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Some analytical solutions for Saint-Venant torsion of non-homogeneous cylindrical bars

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Abstract

The Saint-Venant torsion problem of linearly elastic cylindrical bars with solid and hollow cross-section is treated. The shear modulus of the non-homogeneous bar is a given function of the Prandtl's stress function of considered cylindrical bar when its material is homogeneous. The solution of the torsional problem of non-homogeneous bar is expressed in terms of the torsional and Prandtl's stress functions of homogeneous bar having the same cross-section as the non-homogeneous bar.

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1. Introduction

Whilst the Saint-Venant's torsional problem of cylindrical bars is a classical one in the field of elasticity, there has been recently a growing interest in the context of non-homogeneous and/or anisotropic bars (Arghavan and Hematiyan, 2008; Batra, 2006; Horgan and Chan, 1999; Horgan, 2007; Rooney and Ferrari, 1995; Ueda et al., 2002; Yasusi and Shigeyasu, 2000). In this paper a method is presented to find some analytical solutions for the Saint-Venant torsion of non-homogeneous elastic members. By the proposed method the existing solutions of homogeneous torsion problems are employed to find solutions of the corresponding non-homogeneous problems. An integral transformation similar to the Kirchhoff transformation in non-linear heat conduction (Carslaw and Jager, 1959) is used to develop the formulation. In the conventional problem of torsion of non-homogeneous bars, the shear modulus is considered to be an arbitrary non-negative function of cross-sectional coordinates (x, y) is known, but in this study, shear modulus is considered to be a smooth function of Prandtl's stress function of corresponding homogeneous problem. So that, the developed method is not general, but since by this formulation, some analytical solutions of torsional problems of non-homogeneous bars are found which are remarkable and useful specially for benchmarking purposes.

The present paper is a contribution to the existing exact benchmark solutions of non-homogeneous, isotropic, linearly elastic, twisted cylindrical bars with simply or multiply connected cross-sections. The shear modulus of the non-homogeneous bar is a given function of the Prandtl's stress function of the considered cylindrical bar when its material is homogeneous. The solution of the torsional problem of the non-homogeneous bar is expressed in terms of torsional and Prandtl's stress functions of a homogeneous bar which has the same

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cross-section as the non-homogeneous one. In this case by the use of known solutions of Saint-Venant's torsional problem of homogeneous cylindrical bars we can generate the solutions of Saint-Venant's torsional problem for a large set of non-homogeneous bars.

2. A brief review of torsional problems of homogeneous and non-homogeneous bars

Let $B = A \times (0, L)$ be a right cylinder of length L, with its cross-section A which may be simply or multiply connected bounded regular region of $\Re^2$. Let A_1 and A_2 be the bases and $A_3 = \partial A \times (0, L)$ the mantle of B. The cross-section A is multiply connected it has one outer boundary curve a_0 and p inner boundary curve $a_i (i = 1, 2, \dots p)$ as shown in Fig. 1. The Cartesian coordinate frame $Oxyz$ is supposed to be chosen in such a way that axis Oz is parallel to the generators of the cylindrical boundary surface segment A_3 and the plane Oxy contains the terminal cross-section A_1. The position of end cross-section A_2 is given by $z = L$. A point P in $\bar{B} = B \cup \partial B$ $(\partial B = A_1 \cup A_2 \cup A_3)$ is indicated by the vector $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z = \mathbf{R} + z\mathbf{e}_z$, where $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors of the coordinate system $Oxyz$.

We consider the Saint-Venant torsion problem of a homogeneous cylindrical bar. The displacement field $\mathbf{u} = \mathbf{u}(x, y, z)$ in the twisted cylindrical bar is characterized by (Lurje, 1970; Lekhnitskii, 1971; Sokolnikoff, 1956)

$$\mathbf{u} = \vartheta z\mathbf{e}_z \times \mathbf{R} + \vartheta \varphi(x, y)\mathbf{e}_z, \qquad (1)$$

where ϑ is the rate of twist with respect to axial coordinate z and $\varphi = \varphi(x, y)$ is the torsional function. For homogeneous bar $\varphi = \varphi(x, y)$ is a solution of the next Neumann type boundary value problem:

$$\Delta \varphi = 0 \quad \text{in } A, \qquad (2)$$

$$\mathbf{n} \cdot (\nabla \varphi + \mathbf{e}_z \times \mathbf{R}) = 0 \quad \text{on } \partial A. \qquad (3)$$

In Eq. (1), the cross is the sign of the vectorial product and in Eq. (3), scalar product of two vectors is indicated by dot and ∇ is the gradient (del) operator Δ is the Laplace operator that is

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y, \quad \Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$\mathbf{n}$ is the outer unit normal vector to boundary curve ∂A (Fig. 1).

