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► **To cite this version:**

Catalin Badea, Michel Crouzeix. Numerical radius and distance from unitary operators. *Operators and Matrices*, 2013, 7 (2), pp.285-292. <10.7153/oam-07-16>. <hal-00634780v2>

HAL Id: hal-00634780

<https://hal.archives-ouvertes.fr/hal-00634780v2>

Submitted on 24 Jan 2012

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Numerical radius and distance from unitary operators

Catalin BADEA and Michel CROUZEIX

Abstract

Denote by $w(A)$ the numerical radius of a bounded linear operator A acting on Hilbert space. Suppose that A is invertible and that $w(A) \leq 1+\varepsilon$ and $w(A^{-1}) \leq 1+\varepsilon$ for some $\varepsilon \geq 0$. It is shown that $\inf\{\|A-U\| : U \text{ unitary}\} \leq c\varepsilon^{1/4}$ for some constant $c > 0$. This generalizes a result due to J.G. Stampfli, which is obtained for $\varepsilon = 0$. An example is given showing that the exponent $1/4$ is optimal. The more general case of the operator ρ -radius $w_\rho(\cdot)$ is discussed for $1 \leq \rho \leq 2$.

1 Introduction and statement of the results

Let H be a complex Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the associated norm $\|\cdot\|$. We denote by $\mathcal{B}(H)$ the C^* -algebra of all bounded linear operators on H equipped with the operator norm

$$\|A\| = \sup\{\|Ah\| : h \in H, \|h\| = 1\}.$$

It is easy to see that unitary operators can be characterized as invertible contractions with contractive inverses, i.e. as operators A with $\|A\| \leq 1$ and $\|A^{-1}\| \leq 1$. More generally, if $A \in \mathcal{B}(H)$ is invertible then

$$\inf\{\|A-U\| : U \text{ unitary}\} = \max\left(\|A\| - 1, 1 - \frac{1}{\|A^{-1}\|}\right).$$

We refer to [6, Theorem 1.3] and [9, Theorem 1] for a proof of this equality using the polar decomposition of bounded operators. It also follows from this proof that if $A \in \mathcal{B}(H)$ is an invertible operator satisfying $\|A\| \leq r$ and $\|A^{-1}\| \leq r$ for some $r \geq 1$, then there exists a unitary operator $U \in \mathcal{B}(H)$ such that $\|A-U\| \leq r-1$.

The numerical radius of the operator A is defined by

$$w(A) = \sup\{|\langle Ah, h \rangle| : h \in H, \|h\| = 1\}.$$

Stampfli has proved in [8] that numerical radius contractivity of A and of its inverse A^{-1} , that is $w(A) \leq 1$ and $w(A^{-1}) \leq 1$, imply that A is unitary. We define a function $\psi(r)$ for $r \geq 1$ by

$$\psi(r) = \sup\{\|A\| : A \in \mathcal{B}(H), w(A) \leq r, w(A^{-1}) \leq r\},$$

the supremum being also considered over all Hilbert spaces H . Then the conditions $w(A) \leq r$ and $w(A^{-1}) \leq r$ imply $\max(\|A\|-1, 1-\|A^{-1}\|^{-1}) \leq \max(\|A\|-1, \|A^{-1}\|-1) \leq \psi(r)-1$, hence the existence of a unitary operator U such that $\|A-U\| \leq \psi(r)-1$. We have the two-sided estimate

$$r + \sqrt{r^2 - 1} \leq \psi(r) \leq 2r.$$

The upper bound follows from the well-known inequalities $w(A) \leq \|A\| \leq 2w(A)$, while the lower bound is obtained by choosing $H = \mathbb{C}^2$ and

$$A = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \quad \text{with } y = \sqrt{r^2 - 1},$$

in the definition of ψ . Indeed, we have $A = A^{-1}$, $w(A) = \sqrt{1+y^2} = r$, and $\|A\| = y + \sqrt{1+y^2} = r + \sqrt{r^2-1}$.

Our first aim is to improve the upper estimate.

