Detection of non-constant long memory parameter

Frédéric Lavancier, Remigijus Leipus, Anne Philippe, Donatas Surgailis

Detection of non-constant long memory parameter*

Frédéric Lavancier[1] (frederic.lavancier@univ-nantes.fr,
Remigijus Leipus[2,3] (remigijus.leipus@mif.vu.lt),
Anne Philippe[1] (anne.philippe@univ-nantes.fr),
Donatas Surgailis[2,3] (donatas.surgailis@mii.vu.lt)

[1] *Laboratoire de Mathématiques Jean Leray, Université de Nantes, France,*
[2] *Faculty of Mathematics and Informatics, Vilnius University, Lithuania,*
[3] *Institute of Mathematics and Informatics, Vilnius University, Lithuania*

Abstract This article deals with detection of nonconstant long memory parameter in time series. The null hypothesis presumes stationary or nonstationary time series with constant long memory parameter, typically an $I(d)$ series with $d > -.5$. The alternative corresponds to an increase in persistence and includes in particular an abrupt or gradual change from $I(d_1)$ to $I(d_2)$, $-.5 < d_1 < d_2$. We discuss several test statistics based on the ratio of forward and backward sample variances of the partial sums. The consistency of the tests is proved under a very general setting. We also study the behavior of these test statistics for some models with changing memory parameter. A simulation study shows that our testing procedures have good finite sample properties and turn out to be more powerful than the KPSS-based tests considered in some previous works.

1 Introduction

The present paper discusses statistical tests for detection of non-constant memory parameter of time series versus the null hypothesis that this parameter has not changed over time. As a particular case, our framework includes testing the null hypothesis that the observed series is $I(d)$ with constant $d > -.5$, against the alternative hypothesis that d has changed, together with a rigorous formulation of the last change. This kind of testing procedure is the basis to study the dynamics of persistence, which is a major question in economy (see Kumar and Okimoto (2007), Hassler and Nautz (2008), Kruse (2008)).

In a parametric setting and for stationary series ($|d| < .5$), the problem of testing for a single change of d was first investigated by Beran and Terrin (1996), Horváth and Shao (1999), Horváth (2001), Yamaguchi (2011) (see also Lavielle and Ludeña (2000), Kokoszka and Leipus (2003)). Typically, the sample is partitioned into two parts and d is estimated on each part. The test statistic is obtained by maximizing the difference of these estimates over all such partitions. A similar approach for detecting

*The second and fourth authors are supported by a grant (No. MIP-11155) from the Research Council of Lithuania.

multiple changes of d was used in Shimotsu (2006) and Bardet and Kammoun (2008) in a more general semiparametric context.

The above approach for testing against changes of d appears rather natural although applies to abrupt changes only and involves (multiple) estimation of d which is not very accurate if the number of observations between two change-points is not large enough; moreover, estimates of d involve bandwidth or some other tuning parameters and are rather sensitive to the short memory spectrum of the process.

On the other hand, some regression-based Lagrange Multiplier procedures have been recently discussed in Hassler and Meller (2009) and Martins and Rodrigues (2010). The series is first filtered by $(1-L)^d$, where L is the lag operator and d is the long memory parameter under the null hypothesis, then the resulting series is subjected to a (augmented) Lagrange Multiplier test for fractional integration, following the pioneer works by Robinson (1991, 1994). The filtering step can be done only approximatively and involves in practice an estimation of d. This is certainly the main reason for the size distortion that can be noticed in the simulation study displayed in Martins and Rodrigues (2010).

In a nonparametric set up, following Kim (2000), Kim et al. (2002) proposed several tests (hereafter referred to as Kim's tests), based on the ratio

$$\mathcal{K}_n(\tau) := \frac{U^*_{n-\lfloor n\tau\rfloor}(X)}{U_{\lfloor n\tau\rfloor}(X)}, \qquad \tau \in [0,1], \tag{1.1}$$

where

$$U_k(X) := \tfrac{1}{k^2}\textstyle\sum_{j=1}^k \big(S_j - \tfrac{j}{k}S_k\big)^2, \qquad U^*_{n-k}(X) := \tfrac{1}{(n-k)^2}\textstyle\sum_{j=k+1}^n \big(S^*_{n-j+1} - \tfrac{n-j+1}{n-k}S^*_{n-k}\big)^2 \tag{1.2}$$

are estimates of the second moment of forward and backward de-meaned partial sums

$$\frac{1}{k^{1/2}}\Big(S_j - \frac{j}{k}S_k\Big),\ j=1,\dots,k \qquad \text{and} \qquad \frac{1}{(n-k)^{1/2}}\Big(S^*_{n-j+1} - \frac{n-j+1}{n-k}S^*_{n-k}\Big),\ j=k+1,\dots,n,$$

on intervals $[1,2,\dots,k]$ and $[k+1,\dots,n]$, respectively. Here and below, given a sample $X = (X_1,\dots,X_n)$,

$$S_k := \sum_{j=1}^k X_j, \qquad S^*_{n-k} := \sum_{j=k+1}^n X_j$$

denote the forward and backward partial sums processes. Originally developed to test for a change from $I(0)$ to $I(1)$ (see also Busetti and Taylor (2004), Kim et al. (2002)), Kim's statistics were extended in Hassler and Scheithauer (2011) to detect a change from $I(0)$ to $I(d)$, $d > 0$. A related, though different approach based on the so-called CUSUM statistics, was used in Leybourne et al. (2007) and Sibbertsen and Kruse (2009) to test for a change from stationarity ($d_1 < .5$) to nonstationarity ($d_2 > .5$), or vice versa.

The present work extends Kim's approach to detect an abrupt or gradual change from $I(d_1)$ to $I(d_2)$, for any $-.5 < d_1 < d_2$ with exception of values $d_1, d_2 \in \{.5, 1.5, \dots\}$ (see Remark 3.2 for an explanation of the last restriction). This includes both stationary and nonstationary null (no-change) hypothesis which is important for applications since nonstationary time series with $d > .5$ are common in economics. Although our asymptotic results (Propositions 3.1, 4.1 and Corollary 3.1) are valid for the original Kim's statistics, see Remark 4.3, we modify Kim's ratio (1.1), by replacing the second sample moments $U_k(X)$, $U^*_{n-k}(X)$ in (1.2) of backward and forward partial sums by the corresponding empirical variances $V_k(X)$, $V^*_{n-k}(X)$ defined at (3.1) below. This modification is similar to the difference between the KPSS and the V/S tests, see Giraitis et al. (2003), and leads to a more powerful testing procedure (see Tables 1–2). It is important to note that the ratio-based statistics discussed in our paper, as well as the original Kim's statistics, do not require an estimate of d and do not depend on any tuning parameter apart from the choice of the testing interval $\mathcal{T} \subset (0,1)$. However, the limiting law under the null hypothesis depends on d, hence the computation of the quantile defining the critical region requires a weakly consistent estimate of the memory parameter d.

The paper is organized as follows. Section 2 contains formulations of the null and alternative hypotheses, in terms of joint convergence of forward and backward partial sums processes, and describes a class of $I(d)$ processes which satisfy the null hypothesis. Section 3 introduces the ratio statistics W_n, I_n and R_n and derives their limit distribution under the null hypothesis. Section 4 displays theoretical results, from which the consistency of our testing procedures is derived. Section 5 discusses the behavior of our statistics under alternative hypothesis. Some fractionally integrated models with constant or changing memory parameter are considered and the behavior of the above statistics for such models is studied. Section 6 extends the tests of Section 3 to the case when observations contain a linear trend. Section 7 contains simulations of empirical size and power of our testing procedures. All proofs are collected in Section 8.

2 The null and alternative hypotheses

Let $X = (X_1, \dots, X_n)$ be a sample from a time series $\{X_j\} = \{X_j, j = 1, 2, \dots\}$. Additional assumptions about $\{X_j\}$ will be specified later. Recall the definition of forward and backward partial sums processes of X:

$$S_k = S_k(X) = \sum_{j=1}^{k} X_j, \qquad S^*_{n-k} = S^*_{n-k}(X) = \sum_{j=k+1}^{n} X_j.$$

Note that backward sums can be expressed via forward sums, and vice versa: $S^*_{n-k} = S_n - S_k$, $S_k = S^*_n - S^*_{n-k}$.

For $0 \le a < b \le 1$, let us denote by $D[a,b]$ the Skorokhod space of all cadlag (i.e. right-continuous with left limits) real-valued functions defined on interval $[a,b]$. In this article, the space $D[a,b]$ and the product space $D[a_1,b_1] \times D[a_2,b_2]$, for any $0 \le a_i < b_i \le 1$, $i = 1,2$, are all endowed with the uniform topology and the σ-field generated by the open balls (see Pollard (1984)). The weak convergence of random elements in such spaces is denoted '$\longrightarrow_{D[a,b]}$' and '$\longrightarrow_{D[a_1,b_1]\times D[a_2,b_2]}$', respectively; the weak convergence of finite-dimensional distributions is denoted '$\longrightarrow_{fdd}$'; the convergence in law and in probability of random variables are denoted '$\longrightarrow_{\rm law}$' and '$\longrightarrow_p$', respectively.

The following hypotheses are clear particular cases of our more general hypotheses $\mathbf{H_0}$, $\mathbf{H_1}$ specified later. The null hypothesis below involves the classical type I fractional Brownian motion in the limit behavior of the partial sums, which is typical for linear models with long memory. Recall that a type I fractional Brownian motion $B^{\rm I}_{d+.5} = \{B^{\rm I}_{d+.5}(\tau), \tau \ge 0\}$ with Hurst parameter $H = d + .5 \in (0,2)$,

$H \neq 1$ is defined by

$$B^{\mathrm{I}}_{d+.5}(\tau) \quad := \quad \begin{cases} \frac{1}{\Gamma(d+1)} \int_{-\infty}^{\tau} \left((\tau-u)^d - (-u)_+^d\right) \mathrm{d}B(u), & -.5 < d < .5, \\ \int_0^{\tau} B^{\mathrm{I}}_{d-.5}(u)\mathrm{d}u, & .5 < d < 1.5, \end{cases} \tag{2.1}$$

where $(-u)_+ := (-u) \vee 0$ and $\{B(u), u \in \mathbb{R}\}$ is a standard Brownian motion with zero mean and variance $\mathrm{E}B^2(u) = |u|$. Let $\lfloor x \rfloor$ denote the integer part of the real number $x \in \mathbb{R}$.

$\mathbf{H_0[I]}$: There exist $d \in (-.5, 1.5)$, $d \neq .5$, $\kappa > 0$ and a normalization A_n such that

$$n^{-d-.5}\big(S_{\lfloor n\tau \rfloor} - \lfloor n\tau \rfloor A_n\big) \quad \longrightarrow_{D[0,1]} \quad \kappa B^{\mathrm{I}}_{d+.5}(\tau), \quad n \to \infty. \tag{2.2}$$

$\mathbf{H_1[I]}$: There exist $0 \le \upsilon_0 < \upsilon_1 \le 1$, $d > -.5$, and a normalization A_n such that

$$\Big(n^{-d-.5}\big(S_{\lfloor n\tau_1 \rfloor} - \lfloor n\tau_1 \rfloor A_n\big), \, n^{-d-.5}\big(S^*_{\lfloor n\tau_2 \rfloor} - \lfloor n\tau_2 \rfloor A_n\big)\Big) \quad \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \quad \big(0, Z_2(\tau_2)\big), \tag{2.3}$$

as $n \to \infty$, where $\{Z_2(\tau), \tau \in [1-\upsilon_1, 1-\upsilon_0]\}$ is a nondegenerate a.s. continuous Gaussian process.

Here and hereafter, a random element Z of $D[a,b]$ is called *nondegenerate* if it is not identically zero on the interval $[a,b]$ with positive probability, in other words, if $\mathrm{P}(Z(u) = 0, \forall u \in [a,b]) = 0$.

Typically, the null hypothesis $\mathbf{H_0[I]}$ is satisfied by $I(d)$ series (see Definition 5.1). In Section 5.1 we give a general family of linear processes satisfying $\mathbf{H_0[I]}$ including stationary and nonstationary processes. See also Taqqu (1979), Giraitis et al. (2000) and the review paper Giraitis et al. (2009) for some classes of non-linear stationary processes (subordinated Gaussian processes and stochastic volatility models) which satisfy $\mathbf{H_0[I]}$ for $0 < d < .5$. The alternative hypothesis corresponds to the processes changing from $I(d_1)$ to $I(d_2)$ processes (see Section 5.3 for examples).

Let us give a first example based on the well-known FARIMA model.

Example 2.1 A FARIMA$(0,d,0)$ process $\varepsilon_t(d) = \sum_{s=0}^{\infty} \pi_s(d)\zeta_{t-s}$ with $-.5 < d < .5$ satisfies assumption $\mathbf{H_0[I]}$ with $\kappa = 1$, $A_n = 0$. Here, $\pi_s(d), s = 0, 1, \ldots$ are the moving-average coefficients (see (5.12)) and $\{\zeta_t\}$ is a Gaussian white noise with zero mean and unit variance. Moreover, for two different memory parameters $-.5 < d_1 < d_2 < .5$, we can construct a process satisfying $\mathbf{H_1[I]}$ by

$$X_t := \begin{cases} \varepsilon_t(d_1), & t \le \lfloor n\theta^* \rfloor, \\ \varepsilon_t(d_2), & t > \lfloor n\theta^* \rfloor, \end{cases} \tag{2.4}$$

where $\theta^* \in (0,1)$. The process in (2.4) satisfies (2.3) with $d = d_2$, $A_n = 0$, $\upsilon_0 = 0$, $\upsilon_1 = \theta^*$, and $Z_2(\tau) = B^{\mathrm{I}}_{d+.5}(1) - B^{\mathrm{I}}_{d+.5}(\theta^* \vee (1-\tau))$, $\tau \in [0,1]$. In the case of $.5 < d < 1.5$, FARIMA(0,d,0) process in (2.4) is defined by $\varepsilon_t(d) = \sum_{i=1}^{t} Y_i$, where $\{Y_t\}$ is a stationary FARIMA(0,$d-1$,0).

The testing procedures of Section 3 for testing the hypotheses $\mathbf{H_0[I]}$ and $\mathbf{H_1[I]}$ can be extended to more general context. We formulate these 'extended' hypotheses as follows.

$\mathbf{H_0}$: There exist normalizations $\gamma_n \to \infty$ and A_n such that

$$\gamma_n^{-1}\big(S_{\lfloor n\tau \rfloor} - \lfloor n\tau \rfloor A_n\big) \quad \longrightarrow_{D[0,1]} \quad Z(\tau), \tag{2.5}$$

where $\{Z(\tau), \tau \in [0,1]\}$ is a nondegenerate a.s. continuous random process.

$\mathbf{H_1}$: There exist $0 \le \upsilon_0 < \upsilon_1 \le 1$ and normalizations $\gamma_n \to \infty$ and A_n such that

$$\Big(\gamma_n^{-1}\big(S_{\lfloor n\tau_1\rfloor} - \lfloor n\tau_1\rfloor A_n\big),\, \gamma_n^{-1}\big(S^*_{\lfloor n\tau_2\rfloor} - \lfloor n\tau_2\rfloor A_n\big)\Big) \quad \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \quad \big(0, Z_2(\tau_2)\big), \qquad (2.6)$$

where $\{Z_2(\tau), \tau \in [1-\upsilon_1, 1-\upsilon_0]\}$ is a nondegenerate a.s. continuous random process.

Typically, normalization $A_n = \mathrm{E}X_0$ accounts for centering of observations and does not depend on n. Assumptions $\mathbf{H_0}$ and $\mathbf{H_1}$ represent very general forms of the null ('no change in persistence of X') and the alternative ('an increase in persistence of X') hypotheses. Indeed, an increase in persistence of X at time $k_* = \lfloor n\upsilon_1\rfloor$ typically means that forward partial sums $S_j, j \le k_*$ grow at a slower rate γ_{n1} compared with the rate of growth γ_{n2} of backward sums $S^*_j, j \le n - k_*$. Therefore, the former sums tend to a degenerated process $Z_1(\tau) \equiv 0$, $\tau \in [0, \upsilon_1]$ under the normalization $\gamma_n = \gamma_{n2}$. Clearly, $\mathbf{H_0}$ and $\mathbf{H_1}$ are not limited to stationary processes and allow infinite variance processes as well. While these assumptions are sufficient for derivation of the asymptotic distribution and consistency of our tests, they need to be specified in order to be practically implemented. The hypothesis $\mathbf{H_0}[\mathbf{I}]$ presented before is one example of such specification and involves the type I fBm. Another example involving the type II fBm is presented in Section 5.2.

3 The testing procedure

3.1 The test statistics

Analogously to (1.1)–(1.2), introduce the corresponding partial sums' variance estimates

$$\begin{aligned} V_k(X) &:= \frac{1}{k^2}\sum_{j=1}^{k}\Big(S_j - \frac{j}{k}S_k\Big)^2 - \Big(\frac{1}{k^{3/2}}\sum_{j=1}^{k}\Big(S_j - \frac{j}{k}S_k\Big)\Big)^2, \\ V^*_{n-k}(X) &:= \frac{1}{(n-k)^2}\sum_{j=k+1}^{n}\Big(S^*_{n-j+1} - \frac{n-j+1}{n-k}\,S^*_{n-k}\Big)^2 \\ &\quad - \Big(\frac{1}{(n-k)^{3/2}}\sum_{j=k+1}^{n}\Big(S^*_{n-j+1} - \frac{n-j+1}{n-k}\,S^*_{n-k}\Big)\Big)^2 \end{aligned} \qquad (3.1)$$

and the corresponding 'backward/forward variance ratio':

$$\mathcal{L}_n(\tau) := \frac{V^*_{n-\lfloor n\tau\rfloor}(X)}{V_{\lfloor n\tau\rfloor}(X)}, \qquad \tau \in [0,1]. \qquad (3.2)$$

For a given *testing interval* $\mathcal{T} = [\underline{\tau}, \overline{\tau}] \subset (0,1)$, define the analogs of the 'supremum' and 'integral' statistics of Kim (2000):

$$W_n(X) := \sup_{\tau\in\mathcal{T}} \mathcal{L}_n(\tau), \qquad I_n(X) := \int_{\tau\in\mathcal{T}} \mathcal{L}_n(\tau)\mathrm{d}\tau. \qquad (3.3)$$

We also define the analog of the ratio statistic introduced in Sibbertsen and Kruse (2009):

$$R_n(X) := \frac{\inf_{\tau\in\mathcal{T}} V^*_{n-\lfloor n\tau\rfloor}(X)}{\inf_{\tau\in\mathcal{T}} V_{\lfloor n\tau\rfloor}(X)}. \qquad (3.4)$$

This statistic has also the same form as statistic R of Leybourne et al. (2007), formed as a ratio of the minimized CUSUMs of squared residuals obtained from the backward and forward subsamples of X, in the $I(0)/I(1)$ framework. The limit distribution of these statistics is given in Proposition 3.1. To this end, define

$$Z^*(u) \ := \ Z(1) - Z(1-u), \ u \in [0,1] \tag{3.5}$$

and a continuous time analog of the partial sums' variance $V_{\lfloor n\tau \rfloor}(X)$ in (3.1):

$$Q_\tau(Z) \ := \ \frac{1}{\tau^2}\Big[\int_0^\tau \big(Z(u) - \frac{u}{\tau}Z(\tau)\big)^2 \mathrm{d}u - \frac{1}{\tau}\Big(\int_0^\tau \big(Z(u) - \frac{u}{\tau}Z(\tau)\big)\mathrm{d}u\Big)^2\Big]. \tag{3.6}$$

Note $Q_{1-\tau}(Z^*)$ is the corresponding analog of $V^*_{n-\lfloor n\tau \rfloor}(X)$ in the numerators of the statistics in (3.2) and (3.4).

Proposition 3.1 *Assume* $\mathbf{H_0}$. *Then*

$$\Big(\gamma_n^{-1}\big(S_{\lfloor n\tau_1 \rfloor} - \lfloor n\tau_1 \rfloor A_n\big), \gamma_n^{-1}\big(S^*_{\lfloor n\tau_2 \rfloor} - \lfloor n\tau_2 \rfloor A_n\big)\Big) \ \longrightarrow_{D[0,1]\times D[0,1]} \ \big(Z(\tau_1), Z^*(\tau_2)\big). \tag{3.7}$$

Moreover, assume that

$$Q_\tau(Z) > 0 \quad \textit{a.s. for any} \quad \tau \in \mathcal{T}. \tag{3.8}$$

Then

$$\begin{aligned} W_n(X) \ &\longrightarrow_{\text{law}} \ W(Z) \ := \ \sup_{\tau \in \mathcal{T}} \frac{Q_{1-\tau}(Z^*)}{Q_\tau(Z)}, \\ I_n(X) \ &\longrightarrow_{\text{law}} \ I(Z) \ := \ \int_{\tau \in \mathcal{T}} \frac{Q_{1-\tau}(Z^*)}{Q_\tau(Z)} \mathrm{d}\tau, \\ R_n(X) \ &\longrightarrow_{\text{law}} \ R(Z) \ := \ \frac{\inf_{\tau \in \mathcal{T}} Q_{1-\tau}(Z^*)}{\inf_{\tau \in \mathcal{T}} Q_\tau(Z)}. \end{aligned} \tag{3.9}$$

The convergence in (3.7) is an immediate consequence of $\mathbf{H_0}$, while the fact that (3.7) and (3.8) imply (3.9) is a consequence of Proposition 4.1 stated in Section 4.