Fig. 1

The displacement field for non-homogeneous twisted beam when the shear modulus does not depend on the axial coordinate z has the same form as given by Eq. (1) and in this case the rate of twist is denoted by Θ. For non-homogeneous beam if the shear modulus G is a smooth function of the cross-sectional coordinates x and y then the torsional function, which is denoted by Φ is a solution of the next boundary value problem (Kolchin, 1971; Lekhnitskii, 1971)

$$\nabla \cdot [G(\nabla \Phi + \mathbf{e}_z \times \mathbf{R})] = 0 \quad \text{in } A, \qquad (4)$$

$$\mathbf{n} \cdot \left(\nabla \Phi + \mathbf{e}_z \times \mathbf{R} \right) = 0 \quad \text{on} \quad \partial A. \tag{5}$$

The solution of torsional problem can be formulated by the use of Prandtl's stress function for homogeneus bar with shear modulus g as (Lurje, 1970; Sokolnikoff; 1956)

$$\mathbf{t}_z = t_{xz}\mathbf{e}_x + t_{yz}\mathbf{e}_y = g\vartheta \nabla u \times \mathbf{e}_z, \tag{6}$$

where t_{xz}, t_{yz} are the shearing stresses, $u = u(x, y)$ is the Prandtl's stress function of the considred homogeneous cross-section. In Eq. (6) $u = u(x, y)$ is the solution of the boundary value problem

$$\Delta u = -2 \quad \text{in} \quad A, \tag{7}$$

$$u = 0 \quad \text{on} \quad a_0, \tag{8}$$

$$u = c_i \quad \text{on} \quad a_i \ (i = 1, 2, \ldots \quad p), \tag{9}$$

$$\oint_{a_i} \mathbf{n} \cdot \nabla u \, ds = 2\overline{A}_i \quad (i = 1, 2, \ldots \quad p). \tag{10}$$

The rate of twist is computed from next equation

$$\vartheta = \frac{t}{gk}, \quad k = 2\left(\int_A u \, dA + \sum_{i=1}^{p} c_i \overline{A}_i \right), \tag{11}$$

where t is the applied torque, k is the Saint-Venant torsional constant. In Eq. (10), s is an arc length defined on the boundary curve ∂A (Fig. 1) and $\overline{A}_i$ is the area inclosed by inner boundary curve $a_i (i = 1, 2, \ldots \quad p)$. The connection between φ and u is as follows (Lurje, 1970; Lekhnitskii, 1971)

$$\nabla u \times \mathbf{e}_z = \nabla \varphi + \mathbf{e}_z \times \mathbf{R}. \tag{12}$$

In the case of non-homogeneous bar when $G = G(x, y)$, the solution of the Saint-Venant torsional problem for the shearing stresses T_{xz}, T_{yz} is as follows

$$\mathbf{T}_z = T_{xz}\mathbf{e}_x + T_{yz}\mathbf{e}_y = \Theta \nabla U \times \mathbf{e}_z, \tag{13}$$

where $U = U(x, y)$ is the solution of the following boundary value problem (Kolchin, 1971; Lekhnitskii, 1971)

$$\nabla \cdot \left(\frac{1}{G} \nabla U \right) = -2 \quad \text{in} \quad A, \tag{14}$$

$$U = 0 \quad \text{on} \quad a_0, \tag{15}$$

$$U = C_i \quad \text{on} \quad a_i \ (i = 1, 2, \ldots \quad p), \tag{16}$$

$$\oint_{a_i} \frac{1}{G} \mathbf{n} \cdot \nabla U \, ds = 2\overline{A}_i \quad (i = 1, 2, \ldots \quad p). \tag{17}$$

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For non-homogeneous beam we have (Ecsedi, 1981; 2005)

$$\Theta = \frac{T}{K}, \quad K = 2\left(\int_A U dA + \sum_{i=1}^{p} C_i \bar{A}_i\right) = \int_A \frac{|\nabla U|^2}{G} dA, \tag{18}$$

where K is the torsional rigidity of the considered non-homogeneous cross-section and T is the applied torque. The connection between the torsional function $\Phi = \Phi(x, y)$ and Prandtl's stress function $U = U(x, y)$ is as follows (Kolchin, 1971; Lekhnitskii, 1971)

$$\nabla U \times \mathbf{e}_z = G(\nabla\Phi + \mathbf{e}_z \times \mathbf{R}). \tag{19}$$

3. Solution of the Saint-Venant torsional problem for non-homogeneous bar

It is assumed that the shear modulus of the considered non-homogeneous bar has the form

$$G(x, y) = f\big(u(x, y)\big), \tag{20}$$

where $f = f(\xi)$ is a given positive smooth function. We define a new function $F = F(\xi)$ as

$$F(\xi) = \int_0^{\xi} f(\eta) d\eta. \tag{21}$$

It is clear

$$F(0) = 0, \quad \frac{dF}{d\xi} = F'(\xi) = f(\xi). \tag{22}$$

We remark to formula (20), the Prandtl's stress function $u = u(x, y)$ is positive in A and $c_i > 0$ ($i = 1, 2, \dots \quad p$) (Bandle, 1980) and in the case of functionally graded elastic materials $G = G(x, y)$ is a smooth function of the cross-sectional coordinates x and y. Relations between the solutions of Saint-Venant's torsional problems of homomogeneous and non-homogeneous elastic bars having the same cross-section are formulated in the next theorem.

