Detection of non-constant long memory parameter

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Abstract This article deals with detection of non-constant long memory parameter in time series. The null hypothesis includes stationary or nonstationary time series with constant long memory parameter, in particular $I(d)$ series, $d > -.5$. The alternative corresponds to a change in the long memory parameter and gathers in particular an abrupt or gradual change from $I(d_1)$ to $I(d_2)$, $-.5 < d_1 < d_2$. Various test statistics are considered. They are all based on the ratio of forward and backward sample variances of the partial sums. The consistency of the tests is proved under a very general setting. Moreover, the behavior of the test statistics is studied for some models with changing memory parameter. A simulation study shows that our testing procedures have good finite sample properties and turn out to be more powerful than the KPSS-based tests considered in some previous works.

Keywords: Long memory; change in persistence; ratio test; change point; V/S statistic; fractional integration.

1 Introduction

The present paper discusses statistical tests for detection of non-constant memory parameter of time series versus the null hypothesis that this parameter has not changed over time. As a particular case, our framework includes testing the null hypothesis that the observed series is $I(d)$ with constant $d > -.5$, against the alternative hypothesis that d has changed, together with a rigorous formulation of the last change. This kind of testing procedure is the basis to study the dynamics of persistence, which is a major question in economy (see Kumar and Okimoto (2007), Hassler and Nautz (2008), Kruse (2008)).

In a parametric setting and for stationary series ($|d| < .5$), the problem of testing for a single change of d was first investigated by Beran and Terrin (1996), Horváth and Shao (1999), Horváth (2001), Yamaguchi (2011) (see also Lavielle and Ludeña (2000), Kokoszka and Leipus (2003)). Typically, the sample is partitioned into two parts and d is estimated on each part. The test statistic is obtained by maximizing the difference of these estimates over all such partitions. A similar approach for detecting

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multiple changes of d was used in Shimotsu (2006) and Bardet and Kammoun (2008) in a more general semiparametric context.

The above approach for testing against changes of d appears rather natural although applies to abrupt changes only and involves (multiple) estimation of d which is not very accurate if the number of observations between two change-points is not large enough; moreover, estimates of d involve bandwidth or some other tuning parameters and are rather sensitive to the short memory spectrum of the process.

On the other hand, some regression-based Lagrange Multiplier procedures have been recently discussed in Hassler and Meller (2009) and Martins and Rodrigues (2010). The series is first filtered by $(1-L)^d$, where L is the lag operator and d is the long memory parameter under the null hypothesis, then the resulting series is subjected to a (augmented) Lagrange Multiplier test for fractional integration, following the pioneer works by Robinson (1991, 1994). The filtering step can be done only approximatively and involves in practice an estimation of d. This is certainly the main reason for the size distortion that can be noticed in the simulation study displayed in Martins and Rodrigues (2010).

In a nonparametric set up, Kim (2000) proposed several tests based on the ratio

$$\mathcal{K}_n(\tau) := \frac{U^*_{n-[n\tau]}(X)}{U_{[n\tau]}(X)}, \qquad \tau \in [0,1], \tag{1.1}$$

where

$$U_k(X) := \tfrac{1}{k^2}\textstyle\sum_{j=1}^k \big(S_j - \tfrac{j}{k}S_k\big)^2, \qquad U^*_{n-k}(X) := \tfrac{1}{(n-k)^2}\textstyle\sum_{j=k+1}^n \big(S^*_{n-j+1} - \tfrac{n-j+1}{n-k}S^*_{n-k}\big)^2 \tag{1.2}$$

are estimates of the second moment of forward and backward de-meaned partial sums

$$\frac{1}{k^{1/2}}\Big(S_j - \frac{j}{k}S_k\Big),\ j=1,\dots,k \qquad \text{and} \qquad \frac{1}{(n-k)^{1/2}}\Big(S^*_{n-j+1} - \frac{n-j+1}{n-k}S^*_{n-k}\Big),\ j=k+1,\dots,n,$$

on intervals $[1,2,\dots,k]$ and $[k+1,\dots,n]$, respectively. Here and below, given a sample $X = (X_1,\dots,X_n)$,

$$S_k := \sum_{j=1}^k X_j, \qquad S^*_{n-k} := \sum_{j=k+1}^n X_j$$

denote the forward and backward partial sums processes. Originally developed to test for a change from $I(0)$ to $I(1)$ (see also Busetti and Taylor (2004), Kim *et al.* (2002)), Kim's statistics were extended in Hassler and Scheithauer (2011), to detect a change from $I(0)$ to $I(d)$, $d > 0$. A related, though different approach based on the so-called CUSUM statistics, was used in Leybourne *et al.* (2007) and Sibbertsen and Kruse (2009) to test for a change from stationarity ($d_1 < .5$) to nonstationarity ($d_2 > .5$), or vice versa.

The present work extends Kim's approach to detect a change from $I(d_1)$ to $I(d_2)$, for any $-.5 < d_1 < d_2$, where $d_1, d_2 \neq .5, 1.5, \dots$, or vice versa. This includes both stationary and nonstationary null (no-change) hypothesis which is important for applications since nonstationary time series with $d > .5$ are common in economics. Although our asymptotic results (Propositions 3.1, 4.1 and Corollary 3.2) are valid for the original Kim's statistics, see Remark 4.3, we modify Kim's ratio (1.1), by replacing the second sample moments $U_k(X)$, $U^*_{n-k}(X)$ in (1.2) of backward and forward partial sums by the corresponding empirical variances $V_k(X)$, $V^*_{n-k}(X)$ defined at (3.1) below. This modification is similar to the difference between the KPSS and the V/S tests, see Giraitis *et al.* (2003), and leads to a more powerful testing procedure (see Table 1). Let us note, finally, that the ratio-based statistics discussed in our paper, as well as the original Kim's statistics, do not require any estimation of d. So they do not depend on any tuning parameter apart from the choice of the testing interval $\mathcal{T} \subset (0,1)$. However, the limiting law under the null hypothesis involves d, therefore the computation of the quantile defining the critical region requires a weakly consistent estimate of the memory parameter d.

The paper is organized as follows. Section 2 contains formulations of the null and alternative hypotheses, in terms of joint convergence of forward and backward partial sums processes, and describes a class of $I(d)$ processes which satisfy the null hypothesis. Section 3 introduces the change-point statistics W_n, I_n and R_n and derives their limit distribution under the null hypothesis. Section 4 displays theoretical results, from which the consistency of our testing procedures is derived. Section 5 discusses the behavior of our statistics under alternative hypothesis. Some fractionally integrated models with constant or changing memory parameter are considered and the behavior of change-point statistics for such models is studied. Section 6 contains simulations of empirical size and power of our testing procedures. All proofs are collected in Section 7.

2 The null and alternative hypotheses

Let $X = (X_1, \cdots, X_n)$ be a sample from a time series $\{X_j\} = \{X_j, j = 1, 2, \dots\}$. Additional assumptions about $\{X_j\}$ will be specified later. Recall the definition of forward and backward partial sums processes of X:

$$S_k = S_k(X) = \sum_{j=1}^{k} X_j, \qquad S^*_{n-k} = S^*_{n-k}(X) = \sum_{j=k+1}^{n} X_j.$$

Note that backward sums can be expressed via forward sums, and vice versa: $S^*_{n-k} = S_n - S_k$, $S_k = S^*_n - S^*_{n-k}$.

For $0 \le a < b \le 1$, let us denote by $D[a,b]$ the Skorokhod space of cadlag functions (i.e. right continuous with left limits functions). In this article, the space $D[a,b]$ and the product space $D[a_1,b_1] \times D[a_2,b_2]$, for any $0 \le a_i < b_i \le 1$, $i = 1, 2$, are all endowed with the uniform topology and the σ-field generated by the open balls (see Pollard (1984)). The weak convergence of random elements in such spaces will be denoted '$\longrightarrow_{D[a,b]}$' and '$\longrightarrow_{D[a_1,b_1]\times D[a_2,b_2]}$', respectively; the convergence in law and in probability of random variables will be denoted '$\longrightarrow_{\mathrm{law}}$' and '$\longrightarrow_p$', respectively.

The following hypotheses are clear particular cases of our more general hypotheses $\mathbf{H_0}$, $\mathbf{H_1}$ specified later. The null hypothesis below involves the classical type I fractional Brownian motion in the limit behavior of the partial sums, which is typical to a long memory behavior for X. Recall that a fractional Brownian motion $B_{d+.5} = \{B_{d+.5}(\tau), \tau \ge 0\}$ (later also referred to as a type I fBm) with Hurst parameter $H = d + .5 \in (0,2)$, $H \ne 1$ is defined by

$$B_{d+.5}(\tau) \quad := \quad \begin{cases} \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\tau} \left((\tau-u)^d - (-u)_+^d\right) \mathrm{d}B(u), & -.5 < d < .5, \\ \int_0^{\tau} B_{d-.5}(u) \mathrm{d}u, & .5 < d < 1.5, \end{cases} \tag{2.1}$$

where $(-u)_+ := (-u) \vee 0$ and $\{B(u), u \in \mathbb{R}\}$ is a standard Brownian motion with zero mean and variance $\mathrm{E}B^2(u) = u$, $u \ge 0$.

$\mathbf{H_0}[\mathbf{I}]$: There exist $d \in (-.5, 1.5)$, $d \neq .5$, $\kappa > 0$ and a normalization B_n such that

$$n^{-d-.5}\big(S_{[n\tau]} - [n\tau]B_n\big) \quad \longrightarrow_{D[0,1]} \quad \kappa B_{d+.5}(\tau), \quad n \to \infty. \tag{2.2}$$

$\mathbf{H_1}[\mathbf{I}]$: There exist $0 \le \upsilon_0 < \upsilon_1 \le 1$, $d > -.5$, and a normalization B_n such that

$$\Big(n^{-d-.5}\big(S_{[n\tau_1]} - [n\tau_1]B_n\big),\, n^{-d-.5}\big(S^*_{[n\tau_2]} - [n\tau_2]B_n\big)\Big) \quad \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \quad \big(0, Z_2(\tau_2)\big), \tag{2.3}$$

as $n \to \infty$, where $\{Z_2(\tau), \tau \in [1-\upsilon_1, 1-\upsilon_0]\}$ is a nondegenerate a.s. continuous Gaussian process.

Here and hereafter, a random element Z of $D[a,b]$ is called *nondegenerate* if it is not identically zero on the interval $[a,b]$ with positive probability, in other words, if $\mathrm{P}(Z(u) = 0, \forall u \in [a,b]) = 0$.

Typically, the null hypothesis is satisfied by the $I(d)$ $(d > -1/2)$ series (see Definition 5.1). In Section 5.1 we give general family of processes satisfying $\mathbf{H_0}[\mathbf{I}]$ including stationary and nonstationary processes. The alternative hypothesis corresponds to the processes changing from $I(d_1)$ to $I(d_2)$ processes (see Section 5.3 for examples).

Let us give a first example based on the well-known FARIMA model.

Example 2.1 The class of FARIMA(p,d,q) processes with $-.5 < d < .5$ satisfies assumption $\mathbf{H_0}[\mathbf{I}]$. Moreover, from two different memory parameters $d_2 > d_1$, we can construct a process satisfying $\mathbf{H_1}[\mathbf{I}]$ by the following equality

$$X_t = \begin{cases} \sum_{s=0}^{\infty} \psi_s(d_1)\varepsilon_{t-s} & \text{if } t \le n/2, \\ \sum_{s=0}^{\infty} \psi_s(d_2)\varepsilon_{t-s} & \text{if } t > n/2, \end{cases} \tag{2.4}$$

where $(\psi_s(d_i))_{s=0,1,\dots}$, $i = 1,2$ are the coefficients of linear representation of FARIMA(p, d_i, q) processes and (ε_t) a Gaussian white noise with zero mean and unit variance.

The test procedure proposed in Section 3 to detect non-constant long memory parameter can be valid in more general context. We can reformulate the hypotheses as follows.

$\mathbf{H_0}$: There exists normalizations $A_n \to \infty$ and B_n such that

$$A_n^{-1}\big(S_{[n\tau]} - [n\tau]B_n\big) \quad \longrightarrow_{D[0,1]} \quad Z(\tau), \tag{2.5}$$

where $\{Z(\tau), \tau \in [0,1]\}$ is a nondegenerate a.s. continuous random process.

$\mathbf{H_1}$: There exist $0 \le \upsilon_0 < \upsilon_1 \le 1$ and normalizations $A_n \to \infty$ and B_n such that

$$\Big(A_n^{-1}\big(S_{[n\tau_1]} - [n\tau_1]B_n\big),\, A_n^{-1}\big(S^*_{[n\tau_2]} - [n\tau_2]B_n\big)\Big) \quad \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \quad \big(0, Z_2(\tau_2)\big), \tag{2.6}$$

where $\{Z_2(\tau), \tau \in [1-\upsilon_1, 1-\upsilon_0]\}$ is a nondegenerate a.s. continuous random process.