Remark 3.1 As noted previously, the alternative hypothesis $\mathbf{H_1}$ focuses on an *increase* of d, and the statistics (3.3), (3.4) are defined accordingly. It is straightforward to modify our testing procedures to test for a decrease of persistence. In such case, the corresponding test statistics are defined by exchanging forward and backward partial sums, or $V_{\lfloor n\tau \rfloor}(X)$ and $V^*_{n-\lfloor n\tau \rfloor}(X)$:

$$W_n^*(X) \ := \ \sup_{\tau \in \mathcal{T}} \mathcal{L}_n^{-1}(\tau), \quad I_n^*(X) \ := \ \int_{\tau \in \mathcal{T}} \mathcal{L}_n^{-1}(\tau)\mathrm{d}\tau, \quad R_n^*(X) \ := \ \frac{\inf_{\tau \in \mathcal{T}} V_{\lfloor n\tau \rfloor}(X)}{\inf_{\tau \in \mathcal{T}} V^*_{n-\lfloor n\tau \rfloor}(X)}. \tag{3.10}$$

In the case when the direction of the change of d is *unknown*, one can use various combinations of (3.3) and (3.10), e.g. the sums

$$W_n^*(X) + W_n(X), \quad I_n^*(X) + I_n(X), \quad R_n^*(X) + R_n(X),$$

or the maxima

$$\max\{W_n^*(X), W_n(X)\}, \quad \max\{I_n^*(X), I_n(X)\}, \quad \max\{R_n^*(X), R_n(X)\}.$$

The limit distributions of the above six statistics under $\mathbf{H_0}$ follow immediately from Proposition 3.1. However, for a given direction of change, the 'one-sided' tests in (3.3), (3.4) or (3.10) are preferable as they are more powerful.

3.2 Practical implementation for testing $\mathbf{H_0}[\mathbf{I}]$ against $\mathbf{H_1}[\mathbf{I}]$

Under the 'type I fBm null hypothesis' $\mathbf{H_0}[\mathbf{I}]$, the limit distribution of the above statistics follows from Proposition 3.1 with $\gamma_n = n^{d+.5}$ and $Z = \kappa B^{\mathrm{I}}_{d+.5}$. In this case, condition (3.8) is verified and we obtain the following result.

Corollary 3.1 *Assume* $\mathbf{H_0}[\mathbf{I}]$. *Then*

$$W_n(X) \ \longrightarrow_{\mathrm{law}} \ W(B^{\mathrm{I}}_{d+.5}), \quad I_n(X) \ \longrightarrow_{\mathrm{law}} \ I(B^{\mathrm{I}}_{d+.5}), \quad R_n(X) \ \longrightarrow_{\mathrm{law}} \ R(B^{\mathrm{I}}_{d+.5}). \tag{3.11}$$

The process $B^{\mathrm{I}}_{d+.5}$ in (3.11) depends on *unknown* memory parameter d, and so do the upper $\alpha-$quantiles of the r.v.'s in the right-hand sides of (3.11)

$$q^{[\mathrm{I}]}_T(\alpha, d) := \inf\{x : \mathrm{P}(T(B^{\mathrm{I}}_{d+.5}) \le x) \ge 1-\alpha)\}, \tag{3.12}$$

where $T = W, I, R$. Hence, applying the corresponding test, the unknown parameter d in (3.12) is replaced by a consistent estimator $\hat{d}$.

Testing procedure. Reject $\mathbf{H_0}[\mathbf{I}]$, if

$$W_n(X) \ > \ q^{[\mathrm{I}]}_W(\alpha, \hat{d}), \qquad I_n(X) \ > \ q^{[\mathrm{I}]}_I(\alpha, \hat{d}), \qquad R_n(X) \ > \ q^{[\mathrm{I}]}_R(\alpha, \hat{d}), \tag{3.13}$$

respectively, where $\hat{d}$ is a weakly consistent estimator of d:

$$\hat{d} \ \longrightarrow_p \ d, \qquad n \to \infty. \tag{3.14}$$

The fact that the replacement of d by $\hat{d}$ in (3.13) preserves asymptotic significance level α is guaranteed by the continuity of the quantile functions provided by Proposition 3.2 below.

Proposition 3.2 *Let* $d \in (-.5, 1.5)$, $d \ne .5$, $\alpha \in (0,1)$ *and let* $\hat{d}$ *satisfy (3.14). Then*

$$q^{[\mathrm{I}]}_T(\alpha, \hat{d}) \ \longrightarrow_p \ q^{[\mathrm{I}]}_T(\alpha, d), \qquad \textit{for} \quad T \ = \ W, \ I, \ R.$$

We omit the proof of the above proposition since it follows the same lines as in the paper Giraitis et al. (2006, Lemma 2.1) devoted to tests of stationarity based on the V/S statistic.

Several estimators of d can be used in (3.13). See the review paper Bardet et al. (2003) for a discussion of some popular estimators. In our simulations we use the Non-Stationarity Extended Local Whittle Estimator (NELWE) of Abadir et al. (2007), which applies to both stationary ($|d| < .5$) and nonstationary ($d > .5$) cases.

Remark 3.2 The above tests can be straightforwardly extended to $d > 1.5$, $d \ne 2.5, 3.5, \ldots$, provided some modifications. Note that type I fBm for such values of d is defined by iterating the integral in (2.1) (see e.g. Davidson and de Jong (2000)). On the other hand, although type I fBm can be defined for $d = .5, 1.5, \ldots$ as well, these values are excluded from our discussion for the following reasons. Firstly, in such case the normalization γ_n of partial sums process of $I(d)$ processes is different from $n^{d+.5}$ and contains an additional logarithmic factor, see Liu (1998). Secondly and more importantly, for $d = .5$ the limit process $Z(\tau) = B^{\mathrm{I}}_1(\tau) = \tau B^{\mathrm{I}}_1(1)$ is a random line, in which case the limit statistic $Q_\tau(Z)$ in (3.6) degenerates to zero, see also Remark 4.2 below.

4 Consistency and asymptotic power

It is natural to expect that under alternative hypotheses $\mathbf{H_1}$ or $\mathbf{H_1}[\mathbf{I}]$, all three statistics $W_n(X)$, $I_n(X)$, $R_n(X)$ tend to infinity in probability, provided the testing interval $\mathcal{T}$ and the degeneracy interval $[0, \upsilon_1]$ of forward partial sums are embedded: $\mathcal{T} \subset [0, \upsilon_1]$. This is true indeed, see Proposition 4.1 (iii) below, meaning that our tests are consistent. Moreover, it is of interest to determine the rate at which these statistics grow under alternative, or the asymptotic power. The following Proposition 4.1 provides the theoretical background to study the consistency of the tests. It also provides the limit distributions of the test statistics under $\mathbf{H_0}$ since Proposition 3.1 is an easy corollary of Proposition 4.1 (ii).

Proposition 4.1 *(i) Let there exist $0 \le \upsilon_0 < \upsilon_1 \le 1$ and normalizations $\gamma_{ni} \to \infty$ and A_{ni}, $i = 1, 2$ such that*

$$\Big(\gamma_{n1}^{-1}\big(S_{\lfloor n\tau_1\rfloor} - \lfloor n\tau_1\rfloor A_{n1}\big),\, \gamma_{n2}^{-1}\big(S^*_{\lfloor n\tau_2\rfloor} - \lfloor n\tau_2\rfloor A_{n2}\big)\Big) \longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \big(Z_1(\tau_1), Z_2(\tau_2)\big), \quad (4.1)$$

where $\big(Z_1(\tau_1), Z_2(\tau_2)\big)$ is a two-dimensional random process having a.s. continuous trajectories on $[\upsilon_0, \upsilon_1] \times [1-\upsilon_1, 1-\upsilon_0]$. Then

$$\big((n/\gamma_{n1}^2) V_{\lfloor n\tau_1\rfloor}(X),\, (n/\gamma_{n2}^2) V^*_{n-\lfloor n\tau_2\rfloor}(X)\big) \longrightarrow_{D(0,\upsilon_1]\times D[\upsilon_0,1)} \big(Q_{\tau_1}(Z_1), Q_{1-\tau_2}(Z_2)\big). \quad (4.2)$$

Moreover, the limit process $\big(Q_{\tau_1}(Z_1), Q_{1-\tau_2}(Z_2)\big)$ in (4.2) is a.s. continuous on $(\upsilon_0, \upsilon_1] \times [\upsilon_0, \upsilon_1)$.

(ii) Assume, in addition to (i), that $\mathcal{T} \subset \mathcal{U} := [\upsilon_0, \upsilon_1]$ and

$$Q_\tau(Z_1) > 0 \quad \text{a.s. for any} \quad \tau \in \mathcal{T}. \quad (4.3)$$

Then, as $n \to \infty$,

$$\begin{aligned}
(\gamma_{n1}/\gamma_{n2})^2 W_n(X) &\longrightarrow_{\text{law}} \sup_{\tau\in\mathcal{T}} \frac{Q_{1-\tau}(Z_2)}{Q_\tau(Z_1)}, \\
(\gamma_{n1}/\gamma_{n2})^2 I_n(X) &\longrightarrow_{\text{law}} \int_{\tau\in\mathcal{T}} \frac{Q_{1-\tau}(Z_2)}{Q_\tau(Z_1)}\,\mathrm{d}\tau, \\
(\gamma_{n1}/\gamma_{n2})^2 R_n(X) &\longrightarrow_{\text{law}} \frac{\inf_{\tau\in\mathcal{T}} Q_{1-\tau}(Z_2)}{\inf_{\tau\in\mathcal{T}} Q_\tau(Z_1)}.
\end{aligned} \quad (4.4)$$

(iii) Assume, in addition to (i), that $\mathcal{T} \subset \mathcal{U}$, $Z_1(\tau) \equiv 0$, $\tau \in \mathcal{T}$ and the process $\{Q_{1-\tau}(Z_2), \tau \in \mathcal{T}\}$ is nondegenerate. Then

$$(\gamma_{n1}/\gamma_{n2})^2 \left\{ \begin{array}{c} W_n(X) \\ I_n(X) \\ R_n(X) \end{array} \right\} \longrightarrow_p \infty. \quad (4.5)$$

Remark 4.1 Typically, under $\mathbf{H_1}$ relation (4.1) is satisfied with γ_{n2} increasing much faster than γ_{n1} (e.g., $\gamma_{ni} = n^{d_i+.5}$, $i = 1, 2$, $d_1 < d_2$) and then (4.4) imply that $W_n(X)$, $I_n(X)$ and $R_n(X)$ grow as $O_p\big((\gamma_{n2}/\gamma_{n1})^2\big)$. Two classes of fractionally integrated series with changing memory parameter and satisfying (4.1) are discussed in Section 5.

Remark 4.2 Note that $Q_\tau(Z) \geq 0$ by the Cauchy-Schwarz inequality and that $Q_\tau(Z) = 0$ implies $Z(u) - \frac{u}{\tau} Z(\tau) = a$ for all $u \in [0, \tau]$ and some (random) $a = a(\tau)$. In other words, $\mathrm{P}(Q_\tau(Z) = 0) > 0$ implies that for some (possibly, random) constants a and b,

$$\mathrm{P}\big(Z(u) = a + \frac{u}{\tau} b, \forall\, u \in [0, \tau]\big) \; > \; 0. \tag{4.6}$$

Therefore, condition (4.3) implicitly excludes situations as in (4.6), with $a \neq 0$, $b \neq 0$, which may arise under the null hypothesis $\mathbf{H_0}$, if $A_n = 0$ in (2.5) whereas the X_j's have nonzero mean.

Remark 4.3 All the results in Sections 3 and 5 hold for Kim's statistics in (7.1), defined by replacing $V_{\lfloor n\tau \rfloor}(X), V^*_{n-\lfloor n\tau \rfloor}(X)$ in (3.3), (3.4) by $U_{\lfloor n\tau \rfloor}(X), U^*_{n-\lfloor n\tau \rfloor}(X)$ as given in (1.2), with the only difference that the functional $Q_\tau(Z)$ in the corresponding statements must be replaced by its counterpart $\widetilde{Q}_\tau(Z) := \tau^{-2} \int_0^\tau \big(Z(u) - \frac{u}{\tau} Z(\tau)\big)^2 \mathrm{d}u$, cf. (3.6).

5 Application to fractionally integrated processes

This section discusses the convergence of forward and backward partial sums for some fractionally integrated models with constant or changing memory parameter and the behavior of statistics W_n, I_n, R_n for such models.

5.1 Type I fractional Brownian motion and the null hypothesis $\mathbf{H_0}[\mathbf{I}]$

It is well-known that type I fBm arises in the scaling limit of $d-$integrated, or $I(d)$, series with i.i.d. or martingale difference innovations. See Davydov (1970), Peligrad and Utev (1997), Marinucci and Robinson (1999), Bružaitė and Vaičiulis (2005) and the references therein.

A formal definition of $I(d)$ process (denoted $\{X_t\} \sim I(d)$) for $d > -.5$, $d \neq .5, 1.5, \ldots$ is given below. Let $\mathrm{MD}(0,1)$ be the class of all stationary ergodic martingale differences $\{\zeta_s, s \in \mathbb{Z}\}$ with unit variance $\mathrm{E}[\zeta_0^2] = 1$ and zero conditional expectation $\mathrm{E}[\zeta_s | \mathcal{F}_{s-1}] = 0$, $s \in \mathbb{Z}$, where $\{\mathcal{F}_s, s \in \mathbb{Z}\}$ is a nondecreasing family of $\sigma-$fields.

Definition 5.1 *(i) Write* $\{X_t\} \sim I(0)$ *if*

$$X_t = \sum_{j=0}^{\infty} a_j \zeta_{t-j}, \qquad t \in \mathbb{Z} \tag{5.1}$$

is a moving average with martingale difference innovations $\{\zeta_j\} \in \mathrm{MD}(0,1)$ *and summable coefficients* $\sum_{j=0}^\infty |a_j| < \infty$, $\sum_{j=0}^\infty a_j \neq 0$.
(ii) Let $d \in (-.5, .5) \backslash \{0\}$. *Write* $\{X_t\} \sim I(d)$ *if* $\{X_t\}$ *is a fractionally integrated process*

$$X_t = (1-L)^{-d} Y_t = \sum_{j=0}^{\infty} \pi_j(d) Y_{t-j}, \qquad t \in \mathbb{Z}, \tag{5.2}$$

where $Y_t = \sum_{j=0}^\infty a_j \zeta_{t-j}$, $\{Y_t\} \sim I(0)$ *and* $\{\pi_j(d), j \geq 0\}$ *are the coefficients of the binomial expansion* $(1-z)^{-d} = \sum_{j=0}^\infty \pi_j(d) z^j$, $|z| < 1$.
(iii) Let $d > .5$ *and* $d \neq 1.5, 2.5, \ldots$. *Write* $\{X_t\} \sim I(d)$ *if* $X_t = \sum_{j=1}^t Y_j$, $t = 1, 2, \ldots$, *where* $\{Y_t\} \sim I(d-1)$.

In the above definition, $\{X_t\} \sim I(d)$ for $d > .5$ is recursively defined for $t = 1, 2, \ldots$ only, as a $p-$times integrated stationary $I(d-p)$ process, where $p = \lfloor d + .5 \rfloor$ is the integer part of $d + .5$, and therefore $\{X_t\}$ has stationary increments of order p. A related definition of $I(d)$ process involving initial values $X_{-i}, i = 0, 1, \ldots$ is given in (5.8) below. From Definition 5.1 it also follows that an $I(d)$ process can be written as a weighted sum of martingale differences $\{\zeta_s\} \in \mathrm{MD}(0,1)$, for instance:

$$X_t = \begin{cases} \sum_{s \le t} (a \star \pi(d))_{t-s} \zeta_s, & -.5 < d < .5, \\ \sum_{s \le t} \sum_{1 \vee s \le j \le t} (a \star \pi(d-1))_{j-s} \zeta_s, & .5 < d < 1.5, \end{cases} \qquad t = 1, 2, \ldots, \tag{5.3}$$

where $(a \star \pi(d))_j := \sum_{i=0}^{j} a_i \pi_{j-i}(d)$, $j \ge 0$ is the convolution of the sequences $\{a_j\}$ and $\{\pi_j(d)\}$.

Proposition 5.1 *(i) Let $\{X_t\} \sim I(d)$ for some $d \in (-.5, 1.5)$, $d \neq .5$. If $d \in (-.5, 0]$, assume in addition $\mathrm{E}|\zeta_1|^p < \infty$, for some $p > 1/(.5 + d)$. Then (2.2) holds with $A_n = 0$, $\kappa = \sum_{i=0}^{\infty} a_i$.*

(ii) Let $\{\sigma_s, s \in \mathbb{Z}\}$ be an almost periodic sequence such that $\bar\sigma^2 := \lim_{n\to\infty} n^{-1} \sum_{s=1}^{n} \sigma_s^2 > 0$. Let $\{X_t\}$ be defined as in (5.3), where ζ_s, $s \in \mathbb{Z}$ are replaced by $\sigma_s \zeta_s$, $s \in \mathbb{Z}$ and where d and $\{\zeta_s\}$ satisfy the conditions in (i). Then (2.2) holds with $A_n = 0, \kappa = \bar\sigma \sum_{i=0}^{\infty} a_i$.

The proof of Proposition 5.1 can be easily reduced to the case $a_j = \kappa \delta_j$, where $\delta_j = \mathbf{1}(j = 0)$ is the delta-function. Indeed,

$$\mathrm{E}\Big(\sum_{j=1}^{n} (X_j - X_j^\dagger) \Big)^2 = o\big(n^{2d+1}\big), \tag{5.4}$$

where $X_j^\dagger := \kappa(1-L)^{-d}\zeta_j \quad (-.5 < d < .5)$ and $X_j^\dagger := \kappa \sum_{k=1}^{j} (1-L)^{-(d-1)} \zeta_k$ $(.5 < d < 1.5)$ is (integrated) FARIMA$(0, d, 0)$ process. The proof of the approximation (5.4) is given in Section 8. The proof of Proposition 5.1 is omitted in view of (5.4) and since similar results under slightly different hypotheses on the innovations $\{\zeta_s\}$ can be found in Bružaitė and Vaičiulis (2005), Chan and Terrin (1995), Davidson and de Jong (2000), Giraitis et al. (2012), and elsewhere. In particular, the proof of the tightness in $D[0,1]$ given in Giraitis et al. (2012, Proposition 4.4.4) carries over to martingale difference innovations, see also Bružaitė and Vaičiulis (2005, Theorem 1.2), while part (ii) follows similarly to Bružaitė and Vaičiulis (2005, Theorem 1.1), using the fact that the sequence $\{\sigma_s \zeta_s\}$ satisfies the martingale central limit theorem: $n^{-1/2} \sum_{s=1}^{\lfloor n\tau \rfloor} \sigma_s \zeta_s \longrightarrow_{fdd} \bar\sigma B(\tau)$. Note that the linear process $\{X_t\}$ in Proposition 5.1 (ii) with heteroscedastic noise $\{\sigma_s \zeta_s\}$ is nonstationary even if $|d| < .5$.

5.2 Type II fractional Brownian motion and the null hypothesis $\mathbf{H_0}$[II]

Definition 5.2 *A type II fractional Brownian motion with parameter $d > -.5$ is defined by*

$$B^{\mathrm{II}}_{d+.5}(\tau) := \frac{1}{\Gamma(d+1)} \int_0^\tau (\tau - u)^d \mathrm{d}B(u), \qquad \tau \ge 0, \tag{5.5}$$

where $\{B(u), u \ge 0\}$ is a standard Brownian motion with zero mean and variance $\mathrm{E}B^2(u) = u$.

A type II fBm shares many properties of type I fBm except that it has nonstationary increments, however, for $|d| < .5$ increments at time τ of type II fBm tend to those of type I fBm when $\tau \to \infty$. Davidson and Hashimzade (2009) discussed distinctions between the distributions of type I and type II fBms. Convergence to type II fBm of partial sums of fractionally integrated processes was studied in Marinucci and Robinson (1999). See also Marinucci and Robinson (2000), Davidson and de Jong (2000), Leipus and Surgailis (2010).

Type II fBm may serve as the limit process in the following specification of the null hypothesis $\mathbf{H_0}$.

$\mathbf{H_0}$[**II**]: *There exist* $d > -.5$, $\kappa > 0$ *and a normalization* A_n *such that*

$$n^{-d-.5}\big(S_{\lfloor n\tau\rfloor} - \lfloor n\tau\rfloor A_n\big) \quad \longrightarrow_{D[0,1]} \quad \kappa B^{\rm II}_{d+.5}(\tau). \tag{5.6}$$

The alternative hypothesis to $\mathbf{H_0}$[**II**] can be again $\mathbf{H_1}$[**I**] of Section 2.

