar arguments.

According to proposition 2.2, there exist two non-decreasing continuous functions $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ and a real-valued measurable function $Z$ such that $X = u(Z)$ and $Y = v(Z)$. Let $t \in (0,1)$. As the function $u + v$ is non-decreasing and continuous, we can apply lemma 2.3 to obtain

$$r^+_{X+Y}(t) = r^+_{(u+v)(Z)}(t) = (u+v)r^+_Z(t) = u\left(r^+_Z(t)\right) + v\left(r^+_Z(t)\right).$$

It follows from lemma 2.3 (applied with $f = u$ and with $f = v$) that $u\left(r^+_Z(t)\right) = r^+_{u(Z)}(t)$ and $v\left(r^+_Z(t)\right) = r^+_{v(Z)}(t)$ which concludes the proof.

□

**The counter-example of remark 4.1:**

Indeed, let $(\Omega, \mathcal{F}, P)$ be an atomless probability space. Let $\psi(x) := x^\alpha, \forall x \in [0,1]$ and $\phi(x) := x^\beta, \forall x \in [0,1]$ where $\alpha \in (0,1)$ and $\beta > \frac{1}{\alpha}$. Let us define a capacity $\mu$ by $\mu := \phi \circ P$ and a functional $\rho$ by $\rho(X) := \mathbb{E}_{\psi \circ \mu}(X)$, $\forall X \in \chi$. The space $(\Omega, \mathcal{F}, \mu)$ satisfies the assumptions of theorem 4.3. Moreover, by applying theorem 3.3 (the distortion function $\psi$ being concave), we obtain that the functional $\rho$ is a monetary risk measure satisfying the properties of comonotonic additivity and consistency with respect to the $\leq_{sl,\mu}$ −relation. However, the functional $\rho$ is not convex. The lack of convexity of $\rho$ can be deduced from the fact that $\rho$ can be represented as a Choquet integral with respect to the distorted probability $(\psi \circ \phi) \circ P$ where $\psi \circ \phi$ is a distortion function which is not concave (cf. proposition 4.69 and theorem 4.88 in Föllmer and Schied 2004).

**The counter-example of remark 4.2:**

In the framework of the counter-example of subsection 3.4, let us define a functional $\rho : \chi \to \mathbb{R}$ by $\rho(X) = \mathbb{E}_{\psi \circ \mu}(X), \forall X \in \chi$ where the distortion function $\psi$ and the capacity $\mu$ are the same as in the counter-example of subsection 3.4. We note that the space $(\Omega, \mathcal{F}, \mu)$ of the counter-example of subsection 3.4 satisfies the assumptions of theorem 4.3. Being a generalized distortion risk measure, the functional $\rho$ satisfies the properties of comonotonic additivity and consistency with respect to the $\leq_{mon,\mu}$ −relation (cf. remark 3.6). Moreover, the capacity $\psi \circ \mu$ being concave, the functional $\rho$ is convex. However, $\rho$ is not consistent with respect to the $\leq_{sl,\mu}$ −relation as the distortion function $\psi$ is not concave. The last statement can be easily deduced from theorem 3.2 and remark 3.11.

**Proof of proposition 5.1:** Using the definition of the lower quantile function $r^{-}_{X,\mu}$ and the definition of the distribution function $G_{X,\mu}$, as well as the particular form of the capacity $\mu$, we compute

$$\begin{aligned} r^{-}_{X,\mu}(t) &= \sup\{x \in \mathbb{R} : G_{X,\mu}(x) < t\} = \\ &= \sup\{x \in \mathbb{R} : \mu(X > x) > 1 - t\} = \\ &= \sup\{x \in \mathbb{R} : \phi(P(X > x)) > 1 - t\}. \end{aligned}$$

Now, the function $\phi$ being continuous by assumption, the following equivalence holds true

$$\phi(a) \leq t \text{ if and only if } a \leq \check{\phi}(t). \tag{A.4}$$

This observation implies that

$$\sup\{x \in \mathbb{R} : \phi(P(X > x)) > 1 - t\} = \sup\{x \in \mathbb{R} : P(X > x) > \check{\phi}(1 - t)\}.$$

Finally, it follows from the definition of the distribution function (with respect to $P$) $F_X$ and the definition of the lower quantile function (with respect to $P$) $q_X^-$ that

$$\begin{aligned}\sup\{x \in \mathbb{R} : P(X > x) > \check{\phi}(1 - t)\} &= \sup\{x \in \mathbb{R} : F_X(x) < 1 - \check{\phi}(1 - t)\} = \\ &= q_X^-(1 - \check{\phi}(1 - t))\end{aligned}$$

which concludes the proof.

□

**Proof of proposition 5.2:** Let $t \in (0, 1)$. The definitions of the lower quantile function $r_{X,\mu}^-$ and of the distribution function $G_{X,\mu}$, as well as the particular form of the capacity $\mu$ lead to the following equalities:

$$\begin{aligned}r_{X,\mu}^-(t) &= \sup\{x \in \mathbb{R} : G_{X,\mu}(x) < t\} = \\ &= \sup\{x \in \mathbb{R} : 1 - \sup_{P \in \mathcal{P}} P(X > x) < t\} = \\ &= \sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} \big(1 - P(X > x)\big) < t\}.\end{aligned}$$

Therefore,

$$r_{X,\mu}^-(t) = \sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\},$$

where $F_{X,P}$ denotes the distribution function of $X$ with respect to the probability $P$. In order to establish the desired result, it suffices to prove that

$$\sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\} = \sup_{P \in \mathcal{P}} q_{X,P}^-(t). \tag{A.5}$$

Let us first prove the inequality $\sup\{x \in \mathbb{R} : \inf_{P \in \mathcal{P}} F_{X,P}(x) < t\} \leq \sup_{P \in \mathcal{P}} q_{X,P}^-(t)$. Let $x \in \mathbb{R}$ be such that $\inf_{P \in \mathcal{P}} F_{X,P}(x) < t$. Then, there exists $P_x \in \mathcal{P}$ such that $F_{X,P_x}(x) < t$. This inequality and the definition of the lower quantile function $q_{X,P_x}^-$ lead to $x \leq q^{-}_{X,P_x}(t)$. Thus, $x \leq \sup_{P\in\mathcal{P}} q^{-}_{X,P}(t)$.

Let us prove the converse inequality, namely $\sup\{x \in \mathbb{R} : \inf_{P\in\mathcal{P}} F_{X,P}(x) < t\} \geq \sup_{P\in\mathcal{P}} q^{-}_{X,P}(t)$. Let $P \in \mathcal{P}$ and let $x_P \in \mathbb{R}$ be such that $F_{X,P}(x_P) < t$. Then, $x_P$ satisfies $\inf_{Q\in\mathcal{P}} F_{X,Q}(x_P) < t$. Therefore, $x_P \leq \sup\{y \in \mathbb{R} : \inf_{Q\in\mathcal{P}} F_{X,Q}(y) < t\}$. This inequality and the definition of the lower quantile function $q^{-}_{X,P}$ imply $q^{-}_{X,P}(t) \leq \sup\{y \in \mathbb{R} : \inf_{Q\in\mathcal{P}} F_{X,Q}(y) < t\}$. The probability $P \in \mathcal{P}$ being arbitrary, the proof is thus conluded.

□

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