Typically, normalization $B_n = \mathrm{E}X_0$ accounts for centering of observations and does not depend on n. Assumptions $\mathbf{H_0}$ and $\mathbf{H_1}$ represent very general forms of the null ('no change in persistence of X') and the alternative ('an increase in persistence of X') hypotheses. Indeed, an increase in persistence of X at time $k_* = [n\upsilon_1]$ typically means that forward partial sums $S_j, j \le k_*$ grow at a slower rate A_{n1} compared with the rate of growth A_{n2} of backward sums $S^*_j, j \le n - k_*$. Therefore, the former sums tend to a degenerated process $Z_1(\tau) \equiv 0$, $\tau \in [0, \upsilon_1]$ under the normalization $A_n = A_{n2}$. Clearly, $\mathbf{H_0}$ and $\mathbf{H_1}$ are not limited to stationary processes and allow infinite variance processes as well. While these assumptions are sufficient for derivation of the asymptotic distribution and consistency of our tests, they need to be specified in order to be practically implemented. The hypothesis $\mathbf{H_0}[\mathbf{I}]$ presented before is one example of such specification and involves the type I fBm. Another example involving the type II fBm is presented in Section 5.2.

3 The testing procedure

3.1 The test statistics

Analogously to (1.1)–(1.2), introduce the corresponding partial sums' variance estimates

$$\begin{aligned} V_k(X) &:= \frac{1}{k^2}\sum_{j=1}^{k}\Big(S_j - \frac{j}{k}S_k\Big)^2 - \Big(\frac{1}{k^{3/2}}\sum_{j=1}^{k}\Big(S_j - \frac{j}{k}S_k\Big)\Big)^2, \qquad (3.1)\\ V^*_{n-k}(X) &:= \frac{1}{(n-k)^2}\sum_{j=k+1}^{n}\Big(S^*_{n-j+1} - \frac{n-j+1}{n-k}S^*_{n-k}\Big)^2 \\ &\quad - \Big(\frac{1}{(n-k)^{3/2}}\sum_{j=k+1}^{n}\Big(S^*_{n-j+1} - \frac{n-j+1}{n-k}S^*_{n-k}\Big)\Big)^2 \end{aligned}$$

and the corresponding "backward/forward variance ratio":

$$\mathcal{L}_n(\tau) := \frac{V^*_{n-[n\tau]}(X)}{V_{[n\tau]}(X)}, \qquad \tau \in [0,1]. \qquad (3.2)$$

For a given *testing interval* $\mathcal{T} = [\underline{\tau}, \overline{\tau}] \subset (0,1)$, define the analogs of the 'supremum' and 'integral' statistics of Kim (2000):

$$W_n(X) := \sup_{\tau\in\mathcal{T}} \mathcal{L}_n(\tau), \qquad I_n(X) := \int_{\tau\in\mathcal{T}} \mathcal{L}_n(\tau)\mathrm{d}\tau. \qquad (3.3)$$

We also define the analog of the ratio statistic introduced in Sibbertsen and Kruse (2009):

$$R_n(X) := \frac{\inf_{\tau\in\mathcal{T}} V^*_{n-[n\tau]}(X)}{\inf_{\tau\in\mathcal{T}} V_{[n\tau]}(X)}. \qquad (3.4)$$

This statistic has also the same form as statistic R of Leybourne *et al.* (2007), formed as a ratio of the minimized CUSUMs of squared residuals obtained from the backward and forward subsamples of X, in the $I(0)/I(1)$ framework.

The limit distribution of these statistics is given in Proposition 3.1. To this end, define

$$Z^*(u) := Z(1) - Z(1-u), \ u \in [0,1] \qquad (3.5)$$

and a continuous time analog of the partial sums' variance $V_{[n\tau]}(X)$ in (3.1):

$$Q_\tau(Z) := \frac{1}{\tau^2}\Big[\int_0^\tau \big(Z(u) - \frac{u}{\tau}Z(\tau)\big)^2\mathrm{d}u - \frac{1}{\tau}\Big(\int_0^\tau \big(Z(u) - \frac{u}{\tau}Z(\tau)\big)\mathrm{d}u\Big)^2\Big]. \qquad (3.6)$$

Note $Q_{1-\tau}(Z^*)$ is the corresponding analog of $V^*_{n-[n\tau]}(X)$ in the numerators of the statistics in (3.2) and (3.4).

Proposition 3.1 *Assume* $\mathbf{H_0}$. *Then*

$$\Big(A_n^{-1}\big(S_{[n\tau_1]}-[n\tau_1]B_n\big),\, A_n^{-1}\big(S^*_{[n\tau_2]}-[n\tau_2]B_n\big)\Big) \quad \longrightarrow_{D[0,1]\times D[0,1]} \quad \big(Z(\tau_1), Z^*(\tau_2)\big). \tag{3.7}$$

Moreover, assume that

$$Q_\tau(Z) > 0 \quad \textit{a.s. for any} \quad \tau \in \mathcal{T}. \tag{3.8}$$

Then

$$\begin{aligned} W_n(X) \quad &\longrightarrow_{\rm law} \quad W(Z) \;:=\; \sup_{\tau\in\mathcal{T}} \frac{Q_{1-\tau}(Z^*)}{Q_\tau(Z)}, \\ I_n(X) \quad &\longrightarrow_{\rm law} \quad I(Z) \;:=\; \int_{\tau\in\mathcal{T}} \frac{Q_{1-\tau}(Z^*)}{Q_\tau(Z)}\,\mathrm{d}\tau, \\ R_n(X) \quad &\longrightarrow_{\rm law} \quad R(Z) \;:=\; \frac{\inf_{\tau\in\mathcal{T}} Q_{1-\tau}(Z^*)}{\inf_{\tau\in\mathcal{T}} Q_\tau(Z)}. \end{aligned} \tag{3.9}$$

The convergence in (3.7) is an immediate consequence of $\mathbf{H_0}$, while the fact that (3.7) and (3.8) imply (3.9) is a consequence of Proposition 4.1 stated in Section 4.

Remark 3.1 As noted previously, the alternative hypothesis $\mathbf{H_1}$ focuses on an *increase* of d, and the statistics (3.3), (3.4) are defined accordingly. It is straightforward to modify our testing procedures to test for a decrease of persistence. In such case, the corresponding test statistics are defined by exchanging forward and backward partial sums, or $V_{[n\tau]}(X)$ and $V^*_{n-[n\tau]}(X)$:

$$W_n^*(X) \;:=\; \sup_{\tau\in\mathcal{T}} \mathcal{L}_n^{-1}(\tau), \quad I_n^*(X) \;:=\; \int_{\tau\in\mathcal{T}} \mathcal{L}_n^{-1}(\tau)\mathrm{d}\tau, \quad R_n^*(X) \;:=\; \frac{\inf_{\tau\in\mathcal{T}} V_{[n\tau]}(X)}{\inf_{\tau\in\mathcal{T}} V^*_{n-[n\tau]}(X)}. \tag{3.10}$$

3.2 Practical implementation for testing $\mathbf{H_0}[\mathbf{I}]$ against $\mathbf{H_1}[\mathbf{I}]$

Under the 'type I fBm null hypothesis' $\mathbf{H_0}[\mathbf{I}]$, the limit distribution of the above statistics follows from Proposition 3.1 with $A_n = n^{d+.5}$, $Z = \kappa B_{d+.5}$. In this case, condition (3.8) is verified and we obtain the following result.

Corollary 3.2 *Assume* $\mathbf{H_0}[\mathbf{I}]$. *Then*

$$W_n(X) \quad \longrightarrow_{\rm law} \quad W(B_{d+.5}), \quad I_n(X) \quad \longrightarrow_{\rm law} \quad I(B_{d+.5}), \quad R_n(X) \quad \longrightarrow_{\rm law} \quad R(B_{d+.5}). \tag{3.11}$$

The process $B_{d+.5}$ in (3.11) depends on *unknown* memory parameter d, and so do the upper $\alpha-$quantiles of the r.v.s on the right-hand side of (3.11)

$$q_T^{[\mathrm{I}]}(\alpha, d) := \inf\{x : \mathrm{P}(T(B_{d+.5}) \le x) \ge 1-\alpha)\}, \tag{3.12}$$

where $T = W, I, R$. Hence, applying the corresponding test, the unknown parameter d in (3.12) is replaced by a consistent estimator $\hat{d}$.

<u>Testing procedure.</u> Reject $\mathbf{H_0}[\mathbf{I}]$, if

$$W_n(X) \;>\; q_W^{[\mathrm{I}]}(\alpha, \hat{d}), \qquad I_n(X) \;>\; q_I^{[\mathrm{I}]}(\alpha, \hat{d}), \qquad R_n(X) \;>\; q_R^{[\mathrm{I}]}(\alpha, \hat{d}), \tag{3.13}$$

respectively, where $\hat{d}$ is a weakly consistent estimator of d:

$$\hat{d} \;\longrightarrow_p\; d, \qquad n \to \infty. \tag{3.14}$$

The fact that the replacement of d by $\hat{d}$ in (3.13) preserves asymptotic significance level α is guaranteed by the continuity of the quantile functions provided by Proposition 3.3 below.

Proposition 3.3 *Let $d \in (-.5, 1.5)$, $d \neq .5$, $\alpha \in (0,1)$ and let $\hat{d}$ satisfy (3.14). Then*

$$q_T^{[\mathrm{I}]}(\alpha, \hat{d}) \longrightarrow_p q_T^{[\mathrm{I}]}(\alpha, d), \qquad \textit{for} \quad T = W,\, I,\, R.$$

The proof follows the same lines as in Giraitis et al. (2006, Lemma 2.1) for the test of stationarity based on the V/S statistic, we omit it.

Several estimators of d can be used in (3.13). See the review paper Bardet *et al.* (2003) for discussion of some popular estimators. In our simulations we use the Non-Stationarity Extended Local Whittle Estimator (NELWE) of Abadir *et al.* (2007), which applies to both the stationary ($|d| < .5$) and nonstationary ($d > .5$) cases.

Remark 3.2 The above testing procedure can be straightforwardly extended to the case $d > 1.5$, $d \neq 2.5, 3.5, \ldots$, provided some slight modifications. In particular, the integral representation of the type I fractional Brownian motion (2.1) involves in this case multiple integrals.

4 Consistency and asymptotic power

It is natural to expect that under alternative hypotheses $\mathbf{H_1}$ or $\mathbf{H_1}[\mathbf{I}]$, all three statistics $W_n(X)$, $I_n(X)$, $R_n(X)$ tend to infinity in probability, provided the testing interval $\mathcal{T}$ and the degeneracy interval $[0, \upsilon_1]$ of forward partial sums are embedded: $\mathcal{T} \subset [0, \upsilon_1]$. This is true indeed, see Proposition 4.1 (iii) below, meaning that our tests are consistent. Moreover, it is of interest to determine the rate at which these statistics grow under alternative, or the asymptotic power.

The following Proposition 4.1 provides the theoretical background to study the consistency of the tests. It also provides the limit distributions of the test statistics under $\mathbf{H_0}$ since Proposition 3.1 is an easy corollary of Proposition 4.1 (ii).

Proposition 4.1 *(i) Let there exist $0 \le \upsilon_0 < \upsilon_1 \le 1$ and normalizations $A_{ni} \to \infty$ and B_{ni}, $i = 1, 2$ such that*

$$\Big(A_{n1}^{-1}\big(S_{[n\tau_1]} - [n\tau_1]B_{n1}\big),\, A_{n2}^{-1}\big(S^*_{[n\tau_2]} - [n\tau_2]B_{n2}\big)\Big) \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \big(Z_1(\tau_1), Z_2(\tau_2)\big), \quad (4.1)$$

where $\big(Z_1(\tau_1), Z_2(\tau_2)\big)$ is a two-dimensional random process having a.s. continuous trajectories on $[\upsilon_0, \upsilon_1] \times [1-\upsilon_1, 1-\upsilon_0]$. Then

$$\big((n/A_{n1}^2)V_{[n\tau_1]}(X),\, (n/A_{n2}^2)V^*_{n-[n\tau_2]}(X)\big) \longrightarrow_{D(0,\upsilon_1]\times D[\upsilon_0,1)} \big(Q_{\tau_1}(Z_1), Q_{1-\tau_2}(Z_2)\big). \qquad (4.2)$$

Moreover, the limit process $\big(Q_{\tau_1}(Z_1), Q_{1-\tau_2}(Z_2)\big)$ in (4.2) is a.s. continuous on $(\upsilon_0, \upsilon_1] \times [\upsilon_0, \upsilon_1)$.