Theorem 1.1. *Let $r \geq 1$. Then*

$$\psi(r) \leq X(r) + \sqrt{X(r)^2-1}, \quad \text{with } X(r) = r + \sqrt{r^2-1}. \quad (1)$$

The estimate given in Theorem 1.1 is more accurate than $\psi(r) \leq 2r$ for r close to 1, more precisely for $1 \leq r \leq 1.0290855\dots$. It also gives $\psi(1) = 1$ (leading to Stampfli's result) and the following asymptotic estimate.

Corollary 1.2. *We have*

$$\psi(1+\varepsilon) \leq 1 + \sqrt[4]{8\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \rightarrow 0.$$

Our second aim is to prove that the exponent $1/4$ in Corollary 1.2 is optimal. This is a consequence of the following result.

Theorem 1.3. *Let n be a positive integer of the form $n = 8k + 4$. There exists a $n \times n$ invertible matrix A_n with complex entries such that*

$$w(A_n) \leq \frac{1}{\cos \frac{\pi}{n}}, \quad w(A_n^{-1}) \leq \frac{1}{\cos \frac{\pi}{n}}, \quad \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Indeed, Theorem 1.3 implies that

$$\psi\left(\frac{1}{\cos \frac{\pi}{n}}\right) \geq \|A_n\| = 1 + \frac{1}{8\sqrt{n}}.$$

Taking $1+\varepsilon = 1/\cos \frac{\pi}{n} = 1 + \frac{\pi^2}{2n^2} + O(\frac{1}{n^4})$, we see that the exponent $\frac{1}{4}$ cannot be improved.

More generally, we can consider for $\rho \geq 1$ the ρ -radius $w_\rho(A)$ introduced by Sz.-Nagy and Foias (see [5, Chapter 1] and the references therein). Consider the class \mathcal{C}_ρ of operators $T \in \mathcal{B}(H)$ which admit unitary ρ -dilations, i.e. there exist a super-space $\mathcal{H} \supset H$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$T^n = \rho P U^n P^*, \quad \text{for } n = 1, 2, \dots$$

Here P denotes the orthogonal projection from \mathcal{H} onto H . Then the operator ρ -radius is defined by

$$w_\rho(A) = \inf\{\lambda > 0; \lambda^{-1}A \in \mathcal{C}_\rho\}.$$

From this definition it is easily seen that $r(A) \leq w_\rho(A) \leq \rho\|A\|$, where $r(A)$ denotes the spectral radius of A . Also, $w_\rho(A)$ is a non-increasing function of ρ . Another equivalent definition follows from [5, Theorem 11.1]:

$$w_\rho(A) = \sup_{h \in \mathcal{E}_\rho} \left\{ \left(1 - \frac{1}{\rho}\right) |\langle Ah, h \rangle| + \sqrt{\left(1 - \frac{1}{\rho}\right)^2 |\langle Ah, h \rangle|^2 + \left(\frac{2}{\rho} - 1\right) \|Ah\|^2} \right\}, \quad \text{with}$$

$$\mathcal{E}_\rho = \{h \in H; \|h\| = 1 \text{ and } (1 - \frac{1}{\rho})^2 |\langle Ah, h \rangle|^2 - (1 - \frac{2}{\rho}) \|Ah\|^2 \geq 0\}.$$

Notice that $\mathcal{E}_\rho = \{h \in H; \|h\| = 1\}$ whenever $1 \leq \rho \leq 2$. This shows that $w_1(A) = \|A\|$, $w_2(A) = w(A)$ and $w_\rho(A)$ is a convex function of A if $1 \leq \rho \leq 2$.

We now define a function $\psi_\rho(r)$ for $r \geq 1$ by

$$\psi_\rho(r) = \sup\{\|A\|; A \in \mathcal{B}(H), w_\rho(A) \leq r, w_\rho(A^{-1}) \leq r\}.$$

As before, the conditions $w_\rho(A) \leq r$ and $w_\rho(A^{-1}) \leq r$ imply the existence of a unitary operator U such that $\|A-U\| \leq \psi_\rho(r)-1$, and we have $\psi_\rho(r) \leq \rho r$. We will generalize the estimate (1) from Theorem 1.1 by proving, for $1 \leq \rho \leq 2$, the following result.