Proposition 5.1 can be extended to type II fBm convergence in (5.6) as follows. Introduce a 'truncated' $I(0)$ process

$$Y_t := \begin{cases} \sum_{j=0}^t a_j\zeta_{t-j}, & t = 1,2,\ldots, \\ 0, & t = 0,-1,-2,\ldots, \end{cases} \tag{5.7}$$

where $\{a_j\}$ and $\{\zeta_s\}$ are the same as in (5.1). Following Johansen and Nielsen (2010), for $d > 0$ consider a $d-$integrated process $\{X_t, t = 1,2,\ldots\}$ with given initial values $\{X^0_{-i}, i = 0,1,\ldots\}$ as defined by

$$X_t \quad = \quad (1-L)_+^{-d}Y_t \ + \ (1-L)_+^{-d}(1-L)_-^{d}X_t^0, \qquad t = 1,2,\ldots, \tag{5.8}$$

where $\{Y_t\}$ is defined in (5.7) and the operators $(1-L)^d_\pm$ are defined through corresponding 'truncated' binomial expansions:

$$(1-L)_+^d Z_t \ := \ \sum_{j=0}^{t-1}\pi_j(-d)L^jZ_t, \qquad (1-L)_-^d Z_t \ := \ \sum_{j=t}^{\infty}\pi_j(-d)L^jZ_t \ = \ \sum_{i=0}^{\infty}\pi_{t+i}(-d)L^iZ_0,$$

$t = 1,2,\ldots$. Note that the term $(1-L)_+^{-d}(1-L)_-^dX_t^0$ in (5.8) depends on initial values $\{X^0_{-i}, i = 0,1,\ldots\}$ only. The choice of zero initial values $X^0_{-i} = 0$, $i = 0,1,\ldots$ in (5.8) leads to type II process $X_t = (1-L)_+^{-d}Y_t$, more explicitly,

$$X_t \quad = \quad \sum_{s=1}^t (a\star\pi(d))_{t-s}\zeta_s. \tag{5.9}$$

In general, $\{X^0_{-i}\}$ can be deterministic or random variables satisfying mild boundedness conditions for the convergence of the series $(1-L)_-^dX_t^0$.

Proposition 5.2 *(i) Let* $\{X_t\}$ *be defined in (5.8), with* $\{Y_t\}$ *as in (5.7) and initial values* $\{X^0_{-i}\}$ *satisfying for* $d > .5$

$$\sup_{i\ge 0}\mathrm{E}(X^0_{-i})^2 < \infty. \tag{5.10}$$

For $-.5 < d \le .5$ *assume that* $X^0_{-i} \equiv 0$. *If* $d \in (-.5, 0]$, *assume in addition* $\mathrm{E}|\zeta_1|^p < \infty$, *for some* $p > 1/(.5+d)$. *Then (5.6) holds with* $A_n = 0$, $\kappa = \sum_{i=0}^\infty a_i$.

(ii) Let $\{\sigma_s, s \ge 1\}$ *be an almost periodic sequence such that* $\bar\sigma^2 := \lim_{n\to\infty} n^{-1}\sum_{s=1}^n\sigma_s^2 > 0$. *Let* $\{X_t\}$ *be defined as in (5.9), where* ζ_s, $s \ge 1$ *are replaced by* $\sigma_s\zeta_s$, $s \ge 1$ *and where* d *and* $\{\zeta_s\}$ *satisfy the conditions in (i). Then (5.6) holds with* $A_n = 0$, $\kappa = \bar\sigma\sum_{i=0}^\infty a_i$.

Remark 5.1 For $d > .5$, Proposition 5.2 (i) implies that any L^2-bounded initial values have no effect on the limit distribution of partial sums of the process in (5.8). As it follows from the proof in Section 8 below, the above statement also remains valid for arbitrary initial values $\{X^0_{-i}\}$ possibly depending on n and growing at a rate $O_p(n^{\lambda/2})$ with some $0 < \lambda < 1 \wedge (2d-1)$, viz., $\sup_{i\ge 0} \mathrm{E}(X^0_{-i})^2 < Cn^\lambda$, $d > 0$.

Similarly to Corollary 3.1, Proposition 3.1 implies the following corollary.

Corollary 5.1 *Let $\{X_t\}$ satisfy the conditions of Proposition 5.2. Then*

$$W_n(X) \longrightarrow_{\mathrm{law}} W(B^{\mathrm{II}}_{d+.5}), \quad I_n(X) \longrightarrow_{\mathrm{law}} I(B^{\mathrm{II}}_{d+.5}), \qquad R_n(X) \longrightarrow_{\mathrm{law}} R(B^{\mathrm{II}}_{d+.5}), \tag{5.11}$$

where $\{B^{\mathrm{II}}_{d+.5}(\tau), \tau \in [0,1]\}$ is a type II fBm as defined in (5.5).

Remark 5.2 Numerical experiments confirm that the upper quantiles $q^{[\mathrm{II}]}_T(\alpha, d)$, $T = W, I, R$, of the limit r.v.s on the r.h.s. of (5.11) are very close to the corresponding upper quantiles $q^{[\mathrm{I}]}_T(\alpha, d)$ of the limiting statistics in (3.11) when d is smaller than 1 (see Figure 1 in the particular case $T = I$). In other words, from a practical point of view, there is not much difference between type I fBM and type II fBm null hypotheses $\mathbf{H_0}[\mathbf{I}]$ and $\mathbf{H_0}[\mathbf{II}]$ in testing for a change of d when $d < 1$.

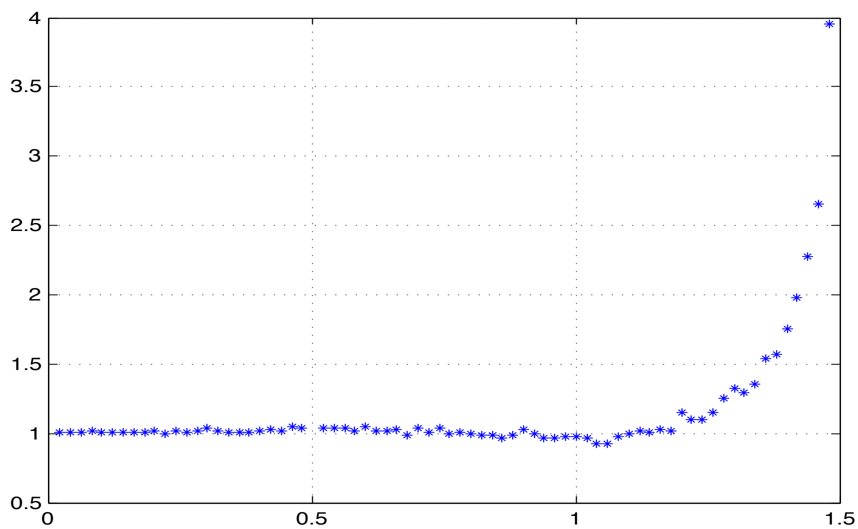

Figure 1: Representation of the ratio $q^{[\mathrm{I}]}_I(0.05, d)/q^{[\mathrm{II}]}_I(0.05, d)$ as function of d, with the choice $\underline{\tau} = 0.1$ and $\overline{\tau} = 0.9$.

5.3 Fractionally integrated models with changing memory parameter

Let us discuss two nonparametric classes of nonstationary time series with time-varying long memory parameter termed 'rapidly changing memory' and 'gradually changing memory'.

Rapidly changing memory. This class is obtained by replacing parameter d by a function $d(t/n) \in [0, \infty)$ in the FARIMA$(0, d, 0)$ filter

$$\pi_j(d) = \frac{d}{1} \cdot \frac{d+1}{2} \cdots \frac{d-1+j}{j} = \frac{\Gamma(d+j)}{j!\Gamma(d)}, \quad j = 1, 2, \ldots, \quad \pi_0(d) := 1. \tag{5.12}$$

Let $d(\tau)$, $\tau \in [0,1]$ be a function taking values in the interval $[0,\infty)$. (More precise conditions on the function $d(\tau)$ will be specified below.) Define

$$\begin{aligned} b_{1,j}(t) &:= \pi_j\big(d(\tfrac{t}{n})\big), \quad j = 0,1,\dots, \\ X_{1,t} &:= \sum_{s=1}^{t} b_{1,t-s}(t)\zeta_s, \qquad t = 1,\dots,n, \end{aligned} \tag{5.13}$$

where the innovations $\zeta_s, s \geq 1$ satisfy the conditions of Definition 5.1. The particular case

$$d(\tau) = \begin{cases} 0, & \tau \in [0,\theta^*], \\ 1, & \tau \in (\theta^*,1] \end{cases} \tag{5.14}$$

for some $0 < \theta^* < 1$, leads to the model

$$X_{1,t} = \begin{cases} \zeta_t, & t = 1,2,\dots,\lfloor\theta^* n\rfloor, \\ \sum_{s=1}^{t}\zeta_s, & t = \lfloor\theta^* n\rfloor + 1,\dots,n, \end{cases} \tag{5.15}$$

which corresponds to transition $I(0) \to I(1)$ at time $\lfloor\theta^* n\rfloor + 1$. A more general step function

$$d(\tau) = \begin{cases} d_1, & \tau \in [0,\theta^*], \\ d_2, & \tau \in (\theta^*,1] \end{cases} \tag{5.16}$$

corresponds to $\{X_{1,t}\}$ changing from $I(d_1)$ to $I(d_2)$ at time $\lfloor\theta^* n\rfloor + 1$.

Gradually changing memory. This class of nonstationary time-varying fractionally integrated processes was defined in Philippe et al. (2006a,b, 2008). Here, we use a truncated modification of these processes with slowly varying memory parameter $d(t/n) \in [0,\infty)$, defined as

$$\begin{aligned} b_{2,j}(t) &:= \frac{d(\frac{t}{n})}{1}\cdot\frac{d(\frac{t-1}{n})+1}{2}\cdots\frac{d(\frac{t-j+1}{n})-1+j}{j}, \quad j = 1,2,\dots, \quad b_{2,0}(t) := 1, \\ X_{2,t} &:= \sum_{s=1}^{t} b_{2,t-s}(t)\zeta_s, \quad t = 1,\dots,n. \end{aligned} \tag{5.17}$$

Contrary to (5.13), the process in (5.17) satisfies an autoregressive time-varying fractionally integrated equation with ζ_t on the right-hand side, see Philippe et al. (2008). In the case when $d(\tau) \equiv d$ is constant function, the coefficients $b_{2,j}(t)$ in (5.17) coincide with FARIMA$(0,d,0)$ coefficients in (5.12) and in this case the processes $\{X_{1,t}\}$ and $\{X_{2,t}\}$ in (5.13) and (5.17) coincide.

To see the difference between these two classes, consider the case of step function in (5.14). Then

$$X_{2,t} = \begin{cases} \zeta_t, & t = 1,2,\dots,\lfloor\theta^* n\rfloor, \\ \sum_{s=\lfloor\theta^* n\rfloor+1}^{t}\zeta_s + \sum_{s=1}^{\lfloor\theta^* n\rfloor}\frac{t-\lfloor\theta^* n\rfloor}{t-s}\zeta_s, & t = \lfloor\theta^* n\rfloor + 1,\dots,n. \end{cases} \tag{5.18}$$

Note $\frac{t-\lfloor\theta^* n\rfloor}{t-s} = 0$ for $t = \lfloor\theta^* n\rfloor$ and monotonically increases with $t \geq \lfloor\theta^* n\rfloor$. Therefore, (5.18) embodies a gradual transition from $I(0)$ to $I(1)$, in contrast to an abrupt change of these regimes in (5.15). The distinction between the two models (5.15) and (5.18) can be clearly seen from the variance behavior:

the variance of $X_{1,t}$ exhibits a jump from 1 to $\lfloor\theta^* n\rfloor + 1 = O(n)$ at time $t = \lfloor\theta^* n\rfloor + 1$, after which it linearly increases with t, while the variance of $X_{2,t}$ changes 'smoothly' with t:

$$\mathrm{Var}(X_{2,t}) \quad = \quad \begin{cases} 1, & t = 1, 2, \ldots, \lfloor\theta^* n\rfloor, \\ (t - \lfloor\theta^* n\rfloor) + \sum_{s=1}^{\lfloor\theta^* n\rfloor} \frac{(t-\lfloor\theta^* n\rfloor)^2}{(t-s)^2}, & t = \lfloor\theta^* n\rfloor + 1, \ldots, n. \end{cases}$$

Similar distinctions between (5.13) and (5.17) prevail also in the case of general 'memory function' $d(\cdot)$: when the memory parameter $d(t/n)$ changes with t, this change gradually affects the lagged ratios in the coefficients $b_{2,j}(t)$ in (5.17), and not all lagged ratios simultaneously as in the case of $b_{1,j}(t)$, see (5.12).

5.4 Asymptotics of change-point statistics for fractionally integrated models with changing memory parameter

In this subsection we study the joint convergence of forward and backward partial sums as in (2.6) for the two models in (5.13) and (5.17) with time-varying memory parameter $d(t/n)$. After the statement of Proposition 5.3 below, we discuss its implications for the asymptotic power of our tests.

Let us specify a class of 'memory functions' $d(\cdot)$. For $0 < d_1 < d_2 < \infty$ and $0 \le \underline{\theta} \le \overline{\theta} \le 1$, introduce the class $\mathcal{D}_{\underline{\theta},\overline{\theta}}(d_1, d_2)$ of left-continuous nondecreasing functions $d(\cdot) \equiv \{d(\tau), \tau \in [0,1]\}$ such that

$$d(\tau) = \begin{cases} d_1, & \tau \in [0, \underline{\theta}], \\ d_2, & \tau \in [\overline{\theta}, 1], \end{cases}, \qquad d_1 < d(\tau) < d_2, \quad \underline{\theta} < \tau < \overline{\theta}. \tag{5.19}$$

The interval $\Theta := [\underline{\theta}, \overline{\theta}]$ will be called the *memory change interval.* Note that for $\underline{\theta} = \overline{\theta} \equiv \theta^*$, the class $\mathcal{D}_{\theta^*,\theta^*}(d_1, d_2)$ consists of a single step function in (5.16). Recall from Section 3 that the interval $\mathcal{T} = [\underline{\tau}, \overline{\tau}]$ in memory change statistics in (3.3) and (3.4) is called the (memory) testing interval. When discussing the behavior of memory tests under alternatives in (5.13), (5.17) with changing memory parameter, the intervals Θ and $\mathcal{T}$ need not coincide since Θ is not known *a priori.*

With a given $d(\cdot) \in \mathcal{D}_{\underline{\theta},\overline{\theta}}(d_1, d_2)$, we associate a function

$$H(u, v) := \begin{cases} \int_u^v \frac{d(x)-d_2}{v-x}\,\mathrm{d}x, & 0 \le u \le v \le 1, \\ 0, & \text{otherwise} \end{cases} \tag{5.20}$$

Note $H(u, v) \le 0$ since $d(x) \le d_2$, $x \in [0, 1]$ and $H(u, v) = 0$ if $\overline{\theta} \le u \le v \le 1$. Define two Gaussian processes $\mathcal{Z}_1$ and $\mathcal{Z}_2$ by

$$\begin{aligned} \mathcal{Z}_1(\tau) &:= \frac{1}{\Gamma(d_2)} \int_0^\tau \Big\{ \int_{\overline{\theta}}^\tau (v-u)_+^{d_2-1} \mathrm{d}v \Big\} \mathrm{d}B(u) \;=\; B^{\mathrm{II}}_{d_2+.5}(\tau) - B^{\mathrm{II}}_{d_2+.5}(\overline{\theta}), \\ \mathcal{Z}_2(\tau) &:= \frac{1}{\Gamma(d_2)} \int_0^\tau \Big\{ \int_{\overline{\theta}}^\tau (v-u)_+^{d_2-1} \mathrm{e}^{H(u,v)} \mathrm{d}v \Big\} \mathrm{d}B(u), \quad \tau > \overline{\theta}, \\ \mathcal{Z}_1(\tau) &= \mathcal{Z}_2(\tau) := 0, \quad \tau \in [0, \overline{\theta}]. \end{aligned} \tag{5.21}$$

The processes $\{\mathcal{Z}_i(\tau), \in [0, 1]\}$, $i = 1, 2$ are well-defined for any $d_2 > -0.5$ and have a.s. continuous trajectories. In the case $\underline{\theta} = \overline{\theta} \equiv \theta^*$ and a step function $d(\cdot)$ in (5.19), $\mathcal{Z}_2(\tau)$ for $\tau > \theta^*$ can be rewritten as

$$\mathcal{Z}_2(\tau) \;=\; \frac{1}{\Gamma(d_2)} \int_0^\tau \Big\{ \int_{\theta^*}^\tau (v-u)_+^{d_1-1} (v-\theta^*)^{d_2-d_1} \mathrm{d}v \Big\} \mathrm{d}B(u). \tag{5.22}$$

Related class of Gaussian processes was discussed in Philippe et al. (2008) and Surgailis (2008).

Proposition 5.3 *Let* $d(\cdot) \in \mathcal{D}_{\underline{\theta},\overline{\theta}}(d_1, d_2)$ *for some* $0 \le d_1 < d_2 < \infty$, $0 \le \underline{\theta} \le \overline{\theta} \le 1$. *Let* $S_{i,k}$ *and* $S^*_{i,n-k}$, $i = 1, 2$ *be the forward and backward partial sums processes corresponding to time-varying fractional filters* $\{X_{i,t}\}$, $i = 1, 2$ *in (5.13), (5.17), with memory parameter* $d(t/n)$ *and standardized i.i.d. innovations* $\{\zeta_j,\, j \ge 1\}$. *Moreover, in the case* $d_1 = 0$ *we assume that* $\mathrm{E}|\zeta_1|^{2+\delta} < \infty$ *for some* $\delta > 0$. *Then*
(i) for any $\theta \in (0, \underline{\theta}]$ *with* $\underline{\theta} > 0$

$$\big(n^{-d_1-.5} S_{i,\lfloor n\tau_1\rfloor}, n^{-d_2-.5} S^*_{i,\lfloor n\tau_2\rfloor}\big) \ \longrightarrow_{D[0,\theta]\times D[0,1-\underline{\tau}]} \ \big(Z_{i,1}(\tau_1), Z_{i,2}(\tau_2)\big), \qquad i = 1, 2, \quad (5.23)$$

where

$$Z_{i,1}(\tau) \ := \ B^{\mathrm{II}}_{d_1+.5}(\tau), \qquad Z_{i,2}(\tau) \ := \ \mathcal{Z}^*_i(\tau) \ = \ \mathcal{Z}_i(1) - \mathcal{Z}_i(1-\tau), \qquad i = 1, 2, \qquad (5.24)$$

and $\mathcal{Z}_i$, $i = 1, 2$ *are defined in (5.21);*
(ii) for any $\theta \in [\underline{\theta}, 1]$, *for any* $d > d(\theta)$, $d_1 < d < d_2$

$$\big(n^{-d-.5} S_{i,\lfloor n\tau_1\rfloor}, n^{-d_2-.5} S^*_{i,\lfloor n\tau_2\rfloor}\big) \ \longrightarrow_{D\lfloor 0,\theta\rfloor\times D[0,1-\underline{\tau}]} \ \big(0, Z_{i,2}(\tau_2)\big), \qquad i = 1, 2, \qquad (5.25)$$

where $Z_{i,2}$, $i = 1, 2$ *are the same as in (5.24).*

The power of our tests depends on whether the testing and the memory change intervals have an empty intersection or not. When $\overline{\tau} < \underline{\theta}$, Proposition 5.3 (i) applies taking $\theta = \overline{\tau}$ and the asymptotic distribution of the memory test statistics for models (5.13) and (5.17) follows from Proposition 4.1 (4.4), with normalization $(\gamma_{n2}/\gamma_{n1})^2 = n^{2(d_1-d_2)} \to 0$, implying the consistency of the tests. But this situation is untypical for practical applications and hence not very interesting. Even less interesting seems the case when a change of memory ends before the start of the testing interval, i.e., when $\overline{\theta} \le \underline{\tau}$. Although the last case is not covered by Proposition 5.3, the limit distribution of the test statistics for models (5.13), (5.17) exists with trivial normalization $(\gamma_{n2}/\gamma_{n1})^2 = 1$ and therefore our tests are inconsistent, which is quite natural in this case.

Let us turn to some more interesting situations, corresponding to the case when the intervals $\mathcal{T}$ and Θ have a nonempty intersection of positive length. There are two possibilities:

Case 1: $\underline{\tau} < \underline{\theta} \le \overline{\tau}$ (a change of memory occurs after the beginning of the testing interval), and

Case 2: $\underline{\theta} \le \underline{\tau} < \overline{\theta}$ (a change of memory occurs before the beginning of the testing interval).

Let us consider Cases 1 and 2 in more detail.