(ii) Assume, in addition to (i), that $\mathcal{T} \subset \mathcal{U} := [\upsilon_0, \upsilon_1]$ and

$$Q_\tau(Z_1) > 0 \quad \textit{a.s. for any} \quad \tau \in \mathcal{T}. \qquad (4.3)$$

Then, as $n \to \infty$,

$$\begin{aligned}
(A_{n1}/A_{n2})^2 W_n(X) &\longrightarrow_{\text{law}} \sup_{\tau\in\mathcal{T}} \frac{Q_{1-\tau}(Z_2)}{Q_\tau(Z_1)},\\
(A_{n1}/A_{n2})^2 I_n(X) &\longrightarrow_{\text{law}} \int_{\tau\in\mathcal{T}} \frac{Q_{1-\tau}(Z_2)}{Q_\tau(Z_1)}\,\mathrm{d}\tau,\\
(A_{n1}/A_{n2})^2 R_n(X) &\longrightarrow_{\text{law}} \frac{\inf_{\tau\in\mathcal{T}} Q_{1-\tau}(Z_2)}{\inf_{\tau\in\mathcal{T}} Q_\tau(Z_1)}.
\end{aligned} \qquad (4.4)$$

(iii) Assume, in addition to (i), that $\mathcal{T} \subset \mathcal{U}$, $Z_1(\tau) \equiv 0$, $\tau \in \mathcal{T}$ and the process $\{Q_{1-\tau}(Z_2), \tau \in \mathcal{T}\}$ is nondegenerate.

Then

$$(A_{n1}/A_{n2})^2 \left\{ \begin{array}{c} W_n(X) \\ I_n(X) \\ R_n(X) \end{array} \right\} \quad \longrightarrow_{\mathrm{p}} \quad \infty. \tag{4.5}$$

Remark 4.1 Typically, under $\mathbf{H_1}$ the relation (4.1) is satisfied with A_{n2} growing much faster than A_{n1} (e.g., $A_{ni} = n^{d_i+.5}$, $i = 1, 2$, $d_1 < d_2$) and (4.4) translates to that $W_n(X)$, $I_n(X)$ and $R_n(X)$ are $O_{\mathrm{p}}\big((A_{n2}/A_{n1})^2\big)$. Two classes of fractionally integrated series with changing memory parameter and satisfying (4.1) are discussed in Section 5.

Remark 4.2 Note that $Q_\tau(Z) \geq 0$ by the Cauchy-Schwarz inequality and that $Q_\tau(Z) = 0$ implies $Z(u) - \frac{u}{\tau}Z(\tau) = a$ for all $u \in [0, \tau]$ and some (random) $a = a(\tau)$. In other words, $\mathrm{P}(Q_\tau(Z) = 0) > 0$ implies that for some (possibly, random) constants a and b,

$$\mathrm{P}\big(Z(u) = a + \frac{u}{\tau}\, b, \, \forall\, u \in [0, \tau]\big) \; > \; 0. \tag{4.6}$$

Therefore, condition (4.3) implicitly excludes situations as in (4.6), with $(a, b) \neq (0, 0)$, which may arise under the null hypothesis $\mathbf{H_0}$, if $B_n = 0$ in (2.5) whereas the X_j's have nonzero mean.

Remark 4.3 All the results of sections 3 and 5 hold for the 'KPSS versions' of the statistics W_n, I_n and R_n defined by replacing $V_{[n\tau]}(X), V^*_{n-[n\tau]}(X)$ in (3.3), (3.4) by $U_{[n\tau]}(X), U^*_{n-[n\tau]}(X)$ as given in (1.2), with the only difference that the functional $Q_\tau(Z)$ in the corresponding statements must be replaced by $\widetilde{Q}_\tau(Z) := \tau^{-2} \int_0^\tau \big(Z(u) - \frac{u}{\tau}Z(\tau)\big)^2 \mathrm{d}u$, cf. (3.6).

5 Application to fractionally integrated processes

This section discusses the convergence of forward and backward partial sums for some fractionally integrated models with constant or changing memory parameter and the behavior of statistics W_n, I_n, R_n for such models.

5.1 Type I fractional Brownian motion and the null hypothesis $\mathbf{H_0}[\mathbf{I}]$

It is well-known that type I fBm arises in the scaling limit of $d-$integrated, or $I(d)$, series with i.i.d. or martingale difference innovations. See Davydov (1970), Peligrad and Utev (1997), Marinucci and Robinson (1999), Bružaitė *et al.* (2005) and the references therein.

A formal definition of $I(d)$ process (denoted $\{X_t\} \sim I(d)$) for $d > -.5$, $d \neq .5, 1.5, \ldots$ is given below.

Definition 5.1 *(i) Write $\{X_t\} \sim I(0)$ if*

$$X_t = \sum_{j=0}^{\infty} a_j \zeta_{t-j}, \qquad t \in \mathbb{Z}$$

is a moving average with i.i.d., zero mean unit variance innovations $\{\zeta_j\}$ *and sumable coefficients* $\sum_{j=0}^{\infty}|a_j|<\infty$, $\sum_{j=0}^{\infty}a_j\neq 0$.

(ii) Let $d\in(-.5,.5), d\neq 0$. *Write* $\{X_t\}\sim I(d)$ *if* $\{X_t\}$ *is a fractionally integrated process*

$$X_t = (1-L)^{-d}Y_j = \sum_{j=0}^{\infty}\psi_j(d)Y_{t-j}, \qquad t\in\mathbb{Z},$$

where $\{Y_t\}\sim I(0)$ *and* $\{\psi_j(d), j\geq 0\}$ *are the coefficients of the binomial expansion* $(1-z)^{-d}=\sum_{j=0}^{\infty}\psi_j(d)z^j$, $|z|<1$.

(iii) Let $d>.5$ *and* $d\neq 1.5,2.5,\ldots$. *Write* $\{X_t\}\sim I(d)$ *if* $X_t=\sum_{j=1}^{t}Y_j$, $t=1,2,\ldots$, *where* $\{Y_t\}\sim I(d-1)$, *and* $X_t=0$, $t=0,-1,-2,\ldots$.

From the above definition it follows that an $I(d)$ process can be written as a moving average (a sum of moving averages) in i.i.d. variables $\{\zeta_s\}$, for instance:

$$X_t \;=\; \begin{cases}\sum_{s\leq t}(a\star\psi(d))_{t-s}\zeta_s, & -.5<d<.5,\\ \sum_{s\leq t}\sum_{1\vee s\leq j\leq t}(a\star\psi(d-1))_{j-s}\zeta_s, & .5<d<1.5,\end{cases} \qquad t=1,2,\ldots, \tag{5.1}$$

where $(a\star\psi(d))_j := \sum_{i=0}^{j}a_i\psi_{j-i}(d)$, $j\geq 0$ is the convolution of the sequences $\{a_j\}$ and $\{\psi_j(d)\}$. Definition 5.1 also implies that $\{X_t\}$ is solution of the fractional equation:

$$\begin{cases}(1-L)^dX_t \;=\; \sum_{j=0}^{\infty}a_j\zeta_{t-j},\ t\in\mathbb{Z}, \quad \text{if } -.5<d<.5,\\ (1-L)^dX_t \;=\; \sum_{j=0}^{\infty}a_j\zeta_{t-j},\ t=1,2,\ldots \text{ and } X_t=0,\ t=-1,-2,\ldots, \quad \text{if} \quad d>.5.\end{cases}$$

Proposition 5.2 *(i) Let* $\{X_t\}\sim I(d)$ *for some* $d\in(-.5,1.5)$, $d\neq .5$. *If* $d\in(-.5,0]$, *assume in addition* $\mathrm{E}|\zeta_0|^p<\infty$, *for some* $p>1/(.5+d)$. *Then (2.2) holds.*

(ii) Let $\{0\leq\sigma_s, s\in\mathbb{Z}\}$ *be an almost periodic sequence such that* $\bar\sigma:=\lim_{n\to\infty}\sum_{s=1}^{n}\sigma_s>0$. *Let* $\{X_t\}$ *be defined as in (5.1), where* ζ_s, $s\in\mathbb{Z}$ *are replaced by* $\sigma_s\zeta_s$, $s\in\mathbb{Z}$ *and where* d *and* $\{\zeta_s\}$ *satisfy the conditions in (i). Then (2.2) holds.*

Proposition 5.2 (i) is proved in Chan and Terrin (1995), while (ii) follows from Philippe *et al.* (2006a). Note that the linear process $\{X_t\}$ in Proposition 5.2 (ii) with heteroscedastic noise $\{\sigma_s\zeta_s\}$ is nonstationary even if $|d|<1/2$. Although type I fBm is well-defined for $d=.5$, theses cases are excluded from $\mathbf{H_0}[\mathbf{I}]$ and Proposition 5.2 since this case is rather peculiar and requires a different normalization (see Liu (2006)).

5.2 Type II fractional Brownian motion and the null hypothesis $\mathbf{H_0}[\mathbf{II}]$

Definition 5.3 *A type II fractional Brownian motion with parameter* $d>-.5$ *is defined by*

$$B^{\mathrm{II}}_{d+.5}(\tau) \;:=\; \frac{1}{\Gamma(d+1)}\int_0^{\tau}(\tau-s)^d\mathrm{d}B(s), \qquad \tau\geq 0, \tag{5.2}$$

where $\{B(s), s\geq 0\}$ *is a standard Brownian motion with zero mean and variance* $\mathrm{E}B^2(s)=s$.

A type II fBm shares many properties of type I fBm except that it has nonstationary increments, however, for $|d| < .5$ increments at time τ of type II fBm tend those of type I fBm when $\tau \to \infty$. Convergence to type II fBm of partial sums of fractionally integrated processes was discussed in Marinucci and Robinson (1999, 2000), Leipus and Surgailis (2010).

Type II fBm may serve as the limit process in the following specification of the null hypothesis $\mathbf{H_0}$.

$\mathbf{H_0}[\mathbf{II}]$: *There exist $d > -.5$, $\kappa > 0$ and a normalization B_n such that*

$$n^{-d-.5}\big(S_{[n\tau]} - [n\tau]B_n\big) \quad \longrightarrow_{D[0,1]} \quad \kappa B^{\mathrm{II}}_{d+.5}(\tau). \tag{5.3}$$

The alternative hypothesis to $\mathbf{H_0}[\mathbf{II}]$ can be again $\mathbf{H_1}$ of Section 2.

Proposition 5.2 can be easily extended to type II fBm convergence in (5.3) as follows. Introduce a "truncated" $I(d)$ process by

$$X_t \ := \ \sum_{s=1}^{t} (a \star \psi(d))_{t-s}\zeta_s, \quad d > -.5, \qquad t = 1, 2, \ldots, \tag{5.4}$$

where the $(a \star \psi(d))_j$ and $\{\zeta_s\}$ are the same as in (5.1).

Put $X_t := 0$, $t = 0, -1, -2, \cdots$. The above definition and (5.4) imply that $\{X_t\}$ is a solution of the fractional equation

$$(1 - L)^d X_t \ = \ Y_t, \qquad t = 1, 2, \ldots,$$

where

$$Y_t := \begin{cases} \sum_{j=0}^{t} a_j \zeta_{t-j}, & t = 1, 2, \ldots, \\ 0, & t = 0, -1, -2, \ldots. \end{cases}$$

Proposition 5.4 *(i) Let $\{X_t\}$ be defined as in (5.4). If $d \in (-.5, 0]$, assume in addition* $\mathrm{E}|\zeta_0|^p < \infty$, *for some $p > 1/(d + .5)$. Then (5.3) holds.*

(ii) Let $\{0 \le \sigma_s, s \ge 1\}$ be an almost periodic sequence such that $\bar\sigma := \lim_{n\to\infty} \sum_{s=1}^{n} \sigma_s > 0$. Let $\{X_t\}$ be defined as in (5.4), where ζ_s, $s \ge 1$ are replaced by $\sigma_s\zeta_s$, $s \ge 1$ and where d and $\{\zeta_s\}$ satisfy the conditions in (i). Then (5.3) holds.

Similarly to Corollary 3.2, Proposition 3.1 implies the following corollary.

Corollary 5.5 *Let $\{X_t\}$ satisfy the conditions of Proposition 5.4. Then*

$$W_n(X) \ \longrightarrow_{\mathrm{law}} \ W(B^{\mathrm{II}}_{d+.5}), \quad I_n(X) \ \longrightarrow_{\mathrm{law}} \ I(B^{\mathrm{II}}_{d+.5}), \qquad R_n(X) \ \longrightarrow_{\mathrm{law}} \ R(B^{\mathrm{II}}_{d+.5}), \tag{5.5}$$

where $\{B^{\mathrm{II}}_{d+.5}(\tau), \tau \in [0, 1]\}$ is a type II fBm as defined in (5.2).