Theorem 1.4. For $1 \leq \rho \leq 2$ we have

$$\psi_\rho(r) \leq X_\rho(r) + \sqrt{X_\rho(r)^2 - 1}, \quad (2)$$

with $X_\rho(r) = \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2r}$.

Corollary 1.5. For $1 \leq \rho \leq 2$ we have

$$\psi_\rho(1+\varepsilon) \leq 1 + \sqrt[4]{8(\rho-1)\varepsilon} + O(\varepsilon^{1/2}), \quad \varepsilon \rightarrow 0.$$

We recover in this way for $1 \leq \rho \leq 2$ the recent result of Ando and Li [2, Theorem 2.3], namely that $w_\rho(A) \leq 1$ and $w_\rho(A^{-1}) \leq 1$ imply A is unitary. The range $1 \leq \rho \leq 2$ coincides with the range of those $\rho \geq 1$ for which $w_\rho(\cdot)$ is a norm. Contrarily to [2], we have not been able to treat the case $\rho > 2$.

The organization of the paper is as follows. In Section 2 we prove Theorem 1.4, which reduces to Theorem 1.1 in the case $\rho = 2$. The proof of Theorem 1.3 which shows the optimality of the exponent $1/4$ in Corollary 1.2 is given in Section 3.

As a concluding remark, we would like to mention that the present developments have been influenced by the recent work of Sano/Uchiyama [7] and Ando/Li [2]. In [3], inspired by the paper of Stampfi [8], we have developed another (more complicated) approach in the case $\rho = 2$.

2 Proof of Theorem 1.4 about ψ_ρ

Let us consider $M = \frac{1}{2}(A + (A^*)^{-1})$; then

$$M^*M - 1 = \frac{1}{4}(A^*A + (A^*A)^{-1} - 2) \geq 0.$$

This implies $\|M^{-1}\| \leq 1$. In what follows $C^{1/2}$ will denote the positive square root of the self-adjoint positive operator C . The relation $(A^*A - 2M^*M + 1)^2 = 4M^*M(M^*M - 1)$ yields

$$A^*A - 2M^*M + 1 \leq 2(M^*M)^{1/2}(M^*M - 1)^{1/2},$$

whence $A^*A \leq ((M^*M)^{1/2} + (M^*M - 1)^{1/2})^2$.

Therefore $\|A\| \leq \|M\| + \sqrt{\|M\|^2 - 1}$.

We now assume $1 \leq \rho \leq 2$. Then $w_\rho(\cdot)$ is a norm and the two conditions $w_\rho(A) \leq r$ and $w_\rho(A^{-1}) \leq r$ imply $w_\rho(M) \leq r$. The desired estimate of $\psi_\rho(r)$ will follow from the following auxiliary result.

Lemma 2.1. Assume $\rho \geq 1$. Then the assumptions $w_\rho(M) \leq r$ and $\|M^{-1}\| \leq 1$ imply $\|M\| \leq X_\rho(r)$.

Proof. The contractivity of M^{-1} implies

$$\|u\| \leq \|Mu\|, \quad (\forall u \in H). \quad (3)$$

As $w_\rho(M) \leq r$, it follows from a generalization by Durszt [4] of a decomposition due to Ando [1], that the operator M can be decomposed as

$$M = \rho r B^{1/2}UC^{1/2},$$

with U unitary, C selfadjoint satisfying $0 < C < 1$, and $B = f(C)$ with $f(x) = (1-x)/(1-\rho(2-\rho)x)^{-1}$. Notice that f is a decreasing function on the segment $[0, 1]$ and an involution: $f(f(x)) = x$. Let $[\alpha, \beta]$ be the smallest segment containing the spectrum of C . Then $[\sqrt{\alpha}, \sqrt{\beta}]$ is the smallest segment

containing the spectrum of $C^{1/2}$ and $[\sqrt{f(\beta)}, \sqrt{f(\alpha)}]$ is the smallest segment containing the spectrum of $B^{1/2}$. We have

$$\|u\| \leq \|Mu\| \leq \rho r \sqrt{f(\alpha)} \|C^{1/2}u\|, \quad (\forall u \in H).$$