Case 1. Let $\widetilde{\mathcal{T}} := [\underline{\tau}, \underline{\theta}] \subset \mathcal{T}$. Introduce the following 'dominated' (see (5.27)) statistics:

$$\begin{aligned}\widetilde{W}_n(X) \ &:= \ \sup_{\tau\in\widetilde{\mathcal{T}}} \frac{V^*_{n-\lfloor n\tau\rfloor}(X)}{V_{\lfloor n\tau\rfloor}(X)}, \qquad \widetilde{I}_n(X) \ := \ \int_{\widetilde{\mathcal{T}}} \frac{V^*_{n-\lfloor n\tau\rfloor}(X)}{V_{\lfloor n\tau\rfloor}(X)}\,\mathrm{d}\tau, \qquad (5.26)\\ \widetilde{R}_n(X) \ &:= \ \frac{\inf_{\tau\in\mathcal{T}} V^*_{n-\lfloor n\tau\rfloor}(X)}{\inf_{\tau\in\widetilde{\mathcal{T}}} V_{\lfloor n\tau\rfloor}(X)}.\end{aligned}$$

Clearly,

$$W_n(X) \ge \widetilde{W}_n(X), \qquad I_n(X) \ge \widetilde{I}_n(X), \qquad R_n(X) \ge \widetilde{R}_n(X), \qquad \text{a.s.} \qquad (5.27)$$

The limit distribution of (5.26) for models (5.13) and (5.17) can be derived from propositions 4.1 and 5.3 (i) choosing $\theta = \underline{\theta}$. In particular, it follows that $n^{2(d_1-d_2)}\widetilde{W}_n(X_i)$, $n^{2(d_1-d_2)}\widetilde{I}_n(X_i)$, and $n^{2(d_1-d_2)}\widetilde{R}_n(X_i)$, $i = 1, 2$ tend, in distribution, to the corresponding limits in (4.4), with $\mathcal{T}$ replaced by $\widetilde{\mathcal{T}}$ and $Z_1 = Z_{i,1}$, $Z_2 = Z_{i,2}$, $i = 1, 2$ as defined in (5.24). Moreover, it can be shown that $n^{-2d_1}V_{\lfloor n\tau\rfloor}(X_i) \longrightarrow_p \infty$ for any $\tau \in \mathcal{T} \setminus \widetilde{\mathcal{T}}$. Therefore, in Case 1, the limit distributions of the original statistics in (3.3) and the 'dominated' statistics in (5.26) coincide.

Case 2. In this case, define $\widetilde{\mathcal{T}} := [\underline{\tau}, \tilde{\theta}] \subset \mathcal{T}$, where $\tilde{\theta} \in (\underline{\tau}, \overline{\theta})$ is an inner point of the interval $[\underline{\tau}, \overline{\theta}]$. Let $\widetilde{W}_n(X)$, $\widetilde{I}_n(X)$, $\widetilde{R}_n(X)$ be defined as in (5.26). Obviously, relations (5.27) hold as in the previous case. Since the memory parameter increases on the interval $\widetilde{\mathcal{T}}$, the limit distribution of the process $V_{\lfloor n\tau\rfloor}(X_i)$, $\tau \in \widetilde{\mathcal{T}}$ in the denominator of the statistics is not identified from Proposition 4.1 (ii). Nevertheless in this case we can use Propositions 4.1 (iii) and 5.3 (ii) to obtain a robust rate of growth of the memory statistics in (5.26) and (3.3). Indeed from Proposition 5.3 (ii) with $\theta = \tilde{\theta}$, we have that $n^{-2d}V_{\lfloor n\tau\rfloor}(X_i) \longrightarrow_{D(0,\tilde{\theta}]} 0$ for any $d_2 > d > d(\tilde{\theta})$ and hence $n^{2(d-d_2)}W_n(X_i)$, $n^{2(d-d_2)}I_n(X_i)$ and $n^{2(d-d_2)}R_n(X_i)$, $i = 1, 2$ tend to infinity, in probability.

6 Testing in the presence of linear trend

The tests discussed in Section 3 can be further developed to include the presence of a linear trend. In such a case, partial sums $S_{\lfloor n\tau\rfloor}$ may grow as a second-order polynomial of $\lfloor n\tau\rfloor$ (see Example 6.1 below). Then, the null and alternative hypotheses have to be modified, as follows.

$\mathbf{H}_0^{\mathrm{trend}}$: There exists normalizations $\gamma_n \to \infty$, A_n, B_n, such that

$$\gamma_n^{-1}\big(S_{\lfloor n\tau\rfloor} - \lfloor n\tau\rfloor A_n - \lfloor n\tau\rfloor^2 B_n\big) \quad \longrightarrow_{D[0,1]} \quad Z(\tau), \tag{6.1}$$

where $\{Z(\tau), \tau \in [0,1]\}$ is a nondegenerate a.s. continuous random process.

$\mathbf{H}_1^{\mathrm{trend}}$: There exist $0 \le \upsilon_0 < \upsilon_1 \le 1$ and normalizations $\gamma_n \to \infty$, A_n, B_n, such that

$$\begin{aligned} &\Big(\gamma_n^{-1}\big(S_{\lfloor n\tau_1\rfloor} - \lfloor n\tau_1\rfloor A_n - \lfloor n\tau\rfloor^2 B_n\big),\, \gamma_n^{-1}\big(S^*_{\lfloor n\tau_2\rfloor} - \lfloor n\tau_2\rfloor^* A_n - \lfloor n\tau_2\rfloor^{2*} B_n\big)\Big) \\ &\longrightarrow_{D[0,\upsilon_1]\times D[0,1-\upsilon_0]} \ \big(0, Z_2(\tau_2)\big), \end{aligned} \tag{6.2}$$

where $\{Z_2(\tau), \tau \in [1-\upsilon_1, 1-\upsilon_0]\}$ is a nondegenerate a.s. continuous random process, $\lfloor n\tau\rfloor^* := n - \lfloor n(1-\tau)\rfloor = \lfloor n\tau\rfloor$, $\lfloor n\tau\rfloor^{2*} := n^2 - \lfloor n(1-\tau)\rfloor^2$.

Example 6.1 Consider a process $\{X_t\}$ defined as in Example 2.1 from equation (2.4). We construct the process $\{\mathcal{X}_t\}$ by adding to $\{X_t\}$ an additive linear trend:

$$\mathcal{X}_t = X_t + a + bt, \tag{6.3}$$

where a, b are some coefficients.

When $d_1 = d_2 = d$, we have $\mathcal{X}_t = \varepsilon_t(d) + a + bt$ and $\{\mathcal{X}_t\}$ satisfies the hypothesis $\mathbf{H}_0^{\mathrm{trend}}$ with

$B_n = \frac{b}{2}, A_n = a + \frac{b}{2}$ and $Z = B^{\mathrm{I}}_{d+.5}$. Indeed,

$$
\begin{aligned}
& n^{-d-.5}\Big(S_{\lfloor n\tau\rfloor}(\mathcal{X}) - \lfloor n\tau\rfloor(a+\frac{b}{2}) - \lfloor n\tau\rfloor^2\frac{b}{2}\Big) \\
&= n^{-d-.5}S_{\lfloor n\tau\rfloor}(\varepsilon(d)) + n^{-d-.5}\Big\{a\lfloor n\tau\rfloor + b\sum_{j=1}^{\lfloor n\tau\rfloor} j - \lfloor n\tau\rfloor\Big(a+\frac{b}{2}\Big) - \lfloor n\tau\rfloor^2\frac{b}{2}\Big\} \\
&= n^{-d-.5}S_{\lfloor n\tau\rfloor}(\varepsilon(d)) \longrightarrow_{D[0,1]} B^{\mathrm{I}}_{d+.5}(\tau), \quad n\to\infty.
\end{aligned}
$$

Under linear trend, the test statistics of Section 3.1 have to be modified, as follows. For a fixed $1 \le k \le n$, let

$$\widehat{X}_j := X_j - \hat{a}_k(X) - \hat{b}_k(X)j, \qquad 1 \le j \le k$$

denote the residuals from the least-squares regression of $(X_j)_{1\le j\le k}$ on $(a+bj)_{1\le j\le k}$. Similarly, let

$$\widehat{X}^*_j := X_j - \hat{a}^*_{n-k}(X) - \hat{b}^*_{n-k}(X)j, \qquad k < j \le n$$

denote the residuals from the least-squares regression of $(X_j)_{k<j\le n}$ on $(a+bj)_{k<j\le n}$. The corresponding intercept and slope coefficients are defined through $(\hat{a}_k(X), \hat{b}_k(X)) := \mathrm{argmin}\big(\sum_{j=1}^k (X_j - a - bj)^2\big)$, $(\hat{a}^*_{n-k}(X), \hat{b}^*_{n-k}(X)) := \mathrm{argmin}\big(\sum_{j=k+1}^n (X_j - a - bj)^2\big)$. The variance estimates of de-trended forward and backward partial sums are defined by

$$
\begin{aligned}
\mathcal{V}_k(X) &:= \frac{1}{k^2}\sum_{j=1}^k (\widehat{S}_j)^2 - \Big(\frac{1}{k^{3/2}}\sum_{j=1}^k \widehat{S}_j\Big)^2, \\
\mathcal{V}^*_{n-k}(X) &:= \frac{1}{(n-k)^2}\sum_{j=k+1}^n \big(\widehat{S}^*_{n-j+1}\big)^2 - \Big(\frac{1}{(n-k)^{3/2}}\sum_{j=k+1}^n \widehat{S}^*_{n-j+1}\Big)^2,
\end{aligned}
\tag{6.4}
$$

where

$$\widehat{S}_j := \sum_{i=1}^j \widehat{X}_i = \sum_{i=1}^j (X_i - \hat{a}_k(X) - \hat{b}_k(X)i), \quad \widehat{S}^*_{n-j+1} := \sum_{i=j}^n \widehat{X}^*_i = \sum_{i=j}^n (X_i - \hat{a}^*_{n-k}(X) - \hat{b}^*_{n-k}(X)i),$$

cf. (3.1). Replacing $V_n(X), V^*_{n-k}(X)$ in (3.3)–(3.4) by the corresponding quantities $\mathcal{V}_n(X), \mathcal{V}^*_{n-k}(X)$, the statistics in presence of a linear trend are given by

$$\mathcal{W}_n(X) := \sup_{\tau\in\mathcal{T}} \frac{\mathcal{V}^*_{n-\lfloor n\tau\rfloor}(X)}{\mathcal{V}_{\lfloor n\tau\rfloor}(X)}, \quad \mathcal{I}_n(X) := \int_{\tau\in\mathcal{T}} \frac{\mathcal{V}^*_{n-\lfloor n\tau\rfloor}(X)}{\mathcal{V}_{\lfloor n\tau\rfloor}(X)}\mathrm{d}\tau, \quad \mathcal{R}_n(X) := \frac{\inf_{\tau\in\mathcal{T}}\mathcal{V}^*_{n-\lfloor n\tau\rfloor}(X)}{\inf_{\tau\in\mathcal{T}}\mathcal{V}_{\lfloor n\tau\rfloor}(X)}. \tag{6.5}$$

Note that (6.5) agree with (3.3)–(3.4) if no trend is assumed (i.e. b is known and equal to zero).

Under the null hypothesis $\mathbf{H_0^{\mathrm{trend}}}$ the distributions of (6.5) can be obtained similarly to that of (3.3)–(3.4). The following proposition is the analog of the corresponding result (4.2) of Proposition 4.1 (ii) for (6.4).

Proposition 6.1 *Under the hypothesis* $\mathbf{H}_0^{\mathrm{trend}}$,

$$\big((n/\gamma_n^2)\mathcal{V}_{\lfloor n\tau_1\rfloor}(X), (n/\gamma_n^2)\mathcal{V}^*_{n-\lfloor n\tau_2\rfloor}(X)\big) \longrightarrow_{D(0,1]\times D[0,1)} \big(\mathcal{Q}_{\tau_1}(Z), \mathcal{Q}_{1-\tau_2}(Z^*)\big), \tag{6.6}$$

where Z^* *is defined in (3.5) and*

$$
\begin{aligned}
\mathcal{Q}_\tau(Z) &:= \frac{1}{\tau^2}\Big[\int_0^\tau \mathcal{Z}(u,\tau)^2\mathrm{d}u - \frac{1}{\tau}\Big(\int_0^\tau \mathcal{Z}(u,\tau)\mathrm{d}u\Big)^2\Big], \\
\mathcal{Z}(u,\tau) &:= Z(u) + 2Z(\tau)\frac{u}{\tau}\Big(1 - \frac{3u}{2\tau}\Big) + 6\Big(\frac{1}{\tau}\int_0^\tau Z(v)\mathrm{d}v\Big)\frac{u}{\tau}\Big(1 - \frac{u}{\tau}\Big).
\end{aligned}
\tag{6.7}
$$

Note that the process $\{\mathcal{Z}(u,\tau), u \in [0,\tau]\}$ defined in (6.7) satisfies $\mathcal{Z}(0,\tau) = \mathcal{Z}(\tau,\tau) = 0$ and $\int_0^\tau \mathcal{Z}(u,\tau)\mathrm{d}u = 0$. In the case of Brownian motion $Z = B$, $\{\mathcal{Z}(u,1), u \in [0,1]\}$ is known as the second level Brownian bridge (see MacNeill (1978)). Extension of Proposition 3.1 to the modified statistics W_n, I_n, R_n with the ratio (3.2) replaced by $\mathcal{V}^*_{n-\lfloor n\tau\rfloor}(X)/\mathcal{V}_{\lfloor n\tau\rfloor}(X)$ is straightforward. Clearly, $\mathcal{Z}(u,\tau)$ in (6.7) is different from the corresponding process $Z(u) - \frac{u}{\tau}Z(\tau)$ in (3.6) and therefore the 'de-trended' tests $\mathcal{W}_n, \mathcal{I}_n, \mathcal{R}_n$ have different critical regions from those in (3.13). We also note that $\{\mathcal{Z}(u,\tau)\}$ can be heuristically defined as the residual process $\{\mathcal{Z}(u,\tau) = Z(u) - \hat{a}_\tau u - \frac{\hat{b}_\tau}{2}u^2\}$ from the least squares regression of $(\mathrm{d}Z(u)/\mathrm{d}u)_{u\in[0,\tau]}$ onto $(a+bu)_{u\in[0,\tau]}$, with $(\hat{a}_\tau, \hat{b}_\tau)$ minimizing the integral $\int_0^\tau (\frac{\mathrm{d}Z(u)}{\mathrm{d}u} - a - bu)^2\mathrm{d}u$. Indeed, the above minimization problem leads to linear equations $\int_0^\tau (\frac{\mathrm{d}Z(u)}{\mathrm{d}u} - a - bu)\mathrm{d}u = 0$, $\int_0^\tau (\frac{\mathrm{d}Z(u)}{\mathrm{d}u} - a - bu)u\mathrm{d}u = 0$, or

$$Z(\tau) = a\tau + b\frac{\tau^2}{2}, \qquad \tau Z(\tau) - \int_0^\tau Z(u)\mathrm{d}u \;=\; a\frac{\tau^2}{2} + b\frac{\tau^3}{3},$$

where we used $\int_0^\tau u\frac{\mathrm{d}Z(u)}{\mathrm{d}u}\mathrm{d}u = \int_0^\tau u\mathrm{d}Z(u) = \tau Z(\tau) - \int_0^\tau Z(u)\mathrm{d}u$. Solving the above equations leads to the same $\hat{a}_\tau = -\frac{2}{\tau}Z(\tau) + \frac{6}{\tau^2}\int_0^\tau Z(u)\mathrm{d}u$, $\hat{b}_\tau = \frac{6}{\tau^2}Z(\tau) - \frac{12}{\tau^3}\int_0^\tau Z(u)\mathrm{d}u$ as in (8.18) below. The resulting expression of $\mathcal{Z}(u,\tau) = Z(u) - \hat{a}_\tau u - \frac{\hat{b}_\tau}{2}u^2$ agrees with (6.7).

7 Simulation study

In this section we compare from numerical experiments the finite-sample performance of the three test statistics in (3.13) for testing $\mathbf{H_0[I]}$ against $\mathbf{H_1[I]}$ with nominal level $\alpha = 5\%$. A comparison with the Kim's tests based on the ratio (1.1), is also provided.

The main steps to implement the testing procedures defined in (3.13) are the following:

- We choose $\overline{\tau} = 1 - \underline{\tau}$ for $\underline{\tau} \in (0,1)$ which defines the testing region $\mathcal{T} := [\underline{\tau}, 1-\underline{\tau}]$. Sensitivity to the choice of $\underline{\tau}$ is also explored;
- For each simulated sample $X_1, \ldots, X_n$, we estimate the parameter d using the NELWE of Abadir et al. (2007) as the estimate of d. Following the recommendation in Abadir et al. (2007), the bandwidth parameter in the above estimate is chosen to be $\lfloor\sqrt{n}\rfloor$;
- The quantiles $q_T^{[\mathrm{I}]}(.05, d)$ in the critical regions (3.13), as functions of d, for $T = W, R, I$, and for chosen values of $\underline{\tau}$, are approximated by extensive Monte Carlo experiments. The integral appearing in the definition of $T = I$ in (3.9) is approximated by a Riemann sum. See also Hassler and Scheithauer (2008) on approximation of similar quantiles. The quantile graph for $T = I$ and three different values of $\underline{\tau}$ is shown in Figure 2.

7.1 Empirical comparisons of the tests I_n, R_n, W_n, I_n^{Kim}, R_n^{Kim}, and W_n^{Kim}

In this section we compare the test procedures based on our statistics I_n, R_n, W_n and the corresponding Kim's statistics

$$I_n^{Kim} := \int_{\mathcal{T}} \mathcal{K}_n(\tau)\mathrm{d}\tau, \quad W_n^{Kim} := \sup_{\tau\in\mathcal{T}} \mathcal{K}_n(\tau) \quad \text{and} \quad R_n^{Kim} := \frac{\inf_{\tau\in\mathcal{T}} U^*_{n-\lfloor n\tau\rfloor}(X)}{\inf_{\tau\in\mathcal{T}} U_{\lfloor n\tau\rfloor}(X)}, \tag{7.1}$$

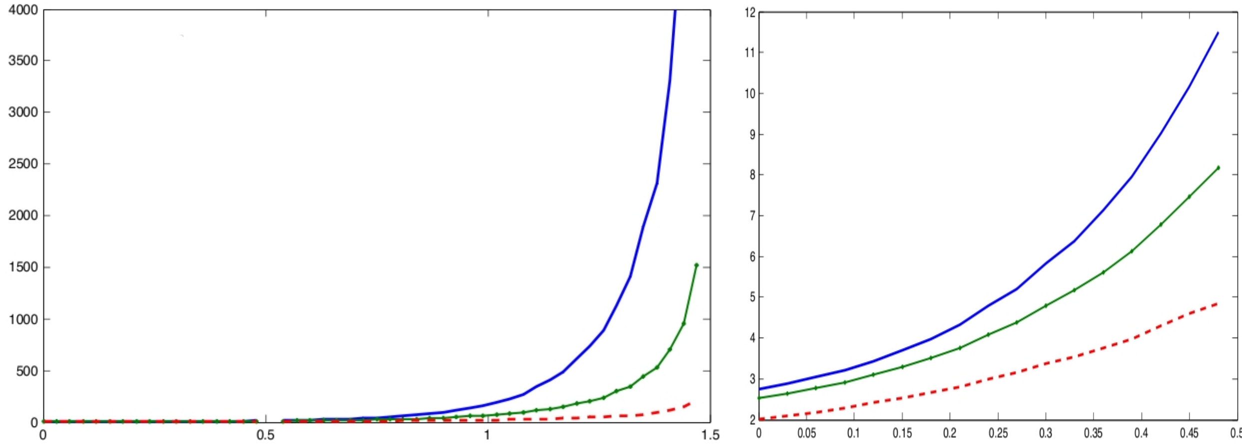

Figure 2: The upper 5% quantile of $I(B^{\mathrm{I}}_{d+.5})$ as a function of d, for $\underline{\tau} = .05$ (solid graph); $\underline{\tau} = .1$ (solid graph with points); $\underline{\tau} = .2$ (dashed graph). The right plot is a zoom in of the left plot in the region $d \in [0, .5)$.

where $\mathcal{K}_n(\tau)$, $U_k(X)$ and $U^*_{n-k}(X)$ are defined in (1.1) and (1.2).

The empirical size of the above tests is evaluated by simulating the FARIMA(0,d,0) process of Example 2.1 for $d = 0,\ 0.1,\ 0.2,\ 0.3,\ 0.4$. The empirical power is estimated by simulating the FARIMA process of Example 2.1, (2.4) with of $d_1, d_2 \in \{0,\ 0.1,\ 0.2,\ 0.3,\ 0.4\}$ and the change-point of d in the middle of the sample ($\theta^* = 0.5$).

Tables 1 and 2 display the estimated level and power based on 10^4 replications of the testing procedures, for respective sample sizes $n = 500$ and $n = 5000$. These results show that for all six statistics and three values of $\underline{\tau} = 0.05, 0.1, 0.2$, the estimated level is close to the nominal level 5%. We also observe that while the performance of the tests I_n and I_n^{Kim} does not much depend on $\underline{\tau}$, the last property is not shared by W_n^{Kim}, R_n^{Kim}, W_n and R_n .