Remark 5.1 Numerical experiments confirm that the upper quantiles $q^{[\mathrm{II}]}_T(\alpha, d)$, $T = W, I, R$, of the limit r.v.s on the r.h.s. of (5.5) are very close to the corresponding upper quantiles $q^{[\mathrm{I}]}_T(\alpha, d)$ of the limiting statistics in (3.11) when d is smaller than 1 (see Figure 1 in the particular case $T = I$). In other words, from a practical point of view, there is not much difference between type I fBM and type II fBm null hypotheses $\mathbf{H_0}[\mathbf{I}]$ and $\mathbf{H_0}[\mathbf{II}]$ in testing for a change of d when $d < 1$.

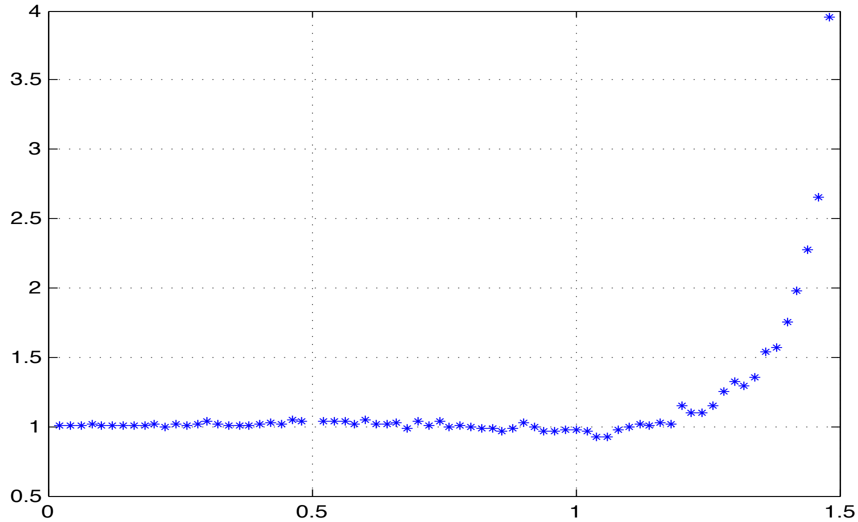

Figure 1: Representation of the ratio $q_I^{[\mathrm{I}]}(0.05, d)/q_I^{[\mathrm{II}]}(0.05, d)$ as function of d, with the choice $\underline{\tau} = 0.1$ and $\overline{\tau} = 0.9$.

5.3 Fractionally integrated models with changing memory parameter

Let us discuss two nonparametric classes of nonstationary time series with time-varying long memory parameter termed "rapidly changing memory" and "gradually changing memory".

Rapidly changing memory. This class is obtained by replacing parameter d by a function $d(t/n) \in [0, \infty)$ in the FARIMA$(0, d, 0)$ filter

$$\psi_j(d) \quad = \quad \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} \quad = \quad \frac{\Gamma(d+j)}{j!\Gamma(d)} \quad (j \ge 1), \quad \psi_0(d) := 1. \tag{5.6}$$

Let $d(\tau)$, $\tau \in [0, 1]$ be a function taking values in the interval $[0, \infty)$. (More precise conditions on the function $d(\tau)$ will be specified below.) Define

$$\begin{aligned} b_{1,j}(t) &:= \psi_j\big(d(\tfrac{t}{n})\big), \quad j = 0, 1, \ldots, \\ X_{1,t} &:= \sum_{s=1}^{t} b_{1,t-s}(t)\zeta_s, \qquad t = 1, \ldots, n, \end{aligned} \tag{5.7}$$

where the innovations $\zeta_s, s \ge 1$ satisfy the conditions of Definition 5.1. The particular case

$$d(\tau) = \begin{cases} 0, & \tau \in [0, \theta^*], \\ 1, & \tau \in (\theta^*, 1] \end{cases} \tag{5.8}$$

for some $0 < \theta^* < 1$, leads to the model

$$X_{1,t} \quad = \quad \begin{cases} \zeta_t, & t = 1, 2, \ldots, [\theta^* n], \\ \sum_{s=1}^{t} \zeta_s & t = [\theta^* n] + 1, \ldots, n, \end{cases} \tag{5.9}$$

which corresponds to transition $I(0) \to I(1)$ at time $[\theta^* n] + 1$. A more general step function

$$d(\tau) = \begin{cases} d_1, & \tau \in [0, \theta^*], \\ d_2, & \tau \in (\theta^*, 1] \end{cases} \tag{5.10}$$

corresponds to $X_{1,t}$ changing from $I(d_1)$ to $I(d_2)$ at time $[\theta^* n] + 1$.

Gradually changing memory. This class of nonstationary time-varying fractionally integrated processes was defined in Philippe et al. (2006a, 2006b, 2008). Here, we use a truncated modification of these processes with slowly varying memory parameter $d(t/n) \in [0, \infty)$, defined as

$$\begin{aligned} b_{2,j}(t) &:= \frac{d(\frac{t}{n})}{1} \cdot \frac{d(\frac{t-1}{n}) + 1}{2} \cdots \frac{d(\frac{t-j+1}{n}) - 1 + j}{j}, \ j = 1, 2, \dots, \quad b_{2,0}(t) := 1, \\ X_{2,t} &:= \sum_{s=1}^{t} b_{2,t-s}(t) \zeta_s, \quad t = 1, \dots, n. \end{aligned} \tag{5.11}$$

Contrary to (5.7), the process in (5.11) satisfies an autoregressive time-varying fractionally integrated equation with ζ_t on the right-hand side, see Philippe *et al.* (2008). In the case when $d(\tau) \equiv d$ is constant function, the coefficients $b_{2,j}(t)$ in (5.11) coincide with FARIMA$(0, d, 0)$ coefficients in (5.6) and in this case the processes $\{X_{1,t}\}$ and $\{X_{2,t}\}$ in (5.7) and (5.11) coincide.

To see the difference between these two classes, consider the case of step function in (5.8). Then

$$X_{2,t} = \begin{cases} \zeta_t, & t = 1, 2, \dots, [\theta^* n], \\ \sum_{s=[\theta^* n]+1}^{t} \zeta_s + \sum_{s=1}^{[\theta^* n]} \frac{t-[\theta^* n]}{t-s} \zeta_s, & t = [\theta^* n] + 1, \dots, n. \end{cases} \tag{5.12}$$

Note $\frac{t-[\theta^* n]}{t-s} = 0$ for $t = [\theta^* n]$ and monotonically increases with $t \geq [\theta^* n]$. Therefore, (5.12) embodies a gradual transition from $I(0)$ to $I(1)$, in contrast to an abrupt change of these regimes in (5.9). The distinction between the two models (5.9) and (5.12) can be clearly seen from the variance behavior: the variance of $X_{1,t}$ exhibits a jump from 1 to $[\theta^* n] + 1 = O(n)$ at time $t = [\theta^* n] + 1$, after which it linearly increases with t, while the variance of $X_{2,t}$ changes "smoothly" with t:

$$\mathrm{Var}(X_{2,t}) = \begin{cases} 1, & t = 1, 2, \dots, [\theta^* n], \\ (t - [\theta^* n]) + \sum_{s=1}^{[\theta^* n]} \frac{(t-[\theta^* n])^2}{(t-s)^2}, & t = [\theta^* n] + 1, \dots, n. \end{cases}$$

Similar distinctions between (5.7) and (5.11) prevail also in the case of general "memory function" $d(\cdot)$: when the memory parameter $d(t/n)$ changes with t, this change gradually affects the lagged ratios in the coefficients $b_{2,j}(t)$ in (5.11), and not all lagged ratios simultaneously as in the case of $b_{1,j}(t)$, see (5.6).

5.4 Asymptotics of change-point statistics for fractionally integrated models with changing memory parameter

In this subsection we study the joint convergence of forward and backward partial sums as in (2.6) for the two models in (5.7) and (5.11) with time-varying memory parameter $d(t/n)$. After the statement of Proposition 5.6 below, we discuss its implications for the asymptotic power of our tests.

Let us specify a class of "memory function" $d(\cdot)$. For $-.5 < d_1 < d_2 < \infty$ and $0 \leq \underline{\theta} \leq \overline{\theta} \leq 1$, introduce the class $\mathcal{D}_{\underline{\theta},\overline{\theta}}(d_1, d_2)$ of left-continuous nondecreasing functions $d(\cdot) \equiv \{d(\tau), \tau \in [0, 1]\}$ such that

$$d(\tau) = \begin{cases} d_1, & \tau \in [0, \underline{\theta}], \\ d_2, & \tau \in [\overline{\theta}, 1], \end{cases}, \qquad d_1 < d(\tau) < d_2, \quad \underline{\theta} < \tau < \overline{\theta}. \tag{5.13}$$

The interval $\Theta := [\underline{\theta}, \overline{\theta}]$ will be called the *memory change interval.* Note that for $\underline{\theta} = \overline{\theta} \equiv \theta^*$, the class $\mathcal{D}_{\theta^*,\theta^*}(d_1, d_2)$ consists of a single step function in (5.10). Recall from Section 3 that the interval $\mathcal{T} = [\underline{\tau}, \overline{\tau}]$ in memory change statistics in (3.3) and (3.4) is called the (memory) testing interval.

When discussing the behavior of memory tests under alternatives in (5.7), (5.11) with changing memory parameter, the intervals Θ and $\mathcal{T}$ need not coincide since Θ is not known *a priori.*

With a given $d(\cdot) \in \mathcal{D}_{\underline{\theta},\overline{\theta}}(d_1, d_2)$, we associate a function

$$H(u, v) := \int_u^v \frac{d(x) - d_2}{v - x}\, \mathrm{d}x, \qquad 0 \le u \le v \le 1. \tag{5.14}$$

Note $H(u, v) \le 0$ since $d(x) \le d_2$, $x \in [0, 1]$ and $H(u, v) = 0$ if $\overline{\theta} \le u \le v \le 1$. Define two Gaussian processes $\mathcal{Z}_1$ and $\mathcal{Z}_2$ by

$$\begin{aligned}
\mathcal{Z}_1(\tau) &:= \frac{1}{\Gamma(d_2)} \int_0^\tau \Big\{ \int_{\overline{\theta}}^\tau (v - u)_+^{d_2 - 1} \mathrm{d}v \Big\} \mathrm{d}B(u) \;=\; B^{\mathrm{II}}_{d_2+.5}(\tau) - B^{\mathrm{II}}_{d_2+.5}(\overline{\theta}), && (5.15)\\
\mathcal{Z}_2(\tau) &:= \frac{1}{\Gamma(d_2)} \int_0^\tau \Big\{ \int_{\overline{\theta}}^\tau (v - u)_+^{d_2 - 1} \mathrm{e}^{H(u,v)} \mathrm{d}v \Big\} \mathrm{d}B(u), \quad \tau > \overline{\theta},\\
\mathcal{Z}_1(\tau) &= \mathcal{Z}_2(\tau) \;:=\; 0, \quad \tau \in [0, \overline{\theta}].
\end{aligned}$$

The processes $\{\mathcal{Z}_i(\tau), \in [0, 1]\}$, $i = 1, 2$ are well-defined for any $d_2 > -0.5$ and have a.s. continuous trajectories.

In the case $\underline{\theta} = \overline{\theta} \equiv \theta^*$ and a step function $d(\cdot)$ in (5.13), $\mathcal{Z}_2(\tau)$ for $\tau > \overline{\theta}$ can be rewritten as

$$\mathcal{Z}_2(\tau) = \frac{1}{\Gamma(d_2)} \int_0^\tau \Big\{ \int_{\overline{\theta}}^\tau (v - u)^{d_1 - 1} (v - \overline{\theta})^{d_2 - d_1} \mathrm{d}v \Big\} \mathrm{d}B(u).$$

Related class of Gaussian processes was discussed in Philippe *et al.* (2008) and Surgailis (2008).