Choosing a sequence u_n of norm-one vectors ($\|u_n\| = 1$) such that $\|C^{1/2}u_n\|$ tends to $\sqrt{\alpha}$, we first get $1 \leq \rho r \sqrt{\alpha f(\alpha)}$, i.e. $1 - (2 + \rho r^2 - \rho)\rho\alpha + \rho^2 r^2 \alpha^2 \leq 0$. Consequently we have

$$\frac{2 + \rho r^2 - \rho - \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2} \leq \alpha \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2},$$

and by $\alpha = f(f(\alpha))$

$$\frac{2 + \rho r^2 - \rho - \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2} \leq f(\alpha) \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2}.$$

Similarly, noticing that $\|(M^*)^{-1}\| \leq 1$, $M^* = \rho r C^{1/2}U^*B^{1/2}$ and $C = f(B)$, we obtain

$$\frac{2 + \rho r^2 - \rho - \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2} \leq \beta \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2\rho r^2}.$$

Therefore

$$\|M\| \leq \rho r \|B^{1/2}\| \|C^{1/2}\| = \rho r \sqrt{f(\alpha)\beta} \leq \frac{2 + \rho r^2 - \rho + \sqrt{(2 + \rho r^2 - \rho)^2 - 4r^2}}{2r}.$$

This shows that $\|M\| \leq X_\rho(r)$. □

3 The exponent $1/4$ is optimal (Proof of Theorem 1.3)

Consider the family of $n \times n$ matrices $A = DBD$, defined for $n = 8k + 4$, by

$$\begin{aligned} D &= \text{diag}(e^{i\pi/2n}, \dots, e^{(2\ell-1)i\pi/2n}, \dots, e^{(2n-1)i\pi/2n}), \\ B &= I + \frac{1}{2n^{3/2}}E, \quad \text{where } E \text{ is a matrix whose entries are defined as} \\ e_{ij} &= 1 \quad \text{if } 3k + 2 \leq |i - j| \leq 5k + 2, \quad e_{ij} = 0 \quad \text{otherwise.} \end{aligned}$$

We first remark that $\|A\| = \|B\| = 1 + \frac{1}{8\sqrt{n}}$. Indeed, B is a symmetric matrix with non negative entries, $Be = (1 + \frac{1}{8\sqrt{n}})e$ with $e^T = (1, 1, 1, \dots, 1)$. Thus $\|B\| = r(B) = 1 + \frac{1}{8\sqrt{n}}$ by the Perron-Frobenius theorem.

Consider now the permutation matrix P defined by $p_{ij} = 1$ if $i = j + 1$ modulo n and $p_{ij} = 0$ otherwise and the diagonal matrix $\Delta = \text{diag}(1, \dots, 1, -1)$. Then $P^{-1}DP = e^{i\pi/n}\Delta D$ and $P^{-1}EP = E$, whence $(P\Delta)^{-1}AP\Delta = e^{2i\pi/n}A$. Since $P\Delta$ is a unitary matrix, the numerical range $W(A) = \{ \langle Au, u \rangle, \|u\| = 1 \}$ of A satisfies $W(A) = W((P\Delta)^{-1}AP\Delta) = e^{2i\pi/n}W(A)$. This shows that the numerical range of A is invariant by the rotation of angle $2\pi/n$ centered in 0, and the same property also holds for the numerical range of A^{-1} .

We postpone the proof of the estimates $\|\frac{1}{2}(A + A^*)\| \leq 1$ and $\|\frac{1}{2}(A^{-1} + (A^{-1})^*)\| \leq 1$ to later sections. Using these estimates, we obtain that the numerical range $W(A)$ is contained in the half-plane $\{z; \text{Re } z \leq 1\}$, whence in the regular n -sided polygon given by the intersection of the half-planes $\{z; \text{Re}(e^{2i\pi k/n}z) \leq 1\}$, $k = 1, \dots, n$. Consequently $w(A) \leq 1/\cos(\pi/n)$. The proof of $w(A^{-1}) \leq 1/\cos(\pi/n)$ is similar.