Tables 1 and 2 suggest that when $\underline{\tau}$ is small, I_n clearly outperforms the remaining five tests. As $\underline{\tau}$ increases, the performance of R_n becomes comparable to that of I_n, while W_n, I_n^{Kim}, W_n^{Kim} and R_n^{Kim} still remain less efficient. Clearly, it make sense to choose $\underline{\tau}$ as small as possible, since none of these tests can detect a change-point that occurs outside the testing interval $\mathcal{T} = [\underline{\tau}, 1 - \underline{\tau}]$.

In conclusion, given the choice $\underline{\tau} = 0.05$ (or $\underline{\tau} = 0.1$), the statistic I_n seems preferable to W_n and R_n and the three Kim's statistics I_n^{Kim}, R_n^{Kim}, and W_n^{Kim}.

7.2 Further simulation results pertaining to the test I_n

As noted above, the results in Subsection 7.1 suggest that I_n is favorable among the six tests in the case when the observations follow a 'pure' FARIMA(0,d,0) model with a possible jump of d for $d \in [0, .5)$. Here, we explore the performance of I_n when the observations are simulated from other classes of processes following the hypotheses $\mathbf{H_0}[\mathbf{I}]$ and $\mathbf{H_1}[\mathbf{I}]$.

Table 3 extends the results of Tables 1 and 2 to a larger interval of d values, viz., $0 < d_1 \le d_2 < 1.5$. Recall that, in accordance with Definition 5.1 (iii), for $d > .5$ the FARIMA$(0, d, 0)$ process is defined by

$$X_t = \sum_{i=1}^{t} Y_i, \qquad t = 1, \dots, n, \tag{7.2}$$

where $\{Y_t\}$ is a stationary FARIMA$(0, d-1, 0)$.

Figure 3 shows some trajectories simulated from model (2.4) with fixed $d_2 - d_1 = 0.3$ and three different values of d_1, with the change-point in the middle of the sample. From visual inspection of these paths, it seems that it is more difficult to detect a change in the memory parameter when $0 \le d_1 < d_2 < .5$ or $.5 < d_1 < d_2$ (top or bottom graphs) than when $d_1 < .5 < d_2$ (middle graph). Note that the top and bottom graphs of Figure 3 correspond to d_1, d_2 belonging to the same 'stationarity interval' (either $[0, .5)$ or $(.5, 1.5)$) and the middle graph to d_1, d_2 falling into different 'stationarity intervals' $[0, .5)$ and $(.5, 1.5)$. The above visual observation is indeed confirmed in Table 3. The last table also shows that when the difference $d_2 - d_1$ is fixed, the test I_n is more powerful in the region $0 \le d_1 < d_2 < .5$ than in $.5 < d_1 < d_2$.

Tables 4 and 5 illustrate the performance of I_n when a positive, resp. negative, autoregressive part is added to the fractional process $\{\varepsilon_t(d_i)\}$ $(i = 1, 2)$ in the model (2.4). These tables show that the performance of the test is essentially preserved, especially when the autoregressive coefficient is positive. However in the case of negative autoregressive coefficient the estimated level is slightly more disturbed (Table 5). Tables 3, 4 and 5 also confirm that I_n is not very sensitive to the choice of the parameter $\underline{\tau}$, i.e. to the length of testing interval.

Finally, we assess the power of the test for fractionally integrated models with changing memory parameters discussed in Section 5.3. Figure 4 presents sample paths of the *rapidly changing memory* model in (5.13) and the *gradually changing memory* model in (5.17), for the same function $d(t/n) = .2 + .6\, t/n$ with the middle point $t = \lfloor n/2 \rfloor$ marking the transition from 'stationarity regime' $d \in [0, .5)$ to 'nonstationarity regime' $d \in (.5, 1)$. The visual impression from Figure 4 is that the above transition is much easier to detect for the rapidly changing memory model than for the gradually changing memory model.

Table 6 displays the estimated power of the test I_n for the rapidly changing memory model (5.13) when $d(\tau) = d_1 + (d_2 - d_1)\tau$, $\underline{\theta} = 0$ and $\overline{\theta} = 1$. The null hypothesis is naturally less often rejected for this model than for the model defined in (2.4), cf. Table 3. However, the estimated power still seems to be satisfactory. Similar simulations under gradually changing memory model (not included in this paper) show that the test has more difficulty to detect this type of changing memory on small samples. However, when n is larger than 500, the difference in the estimated power between the gradually and rapidly memory cases becomes negligible.

7.3 Simulations in the presence of linear trend

In this section we illustrate the performances of the test based on the de-trended statistic $\mathcal{I}_n$ defined in (6.5). This testing procedure is implemented similarly to the previous one. Note that the critical region still depends on the memory parameter d which is estimated as follows: having observed $X_1, \dots, X_n$ we estimate d using NELWE estimate on the residuals from the least-squares regression of $(X_j)_{1 \le j \le n}$ on $(a + bj)_{1 \le j \le n}$.

First we apply the de-trended test on series without trend, namely from model (2.4). Table 7 displays the estimated level and power of the test for this model. The estimated level is close to the nominal level. Moreover the power is close to that obtained in Table 3. Therefore the performances of the testing procedure are preserved even if the estimation of the linear trend was not necessary.

Second we assess the power of this test in presence of a linear trend (see Table 8). Figure 5 presents sample paths of models defined in (6.3) with $a = 1$, $b = .01$, $\theta^* = 1/2$, $n = 500$ and different values of d_1, d_2. For this model, Table 8 summarizes the estimated level and power, which are similar to Table 7.

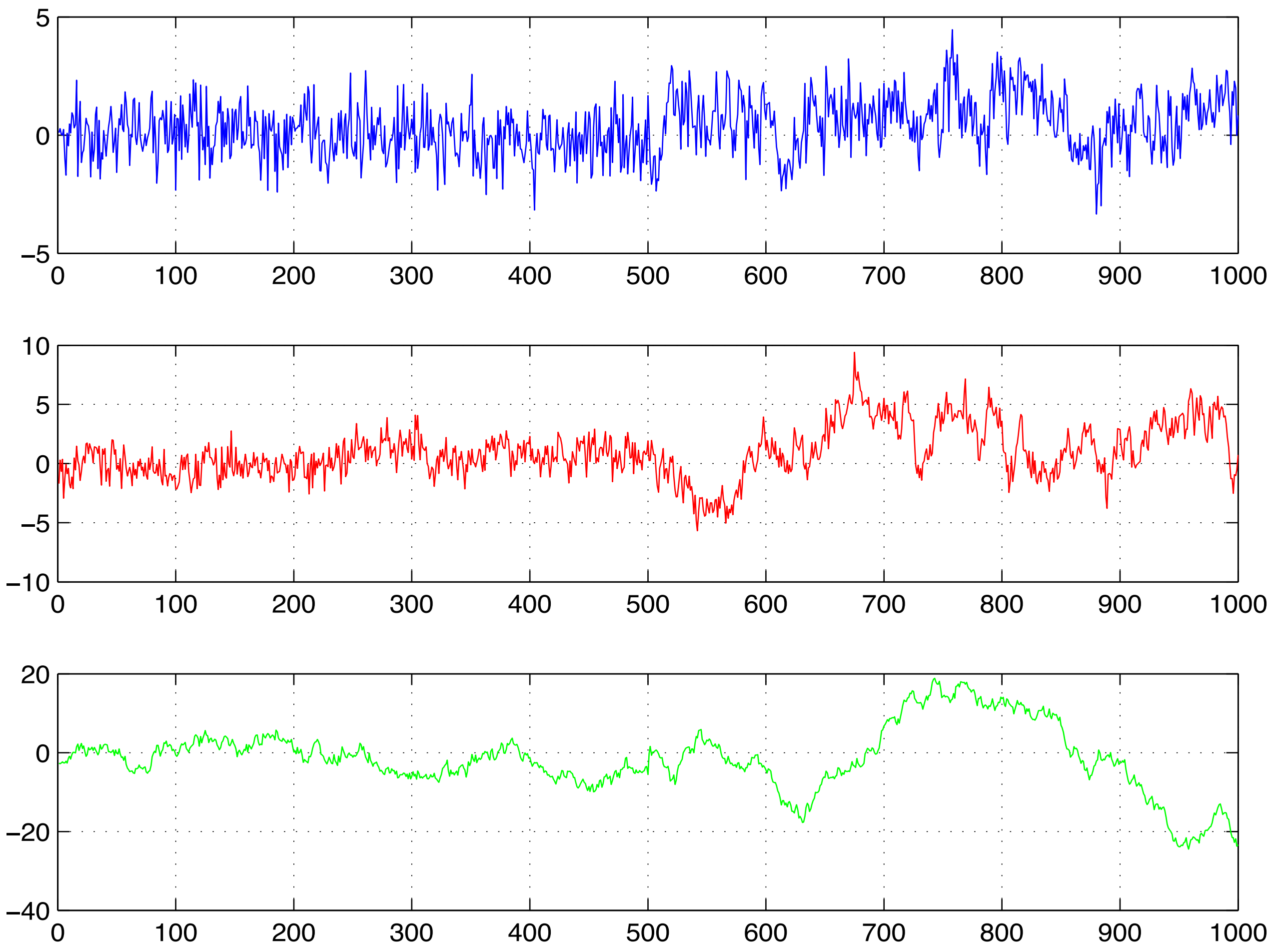

Figure 3: Sample paths simulated from model (2.4) with $\theta^* = 1/2$ for different values of d_1 and d_2: $d_1 = .1$, $d_2 = .4$ (top); $d_1 = .3$, $d_2 = .6$ (middle); $d_1 = .8$, $d_2 = 1.1$ (bottom). The sample size is $n = 1000$.

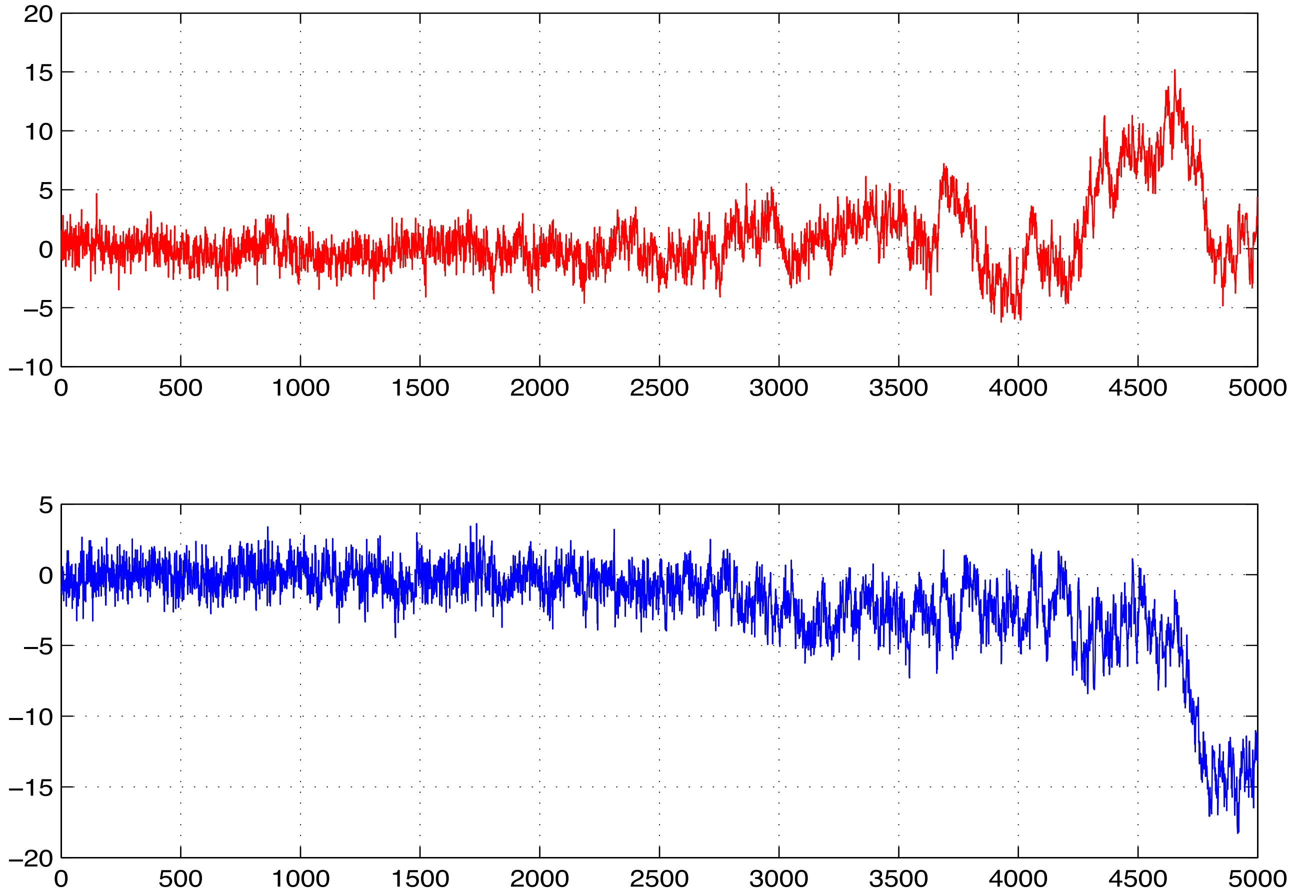

Figure 4: Sample paths simulated from models (5.13) (top) and (5.17) (bottom) with $d(\tau) = .2 + .6\tau$, $\underline{\theta} = 0$ and $\overline{\theta} = 1$.

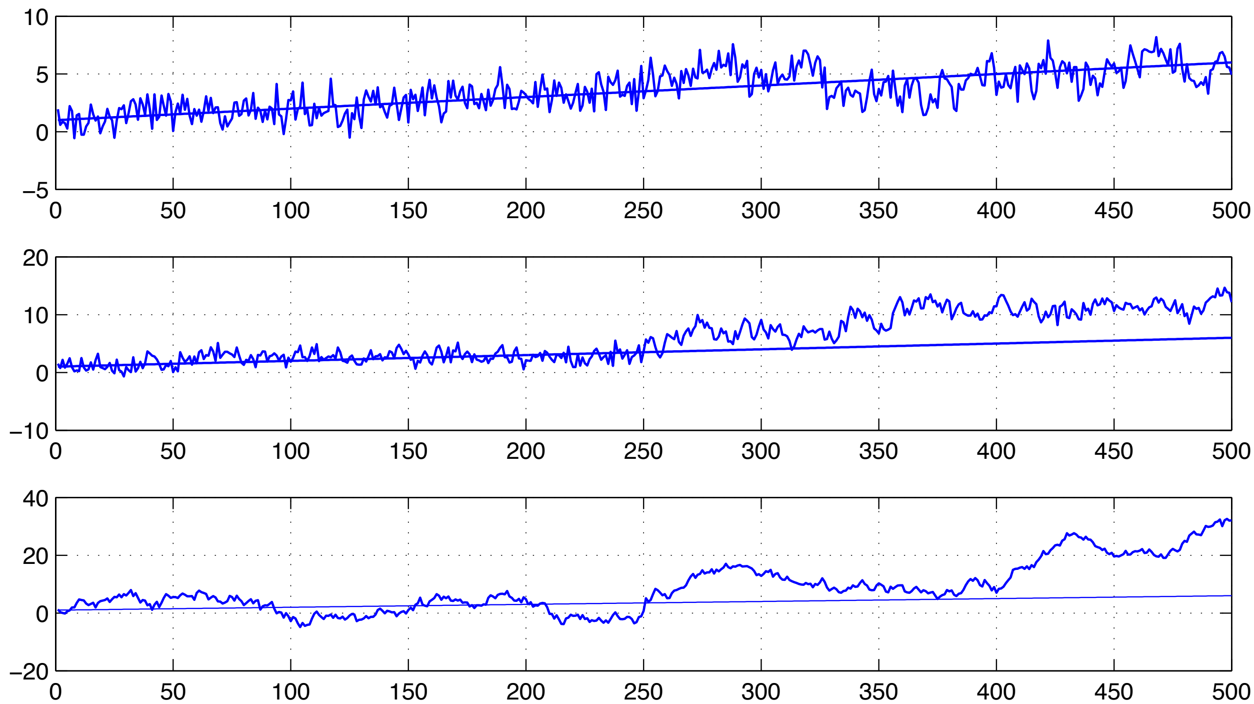

Figure 5: Comparison of trajectories simulated from the model (6.3) with $a = 1$, $b = .01$, change point $\theta^* = 1/2$ and different values of d_1 and d_2: $d_1 = .1$, $d_2 = .4$ (top); $d_1 = .3$, $d_2 = .6$ (middle); $d_1 = .8$, $d_2 = 1.1$ (bottom). The sample size is $n = 500$.

	$\tau = 0.05$					$\tau = 0.1$					$\tau = 0.2$				
d_2 \ d_1	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
W_n statistic															
0	3.6					2.4					2.3				
0.1	10.0	4.4				6.3	2.5				10.4	2.6			
0.2	19.0	10.2	4.4			11.8	5.1	2.8			25.2	9.9	2.7		
0.3	25.9	16.9	9.5	4.9		15.9	9.0	4.7	3.1		45.1	23.3	9.4	2.9	
0.4	31.0	21.6	13.1	8.3	4.8	19.9	12.2	6.8	4.5	2.8	65.1	42	21.0	8.7	2.6
R_n statistic															
0	3.4					2.6					2.5				
0.1	11.2	3.9				8.0	2.9				11.6	2.9			
0.2	24.5	12.0	4.6			17.0	7.2	3.1			28.5	11.3	3.2		
0.3	37.6	23.1	11.5	4.9		27.2	14.7	6.6	3.4		50.6	27.3	11.1	3.4	
0.4	49.5	34.0	19.8	10.6	5.0	41.1	23.7	12.4	6.7	3.0	72.0	49.0	25.5	10.5	3.4
I_n statistic															
0	2.3					2.5					2.9				
0.1	10.4	2.6				11.6	2.9				13.2	3.6			
0.2	25.2	9.9	2.7			28.5	11.3	3.2			31.8	13.4	3.8		
0.3	45.1	23.3	9.4	2.9		50.6	27.3	11.1	3.4		55.0	31.0	13.0	4.0	
0.4	65.1	42.0	21.0	8.7	2.6	72.0	49.0	25.5	10.5	3.4	77.1	54.3	29.6	12.4	4.1
W_n^{Kim} statistic															
0	3.1					3.3					3.3				
0.1	6.6	3.2				8.3	3.7				10.3	4.2			
0.2	10.7	6.2	3.3			14.6	7.4	3.6			20.2	9.5	4.4		
0.3	16.5	9.0	5.3	3.1		25.2	13.6	7.0	3.8		36.6	20	9.4	4.4	
0.4	24.4	15.5	9.5	5.7	3.4	41.7	26.4	15.7	7.7	4.2	58.3	37.9	21.7	10.0	4.8
R_n^{Kim} statistic															
0	3.7					3.3					3.3				
0.1	9.4	4.3				9.9	4.4				10.7	4.3			
0.2	17.1	9.5	4.7			21.1	11.1	4.7			24.1	11.6	4.5		
0.3	25.8	15.3	8.8	4.6		34.9	20.6	10.6	5.1		42.4	24.2	11.7	4.8	
0.4	31.4	22.1	13.6	8.4	4.9	47.4	32.3	19.1	10.3	5.0	58.8	40.2	22.5	11.7	4.9
I_n^{Kim} statistic															
0	2.6					2.8					3.1				
0.1	8.6	3.2				9.5	3.5				10.5	4.0			
0.2	18.7	8.3	3.3			20.8	9.1	3.5			23.1	10.4	4.0		
0.3	35.0	18.3	8.4	3.1		38.9	21.0	9.3	3.4		42.0	23.4	10.6	4.0	
0.4	57.0	37.3	20.5	9.0	3.9	62.3	41.3	23.5	10.2	4.2	64.8	43.8	24.5	11.3	4.4

Table 1: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the tests I_n, W_n, R_n, I_n^{Kim}, W_n^{Kim}, R_n^{Kim}. The nominal level is $\alpha = 5\%$. The samples are simulated from model (2.4) with $\theta^* = 1/2$. The sample size is 500 and the number of independent replications is 10^4.