Proposition 5.6 *Let $d(\cdot) \in \mathcal{D}_{\underline{\theta},\overline{\theta}}(d_1, d_2)$ for some $0 \le d_1 < d_2 < \infty$, $0 \le \underline{\theta} \le \overline{\theta} \le 1$. Let $S_{i,k}$ and $S^*_{i,n-k}$, $i = 1, 2$ be the forward and backward partial sums processes corresponding to time-varying fractional filters $\{X_{i,t}\}$, $i = 1, 2$ in (5.7), (5.11), with memory parameter $d(t/n)$ and standardized i.i.d. innovations $\{\zeta_j,\, j \ge 1\}$ as in Definition 5.1. Moreover, in the case $d_1 = 0$ we assume that* $\mathrm{E}|\zeta_0|^{2+\delta} < \infty$ *for some $\delta > 0$. In addition,*
(i) Let $\underline{\theta} > 0$. For any $\theta \in (0, \underline{\theta}]$,

$$\big(n^{-d_1 - .5} S_{i,[n\tau_1]}, n^{-d_2 - .5} S^*_{i,[n\tau_2]}\big) \longrightarrow_{D[0,\theta] \times D[0,1-\underline{\tau}]} \big(Z_{i,1}(\tau_1), Z_{i,2}(\tau_2)\big), \qquad i = 1, 2, \tag{5.16}$$

where

$$Z_{i,1}(\tau) := B^{\mathrm{II}}_{d_1+.5}(\tau), \qquad Z_{i,2}(\tau) := \mathcal{Z}^*_i(\tau) = \mathcal{Z}_i(1) - \mathcal{Z}_i(1 - \tau), \qquad i = 1, 2, \tag{5.17}$$

and $\mathcal{Z}_i$, $i = 1, 2$ are defined in (5.15).
(ii) For any $\theta \in [\underline{\theta}, 1]$, for any $d > d(\theta)$, $d_1 < d < d_2$

$$\big(n^{-d - .5} S_{i,[n\tau_1]}, n^{-d_2 - .5} S^*_{i,[n\tau_2]}\big) \longrightarrow_{D[0,\theta] \times D[0,1-\underline{\tau}]} \big(0, Z_{i,2}(\tau_2)\big), \qquad i = 1, 2, \tag{5.18}$$

where $Z_{i,2}$, $i = 1, 2$ are the same as in (5.17).

The power of our tests depends on whether the testing and the memory change intervals have an empty intersection or not. When $\overline{\tau} < \underline{\theta}$, Proposition 5.6 (i) applies taking $\theta = \overline{\tau}$ and the asymptotic distribution of the memory test statistics for models (5.7) and (5.11) follows from Proposition 4.1 (4.4), with normalization $(A_{n2}/A_{n1})^2 = n^{2(d_1-d_2)} \to 0$, implying the consistency of the tests. But this situation is untypical for practical applications and hence not very interesting. Even less interesting seems the case when a change of memory ends before the start of the testing interval, i.e., when $\overline{\theta} \le \underline{\tau}$. Although the last case is not covered by Proposition 5.6, the limit distribution of the test statistics for models (5.7), (5.11) exists with trivial normalization $(A_{n2}/A_{n1})^2 = 1$ and therefore our tests are inconsistent, which is quite natural in this case.

Let us turn to some more interesting situations, corresponding to the case when the intervals $\mathcal{T}$ and Θ have a nonempty intersection of positive length. There are two possibilities:

Case 1: $\underline{\tau} < \underline{\theta} \le \overline{\tau}$ (a change of memory occurs after the beginning of the testing interval), and

Case 2: $\underline{\theta} \le \underline{\tau} < \overline{\theta}$ (a change of memory occurs before the beginning of the testing interval).

Let us consider the two cases 1 and 2 in more detail.

Case 1. Let $\widetilde{\mathcal{T}} := [\underline{\tau}, \underline{\theta}] \subset \mathcal{T}$. Introduce the following "dominated" (see (5.20)) statistics:

$$\begin{aligned} \widetilde{W}_n(X) &:= \sup_{\tau \in \widetilde{\mathcal{T}}} \frac{V^*_{n-[n\tau]}(X)}{V_{[n\tau]}(X)}, \qquad \widetilde{I}_n(X) := \int_{\widetilde{\mathcal{T}}} \frac{V^*_{n-[n\tau]}(X)}{V_{[n\tau]}(X)} \, \mathrm{d}\tau, \\ \widetilde{R}_n(X) &:= \frac{\inf_{\tau \in \mathcal{T}} V^*_{n-[n\tau]}(X)}{\inf_{\tau \in \widetilde{\mathcal{T}}} V_{[n\tau]}(X)}. \end{aligned} \tag{5.19}$$

Clearly,

$$W_n(X) \ge \widetilde{W}_n(X), \qquad I_n(X) \ge \widetilde{I}_n(X), \qquad R_n(X) \ge \widetilde{R}_n(X), \qquad \text{a.}s. \tag{5.20}$$

The limit distribution of (5.19) for models (5.7) and (5.11) can be derived from propositions 4.1 and 5.6 (i) choosing $\theta = \underline{\theta}$. In particular, it follows that $n^{2(d_1-d_2)}\widetilde{W}_n(X_i)$, $n^{2(d_1-d_2)}\widetilde{I}_n(X_i)$, and $n^{2(d_1-d_2)}\widetilde{R}_n(X_i)$, $i = 1, 2$ tend, in distribution, to the corresponding limits in (4.4), with $\mathcal{T}$ replaced by Θ and $Z_1 = Z_{i,1}$, $Z_2 = Z_{i,2}$, $i = 1, 2$ as defined in (5.17). Moreover, it can be shown that $n^{-2d_1} V_{[n\tau]}(X_i) \longrightarrow_p \infty$ for any $\tau \in \mathcal{T} \setminus \Theta$. Therefore one can expect that in case 1, the limit distributions of the original statistics in (3.3) and the "dominated" statistics in (5.19) coincide.

Case 2. In this case, define $\widetilde{\mathcal{T}} := [\underline{\tau}, \tilde{\theta}] \subset \mathcal{T}$, where $\tilde{\theta} \in (\underline{\tau}, \overline{\theta})$ is an inner point of the interval $[\underline{\tau}, \overline{\theta}]$. Let $\widetilde{W}_n(X), \widetilde{I}_n(X), \widetilde{R}_n(X)$ be defined as in (5.19). Obviously, relations (5.20) hold as in the previous case. Since the memory parameter increases on the interval $\widetilde{\mathcal{T}}$, the limit distribution of the process $V_{[n\tau]}(X_i), \tau \in \widetilde{\mathcal{T}}$ in the denominator of the statistics is not identified from Proposition 4.1 (ii). Nevertheless in this case we can use Propositions 4.1 (iii) and 5.6 (ii) to obtain a robust rate of growth of the memory statistics in (5.19) and (3.3). Indeed from Proposition 5.6 (ii) with $\theta = \tilde{\theta}$, we have that $n^{-2d} V_{[n\tau]}(X_i) \longrightarrow_{D(0,\tilde{\theta}]} 0$ for any $d_2 > d > d(\tilde{\theta})$ and hence $n^{2(d-d_2)} W_n(X_i), n^{2(d-d_2)} I_n(X_i)$ and $n^{2(d-d_2)} R_n(X_i)$, $i = 1, 2$ tend to infinity, in probability.

6 Simulation study

In this section we compare from numerical experiments the performance of the different test statistics in (3.13) for testing $\mathbf{H_0[I]}$ against $\mathbf{H_1}$ with nominal level $\alpha = 5\%$. A comparison with the Kim's approach, based on the ratio (1.1), is also provided.

The different steps to implement the testing procedures defined in (3.13) are the following: having observed $X_1, \ldots, X_n$

- we choose $\overline{\tau} = 1 - \underline{\tau}$ for $\underline{\tau} \in (0, 1)$ which defines the testing region $\mathcal{T} := [\underline{\tau}, 1 - \underline{\tau}]$. The sensitivity to the parameter $\underline{\tau}$ will be explored.

- we estimate the parameter d which appears in the limit distributions in $\mathbf{H_0}[\mathbf{I}]$. We choose $\hat{d}_n$ as the fully extended local Whittle (FELW) estimate (see Abadir *et al.* (2007) for the definition and its convergence properties). The FELW estimate can be replaced by any estimator which converges to d in probability (see Proposition 3.3).

- to deduce the critical regions (3.13), we approximate the quantile function $q_T^{[\mathrm{I}]}(.05,\ d)$, for $T = W,\ R,\ I$, as a function of d and for different values of $\underline{\tau}$, by extensive Monte Carlo experiments. See Figure 2 in the particular case $T = I$.

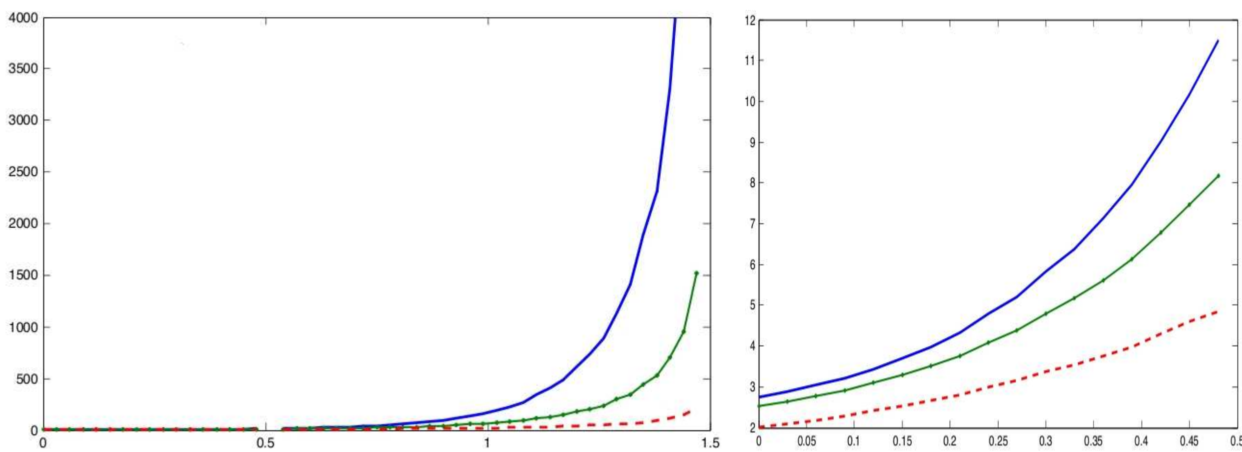

Figure 2: Upper quantile of order 5% of $I(B_{d+.5})$ as a function of d, for three different values of $\underline{\tau}$: [solid line] $\underline{\tau} = 5\%$; [solid line with points] $\underline{\tau} = 10\%$; [dashed line] $\underline{\tau} = 20\%$. The right plot is a zoom in on the region $d \in [0, .5)$.

6.1 Comparison of the test statistics

In this part we compare the test procedures based on our statistics I_n, R_n, W_n and the Kim's statistic associated to the integral functional

$$I_n^{Kim} = \int_{\mathcal{T}} \mathcal{K}_n(\tau) \mathrm{d}\tau, \tag{6.1}$$

where $\mathcal{K}_n(\tau)$ is defined in (1.1).

To evaluate the properties of the tests under the null hypothesis $\mathbf{H_0}[\mathbf{I}]$, we consider the class of FARIMA(0,d,0) processes with $d \in [0, .5[$. Then we estimate the power in presence of a change point in the long memory parameter at $n/2$, i.e. we assume

$$X_t = \begin{cases} \sum_{s=0}^{\infty} \psi_s(d_1) \varepsilon_{t-s} & \text{if } t \le n/2, \\ \sum_{s=0}^{\infty} \psi_s(d_2) \varepsilon_{t-s} & \text{if } t > n/2, \end{cases} \tag{6.2}$$

where $(\psi_s(d))_{s=0,1,\dots}$ is defined in (5.6) and (ε_t) is a Gaussian white noise with zero mean and unit variance.

Table 1 displays the estimated level and power based on 10^4 replications of the testing procedures, when $n = 5000$. Table 1 shows that for all $\underline{\tau}$, the estimated level is close to the nominal level for the four statistics. We also observe that the performances of the tests based on I_n and I_n^{Kim} do not depend too much on $\underline{\tau}$; this property is not shared by the tests associated to W_n and R_n.

When $\underline{\tau}$ is small, the most efficient test is clearly the test based on I_n. When $\underline{\tau}$ increases, the performances of the test based on R_n become comparable to the test based on I_n, while W_n and I_n^{Kim} still induce less efficient tests. However it is preferable to choose $\underline{\tau}$ as small as possible, since we can not detect a change point that occurs in $[0, \underline{\tau}] \cup [1 - \underline{\tau}, 1]$.

In conclusion, the choice $\underline{\tau} = 0.05$ (or $\underline{\tau} = 0.1$) and the statistic I_n seems preferable to I_n^{Kim}, W_n and R_n.

6.2 More numerical results for the test based on I_n

The numerical results in Table 1 have led to select the test based on the I_n statistic. Now we explore its performances for various classes of processes under $\mathbf{H_0}[\mathbf{I}]$ and $\mathbf{H_1}$.