3.1 Proof of $\|\frac{1}{2}(A+A^*)\| \leq 1$.

Since the (ℓ, j) -entry of A is $e^{(\ell+j-1)i\frac{\pi}{n}}\left(\delta_{\ell,j} + \frac{e_{\ell,j}}{2n^{3/2}}\right)$, the matrix $\frac{1}{2}(A+A^*)$ is a real symmetric matrix whose (i, j) -entry is $\cos\left((i+j-1)\frac{\pi}{n}\right)\left(\delta_{i,j} + \frac{e_{i,j}}{2n^{3/2}}\right)$. It suffices to show that, for every $u = (u_1, \dots, u_n)^T \in \mathbb{R}^n$, we have $\|u\|^2 - \operatorname{Re}\langle Au, u \rangle \geq 0$. Let $\mathcal{E} = \{(i, j); 1 \leq i, j \leq n, 3k+2 \leq |i-j| \leq 5k+2\}$. The inequality which has to be proved is equivalent to

$$\sum_{i=1}^n 2 \sin^2\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) u_i^2 - \frac{1}{2n^{3/2}} \sum_{i,j \in \mathcal{E}} \cos\left((i+j-1)\frac{\pi}{n}\right) u_i u_j \geq 0.$$

Setting $v_j = u_j \sin\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)$, this may be also written as follows

$$2\|v\|^2 - \langle Mv, v \rangle + \frac{1}{2n^{3/2}} \langle Ev, v \rangle \geq 0, \quad (v \in \mathbb{R}^n). \quad (4)$$

Here M is the matrix whose entries are defined by

$$m_{ij} = \frac{1}{2n^{3/2}} \cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right), \quad \text{if } (i, j) \in \mathcal{E}, \quad m_{ij} = 0 \quad \text{otherwise.}$$

We will see that the Frobenius (or Hilbert-Schmidt) norm of M satisfies $\|M\|_F \leq \sqrt{9/32} < 3/4$. A fortiori, the operator norm of M satisfies $\|M\| \leq \frac{3}{4}$. Together with $\|E\| = n/4$, this shows that $\|M\| + \frac{1}{2n^{3/2}}\|E\| \leq \frac{7}{8}$. Property (4) is now verified.

It remains to show that $\|M\|_F^2 \leq \frac{9}{32}$. First we notice that $m_{ij} = m_{ji} = m_{n+1-i, n+1-j}$, and $m_{ii} = 0$. Hence, with $\mathcal{E}' = \{(i, j) \in \mathcal{E}; i < j \text{ and } i+j \leq n+1\}$,

$$\|M\|_F^2 = 2 \sum_{i < j} |m_{ij}|^2 \leq 4 \sum_{(i,j) \in \mathcal{E}'} |m_{ij}|^2.$$

We have, for $(i, j) \in \mathcal{E}'$,

$$\begin{aligned} 2j \leq i+j+5k+2 \leq n+5k+3 = 13k+7, \quad \text{thus } 3k+3 \leq j \leq \frac{13k+7}{2}, \\ 2i \leq i+j-3k-2 \leq n-3k-1 = 5k+3, \quad \text{thus } 1 \leq i \leq \frac{5k+3}{2}. \end{aligned}$$

This shows that

$$\frac{3\pi}{16} \leq \frac{3k+2}{16k+8}\pi \leq \left(j-\frac{1}{2}\right)\frac{\pi}{n} \leq \frac{13k+6}{16k+8}\pi \leq \pi - \frac{3\pi}{16}, \quad \text{hence } |\cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)| \leq \cot \frac{3\pi}{16} \leq \frac{3}{2}.$$

We also use the estimate $\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \leq n/(\pi(i-\frac{1}{2}))$ and the classical relation $\sum_{i \geq 1} (i-1/2)^{-2} = \pi^2/2$ to obtain

$$\|M\|_F^2 \leq 4 \sum_{(i,j) \in \mathcal{E}'} |m_{ij}|^2 \leq \frac{4}{4n^3} \frac{n^2}{\pi^2} \sum_{i \geq 1} \frac{1}{(i-1/2)^2} (2k+1) \frac{9}{4} = \frac{9}{32}.$$