	$\tau = 0.05$					$\tau = 0.1$					$\tau = 0.2$				
d_2 \ d_1	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4	0	0.1	0.2	0.3	0.4
W_n statistic															
0	3.4					3.6					4.1				
0.1	20.2	3.7				24.2	4.1				28.3	4.5			
0.2	50.1	16.0	3.8			61.1	20.9	4.5			68.6	26.0	4.7		
0.3	74.6	38.5	13.3	3.6		87.1	51.1	17.7	4.1		93.1	62.0	23.6	4.7	
0.4	88.7	64.1	31.2	11.6	3.7	96.6	79.6	44.3	15.0	4.0	99.2	88.8	56.7	20.0	4.4
R_n statistic															
0	3.7					3.9					4.1				
0.1	29.0	5.1				30.9	4.9				31.3	5.0			
0.2	65.0	25.3	4.9			71.0	28.1	4.8			73.1	30.5	4.6		
0.3	85.3	53.8	21.2	5.0		91.7	62.2	24.4	4.9		94.3	67.0	26.5	4.9	
0.4	94.1	75.7	44.8	18.4	4.8	98.0	84.8	54.1	21.7	4.6	99.3	90.3	60.3	24.4	4.5
I_n statistic															
0	2.9					3.2					3.6				
0.1	28.5	3.4				29.5	3.8				30.7	4.1			
0.2	73.4	27.8	3.6			74.0	28.8	3.9			73.7	30.1	4.2		
0.3	95.9	68.0	24.8	3.5		96.3	69.8	26.5	3.7		95.9	69.7	27.9	4.3	
0.4	99.5	92.3	62.5	21.8	3.5	99.7	93.6	65.7	24.1	3.8	99.7	94.1	66.6	25.6	4.0
W_n^{Kim} statistic															
0	5.0					4.9					4.8				
0.1	20.3	5.1				22.0	5.1				23.8	5.2			
0.2	44.9	15.7	4.7			51.1	18.1	5.1			55.2	21.2	5.5		
0.3	68.8	37.2	13.4	4.6		77.6	45.7	16.6	5.0		83.1	52.1	19.5	5.3	
0.4	83.7	60.1	32.4	12.8	4.2	92.2	72.4	41.4	14.9	4.5	96.0	79.7	48.1	18.2	4.6
R_n^{Kim} statistic															
0	3.9					4.2					4.5				
0.1	26.5	4.8				26.5	4.6				26.6	5.1			
0.2	59.6	23.1	4.9			63.2	24.9	5.1			62.7	24.6	5.1		
0.3	81.6	49.9	19.6	5.4		87.0	56.4	22.4	5.2		87.4	57.4	22.6	5.0	
0.4	91.9	72.1	42.0	17.6	4.9	96.5	80.8	50.1	20.4	4.9	97.2	82.6	52.5	20.1	4.5
I_n^{Kim} statistic															
0	3.5					3.7					4.2				
0.1	23.5	3.6				24.1	4.0				24.6	4.3			
0.2	58.4	21.4	4.3			58.5	22.0	4.4			57.4	22.4	4.7		
0.3	86.1	54.8	19.9	4.0		86.8	55.8	20.5	4.2		86.1	55.4	20.8	4.5	
0.4	96.6	82.2	52.1	18.4	4.0	97.3	83.6	53.6	19.2	3.7	97.3	83.4	53.4	19.7	4.0

Table 2: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the tests I_n, W_n, R_n, I_n^{Kim}, W_n^{Kim}, R_n^{Kim}. The nominal level is $\alpha = 5\%$. The samples are simulated from model (2.4) with $\theta^* = 1/2$. The sample size is 5000 and the number of independent replications is 10^4.

τ	d_2 \ d_1	$n=500$: 0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4	$n=5000$: 0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	2.3										2.9									
	0.1	10.4	2.6									28.5	3.4								
	0.2	25.2	9.9	2.7								73.4	27.8	3.6							
	0.3	45.1	23.3	9.4	2.9							95.9	68.0	24.8	3.5						
	0.4	65.1	42.0	21.0	8.7	2.6						99.5	92.3	62.5	21.8	3.5					
	0.6	92.7	85.4	71.8	54.5	35.2	4.9					100	99.9	99.7	97.1	88.0	5.3				
	0.8	94.7	88.8	78.8	63.1	45.2	7.8	6.2				100	100	99.8	98.7	92.6	11.3	5.0			
	1	97.2	94.4	89.2	80.8	67.4	19.8	12.5	6.7			100	100	100	99.9	98.9	45.3	28.1	5.1		
	1.2	98.6	97.6	95.5	91.7	86.9	55.5	44.9	23.5	7.2		100	100	100	100	99.9	87.4	78.6	41.4	5.3	
	1.4	99.5	99.1	98.1	97.4	95.4	84.3	78.8	67.6	44.2	4.9	100	100	100	100	100	98.0	96.4	85.9	59.0	3.3
0.1	0	2.5										3.2									
	0.1	11.6	2.9									29.5	3.8								
	0.2	28.5	11.3	3.2								74.0	28.8	3.9							
	0.3	50.6	27.3	11.1	3.4							96.3	69.8	26.5	3.7						
	0.4	72.0	49.0	25.5	10.5	3.4						99.7	93.6	65.7	24.1	3.8					
	0.6	97.9	94.0	84.5	68.1	47.3	4.9					100	100	99.9	98.7	92.5	5.2				
	0.8	98.8	96.3	90.9	78.4	59.8	9.4	5.9				100	100	99.9	99.4	96.0	11.8	4.9			
	1	99.4	98.8	96.7	92.3	82.4	27.5	16.8	6.2			100	100	100	100	99.7	51.9	34.0	5.1		
	1.2	99.8	99.6	99.0	97.6	95.0	65.1	52.5	26.0	6.3		100	100	100	100	100	91.7	84.2	44.6	5.1	
	1.4	100	99.8	99.7	99.3	98.8	89.7	84.2	70.3	42.4	3.7	100	100	100	100	100	99.1	98.2	88.9	57.8	2.9
0.2	0	2.9										3.6									
	0.1	13.2	3.6									30.7	4.1								
	0.2	31.8	13.4	3.8								73.7	30.1	4.2							
	0.3	55.0	31.0	13	4.0							95.9	69.7	27.9	4.3						
	0.4	77.1	54.3	29.6	12.4	4.1						99.7	94.1	66.6	25.6	4.0					
	0.6	99.3	97.6	91.4	78.3	57.2	5.2					100	100	99.9	99.4	95.4	5.1				
	0.8	99.7	98.9	95.8	87.7	71.8	10.6	5.4				100	100	100	99.8	98.1	11.6	4.9			
	1	99.9	99.8	99.2	97.3	91.7	34.7	21.0	5.8			100	100	100	100	99.9	57.4	39.4	4.6		
	1.2	100	99.9	99.8	99.6	98.7	74.1	60.5	28.6	5.2		100	100	100	100	100	94.8	89.5	48.4	4.9	
	1.4	100	100	99.9	99.9	99.8	94.5	89.6	73.4	41.4	2.7	100	100	100	100	100	99.7	99.3	91.8	57.8	2.7

Table 3: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test I_n with nominal level $\alpha = 5\%$. The samples are simulated from model (2.4) with $\theta^* = 1/2$. The sample sizes are $n = 500$ and $n = 5000$, the number of independent replications is 10^4.

τ	d_2 \ d_1	$n = 500$										$n = 5000$									
		0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	2.6										3.0									
	0.1	10.7	3									28.6	3.6								
	0.2	27.2	10.8	3.4								73.4	26.9	3.6							
	0.3	47.3	24.0	9.1	2.9							95.2	68.3	23.7	3.7						
	0.4	66.4	42.6	22.1	8.1	2.9						99.4	91.5	63.5	21.5	3.4					
	0.6	92.6	84.4	70.6	53.7	35.0	5.0					100	100	99.6	97.4	87.5	5.2				
	0.8	94.6	89.2	79.1	63.5	43.9	8.3	5.9				100	99.9	99.9	98.7	92.2	11.3	5.4			
	1	97.3	94.5	89.0	80.7	67.3	20.2	13.1	6.9			100	100	99.9	99.9	99.0	44.8	28.1	5.5		
	1.2	98.5	97.7	95.4	92.0	86.8	55.4	44.3	23.2	7.1		100	100	100	99.9	99.9	87.3	78.0	43.0	5.7	
	1.4	99.5	99.1	98.4	97.0	95.1	83.4	78.9	67.6	44.7	4.8	100	100	100	100	100	98.2	96.3	84.7	58.9	3.1
0.1	0	2.9										3.3									
	0.1	11.9	3.3									29.5	4.0								
	0.2	29.7	12.2	3.8								73.6	27.8	3.8							
	0.3	52.2	27.4	10.7	3.3							95.7	70.0	24.9	4.0						
	0.4	72.7	48.5	26.1	9.9	3.3						99.6	92.9	66.2	23.5	4.0					
	0.6	97.7	92.9	83.6	67.7	47.2	5.0					100	100	99.9	98.8	92.1	5.1				
	0.8	98.9	96.5	90.6	78.0	58.0	9.4	5.6				100	99.9	99.9	99.5	96.1	11.9	5.2			
	1	99.4	98.7	96.9	92.0	82.5	26.9	16.6	6.5			100	100	100	100	99.7	51.0	34.9	5.4		
	1.2	99.8	99.6	98.9	97.5	94.8	65.4	52.1	25.7	6.2		100	100	100	100	99.9	91.6	84.0	46.0	5.4	
	1.4	99.9	99.9	99.7	99.3	98.6	89.3	84.2	70.1	43.2	3.9	100	100	100	100	100	99.1	98.2	87.9	57.8	2.8
0.2	0	3.2										3.7									
	0.1	13.5	4.0									30.6	4.5								
	0.2	32.8	13.8	4.3								73.4	29.0	4.3							
	0.3	56.5	30.8	12.6	4.0							95.4	70.1	26.6	4.3						
	0.4	77.4	53.6	29.5	12.2	4.2						99.7	93.2	67.4	25.1	4.3					
	0.6	99.4	96.9	90.6	77.6	57.0	5.1					100	100	99.9	99.4	94.6	5.0				
	0.8	99.8	98.9	96	87.4	70.0	10.4	5.3				100	100	100	99.8	98.0	11.7	5.1			
	1	99.9	99.8	99.3	97.0	91.8	33.6	20.3	5.8			100	100	100	100	99.9	56.6	40.5	5.0		
	1.2	99.9	99.9	99.9	99.6	98.6	74.8	60.1	28.6	5.8		100	100	100	100	100	94.7	89.4	49.3	4.9	
	1.4	100	100	100	99.9	99.7	94.1	89.3	73.2	41.4	3.1	100	100	100	100	100	99.7	99.4	91.2	56.9	2.2

Table 4: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test I_n with nominal level $\alpha = 5\%$. The samples are simulated from (2.4) with the FARIMA(0,d,0) processes replaced by FARIMA$(1, d, 0)$ models with the AR parameter .7 and $\theta^* = 1/2$. The sample sizes are $n = 500$ and $n = 5000$, the number of independent replications is 10^4.

		$n = 500$										$n = 5000$									
τ	d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	0.3										1.8									
	0.1	2.4	0.6									24.3	2.7								
	0.2	9.0	3.5	1.1								70.2	25.3	3.1							
	0.3	23.3	11.6	4.7	1.3							94.6	66.5	22.7	3.3						
	0.4	43.5	27.2	13.5	5.1	1.6						99.5	92.7	63.1	22.5	3.7					
	0.6	92.9	87.6	77.4	64.2	46.5	3.5					99.9	100	99.7	97.9	89.3	6.5				
	0.8	94.3	90.5	84.1	72.0	55.3	11.8	6.5				100	99.9	99.8	98.9	93.0	12.6	5.3			
	1	96.7	94.9	91.2	84.4	74.3	25.9	16.8	8.0			100	100	99.9	99.8	99.1	46.3	29.2	5.8		
	1.2	98.7	98.1	96.6	94.4	89.8	61.1	51.4	29.6	9.0		100	100	100	100	99.9	87.8	78.8	42.8	6.0	
	1.4	99.5	99.2	98.6	98.1	96.6	87.2	83.0	71.6	51.1	7.5	100	100	100	100	100	98.1	96.5	86.5	59.9	3.7
0.1	0	0.1										2.0									
	0.1	2.6	0.7									25.9	3.2								
	0.2	11.0	4.5	1.4								72.1	27.5	3.5							
	0.3	29.4	15.2	6.3	1.8							95.4	69.1	24.9	3.9						
	0.4	55.0	37.2	19.2	7.8	2.5						99.6	94.2	66.3	25.0	3.9					
	0.6	98.0	95.0	88.1	77.0	58.7	5.0					100	100	99.9	99.1	93.5	5.7				
	0.8	98.9	97.0	93.8	85.1	69.8	14.9	7.5				100	100	99.9	99.6	96.1	12.8	5.2			
	1	99.5	99.1	97.8	94.8	88.5	35.9	22.6	8.4			100	100	100	99.9	99.8	52.8	35.1	5.6		
	1.2	99.9	99.6	99.4	98.6	96.8	72.7	60.7	32.7	8.8		100	100	100	100	100	92.2	84.4	45.5	5.4	
	1.4	99.9	99.9	99.8	99.6	99.3	92.9	88.2	74.9	49.6	6.1	100	100	100	100	100	99.3	98.2	89.2	58.3	3.1
0.2	0	0.2										2.5									
	0.1	3.7	1.0									27.8	3.9								
	0.2	15.2	6.4	2.1								72.9	29.2	4.1							
	0.3	37.8	20.8	9.4	2.8							95.4	70.4	26.6	4.5						
	0.4	65.4	46.0	26.2	11.3	4.3						99.7	94.5	67.6	26.4	4.3					
	0.6	99.4	98.0	93.4	84.0	66.3	5.9					100	100	100	99.6	95.7	5.4				
	0.8	99.8	99.4	97.4	91.5	78.7	15.2	6.9				100	100	99.9	99.9	97.9	12.1	5.5			
	1	99.9	99.9	99.6	98.6	95.0	42.2	26.0	7.3			100	100	100	100	99.9	57.2	41.3	5.3		
	1.2	100	99.9	99.9	99.8	99.2	80.0	67.4	33.8	7.5		100	100	100	100	100	95.3	89.9	49.3	4.8	
	1.4	100	100	99.9	99.9	99.9	96.2	92.4	77.3	47.2	4.3	100	100	100	100	100	99.7	99.3	92.4	57.7	2.8

Table 5: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test I_n with nominal level $\alpha = 5\%$. The samples are simulated from (2.4) with the FARIMA(0,d,0) processes replaced by FARIMA(1,d,0) models with the AR parameter $-.7$ and $\theta^* = 1/2$. The sample sizes are $n = 500$ and $n = 5000$, the number of independent replications is 10^4.

$\underline{\tau}$	d_2 \ d_1	$n = 500$										$n = 5000$									
		0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	2.4										2.8									
	0.1	6.4	2.7									16.1	4.2								
	0.2	13.9	6.8	3.1								42.3	15.2	3.4							
	0.3	23.3	12.2	6.3	3.0							69.9	36.6	13.1	3.5						
	0.4	33.1	20.4	10.7	5.2	2.9						86.8	61.2	31.8	11.2	3.2					
	0.6	57.2	40.5	27.2	16.6	10.0	4.2					97.8	91.1	74.6	49.7	26.5	4.9				
	0.8	71.6	57.3	43.4	30.1	19.9	7.8	5.9				99.3	97.8	91.9	79.6	59.1	17.1	5.0			
	1	76.1	64.7	51.9	38.3	26.9	11.4	6.0	6.6			99.8	99.1	97.4	92.1	82.8	41.6	11.1	5.3		
	1.2	76.2	68.5	57	45.2	32.6	14.4	6.0	5.0	6.5		99.8	99.5	98.9	96.5	91.5	66.1	26.1	6.7	5.2	
	1.4	74.1	67.2	57.4	47.1	36.2	17.0	6.9	3.3	3.4	3.4	99.8	99.6	99.2	97.7	94.5	78.3	40.4	11.2	3.3	2.3
0.1	0	2.8										3.1									
	0.1	7.1	3.1									16.3	4.2								
	0.2	15.3	7.7	3.6								42.3	15.7	3.8							
	0.3	25.7	14.0	7.1	3.2							69.6	36.5	13.8	3.9						
	0.4	37.0	23.5	12.5	6.2	3.3						86.7	61.8	32.8	11.8	3.3					
	0.6	63.8	46.6	31.8	19.9	11.9	4.5					98.3	92.2	75.8	51.2	27.3	4.7				
	0.8	80.5	66.9	52.0	36.5	24.3	9.5	5.7				99.8	98.8	94.0	82.7	62.3	18.5	4.9			
	1	87.7	78.7	65.4	50.8	36.4	15.2	7.4	6.2			99.9	99.7	98.7	95.0	86.6	45.9	13.1	5.2		
	1.2	90.8	85.2	74.8	62.2	47.7	22.1	8.8	6.3	5.8		99.9	99.9	99.7	98.2	94.7	72.1	29.9	8.4	4.9	
	1.4	90.9	86.0	78.5	67.7	55.1	27.7	10.9	4.9	4.1	2.8	100	99.9	99.8	99.3	97.4	84.7	45.9	13.6	4.4	1.9
0.2	0	3.4										3.6									
	0.1	8.1	3.7									16.3	4.8								
	0.2	17.0	9.0	4.3								39.9	16.0	4.2							
	0.3	27.7	15.7	8.4	3.9							65.8	35.4	14.0	4.3						
	0.4	39.5	25.8	14.4	7.6	4.1						83.4	59.0	32.7	12.5	3.6					
	0.6	65.6	49.8	34.5	22.1	13.6	4.6					97.7	90.5	73.9	49.7	27.3	4.4				
	0.8	83.1	70.9	56.3	40.4	26.9	10.6	4.9				99.7	98.6	93.1	81.5	60.5	19.9	5.0			
	1	92.0	83.5	71.3	57.2	42.2	18.4	8.6	5.9			100	99.9	98.6	94.3	85.4	45.7	14.8	5.2		
	1.2	95.9	91.3	82.5	70.5	55.6	27.0	11.6	7.0	4.9		99.9	99.9	99.7	98.4	94.3	70.6	30.2	9.9	4.8	
	1.4	97.1	93.6	87.3	77.1	64.6	33.3	14.4	6.4	5.1	2.7	100	99.9	99.9	99.4	97.3	82.9	43.7	14.8	5.8	2.3

Table 6: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the test I_n with nominal level $\alpha = 5\%$. The samples are simulated from model (5.13) with $d(\tau) = d_1 + (d_2 - d_1)\tau$, $\underline{\theta} = 0$ and $\overline{\theta} = 1$. The sample sizes are $n = 500$ and $n = 5000$, the number of independent replications is 10^4.

τ	d_2 \ d_1	$n=500$: 0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4	$n=5000$: 0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	2.4										3.0									
	0.1	11.5	3.3									32.8	3.6								
	0.2	30.9	12.8	3.3								79.2	31.5	4.0							
	0.3	53.4	30.5	12.9	4.3							97.2	72.9	28.5	4.3						
	0.4	71.5	50.7	28.3	12.8	4.6						99.7	94.0	67.6	25.5	4.4					
	0.6	88.7	80.7	68.2	52.7	36.4	6.2					100	99.9	99.7	97.2	87.2	5.6				
	0.8	91.2	84.2	73.8	60.7	43.8	8.1	5.4				100	100	99.9	98.8	92.6	11.1	4.6			
	1	93.9	88.9	81.8	71.3	57.8	16.6	10.2	4.0			100	100	100	99.9	99.4	47.0	27.0	4.7		
	1.2	97.0	94.6	90.0	83.1	73.7	33.7	24.4	14.7	4.0		100	100	100	100	99.9	86.2	73.8	32.7	4.3	
	1.4	99.5	98.9	97.6	95.2	90.8	62.0	50.7	34.1	31.9	4.9	100	100	100	100	100	98.5	95.5	75.6	47.6	4.3
0.1	0	2.7										3.3									
	0.1	12.6	3.6									33.7	3.9								
	0.2	33.8	14.1	3.8								79.8	32.6	4.2							
	0.3	57.8	33.8	14.1	4.7							97.3	74.6	30.1	4.4						
	0.4	77.2	56.4	32.3	14.7	4.7						99.8	95.4	71.2	27.4	4.6					
	0.6	96.4	91.2	81.9	65.7	46.0	5.5					100	100	99.9	98.9	92.2	5.4				
	0.8	97.9	94.9	87.9	75.8	58.2	9.4	5.1				100	100	100	99.7	96.6	12.5	4.6			
	1	99.3	98.0	95.0	89.2	78.3	23.5	13.2	4.3			100	100	100	99.9	99.9	55.4	33.1	5.0		
	1.2	99.8	99.5	98.6	96.4	92.5	51.7	37.1	17.8	4.3		100	100	100	100	100	92.8	84.2	39.0	4.4	
	1.4	99.9	99.9	99.8	99.6	98.6	82.1	71.0	45.9	33.2	4.7	100	100	100	100	100	99.7	98.7	85.5	49.5	4.5
0.2	0	3.2										3.8									
	0.1	14.1	4.4									34.0	4.3								
	0.2	36.0	15.7	4.5								79.2	32.7	4.6							
	0.3	61.0	37.0	15.6	5.3							97.0	74.7	30.7	4.9						
	0.4	81.3	60.4	35.2	15.9	5.4						99.8	95.7	72.9	29.2	4.6					
	0.6	99.2	96.3	89.3	75.9	55.4	5.5					100	100	100	99.6	95.2	5.2				
	0.8	99.6	98.8	95.1	86.3	70.2	13.0	5.3				100	100	100	99.9	98.4	15.3	4.8			
	1	99.9	99.7	99.2	96.7	89.7	32.9	19.6	5.1			100	100	100	100	99.9	61.8	40.0	5.3		
	1.2	100	100	99.9	99.4	98.4	66.8	49.9	22.5	4.6		100	100	100	100	100	96.1	89.7	44.4	5.1	
	1.4	100	100	100	99.9	99.9	92.3	83.5	56.4	34.1	5.1	100	100	100	100	100	99.9	99.7	90.9	51.4	4.7

Table 7: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the de-trended test based on $\mathcal{I}_n$ with nominal level $\alpha = 5\%$. The samples are simulated from the model (2.4) with $\theta^* = 1/2$. The sample sizes are $n = 500$ and $n = 5000$, the number of independent replications is 10^4.