Table 2 extends the study in Table 1 to $0 < d_1 \leq d_2 < 1.5$. For $d > .5$, FARIMA$(0, d, 0)$ processes are simulated using the following representation

$$X_t = X_0 + \sum_{i=1}^{t} Y_i, \tag{6.3}$$

where (Y_t) is a stationary FARIMA$(0, d-1, 0)$.

Figure 3 represents some trajectories simulated from model (6.2) with $d_2 - d_1 = 0.3$ and different values of d_1. From the observation of these realizations, it seems more difficult to detect a change in the memory parameter when $0 \leq d_1 < d_2 < .5$ or $.5 < d_1 < d_2$, than when $d_1 < .5 < d_2$. Table 2 confirms this. Moreover, fixing the difference $d_2 - d_1$, it turns out that the test is more powerful in the case $0 \leq d_1 < d_2 < .5$ than in the case $.5 < d_1 < d_2$.

In Tables 3–4, we illustrate the fact that the performances of the test based on I_n are preserved when a positive, resp. negative, autoregressive part is added to the model (6.2).

Tables 2, 3 and 4 confirm that I_n is not very sensitive to the testing interval parameter $\underline{\tau}$.

Finally, we assess the power of the test in presence of fractionally integrated models with changing memory parameters presented in Section 5.3. Figure 4 first provides trajectories of the *rapidly changing memory* model defined in (5.7) and of the *gradually changing memory* model defined in (5.11), for the same function $d(t/n) = .2 + .6\ t/n$. From this representation, it is clearly easier to realize that $d(t/n) > 1/2$ when $t \geq n/2$ for the *rapidly changing memory* model than for the *gradually changing memory* model.

Table 5 displays the estimated power of the test for the *rapidly changing memory* model (5.7) when $d(\tau) = d_1 + (d_2 - d_1)\tau$, $\underline{\theta} = 0$ and $\overline{\theta} = 1$. The null hypothesis is naturally less often rejected for this model than for the model defined in (6.2). However the power is still satisfactory. Similar simulations (omitted here) show that the test has more difficulty to detect *gradually changing memory* than *rapidly changing memory* for small samples, but the difference becomes negligible for $n = 5000$.

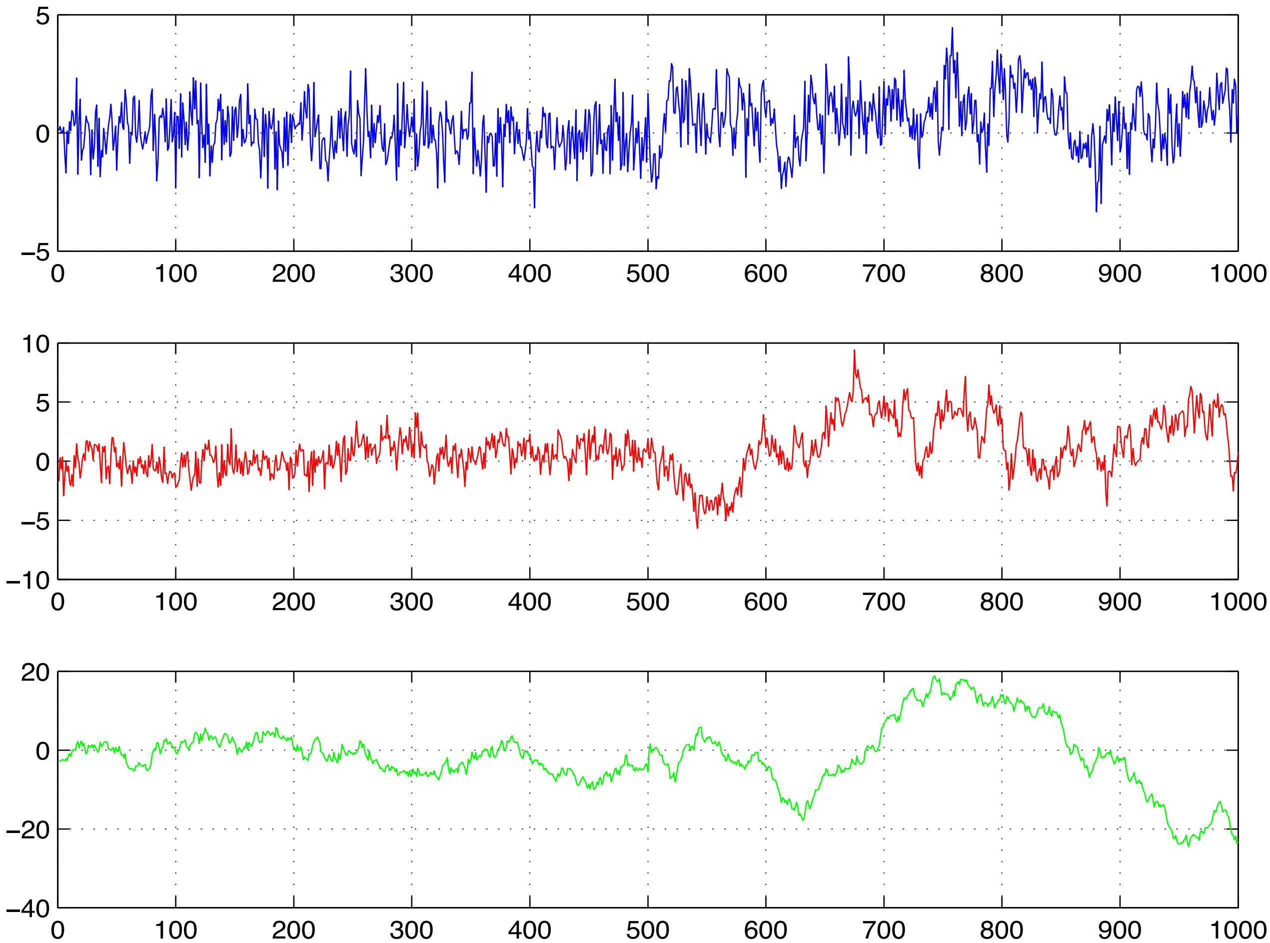

Figure 3: Comparison of trajectories simulated from the model (6.2) for different values of (d_1, d_2): [top] $(.1, .4)$ [middle] $(.3, .6)$ and [bottom] $(.8, 1.1)$. The sample size is $n = 1000$.

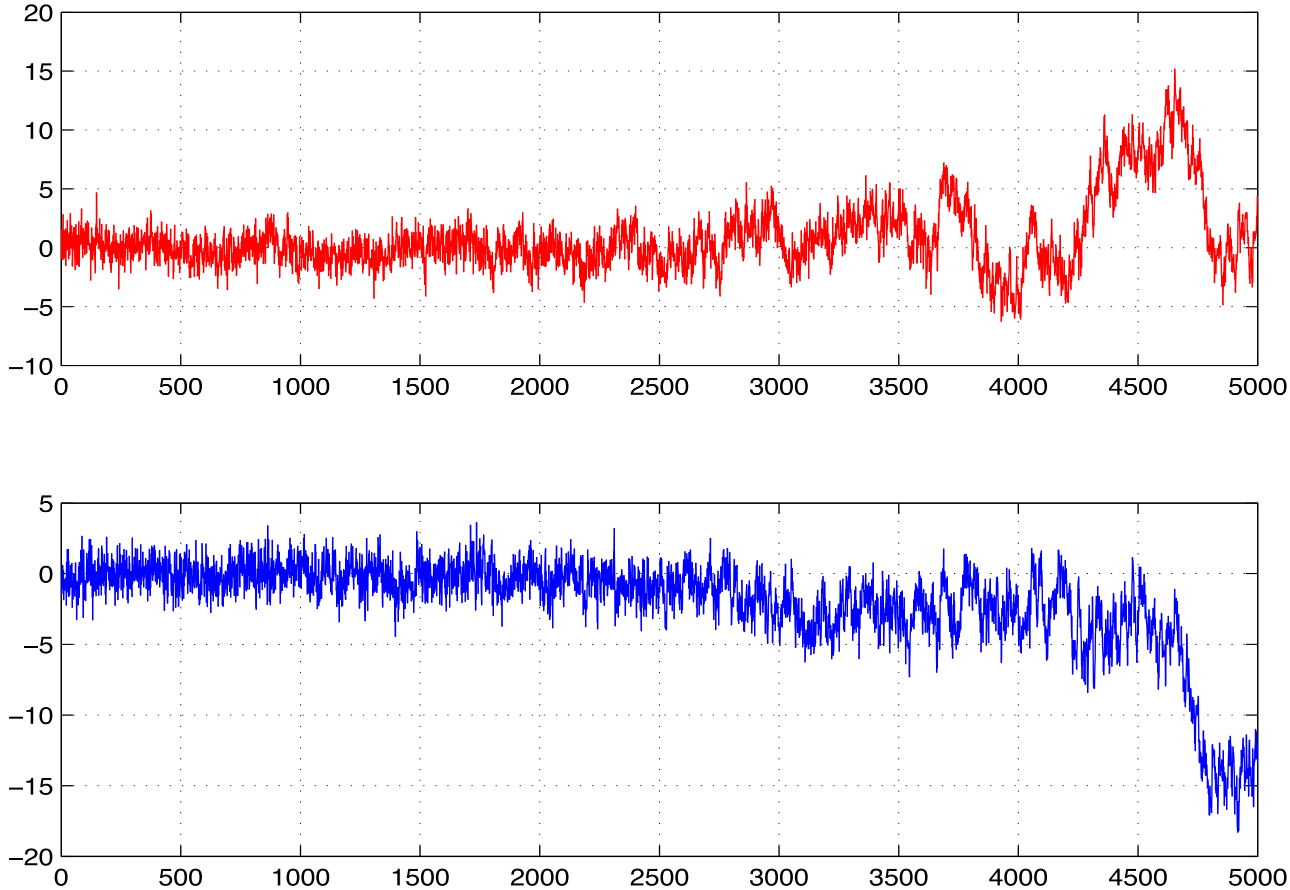

Figure 4: Comparison of trajectories simulated from the model (5.7) [top] and (5.11) [bottom] with $d(\tau) = .2 + .6\tau$, $\underline{\theta} = 0$ and $\overline{\theta} = 1$.

	$\underline{\tau} = 0.05$					$\underline{\tau} = 0.1$					$\underline{\tau} = 0.2$				
d_2 \ d_1	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
W_n statistic															
0	3.4					3.6					4.1				
0.1	20.2	3.7				24.2	4.1				28.3	4.5			
0.2	50.1	16	3.8			61.1	20.9	4.5			68.6	26	4.7		
0.3	74.6	38.5	13.3	3.6		87.1	51.1	17.7	4.1		93.1	62	23.6	4.7	
0.4	88.7	64.1	31.2	11.6	3.7	96.6	79.6	44.3	15	4	99.2	88.8	56.7	20	4.4
R_n statistic															
0	3.7					3.9					4.1				
0.1	29	5.1				30.9	4.9				31.3	5			
0.2	65	25.3	4.9			71	28.1	4.8			73.1	30.5	4.6		
0.3	85.3	53.8	21.2	5		91.7	62.2	24.4	4.9		94.3	67	26.5	4.9	
0.4	94.1	75.7	44.8	18.4	4.8	98	84.8	54.1	21.7	4.6	99.3	90.3	60.3	24.4	4.5
I_n statistic															
0	2.9					3.2					3.6				
0.1	28.5	3.4				29.5	3.8				30.7	4.1			
0.2	73.4	27.8	3.6			74	28.8	3.9			73.7	30.1	4.2		
0.3	95.9	68	24.8	3.5		96.3	69.8	26.5	3.7		95.9	69.7	27.9	4.3	
0.4	99.5	92.3	62.5	21.8	3.5	99.7	93.6	65.7	24.1	3.8	99.7	94.1	66.6	25.6	4
I_n^{Kim} statistic															
0	3.5					3.7					4.2				
0.1	23.5	3.6				24.1	4				24.6	4.3			
0.2	58.4	21.4	4.3			58.5	22	4.4			57.4	22.4	4.7		
0.3	86.1	54.8	19.9	4		86.8	55.8	20.5	4.2		86.1	55.4	20.8	4.5	
0.4	96.6	82.2	52.1	18.4	4	97.3	83.6	53.6	19.2	3.7	97.3	83.4	53.4	19.7	4

Table 1: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the tests based on the three statistics I_n, W_n, R_n and the Kim's statistic (6.1). The nominal level is $\alpha = 5\%$. The samples are simulated from model (6.2). The sample size is 5000 and the number of independent replications is 10^4.