3.2 Proof of $\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\| \leq 1$.

We start from

$$\begin{aligned} (A^{-1})^* &= D\left(1 + \frac{1}{2n^{3/2}}E\right)^{-1}D \\ &= D^2 - \frac{1}{2n^{3/2}}DED + \frac{1}{4n^3}DE^2\left(1 + \frac{1}{2n^{3/2}}E\right)^{-1}D, \end{aligned}$$

and we want to show that $\|u\|^2 - \operatorname{Re}\langle A^{-1}u, u \rangle \geq 0$. As previously, we set $v_j = u_j \sin\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)$. The inequality $\|\frac{1}{2}(A^{-1}+(A^{-1})^*)\| \leq 1$ is equivalent to

$$2\|v\|^2 - \langle (M_1 + M_2 + M_3 + M_4)v, v \rangle \geq 0, \quad (v \in \mathbb{R}^n).$$

Here the entries of the matrices M_p , $1 \leq p \leq 4$, are given by

$$\begin{aligned}(m_1)_{ij} &= -\frac{1}{2n^{3/2}} \cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right) e_{ij}, \\(m_2)_{ij} &= \frac{1}{2n^{3/2}} e_{ij}, \\(m_3)_{ij} &= \frac{1}{4n^3} \cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right) \cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right) f_{ij}, \\(m_4)_{ij} &= -\frac{1}{4n^3} f_{ij},\end{aligned}$$

e_{ij} and f_{ij} respectively denoting the entries of the matrices E and $F = E^2(1 + \frac{1}{2n^{3/2}}E)^{-1}$. Noticing that $M_1 = -M$, we have $\|M_1\| \leq \frac{3}{4}$, $\|M_2\| = \frac{1}{8\sqrt{n}}$, $\|F\| \leq \frac{n^2/16}{1-1/(8\sqrt{n})} \leq \frac{n^2}{14}$ and $\|M_4\| = \frac{1}{4n^3} \|F\|$. Now we use

$$\|M_3\|^2 \leq \|M_3\|_F^2 \leq \frac{1}{16n^6} \max_{i,j} |f_{ij}|^2 \sum_{i,j} |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2 |\cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2,$$

together with

$$\begin{aligned}\sum_{i,j} |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2 |\cot\left(\left(j-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2 &= \left(\sum_{i=1}^n |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2\right)^2 \\ &\leq 4 \left(\sum_{i=1}^{n/2} |\cot\left(\left(i-\frac{1}{2}\right)\frac{\pi}{n}\right)|^2\right)^2 \leq n^4,\end{aligned}$$

to obtain

$$\|M_3\| \leq \frac{1}{4n} \max_{i,j} |f_{ij}|.$$

Using the notation $\|E\|_\infty := \max\{\|Eu\|_\infty; u \in \mathbb{C}^n, \|u\|_\infty \leq 1\}$ for the operator norm induced by the maximum norm in \mathbb{C}^d , it holds $\|E\|_\infty = n/4$, whence $\|\frac{1}{2n^{3/2}}E\|_\infty \leq 1/8$ and thus $\|(1 + \frac{1}{2n^{3/2}}E)^{-1}\|_\infty \leq \frac{1}{1-1/8} = \frac{8}{7}$. This shows that

$$\max_{i,j} |f_{ij}| \leq \|(1 + \frac{1}{2n^{3/2}}E)^{-1}\|_\infty \max_{i,j} |e_{ij}^2| \leq \frac{2n}{7},$$

by denoting e_{ij}^2 the entries of the matrix E^2 and noticing that $\max_{i,j} |e_{ij}^2| = n/4$. Finally, we obtain $\|M_3\| \leq \frac{1}{14}$ and $\|M_1 + M_2 + M_3 + M_4\| \leq \frac{3}{4} + \frac{1}{8} + \frac{1}{14} + \frac{1}{56} < 1$.

Acknowledgement. The authors thank the referees for a careful reading of the article and for their useful remarks and suggestions. The first-named author was supported in part by ANR Projet Blanc DYNOP.

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