		$n = 500$										$n = 5000$									
τ	d_2 \ d_1	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4	0	0.1	0.2	0.3	0.4	0.6	0.8	1	1.2	1.4
0.05	0	2.3										3.1									
	0.1	11.4	3.0									33.0	4.0								
	0.2	30.1	12.2	3.7								79.1	30.7	4.5							
	0.3	53.6	30.1	13.3	4.2							96.9	73.7	28.6	4.6						
	0.4	71.9	49.6	28.3	13.0	4.8						99.7	94.1	67.4	25.4	4.6					
	0.6	89.1	80.0	68.4	53.2	35.9	6.1					100	99.9	99.7	97.5	86.8	5.5				
	0.8	91.1	83.9	73.6	59.8	44.6	7.5	5.8				100	100	99.9	99.1	93.1	11.0	4.5			
	1	93.8	89.4	82.0	70.5	57.5	16.3	10.5	4.0			100	100	100	99.9	99.2	46.4	26.0	4.9		
	1.2	97.3	94.6	90.6	83.0	74.5	34.1	24.3	14.5	4.0		100	100	100	100	99.9	87.0	73.7	32.6	4.4	
	1.4	99.6	99.0	97.8	94.7	90.6	62.6	50.9	34.5	32.6	4.9	100	100	100	100	100	98.3	95.7	76.0	47.1	4.2
0.1	0	2.7										3.4									
	0.1	12.8	3.3									34.0	4.3								
	0.2	32.7	13.5	4.0								79.7	32.1	4.7							
	0.3	58.2	33.3	14.8	4.6							97.4	75.5	29.7	4.6						
	0.4	78.0	55.8	32.4	14.5	4.9						99.9	95.4	70.9	27.5	4.7					
	0.6	96.7	91.1	81.4	65.7	45.5	5.5					100	100	99.9	99.1	91.7	5.3				
	0.8	97.8	94.7	88.1	75.6	58.6	8.8	5.3				100	100	100	99.7	96.5	12.5	4.6			
	1	99.2	98.0	95.3	89.0	78.1	23.5	13.6	4.2			100	100	100	100	99.8	54.8	33.2	4.7		
	1.2	99.8	99.5	98.7	96.9	92.2	51.6	36.5	18.5	4.1		100	100	100	100	100	93.2	83.9	38.7	4.4	
	1.4	100	99.9	99.9	99.5	98.8	82.2	71.3	46.7	33.6	4.8	100	100	100	100	100	99.7	98.6	85.2	49.3	4.6
0.2	0	3.2										3.9									
	0.1	14.1	4.1									34.4	4.8								
	0.2	35.5	15.0	4.8								78.9	32.7	4.9							
	0.3	61.3	36.0	16.2	5.2							97.3	75.3	30.9	4.7						
	0.4	81.9	59.6	35.6	15.7	5.3						99.8	95.9	72.3	28.8	4.6					
	0.6	99.1	96.3	89.6	76.1	55.4	5.3					100	100	99.9	99.4	94.6	5.4				
	0.8	99.6	98.6	95.1	86.2	70.8	12.5	5.2				100	100	99.9	99.9	98.0	15.6	4.5			
	1	99.9	99.8	99.0	96.6	89.5	33.0	19.6	5.1			100	100	100	100	99.9	60.7	41.1	4.8		
	1.2	100	99.9	99.9	99.7	98.4	67.1	49.2	22.8	4.6		100	100	100	100	100	96.4	89.6	44.7	4.6	
	1.4	100	100	100	99.9	99.9	92.0	83.7	56.4	34.5	5.6	100	100	100	100	100	99.9	99.6	90.9	51.7	4.8

Table 8: Estimated level ($d_1 = d_2$) and power ($d_1 \neq d_2$) (in %) of the de-trended test based on $\mathcal{I}_n$ with nominal level $\alpha = 5\%$. The samples are simulated from the model (6.3) with $a = 1$, $b = .01$ (for sample size $n = 500$) and $a = 1$, $b = .001$ (for sample size $n = 5000$). The number of independent replications is 10^4.

8 Appendix: proofs

Proof of Proposition 4.1. (i) Without loss of generality, we will assume that $A_{n1} = A_{n2} = 0$ in what follows. Write $z_{n1}(\tau) := \gamma_{n1}^{-1} S_{\lfloor n\tau \rfloor}$, $(n/\gamma_{n1}^2) V_{\lfloor n\tau \rfloor}(X) = \sum_{i=1}^{6} U_{ni}(\tau)$, where the terms

$$
\begin{aligned}
U_{n1}(\tau) &:= \big(n^2/\lfloor n\tau\rfloor^2\big)\int_0^{\lfloor n\tau\rfloor/n} z_{n1}^2(u)\mathrm{d}u,\\
U_{n2}(\tau) &:= -2\big(n^2/\lfloor n\tau\rfloor^2\big) z_{n1}(\tau)\int_0^{\lfloor n\tau\rfloor/n} \big(\lfloor nu\rfloor/\lfloor n\tau\rfloor\big) z_{n1}(u)\mathrm{d}u,\\
U_{n3}(\tau) &:= \big(n^2/\lfloor n\tau\rfloor^2\big) z_{n1}^2(\tau)\int_0^{\lfloor n\tau\rfloor/n} \big(\lfloor nu\rfloor/\lfloor n\tau\rfloor\big)^2\mathrm{d}u,\\
U_{n4}(\tau) &:= -\big(n^3/\lfloor n\tau\rfloor^3\big)\Big(\int_0^{\lfloor n\tau\rfloor/n} z_{n1}(u)\mathrm{d}u\Big)^2,\\
U_{n5}(\tau) &:= 2\big(n^3/\lfloor n\tau\rfloor^3\big) z_{n1}(\tau)\Big(\int_0^{\lfloor n\tau\rfloor/n} z_{n1}(u)\mathrm{d}u\Big)\int_0^{\lfloor n\tau\rfloor/n} \big(\lfloor nu\rfloor/\lfloor n\tau\rfloor\big)\mathrm{d}u,\\
U_{n6}(\tau) &:= -\big(n^3/\lfloor n\tau\rfloor^3\big) z_{n1}^2(\tau)\Big(\int_0^{\lfloor n\tau\rfloor/n} \big(\lfloor nu\rfloor/\lfloor n\tau\rfloor\big)\mathrm{d}u\Big)^2
\end{aligned}
$$

tend in distribution, as $n \to \infty$, to the corresponding limit quantities

$$
\begin{aligned}
U_1(\tau) &:= \tau^{-2}\int_0^{\tau} Z_1^2(u)\mathrm{d}u,\\
U_2(\tau) &:= -2\tau^{-2} Z_1(\tau)\int_0^{\tau} (u/\tau) Z_1(u)\mathrm{d}u,\\
U_3(\tau) &:= \tau^{-2} Z_1^2(\tau)\int_0^{\tau} (u/\tau)^2\mathrm{d}u,\\
U_4(\tau) &:= -\tau^{-3}\Big(\int_0^{\tau} Z_1(u)\mathrm{d}u\Big)^2,\\
U_5(\tau) &:= 2\tau^{-3} Z_1(\tau)\Big(\int_0^{\tau} Z_1(u)\mathrm{d}u\Big)\int_0^{\tau} (u/\tau)\mathrm{d}u,\\
U_6(\tau) &:= -\tau^{-3} Z_1^2(\tau)\Big(\int_0^{\tau} (u/\tau)\mathrm{d}u\Big)^2.
\end{aligned}
$$

Note $Q_\tau(Z_1) = \sum_{i=1}^{6} U_i(\tau)$ a.s. for each $\tau \in (0, \upsilon_1]$. The joint convergence

$$
(U_{n1}(\tau), \ldots, U_{n6}(\tau)) \longrightarrow_d (U_1(\tau), \ldots, U_6(\tau)) \tag{8.1}
$$

at each fixed point $\tau \in (0, \upsilon_1]$ can be easily derived from the (marginal) convergence $\gamma_{n1}^{-1} S_{\lfloor n\tau\rfloor} \longrightarrow_{D[0,\upsilon_1]} Z_1(\tau)$ in (4.1). The convergence in (8.1) easily extends to the joint convergence at any finite number of points $0 < \tau_1 < \cdots < \tau_m \le \upsilon_1$. In other words,

$$
(n/\gamma_{n1}^2) V_{\lfloor n\tau\rfloor}(X) \longrightarrow_{\mathrm{fdd}(0,\upsilon_1]} Q_\tau(Z_1). \tag{8.2}
$$

In a similar way,

$$
\gamma_{n2}^{-1} S^*_{\lfloor n\tau\rfloor} \longrightarrow_{D[0,1-\upsilon_0]} Z_2(\tau),
$$

implies

$$
(n/\gamma_{n2}^2) V^*_{n-\lfloor n\tau\rfloor}(X) \longrightarrow_{\mathrm{fdd}[\upsilon_0,1)} Q_{1-\tau}(Z_2). \tag{8.3}
$$

It is clear from the joint convergence in (4.1) that (8.2), (8.3) extend to the joint convergence of finite-dimensional distributions, in other words, that (4.2) holds with $\longrightarrow_{D(0,\upsilon_1]\times D[\upsilon_0,1)}$ replaced by $\longrightarrow_{\mathrm{fdd}(0,\upsilon_1]\times[\upsilon_0,1)}$.

It remains to prove the tightness in $D(0,\upsilon_1]\times D[\upsilon_0,1)$. To this end, it suffices to check the tightness of the marginal processes in (8.2) and (8.3) in the corresponding Skorokhod spaces $D(0,\upsilon_1]$ and $D[\upsilon_0,1)$. See, e.g., Ferger and Vogel (2010), Whitt (1970).

Let us prove the tightness of the l.h.s. in (8.2) in $D(0,\upsilon_1]$, or, equivalently, the tightness in $D[\upsilon,\upsilon_1]$, for any $0<\upsilon<\upsilon_1$. Let $\Upsilon_n(\tau) := (n/\gamma_{n1}^2)V_{\lfloor n\tau\rfloor}(X)$. Since $\{\Upsilon_n(\upsilon), n\ge 1\}$ is tight by (8.2), it suffices to show that for any $\epsilon_1,\epsilon_2>0$ there exist $\delta>0$ and $n_0\ge 1$ such that

$$\mathrm{P}(\omega_\delta(\Upsilon_n)\ge\epsilon_1)\le\epsilon_2, \qquad n\ge n_0, \tag{8.4}$$

where

$$\omega_\delta(x) := \sup\big\{|x(a)-x(b)| : \upsilon\le a<b\le\upsilon_1, a-b<\delta\big\}$$

is the continuity modulus of a function $x\in D[\upsilon,\upsilon_1]$; see Billingsley (1968, Theorem 8.2). Since $\Upsilon_n(\tau)=\sum_{i=1}^6 U_{ni}(\tau)$, it suffices to show (8.4) with Υ_n replaced by $U_{ni}, i=1,\dots,6$, in other words,

$$\mathrm{P}(\omega_\delta(U_{ni})\ge\epsilon_1)\le\epsilon_2, \qquad n\ge n_0, \quad i=1,\dots,6. \tag{8.5}$$

We verify (8.5) for $i=2$ only since the remaining cases follow similarly. Write $U_{n2}(\tau)=\prod_{i=1}^3 H_{ni}(\tau)$, where $H_{n1}(\tau) := -2\big(n^2/\lfloor n\tau\rfloor^2\big)$, $H_{n2}(\tau) := z_{n1}(\tau)$, $H_{n3}(\tau) := \int_0^{\lfloor n\tau\rfloor/n}\big(\lfloor nu\rfloor/\lfloor n\tau\rfloor\big)z_{n1}(u)\mathrm{d}u$. Then $\mathrm{P}(\omega_\delta(U_{n2})\ge\epsilon_1)\le\sum_{i=1}^3\big[\mathrm{P}(\omega_\delta\big(H_{ni})\ge\epsilon_1/(3K)\big)+\mathrm{P}\big(\prod_{j\ne i}\|H_{nj}\|>K\big)\big]$, where $\|x\| := \sup\{|x(a)| : \upsilon\le a\le\upsilon_1\big\}$ is the sup-norm. Relation (4.1) implies that the probability $\mathrm{P}\big(\sum_{i=1}^3\|H_{ni}\|>K\big)$ can be made arbitrary small for all $n>n_0(K)$ by a suitable choice of K. By same relation (4.1) assumed under the uniform topology, for a given ϵ_1/K, we have that $\lim_{\delta\to 0}\limsup_{n\to\infty}\mathrm{P}\big(\omega_\delta(H_{ni})\ge\epsilon_1/K\big)=0$. This proves (8.5) and the functional convergence $(n/\gamma_{n1}^2)V_{\lfloor n\tau\rfloor}(X)\longrightarrow_{D(0,\upsilon_1]} Q_\tau(Z_1)$. The proof of $(n/\gamma_{n2}^2)V^*_{n-\lfloor n\tau\rfloor}(X)\longrightarrow_{D[\upsilon_0,1)} Q_{1-\tau}(Z_2)$ is analogous. This concludes the proof of part (i), since the continuity of the limit process in (4.2) is immediate from continuity of $\big(Z_1(\tau_1),Z_2(\tau_2)\big)$ and the definition of Q_τ in (3.6).

(ii) Note that (4.3) and the a.s. continuity of $\tau\mapsto Q_\tau(Z_1)$ guarantees that $\inf_{\tau\in\mathcal{T}}Q_\tau(Z_1)>0$ a.s. Therefore relations (4.4) follow from (4.2) and the continuous mapping theorem.

(iii) Follows from (4.2) and the fact that $Z_1(\tau)=0$, $\tau\in\mathcal{T}$ implies $Q_\tau(Z_1)=0$, $\tau\in\mathcal{T}$. $\square$

Proof of (5.4). Let first $-.5<d<.5$ and $b_i := (a\star\pi(d))_i-\kappa\pi_i(d)$, $i=0,1,\dots$. Consider the stationary process $\widetilde X_j := X_j-X_j^\dagger=\sum_{i=0}^\infty b_i\zeta_{j-i}$ with spectral density $\tilde f(x)=|\hat a(x)-\kappa|^2 g(x)$, where $\hat a(x)=\sum_{j=0}^\infty a_j\mathrm{e}^{-\mathrm{i}jx}$, $\mathrm{i} := \sqrt{-1}$ and $g(x) := (2\pi)^{-1}|1-\mathrm{e}^{-\mathrm{i}x}|^{-2d}$ is the spectral density of FARIMA$(0,d,0)$. We have

$$\mathrm{E}\Big(\sum_{j=1}^n(X_j-X_j^\dagger)\Big)^2=\int_{-\pi}^{\pi}\tilde f(x)D_n^2(x)\mathrm{d}x, \qquad D_n(x) := \frac{\sin(nx/2)}{\sin(x/2)}. \tag{8.6}$$

Since $\hat a(x)$ is bounded and continuous on $[-\pi,\pi]$, $\hat a(0)=\kappa$, it follows that $\tilde f(x)=o(|x|^{-2d})$ $(x\to 0)$, which in turn implies (5.4) for $-.5<d<.5$; see e.g. Giraitis et al. (2012, proof of Proposition 3.3.1).

Next, let $.5<d<1.5$. Then $X_j-X_j^\dagger=\sum_{k=1}^j\widetilde X_k$, where the stationary process $\widetilde X_k := \sum_{i=0}^\infty((a\star\pi(d-1))_i-\kappa\pi_i(d-1))\zeta_{k-i}$ satisfies $\mathrm{E}\big(\sum_{k=1}^j\widetilde X_k\big)^2\le\epsilon(j)j^{2d-1}$, $\epsilon(j)\to 0$, $(j\to\infty)$, see above. We have

$$\begin{aligned}
\mathrm{E}\Big(\sum_{j=1}^{n}(X_j - X_j^{\dagger})\Big)^2 &= \mathrm{E}\Big(\sum_{j=1}^{n}\sum_{k=1}^{j}\widetilde{X}_k\Big)^2 \le \sum_{j_1,j_2=1}^{n}\Big\{\mathrm{E}\Big(\sum_{k=1}^{j_1}\widetilde{X}_k\Big)^2\mathrm{E}\Big(\sum_{k=1}^{j_2}\widetilde{X}_k\Big)^2\Big\}^{1/2} \\
&\le \Big\{\sum_{j=1}^{n}\sqrt{\epsilon(j)j^{2d-1}}\Big\}^2 = o(n^{2d+1}).
\end{aligned}$$

This completes the proof of (5.4). $\square$

Proof of Proposition 5.2. Note first that the convergence in (5.6) for type II integrated process of (5.9) can be easily established following the proof of Proposition 5.1. Hence, it suffices to show that

$$\sup_{\tau\in[0,1]}\frac{1}{n^{d+.5}}\Big|\sum_{t=1}^{\lfloor n\tau\rfloor}R_t^0\Big| \;\to_p\; 0, \tag{8.7}$$

where $R_t^0 := (1-L)_+^{-d}(1-L)_-^{d}X_t^0 = \sum_{i=0}^{\infty}X_{-i}^0\sum_{j=0}^{t-1}\pi_j(d)\pi_{t-j+i}(-d)$ is the contribution arising from initial values. When $d > 0.5$, using (5.10), the Cauchy-Schwarz inequality and the fact that $|\pi_j(d)| \le Cj^{d-1}$ we obtain

$$\begin{aligned}
\mathrm{E}\Big(\frac{1}{n^{d+.5}}\sum_{t=1}^{n}|R_t^0|\Big)^2 &\le \frac{C}{n^{2d+1}}\Big(\sum_{t=1}^{n}\sum_{i=0}^{\infty}\sum_{j=0}^{t-1}|\pi_j(d)|\,|\pi_{t-j+i}(-d)|\Big)^2 \\
&\le \frac{C}{n^{2d+1}}\Big(\sum_{1\le j<t\le n}j^{d-1}(t-j)^{-d}\Big)^2 \;\to\; 0
\end{aligned}$$

since

$$\sum_{1\le j<t\le n}j^{d-1}(t-j)^{-d} \le \sum_{j=1}^{n}j^{d-1}\sum_{k=1}^{n}k^{-d} \;\le\; C\begin{cases} n^d, & d>1,\\ n\log n, & d=1,\\ n, & \frac{1}{2}<d<1.\end{cases}$$

This proves (8.7). $\square$

Proof of Proposition 5.3. We restrict the proof to the case (i) and $i = 2$, or, equivalently, to the model (5.17), since the remaining cases can be treated similarly. Similarly as in the proof of (4.2), it suffices to prove the joint convergence of finite-dimensional distributions in (5.23) and the functional convergence of marginal processes, viz.,

$$n^{-d_1-.5}S_{2,\lfloor n\tau\rfloor} \;\longrightarrow_{D[0,\theta]}\; Z_{2,1}(\tau), \qquad n^{-d_2-.5}S^{*}_{2,\lfloor n\tau\rfloor} \;\longrightarrow_{D[0,1-\underline{\tau}]}\; Z_{2,2}(\tau). \tag{8.8}$$

Since $X_{2,t} = \sum_{j=0}^{t}\pi_j(d_1)\zeta_{t-j}$, $1 \le t \le \lfloor n\underline{\tau}\rfloor$ has constant memory parameter d_1, the proof of the first convergence in (8.8) to $Z_{2,1}(\tau) = B^{\mathrm{II}}_{d_1+.5}(\tau)$ is standard, and we omit it.

Consider the second convergence in (8.8). It can be rewritten as

$$n^{-d_2-.5}S_{2,\lfloor n\tau\rfloor} \;\longrightarrow_{D[\underline{\tau},1]}\; \mathcal{Z}_2(\tau), \tag{8.9}$$

where $S_{2,\lfloor n\tau\rfloor} = \sum_{t=1}^{\lfloor n\tau\rfloor}X_{2,t} = \sum_{t=1}^{\lfloor n\tau\rfloor}\sum_{s=1}^{t}b_{2,t-s}(t)\zeta_s$. Let us first prove the one-dimensional convergence in (8.9) at a fixed point $\tau \in [\underline{\tau}, 1]$.