τ	d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	2.9									
	0.1	28.5	3.4								
	0.2	73.4	27.8	3.6							
	0.3	95.9	68	24.8	3.5						
	0.4	99.5	92.3	62.5	21.8	3.5					
	0.6	100	99.9	99.7	97.1	88	5.3				
	0.8	100	100	99.8	98.7	92.6	11.3	5			
	1	100	100	100	99.9	98.9	45.3	28.1	5.1		
	1.2	100	100	100	100	99.9	87.4	78.6	41.4	5.3	
	1.4	100	100	100	100	100	98	96.4	85.9	59	3.3
0.1	0	3.2									
	0.1	29.5	3.8								
	0.2	74	28.8	3.9							
	0.3	96.3	69.8	26.5	3.7						
	0.4	99.7	93.6	65.7	24.1	3.8					
	0.6	100	100	99.9	98.7	92.5	5.2				
	0.8	100	100	99.9	99.4	96	11.8	4.9			
	1	100	100	100	100	99.7	51.9	34	5.1		
	1.2	100	100	100	100	100	91.7	84.2	44.6	5.1	
	1.4	100	100	100	100	100	99.1	98.2	88.9	57.8	2.9
0.2	0	3.6									
	0.1	30.7	4.1								
	0.2	73.7	30.1	4.2							
	0.3	95.9	69.7	27.9	4.3						
	0.4	99.7	94.1	66.6	25.6	4					
	0.6	100	100	99.9	99.4	95.4	5.1				
	0.8	100	100	100	99.8	98.1	11.6	4.9			
	1	100	100	100	100	99.9	57.3	39.4	4.6		
	1.2	100	100	100	100	100	94.8	89.5	48.4	4.9	
	1.4	100	100	100	100	100	99.7	99.3	91.8	57	2.7

Table 2: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test based on I_n with nominal level $\alpha = 5\%$. The samples are simulated from model (6.2). The sample size is 5000 and the number of independent replications is 10^4.

τ	d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	3									
	0.1	28.6	3.6								
	0.2	73.4	26.9	3.6							
	0.3	95.2	68.3	23.7	3.7						
	0.4	99.4	91.5	63.5	21.5	3.4					
	0.6	100	100	99.6	97.4	87.5	5.2				
	0.8	100	99.9	99.9	98.7	92.2	11.3	5.4			
	1	100	100	99.9	99.9	99	44.8	28.1	5.5		
	1.2	100	100	100	99.9	99.9	87.3	78	43	5.7	
	1.4	100	100	100	100	100	98.2	96.3	84.7	58.9	3.1
0.1	0	3.3									
	0.1	29.5	4								
	0.2	73.6	27.8	3.8							
	0.3	95.7	70	24.9	4						
	0.4	99.6	92.9	66.2	23.5	4					
	0.6	100	100	99.9	98.8	92.1	5.1				
	0.8	100	99.9	99.9	99.5	96.1	11.9	5.2			
	1	100	100	100	100	99.7	51	34.9	5.4		
	1.2	100	100	100	100	99.9	91.6	84	46	5.4	
	1.4	100	100	100	100	100	99.1	98.2	87.9	57.8	2.8
0.2	0	3.7									
	0.1	30.6	4.5								
	0.2	73.4	29	4.3							
	0.3	95.4	70.1	26.6	4.3						
	0.4	99.7	93.2	67.4	25.1	4.3					
	0.6	100	100	99.9	99.4	94.6	5				
	0.8	100	100	100	99.8	98	11.7	5.1			
	1	100	100	100	100	99.9	56.6	40.5	5		
	1.2	100	100	100	100	100	94.7	89.4	49.3	4.9	
	1.4	100	100	100	100	100	99.7	99.4	91.2	56.9	2.2

Table 3: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test based on I_n with nominal level $\alpha = 5\%$. The samples are simulated from model (6.2), where $(\psi_i(d))$ are FARIMA$(1, d, 0)$ coefficients with AR parameter 0.7. The sample size is 5000 and the number of independent replications is 10^4.

τ	d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	1.8									
	0.1	24.3	2.7								
	0.2	70.2	25.3	3.1							
	0.3	94.6	66.5	22.7	3.3						
	0.4	99.5	92.7	63.1	22.5	3.7					
	0.6	99.9	100	99.7	97.9	89.3	6.5				
	0.8	100	99.9	99.8	98.9	93	12.6	5.3			
	1	100	100	99.9	99.8	99.1	46.3	29.2	5.8		
	1.2	100	100	100	100	99.9	87.8	78.8	42.8	6	
	1.4	100	100	100	100	100	98.1	96.5	86.5	59.9	3.7
0.1	0	2									
	0.1	25.9	3.2								
	0.2	72.1	27.5	3.5							
	0.3	95.4	69.1	24.9	3.9						
	0.4	99.6	94.2	66.3	25	3.9					
	0.6	100	100	99.9	99.1	93.5	5.7				
	0.8	100	100	99.9	99.6	96.1	12.8	5.2			
	1	100	100	100	99.9	99.8	52.8	35.1	5.6		
	1.2	100	100	100	100	100	92.2	84.4	45.5	5.4	
	1.4	100	100	100	100	100	99.3	98.2	89.2	58.3	3.1
0.2	0	2.5									
	0.1	27.8	3.9								
	0.2	72.9	29.2	4.1							
	0.3	95.4	70.4	26.6	4.5						
	0.4	99.7	94.5	67.6	26.4	4.3					
	0.6	100	100	100	99.6	95.7	5.4				
	0.8	100	100	99.9	99.9	97.9	12.1	5.5			
	1	100	100	100	100	99.9	57.2	41.3	5.3		
	1.2	100	100	100	100	100	95.3	89.9	49.3	4.8	
	1.4	100	100	100	100	100	99.7	99.3	92.4	57.7	2.8

Table 4: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test based on I_n with nominal level $\alpha = 5\%$. The samples are simulated from model (6.2), where $(\psi_i(d))$ are FARIMA$(1, d, 0)$ coefficients with AR parameter -0.7. The sample size is 5000 and the number of independent replications is 10^4.

$\underline{\tau}$	d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0										
	0.1	16.1									
	0.2	42.3	15.2								
	0.3	69.9	36.6	13.1							
	0.4	86.8	61.2	31.8	11.2						
	0.6	97.8	91.1	74.6	49.7	26.5					
	0.8	99.3	97.8	91.9	79.6	59.1	17.1				
	1	99.8	99.1	97.4	92.1	82.8	41.6	11.1			
	1.2	99.8	99.5	98.9	96.5	91.5	66.1	26.1	6.7		
	1.4	99.8	99.6	99.2	97.7	94.5	78.3	40.4	11.2	3.3	
0.1	0										
	0.1	16.3									
	0.2	42.3	15.7								
	0.3	69.6	36.5	13.8							
	0.4	86.7	61.8	32.8	11.8						
	0.6	98.3	92.2	75.8	51.2	27.3					
	0.8	99.8	98.8	94	82.7	62.3	18.5				
	1	99.9	99.7	98.7	95	86.6	45.9	13.1			
	1.2	99.9	99.9	99.7	98.2	94.7	72.1	29.9	8.4		
	1.4	100	99.9	99.8	99.3	97.4	84.7	45.9	13.6	4.4	
0.2	0										
	0.1	16.3									
	0.2	39.9	16								
	0.3	65.8	35.4	14							
	0.4	83.4	59	32.7	12.5						
	0.6	97.6	90.5	73.9	49.7	27.3					
	0.8	99.7	98.6	93.1	81.5	60.5	19.9				
	1	100	99.9	98.6	94.3	85.4	45.7	14.8			
	1.2	99.9	99.9	99.7	98.4	94.3	70.6	30.2	9.9		
	1.4	100	99.9	99.9	99.4	97.3	82.9	43.7	14.8	5.8	

Table 5: Estimated power (in %) of the test based on I_n with nominal level $\alpha = 5\%$. The samples are simulated from model (5.7) with $d(\tau) = d_1 + (d_2 - d_1)\tau$, $\underline{\theta} = 0$ and $\overline{\theta} = 1$. The sample size is 5000 and the number of independent replications is 10^4.

7 Appendix: proofs

Proof of Proposition 4.1. (i) Write $z_{n1}(\tau) := A_{n1}^{-1} S_{[n\tau]}$, $(n/A_{n1}^2) V_{[n\tau]}(X) = \sum_{i=1}^{6} U_{ni}(\tau)$, where the terms

$$
\begin{aligned}
U_{n1}(\tau) &:= \big(n^2/[n\tau]^2\big) \int_0^{[n\tau]/n} z_{n1}^2(u)\mathrm{d}u, \\
U_{n2}(\tau) &:= -2\big(n^2/[n\tau]^2\big) z_{n1}(\tau) \int_0^{[n\tau]/n} \big([nu]/[n\tau]\big) z_{n1}(u)\mathrm{d}u, \\
U_{n3}(\tau) &:= \big(n^2/[n\tau]^2\big) z_{n1}^2(\tau) \int_0^{[n\tau]/n} \big([nu]/[n\tau]\big)^2 \mathrm{d}u, \\
U_{n4}(\tau) &:= -\big(n^3/[n\tau]^3\big) \Big(\int_0^{[n\tau]/n} z_{n1}(u)\mathrm{d}u \Big)^2, \\
U_{n5}(\tau) &:= 2\big(n^3/[n\tau]^3\big) z_{n1}(\tau) \Big(\int_0^{[n\tau]/n} z_{n1}(u)\mathrm{d}u \Big) \int_0^{[n\tau]/n} \big([nu]/[n\tau]\big)\mathrm{d}u, \\
U_{n6}(\tau) &:= -\big(n^3/[n\tau]^3\big) z_{n1}^2(\tau) \Big(\int_0^{[n\tau]/n} \big([nu]/[n\tau]\big)\mathrm{d}u \Big)^2
\end{aligned}
$$

tend in distribution, as $n \to \infty$, to the corresponding limit quantities

$$
\begin{aligned}
U_1(\tau) &:= \tau^{-2} \int_0^{\tau} Z_1^2(u)\mathrm{d}u, \\
U_2(\tau) &:= -2\tau^{-2} Z_1(\tau) \int_0^{\tau} (u/\tau) Z_1(u)\mathrm{d}u, \\
U_3(\tau) &:= \tau^{-2} Z_1^2(\tau) \int_0^{\tau} (u/\tau)^2 \mathrm{d}u, \\
U_4(\tau) &:= -\tau^{-3} \Big(\int_0^{\tau} Z_1(u)\mathrm{d}u \Big)^2, \\
U_5(\tau) &:= 2\tau^{-3} Z_1(\tau) \Big(\int_0^{\tau} Z_1(u)\mathrm{d}u \Big) \int_0^{\tau} (u/\tau)\mathrm{d}u, \\
U_6(\tau) &:= -\tau^{-3} Z_1^2(\tau) \Big(\int_0^{\tau} (u/\tau)\mathrm{d}u \Big)^2.
\end{aligned}
$$

Note $Q_\tau(Z_1) = \sum_{i=1}^{6} U_i(\tau)$ a.s. for each $\tau \in (0, \upsilon_1]$. The joint convergence

$$
(U_{n1}(\tau), \ldots, U_{n6}(\tau)) \longrightarrow_d (U_1(\tau), \ldots, U_6(\tau)) \tag{7.1}
$$

at each fixed point $\tau \in (0, \upsilon_1]$ can be easily derived from the (marginal) convergence $A_{n1}^{-1} S_{[n\tau]} \longrightarrow_{D[0,\upsilon_1]} Z_1(\tau)$ in (4.1). The convergence in (7.1) easily extends to the joint convergence at any finite number of points $0 < \tau_1 < \cdots < \tau_m \le \upsilon_1$. In other words,

$$
(n/A_{n1}^2) V_{[n\tau]}(X) \longrightarrow_{\mathrm{fdd}(0,\upsilon_1]} Q_\tau(Z_1). \tag{7.2}
$$

In a similar way,

$$
A_{n2}^{-1} S_{[n\tau]}^{*} \longrightarrow_{D[0,1-\upsilon_0]} Z_2(\tau),
$$

implies

$$
(n/A_{n2}^2) V_{n-[n\tau]}^{*}(X) \longrightarrow_{\mathrm{fdd}[\upsilon_0,1)} Q_{1-\tau}(Z_2). \tag{7.3}
$$

It is clear from the joint convergence in (4.1) that (7.2), (7.3) extend to the joint convergence of finite-dimensional distributions, in other words, that (4.2) holds with $\longrightarrow_{D(0,\upsilon_1]\times D[\upsilon_0,1)}$ replaced by $\longrightarrow_{\mathrm{fdd}(0,\upsilon_1]\times[\upsilon_0,1)}$.

It remains to prove the tightness in $D(0,\upsilon_1]\times D[\upsilon_0,1)$. To this end, it suffices to check the tightness of the marginal processes in (7.2) and (7.3) in the corresponding Skorokhod spaces $D(0,\upsilon_1]$ and $D[\upsilon_0,1)$. See, e.g., Ferger and Vogel (2010), Whitt (1970).