We start with the case $\tau > \overline{\theta}$. Following the scheme of discrete stochastic integrals in Surgailis (2003), rewrite the l.h.s. of (8.9) as a discrete stochastic integral

$$n^{-d_2-.5}S_{2,\lfloor n\tau\rfloor} = \int_0^\tau F_n(u)\mathrm{d}z_n(u) = \int_0^{\overline{\theta}} F_n(u)\mathrm{d}z_n(u) + \int_{\overline{\theta}}^\tau F_n(u)\mathrm{d}z_n(u),$$

where $z_n(u) := n^{-1/2}\sum_{i=1}^{\lfloor nu\rfloor}\zeta_i$ is the partial sum process of standardized i.i.d. r.v.s, tending weakly to a Brownian motion $\{B(u), u \in [0,1]\}$. The integrand F_n in the above integral is equal to

$$\begin{aligned} F_n(u) \;&:=\; n^{-d_2}\sum_{t=\lfloor nu\rfloor}^{\lfloor n\tau\rfloor} b_{2,t-\lfloor nu\rfloor}(t) \\ &=\; \begin{cases} n^{-d_2}\sum_{t=\lfloor nu\rfloor}^{\lfloor n\tau\rfloor} b_{2,t-\lfloor nu\rfloor}(t), & 0 < u \le \overline{\theta}, \\ n^{-d_2}\sum_{t=\lfloor nu\rfloor}^{\lfloor n\tau\rfloor} \pi_{t-\lfloor nu\rfloor}(d_2), & \overline{\theta} < u \le \tau, \end{cases} \end{aligned}$$

where we used the fact that $b_{2,t-\lfloor nu\rfloor}(t) = \pi_{t-\lfloor nu\rfloor}(d_2)$ for $t \ge \lfloor nu\rfloor \ge \lfloor n\overline{\theta}\rfloor$, where $\pi_j(d)$ are the FARIMA coefficients in (5.12). Similarly, the r.h.s. of (8.9) can be written as the sum of two stochastic integrals:

$$\int_0^\tau F(u)\mathrm{d}B(u) = \int_0^{\overline{\theta}} F(u)\mathrm{d}B(u) + \int_{\overline{\theta}}^\tau F(u)\mathrm{d}B(u),$$

where

$$F(u) \;:=\; \begin{cases} \Gamma(d_2)^{-1}\int_{\overline{\theta}}^\tau (v-u)^{d_2-1}\mathrm{e}^{H(u,v)}\mathrm{d}v, & 0 < u \le \overline{\theta}, \\ \Gamma(d_2+1)^{-1}(\tau-u)^{d_2}, & \overline{\theta} < u \le \tau. \end{cases}$$

Accordingly, using the above mentioned criterion in Surgailis (2003, Proposition 3.2) (see also Lemma 2.1 in Bružaitė and Vaičiulis (2005)), the one-dimensional convergence in (8.9) follows from the L^2-convergence of the integrands:

$$\int_0^{\overline{\theta}} |F_n(u) - F(u)|^2\mathrm{d}u \to 0, \qquad \int_{\overline{\theta}}^\tau |F_n(u) - F(u)|^2\mathrm{d}u \to 0. \tag{8.10}$$

The second relation in (8.10) is easy using the properties of FARIMA filters. Denote J_n the first integral in (8.10). The integrand there can be rewritten as

$$F_n(u) - F(u) = \int_u^{\overline{\theta}} n^{1-d_2}b_{2,\lfloor nv\rfloor-\lfloor nu\rfloor}(\lfloor nv\rfloor)\mathrm{d}v + \int_{\overline{\theta}}^\tau G_n(u,v)\mathrm{d}v \,-\, n^{-d_2}(n\tau - \lfloor n\tau\rfloor)b_{2,\lfloor n\tau\rfloor-\lfloor nu\rfloor}(\lfloor n\tau\rfloor), \tag{8.11}$$

where $G_n(u,v) := n^{1-d_2}b_{2,\lfloor nv\rfloor-\lfloor nu\rfloor}(\lfloor nv\rfloor) - \Gamma(d_2)^{-1}(v-u)^{d_2-1}\mathrm{e}^{H(u,v)}$ and $H(u,v)$ is defined at (5.20).

Let us write $b_{2,\lfloor nv\rfloor-\lfloor nu\rfloor}(\lfloor nv\rfloor) = \pi_{\lfloor nv\rfloor-\lfloor nu\rfloor}(d_2)K_n(u,v)$ where

$$K_n(u,v) := \frac{b_{2,\lfloor nv\rfloor-\lfloor nu\rfloor}(\lfloor nv\rfloor)}{\pi_{\lfloor nv\rfloor-\lfloor nu\rfloor}(d_2)} = \prod_{i=1\vee(\lfloor nv\rfloor-\lfloor n\overline{\theta}\rfloor)}^{\lfloor nv\rfloor-\lfloor nu\rfloor} \frac{d\big(\frac{\lfloor nv\rfloor-i+1}{n}\big) - 1 + i}{d_2 - 1 + i}. \tag{8.12}$$

We claim that

$$\lim_{n\to\infty} K_n(u,v) = \mathrm{e}^{H(u,v)}, \qquad 0 \le u \le \overline{\theta} < v \le 1, \tag{8.13}$$

Indeed,

$$K_n(u,v) \;=\; \exp\Big\{\sum_{i=1\vee(\lfloor nv\rfloor-\lfloor n\overline{\theta}\rfloor)}^{\lfloor nv\rfloor-\lfloor nu\rfloor}\log\Big(1-\frac{d_2-d\big(\frac{\lfloor nv\rfloor-i}{n}\big)}{d_2-1+i}\Big)\Big\} \;=\; \mathrm{e}^{H_n(u,v)+R_n(u,v)},$$

where

$$\begin{aligned} H_n(u,v) \;&:=\; n^{-1}\sum_{i=1\vee(\lfloor nv\rfloor-\lfloor n\overline{\theta}\rfloor)}^{\lfloor nv\rfloor-\lfloor nu\rfloor}\frac{d\big(\frac{\lfloor nv\rfloor-i}{n}\big)-d_2}{\frac{d_2-1+i}{n}} \;\to\; H(u,v),\\ R_n(u,v) \;&=\; O\Big(\sum_{i=1\vee(\lfloor nv\rfloor-\lfloor n\overline{\theta}\rfloor)}^{\lfloor nv\rfloor-\lfloor nu\rfloor}\frac{1}{i^2}\Big) \;=\; O\Big(\frac{1}{1\vee(\lfloor nv\rfloor-\lfloor n\overline{\theta}\rfloor)}\Big), \end{aligned}$$

hence $R_n(u,v)\to 0$ for any $v>\overline{\theta}$.

The proof of $J_n\to 0$ in (8.10) then follows from the following arguments. Using on one hand the fact that the ratio $K_n(u,v)$ tends to 0 for $0<u<v\le\overline{\theta}$, on the other hand (8.13), and from the well-known asymptotics $\pi_j(d)\sim\Gamma(d)^{-1}j^{d-1}$, $j\to\infty$ of FARIMA coefficients, it easily follows that $n^{1-d_2}b_{2,\lfloor nv\rfloor-\lfloor nu\rfloor}(\lfloor nv\rfloor)\to 0$ for any $0<u<v\le\overline{\theta}$, and $G_n(u,v)\to 0$ for any $0<u<v\le 1$ fixed. Moreover, the last term in (8.11) obviously tends to 0 because $d_2>0$. Since both sides of (8.13) are nonnegative and bounded by 1, the above convergences extend to the proof of $J_n\to 0$ by the dominated convergence theorem. This proves the convergence of one-dimensional distributions in (8.9) for $\tau>\overline{\theta}$.

For $\underline{\tau}\le\tau\le\overline{\theta}$, the above convergence follows similarly by using the fact that $K_n(u,v)$ tends to 0 for $0<u<v\le\overline{\theta}$.

The proof of the convergence of general finite-dimensional distributions in (8.9), as well as the joint convergence of finite-dimensional distributions in (5.23), can be achieved analogously, by using the Cramèr-Wold device. Finally, the tightness in (8.9) follows by the Kolmorogov criterion (see, e.g. Bružaitė and Vaičiulis (2005), proof of Theorem 1.2 for details). Proposition 5.3 is proved. $\square$

Proof of Proposition 6.1. Consider de-trended observations and their partial sums processes as defined by

$$\varepsilon_j \;:=\; X_j-(A_n-B_n)-2B_nj, \qquad S_k(\varepsilon)\;:=\;\sum_{j=1}^k\varepsilon_j, \qquad S^*_{n-k}(\varepsilon)\;:=\;\sum_{j=k+1}^n\varepsilon_j.$$

Note that $S_k(\varepsilon)=S_k(X)-k(A_n-B_n)-k(k+1)B_n=S_k(X)-kA_n-k^2B_n$ and the null hypothesis $\mathbf{H}_0^{\mathrm{trend}}$ can be rewritten as

$$\gamma_n^{-1}S_{\lfloor n\tau\rfloor}(\varepsilon)\;\longrightarrow_{D[0,1]}\;Z(\tau). \tag{8.14}$$

For a fixed $1\le k<n$, let $(\hat a_k(\varepsilon),\hat b_k(\varepsilon)) := \mathrm{argmin}\big(\sum_{j=1}^k(\varepsilon_j-a-bj)^2\big)$, $(\hat a^*_{n-k}(\varepsilon),\hat b^*_{n-k}(\varepsilon)) := \mathrm{argmin}\big(\sum_{j=k+1}^n(\varepsilon_j-a-bj)^2\big)$ be the corresponding linear regression coefficients. More explicitly,

$$\hat a_k(\varepsilon)\;=\;\frac{\big(\sum_{j=1}^k j\big)\big(\sum_{j=1}^k j\varepsilon_j\big)-\big(\sum_{j=1}^k j^2\big)\big(\sum_{j=1}^k\varepsilon_j\big)}{\big(\sum_{j=1}^k j\big)^2-k\sum_{j=1}^k j^2}, \qquad \hat b_k(\varepsilon)\;:=\;\frac{\big(\sum_{j=1}^k j\big)\big(\sum_{j=1}^k\varepsilon_j\big)-k\sum_{j=1}^k j\varepsilon_j}{\big(\sum_{j=1}^k j\big)^2-k\sum_{j=1}^k j^2} \tag{8.15}$$

It is easy to verify that $\hat{a}_k(\varepsilon) = \hat{a}_k(X) + (B_n - A_n)$, $\hat{b}_k(\varepsilon) = \hat{b}_k(X) - 2B_n$. Hence we obtain the following expression of residual partial sums $\widehat{S}_j = S_j(\widehat{X})$ via de-trended partial sums $S_j(\varepsilon)$ and the above regression coefficients:

$$\widehat{S}_j = S_j(\varepsilon) - j\Big(\hat{a}_k(\varepsilon) + \frac{\hat{b}_k(\varepsilon)}{2}\Big) - j^2\Big(\frac{\hat{b}_k(\varepsilon)}{2}\Big) \tag{8.16}$$

The limit behavior of $\mathcal{V}_{\lfloor n\tau\rfloor}(X)$ follows from the limit behavior of $\gamma_n^{-1}\widehat{S}_{\lfloor nu\rfloor}, u \in [0,\tau]$ similarly as in the proof of Proposition 4.1. The behavior of the first term $\gamma_n^{-1}S_{\lfloor nu\rfloor}(\varepsilon)$ in (8.16) is given in (8.14). It remains to identify the limit regression coefficients $\hat{a}_{\lfloor n\tau\rfloor}(\varepsilon), \hat{b}_{\lfloor n\tau\rfloor}(\varepsilon)$ in (8.16). Clearly the denominator $\big(\sum_{j=1}^k j\big)^2 - k\sum_{j=1}^k j^2 \sim -\frac{k^4}{12}$. The numerators in (8.15) are written in terms of $S_k(\varepsilon)$ and $\sum_{j=1}^k j\varepsilon_j$. From summation by parts and (8.14) we obtain

$$n^{-1}\gamma_n^{-1}\sum_{j=1}^{\lfloor n\tau\rfloor} j\varepsilon_j = n^{-1}\gamma_n^{-1}\Bigg(\lfloor n\tau\rfloor S_{\lfloor n\tau\rfloor}(\varepsilon) - \sum_{j=1}^{\lfloor n\tau\rfloor-1} S_j(\varepsilon)\Bigg) \longrightarrow_{D[0,1]} \tau Z(\tau) - \int_0^\tau Z(v)\mathrm{d}v. \tag{8.17}$$

Relations (8.14), (8.15) and (8.17) entail $(\lfloor nu\rfloor/\gamma_n)\hat{a}_{\lfloor n\tau\rfloor}(\varepsilon) \to_{D[0,\tau]} u\hat{a}_\tau$, $(\lfloor nu\rfloor^2/(2\gamma_n))\hat{b}_{\lfloor n\tau\rfloor}(\varepsilon) \to_{D[0,\tau]} u^2\frac{\hat{b}_\tau}{2}$, where

$$\hat{a}_\tau := -2\Big(\frac{1}{\tau}\Big)Z(\tau) + 6\Big(\frac{1}{\tau}\Big)^2\int_0^\tau Z(v)\mathrm{d}v, \qquad \hat{b}_\tau := 6\Big(\frac{1}{\tau}\Big)^2 Z(\tau) - 12\Big(\frac{1}{\tau}\Big)^3\int_0^\tau Z(v)\mathrm{d}v, \tag{8.18}$$

leading to the convergence $\gamma_n^{-1}\widehat{S}_{\lfloor nu\rfloor} \longrightarrow_{D[0,\tau]} \mathcal{Z}(u,\tau)$, where the limit process is given in (6.7). The remaining details of the proof are similar as in Proposition 4.1. □

References

Abadir, K. M., Distaso, W., and Giraitis, L. (2007). Nonstationarity-extended local Whittle estimation. *Journal of Econometrics*, 141:1353–1384.

Bardet, J.-M. and Kammoun, I. (2008). Detecting abrupt changes of the long-range dependence or the self-similarity of a Gaussian process. *C. R. Acad. Sci. Paris, Ser. I*, 346:789–794.

Bardet, J.-M., Lang, G., Oppenheim, G., Philippe, A., Stoev, S., and Taqqu, M. S. (2003). Semiparametric estimation of the long-range dependence parameter: A survey. In Doukhan, P., Oppenheim, G., and Taqqu, M. S., editors, *Theory and Applications of Long-Range Dependence*, pages 557–577. Birkhäuser, Boston.

Beran, J. and Terrin, N. (1996). Testing for a change of the long-memory parameter. *Biometrika*, 83:627–638.

Billingsley, P. (1968). *Convergence of Probability Measures*. Wiley, New York.

Bružaitė, K. and Vaičiulis, M. (2005). Asymptotic independence of distant partial sums of linear processes. *Lietuvos Matematikos Rinkinys*, 45:479–500.

Busetti, F. and Taylor, A. M. R. (2004). Tests of stationarity against a change in persistence. *Journal of Econometrics*, 123:33–66.

Chan, N. H. and Terrin, N. (1995). Inference for unstable long-memory processes with applications to fractional unit root autoregressions. *Annals of Statistics*, 23:1662–1683.

Davidson, J. and de Jong, R. M. (2000). The functional central limit theorem and weak convergence to stochastic integrals II: fractionally integrated processes. *Econometric Theory*, 16:643–666.

Davidson, J. and Hashimzade, N. (2009). Type I and type II fractional Brownian motions: A reconsideration. *Computational Statistics and Data Analysis*, 53:2089–2106.

Davydov, Y. A. (1970). The invariance principle for stationary processes. *Theory of Probability and its Applications*, 15:487–498.

Ferger, D. and Vogel, D. (2010). Weak convergence of the empirical process and the rescaled empirical distribution function in the Skorokhod product space. *Theory of Probability and its Applications*, 54:609–625.

Giraitis, L., Kokoszka, P., Leipus, R., and Teyssière, G. (2003). Rescaled variance and related tests for long memory in volatility and levels. *Journal of Econometrics*, 112:265–294.

Giraitis, L., Koul, H. L., and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes*. Imperial College Press, London.

Giraitis, L., Leipus, R., and Philippe, A. (2006). A test for stationarity versus trends and unit roots for a wide class of dependent errors. *Econometric Theory*, 22:989–1029.

Giraitis, L., Leipus, R., and Surgailis, D. (2009). ARCH(∞) models and long memory properties. In Andersen, T. G., Davis, R. A., Kreiss, J.-P., and Mikosch, T., editors, *Handbook of Financial Time Series*, pages 71–84. Springer-Verlag, Berlin.

Giraitis, L., Robinson, P. M., and Surgailis, D. (2000). A model for long memory conditional heteroscedasticity. *Annals of Applied Probability*, 10:1002–1024.

Hassler, U. and Meller, B. (2009). Detecting a change in inflation persistence in the presence of long memory: a new approach. Goethe University, Frankfurt, Working Papers.

Hassler, U. and Nautz, D. (2008). On the persistence of the Eonia spread. *Economics Letters*, 101:184–187.

Hassler, U. and Scheithauer, J. (2008). On critical values of tests against a change in persistence. *Oxford Bulletin of Economics and Statistics*, 70:705–710.

Hassler, U. and Scheithauer, J. (2011). Testing against a change from short to long memory. *Statistical Papers*, 52:847–870.

Horváth, L. (2001). Change-point detection in long-memory processes. *Journal of Multivariate Analysis*, 78:218–234.

Horváth, L. and Shao (1999). Limit theorems for quadratic forms with applications to Whittle's estimate. *Annals of Applied Probability*, 9:146–187.

Johansen, S. and Nielsen, M. Ø. (2010). Likelihood inference for a nonstationary fractional autoregressive model. *Journal of Econometrics*, 158:51–66.

Kim, J.-Y. (2000). Detection of change in persistence of a linear time series. *Journal of Econometrics*, 95:97–116.

Kim, J.-Y., Belaire-Franch, J., and Badillo Amador, R. (2002). Corrigendum to "Detection of change in persistence of a linear time series". *Journal of Econometrics*, 109:389–392.

Kokoszka, P. and Leipus, R. (2003). Detection and estimation of changes in regime. In Doukhan, P., Oppenheim, G., and Taqqu, M. S., editors, *Theory and Applications of Long-Range Dependence*, pages 325–337. Birkhäuser, Boston.

Kruse, R. (2008). Rational bubbles and changing degree of fractional integration. Diskussionspapiere der Wirtschaftswissenschaftlichen Fakultät der Universität Hannover, No. dp-394.

Kumar, M. S. and Okimoto, T. (2007). Dynamics of persistence in international inflation rates. *Journal of Money, Credit and Banking*, 39:1457–1479.

Lavielle, M. and Ludeña, C. (2000). The multiple change-points problem for the spectral distribution. *Bernoulli*, 6:845–869.

Leipus, R. and Surgailis, D. (2010). Asymptotics of partial sums of linear processes with changing memory parameter. Preprint.

Leybourne, S., Taylor, R., and Kim, T.-H. (2007). CUSUM of squares-based tests for a change in persistence. *Journal of Time Series Analysis*, 28:408–433.

Liu, M. (1998). Asymptotics of nonstationary fractional integrated series. *Econometric Theory*, 14:641–662.

MacNeill, I. (1978). Properties of sequences of partial sums of polynomial regression residuals with applications to tests for change of regression at unknown times. *Annals of Statistics*, 6:422–433.

Marinucci, D. and Robinson, P. M. (1999). Alternative forms of fractional Brownian motion. *Journal of Statistical Planning and Inference*, 80:111–122.

Marinucci, D. and Robinson, P. M. (2000). Weak convergence of multivariate fractional processes. *Stochastic Processes and their Applications*, 86:103–120.

Martins, L. F. and Rodrigues, P. M. M. (2010). Testing for persistence change in fractionally integrated models: an application to world inflation rates. Banco de Portugal, Economics and Research Department, Working Papers.

Peligrad, M. and Utev, S. (1997). Central limit theorem for linear processes. *Annals of Probability*, 25:443–456.

Philippe, A., Surgailis, D., and Viano, M.-C. (2006a). Almost periodically correlated processes with long-memory. In Bertail, P., Doukhan, P., and Soulier, P., editors, *Dependence in Probability and Statistics*, volume 187 of *Lecture Notes in Statistics*, pages 159–194. Springer-Verlag, New York.

Philippe, A., Surgailis, D., and Viano, M.-C. (2006b). Invariance principle for a class of non stationary processes with long memory. *C. R. Acad. Sci. Paris, Ser. I*, 342:269–274.

Philippe, A., Surgailis, D., and Viano, M.-C. (2008). Time-varying fractionally integrated processes with nonstationary long memory. *Theory of Probability and its Applications*, 52:651–673.

Pollard, D. (1984). *Convergence of Stochastic Processes*. Springer-Verlag, New York.

Robinson, P. M. (1991). Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regressions. *Journal of Econometrics*, 47:67–84.

Robinson, P. M. (1994). Efficient tests of nonstationary hypotheses. *Journal of American Statistical Association*, 89:1420–1437.

Shimotsu, K. (2006). Simple (but effective) tests of long memory versus structural breaks. Queen's University, Department of Economics, Working Papers, No. 1101.

Sibbertsen, P. and Kruse, R. (2009). Testing for a break in persistence under long-range dependencies. *Journal of Time Series Analysis*, 30:263–285.

Surgailis, D. (2003). Non-CLTs: U-statistics, multinomial formula and approximations of multiple Itô-Wiener integrals. In Doukhan, P., Oppenheim, G., and Taqqu, M. S., editors, *Theory and Applications of Long-Range Dependence*, pages 129–142. Birkhäuser, Boston.

Surgailis, D. (2008). Nonhomogeneous fractional integration and multifractional processes. *Stochastic Processes and their Applications*, 118:171–198.

Taqqu, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete*, 50:53–83.

Whitt, W. (1970). Weak convergence of probability measures on the function space $C[0, \infty)$. *Annals of Mathematical Statistics*, 41:939–944.

Yamaguchi, K. (2011). Estimating a change point in the long memory parameter. *Journal of Time Series Analysis*, 32:304–314.