Let us prove the tightness of the l.h.s. in (7.2) in $D(0,\upsilon_1]$, or, equivalently, the tightness in $D[\upsilon,\upsilon_1]$, for any $0<\upsilon<\upsilon_1$. Let $\Upsilon_n(\tau) := (n/A^2_{n1})V_{[n\tau]}(X)$. Since $\{\Upsilon_n(\upsilon), n\ge 1\}$ is tight by (7.2), it suffices to show that for any $\epsilon_1,\epsilon_2>0$ there exist $\delta>0$ and $n_0\ge 1$ such that

$$\mathrm{P}(\omega_\delta(\Upsilon_n)\ge\epsilon_1)\le\epsilon_2, \qquad n\ge n_0, \tag{7.4}$$

where

$$\omega_\delta(x) := \sup\big\{|x(a)-x(b)| : \upsilon\le a<b\le\upsilon_1, a-b<\delta\big\}$$

is the continuity modulus of a function $x\in D[\upsilon,\upsilon_1]$; see Billingsley (1968, Theorem 8.2). Since $\Upsilon_n(\tau)=\sum_{i=1}^6 U_{ni}(\tau)$, it suffices to show (7.4) with Υ_n replaced by $U_{ni}, i=1,\dots,6$, in other words,

$$\mathrm{P}(\omega_\delta(U_{ni})\ge\epsilon_1)\le\epsilon_2, \qquad n\ge n_0, \quad i=1,\cdots,6. \tag{7.5}$$

We verify (7.5) for $i=2$ only since the remaining cases follow similarly. Write $U_{n2}(\tau)=\prod_{i=1}^3 H_{ni}(\tau)$, where $H_{n1}(\tau) := -2\big(n^2/[n\tau]^2\big)$, $H_{n2}(\tau) := z_{n1}(\tau)$, $H_{n3}(\tau) := \int_0^{[n\tau]/n}\big([nu]/[n\tau]\big)z_{n1}(u)\mathrm{d}u$. Then $\mathrm{P}(\omega_\delta(U_{n2})\ge\epsilon_1)\le\sum_{i=1}^3\big[\mathrm{P}\big(\omega_\delta\big(H_{ni}\big)\ge\epsilon_1/(3K)\big)+\mathrm{P}\big(\prod_{j\ne i}\|H_{nj}\|>K\big)\big]$, where $\|x\| := \sup\{|x(a)| : \upsilon\le a\le\upsilon_1\}$ is the sup-norm. Relation (4.1) implies that the probability $\mathrm{P}\big(\sum_{i=1}^3\|H_{ni}\|>K\big)$ can be made arbitrary small for all $n>n_0(K)$ by a suitable choice of K. By same relation (4.1) assumed under the uniform topology, for a given ϵ_1/K, we have that $\lim_{\delta\to 0}\limsup_{n\to\infty}\mathrm{P}\big(\omega_\delta(H_{ni})\ge\epsilon_1/K\big)=0$. This proves (7.5) and the functional convergence $(n/A^2_{n1})V_{[n\tau]}(X)\longrightarrow_{D(0,\upsilon_1]} Q_\tau(Z_1)$. The proof of $(n/A^2_{n2})V^*_{n-[n\tau]}(X)\longrightarrow_{D[\upsilon_0,1)} Q_{1-\tau}(Z_2)$ is analogous. This concludes the proof of part (i), since the continuity of the limit process in (4.2) is immediate from continuity of $(Z_1(\tau_1), Z_2(\tau_2))$ and the definition of Q_τ in (3.6).

(ii) Note that (4.3) and the a.s. continuity of $\tau\mapsto Q_\tau(Z_1)$ guarantees that $\inf_{\tau\in\mathcal{T}}Q_\tau(Z_1)>0$ a.s. Therefore relations (4.4) follow from (4.2) and the continuous mapping theorem.

(iii) Follows from (4.2) and the fact that $Z_1(\tau)=0$, $\tau\in\mathcal{T}$ implies $Q_\tau(Z_1)=0$, $\tau\in\mathcal{T}$. □

Proof of Proposition 5.6. We restrict the proof to the case (i) and $i=2$, or to the model (5.11), since the remaining cases can be treated similarly. Similarly as in the proof of (3.7), it suffices to prove the joint convergence of finite-dimensional distributions in (5.16) and the functional convergence of marginal processes, viz.,

$$n^{-d_1-.5}S_{2,[n\tau]}\ \longrightarrow_{D[0,\theta]}\ Z_{2,1}(\tau), \qquad n^{-d_2-.5}S^*_{2,[n\tau]}\ \longrightarrow_{D[0,1-\underline{\tau}]}\ Z_{2,2}(\tau). \tag{7.6}$$

Since $X_{2,t}=\sum_{j=0}^t\psi_j(d_1)\zeta_{t-j}$, $1\le t\le[n\underline{\tau}]$ has constant memory parameter d_1, the proof of the first convergence in (7.6) to $Z_{2,1}(\tau)=B^{\mathrm{II}}_{d_1+.5}(\tau)$ is standard, and we omit it. Consider the second convergence in (7.6). It can be rewritten as

$$n^{-d_2-.5}S_{2,[n\tau]}\ \longrightarrow_{D[\underline{\tau},1]}\ \mathcal{Z}_2(\tau). \tag{7.7}$$

Consider first the one-dimensional convergence in (7.7) at a fixed point $\tau \in [\underline{\tau}, 1]$. Write $S_{2,[n\tau]} = \sum_{t=1}^{[n\tau]} X_{2,t} = \sum_{t=1}^{[n\tau]} \sum_{s=1}^{t} b_{2,t-s}(t)\zeta_s$. Let $t = [nv]$, $s = [nu]$ and $0 \le u \le \overline{\theta} < v \le 1$. Consider the ratio

$$K_n(u,v) := \frac{b_{2,[nv]-[nu]}([nv])}{\psi_{[nv]-[nu]}(d_2)} = \prod_{i=1\vee([nv]-[n\overline{\theta}])}^{[nv]-[nu]} \frac{d\big(\frac{[nv]-i+1}{n}\big) - 1 + i}{d_2 - 1 + i}, \tag{7.8}$$

where $\psi_j(d)$ are the FARIMA coefficients in (5.6). We claim that

$$\lim_{n\to\infty} K_n(u,v) = \mathrm{e}^{H(u,v)}, \qquad 0 \le u \le \overline{\theta} < v \le 1, \tag{7.9}$$

where $H(u,v)$ is defined at (5.14). Indeed,

$$K_n(u,v) = \exp\Bigg\{ \sum_{i=1\vee([nv]-[n\overline{\theta}])}^{[nv]-[nu]} \log\left(1 - \frac{d_2 - d\big(\frac{[nv]-i}{n}\big)}{d_2 - 1 + i}\right)\Bigg\} = \mathrm{e}^{H_n(u,v)+R_n(u,v)},$$

where

$$\begin{aligned} H_n(u,v) &:= n^{-1} \sum_{i=1\vee([nv]-[n\overline{\theta}])}^{[nv]-[nu]} \frac{d\big(\frac{[nv]-i}{n}\big) - d_2}{\frac{d_2-1+i}{n}} \to H(u,v), \\ R_n(u,v) &= O\Bigg(\sum_{i=1\vee([nv]-[n\overline{\theta}])}^{[nv]-[nu]} \frac{1}{i^2}\Bigg) = O\Big(\frac{1}{1\vee([nv]-[n\overline{\theta}])}\Big), \end{aligned}$$

hence $R_n(u,v) \to 0$ for any $v > \overline{\theta}$.

Let us first consider the case $\tau > \overline{\theta}$. Following the scheme of discrete stochastic integrals in Surgailis (2003), rewrite the l.h.s. of (7.7) as a discrete stochastic integral

$$n^{-d_2-.5} S_{2,[n\tau]} = \int_0^\tau F_n(u)\mathrm{d}z_n(u) = \int_0^{\overline{\theta}} F_n(u)\mathrm{d}z_n(u) + \int_{\overline{\theta}}^\tau F_n(u)\mathrm{d}z_n(u),$$

where $z_n(u) := n^{-1/2}\sum_{i=1}^{[nu]} \zeta_i$ is the partial sum process of standardized i.i.d. r.v.s, tending weakly to a Brownian motion $\{B(u), u \in [0,1]\}$. The integrand F_n in the above integral is equal to

$$\begin{aligned} F_n(u) &:= n^{-d_2} \sum_{t=[nu]}^{[n\tau]} b_{2,t-[nu]}(t) \\ &= \begin{cases} n^{-d_2}\sum_{t=[nu]}^{[n\tau]} b_{2,t-[nu]}(t), & 0 < u \le \overline{\theta}, \\ n^{-d_2}\sum_{t=[nu]}^{[n\tau]} \psi_{t-[nu]}(d_2), & \overline{\theta} < u \le \tau, \end{cases} \end{aligned}$$

where we used the fact that $b_{2,t-[nu]}(t) = \psi_{t-[nu]}(d_2)$ for $t \ge [nu] \ge [n\overline{\theta}]$. Similarly, the r.h.s. of (7.7) can be written as the sum of two stochastic integrals:

$$\int_0^\tau F(u)\mathrm{d}B(u) = \int_0^{\overline{\theta}} F(u)\mathrm{d}B(u) + \int_{\overline{\theta}}^\tau F(u)\mathrm{d}B(u),$$

where

$$F(u) := \begin{cases} \Gamma(d_2)^{-1}\int_{\overline{\theta}}^\tau (v-u)^{d_2-1}\mathrm{e}^{H(u,v)}\mathrm{d}v, & 0 < u \le \overline{\theta}, \\ \Gamma(d_2+1)^{-1}(\tau-u)^{d_2}, & \overline{\theta} < u \le \tau. \end{cases}$$

Accordingly, using the above mentioned criterion in Surgailis (2003, Proposition 3.2) (see also Bružaitė and Vaičiulis (2005, Lemma 2.2)), the one-dimensional convergence in (7.7) follows from the L^2-convergence of the integrands:

$$\int_0^{\overline{\theta}} |F_n(u) - F(u)|^2 \mathrm{d}u \to 0, \qquad \int_{\overline{\theta}}^{\tau} |F_n(u) - F(u)|^2 \mathrm{d}u \to 0. \tag{7.10}$$

The second relation in (7.10) is easy using the properties of FARIMA filters. Denote J_n the first integral in (7.10). Using the above definitions, the integrand there can be rewritten as

$$F_n(u) - F(u) = \int_u^{\overline{\theta}} n^{1-d_2} b_{2,[nv]-[nu]}([nv])\mathrm{d}v + \int_{\overline{\theta}}^{\tau} G_n(u,v)\mathrm{d}v - n^{-d_2}(n\tau - [n\tau])b_{2,[n\tau]-[nu]}([n\tau]), \tag{7.11}$$

where $G_n(u,v) := n^{1-d_2} b_{2,[nv]-[nu]}([nv]) - \Gamma(d_2)^{-1}(v-u)^{d_2-1}\mathrm{e}^{H(u,v)}$.

Recall that from (7.8), $b_{2,[nv]-[nu]}([nv]) = \psi_{[nv]-[nu]}(d_2)K_n(u,v)$. Using on one hand the fact that the ratio $K_n(u,v)$ tends to 0 for $0 < u < v \le \overline{\theta}$, on the other hand (7.9), and from the well-known asymptotics $\psi_j(d) \sim \Gamma(d)^{-1}j^{d-1}$, $j \to \infty$ of FARIMA coefficients, it easily follows that $n^{1-d_2} b_{2,[nv]-[nu]}([nv]) \to 0$ for any $0 < u < v \le \overline{\theta}$, and $G_n(u,v) \to 0$ for any $0 < u < v \le 1$ fixed. Moreover, the last term in (7.11) obviously tends to 0 because $d_2 > 0$. Since both sides of (7.9) are nonnegative and bounded by 1, the above convergences extend to the proof of $J_n \to 0$ by the dominated convergence theorem. This proves the convergence of one-dimensional distributions in (7.7) for $\tau > \overline{\theta}$. For $\underline{\tau} \le \tau \le \overline{\theta}$, the above convergence follows similarly by using the fact that $K_n(u,v)$ tends to 0 for $0 < u < v \le \overline{\theta}$.

The proof of the convergence of general finite-dimensional distributions in (7.7), as well as the joint convergence of finite-dimensional distributions in (5.16), can be achieved analogously, by using the Cramèr-Wald device. Finally, the tightness in (7.7) follows by the Kolmorogov criterion (see, e.g., Bružaitė and Vaičiulis (2005, proof of Theorem 1.2) for details). Proposition 5.6 is proved. □

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