Theorem. Let $G = f\big(u(x, y)\big)$ be. In this case we have

$$U(x, y) = F\big(u(x, y)\big) \quad (x.y) \in A \cup \partial A, \tag{23}$$

$$C_i = F(c_i) \quad (i = 1, 2, \dots \quad p), \tag{24}$$

$$\mathbf{T}_z = \Theta f\big(u(x, y)\big)\nabla u \times \mathbf{e}_z = \frac{T}{K} f\big(u(x, y)\big)\nabla u \times \mathbf{e}_z, \tag{25}$$

$$K = 2\left(\int_A F\big(u(x, y)\big) dA + \sum_{i=1}^{p} \bar{A}_i F(C_i)\right), \tag{26}$$

$$K = \int_A f\big(u(x, y)\big)|\nabla u|^2 dA, \tag{27}$$

$$\Phi(x, y) = \varphi(x, y) + \gamma, \tag{28}$$

where γ is an arbitrary constant.

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Proof. The proof of validity of Eqs. (23), (24) is based on Eqs. (7), (8), (9), (10) and Eqs. (14), (15), (16), (17) and

$$\nabla U = F'\big(u(x,y)\big)\nabla u = f\big(u(x,y)\big)\nabla u = G(x,y)\nabla u. \tag{29}$$

Substitution of Eqs. (23), (24) and (29) into Eqs. (18)$_{1,2}$ gives fomulas (26) and (27). The correctness of Eq. (25) follows from Eqs. (13), (23), (24) and (29). Starting from Eqs. (19), (29) we can write

$$\nabla U \times \mathbf{e}_z = G\nabla u \times \mathbf{e}_z = G\big(\nabla\Phi + \mathbf{e}_z \times \mathbf{R}\big), \tag{30}$$

that is, according to Eq. (12) we have

$$\nabla u \times \mathbf{e}_z = \nabla\Phi + \mathbf{e}_z \times \mathbf{R} = \nabla\varphi + \mathbf{e}_z \times \mathbf{R}. \tag{31}$$

Statement formulated in Eqs. (28) and (31) are just the same.

4. Examples

4.1. *Hollow elliptical cylinder*

Fig. 2 shows the cross-section of the considered elliptical cylinder. Equations of outer and inner boundary curves of hollow elliptical cross-section are

$$\frac{x^2}{(\lambda a)^2} + \frac{y^2}{(\lambda b)^2} - 1 = 0, \tag{32}$$

$$\lambda = \lambda_0 = 1 \quad \text{for} \quad a_0, \quad \lambda = \lambda_1 \quad \text{for} \quad a_1, \quad (0 < \lambda_1 < 1). \tag{33}$$

Fig. 2

The shear modulus of the non-homogeneous bar is as follows

$$G = f(u) = \alpha\exp(\beta u), \quad \alpha > 0, \quad \beta \neq 0. \tag{34}$$

It is evident

$$F(u) = \frac{\alpha}{\beta}\big[\exp(\beta u) - 1\big]. \tag{35}$$

It is known that (Lurje, 1970; Sokolnikoff, 1956)

$$u = u(x,y) = \frac{a^2 b^2}{a^2 + b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right), \tag{36}$$

$$c_1 = \frac{a^2 b^2}{a^2 + b^2}(1 - \lambda_1^2). \tag{37}$$

A simple computation gives

$$C_1 = \frac{\alpha}{\beta}\left(\exp\left[\frac{a^2b^2\beta}{a^2+b^2}(1-\lambda_1^2)\right]-1\right), \tag{38}$$

$$\overline{A}_1 = ab\lambda_1^2\pi. \tag{39}$$

Application of formula (27) and Eqs. (38), (39) leads to the next result for the torsional rigidity

$$K = 2\pi\frac{\alpha}{\beta}ab\left\{\left(\frac{a^2+b^2}{a^2b^2\beta}+\lambda_1^2\right)\exp\left[\frac{a^2b^2\beta}{a^2+b^2}(1-\lambda_1^2)\right]-\left(1+\frac{a^2+b^2}{a^2b^2\beta}\right)\right\}. \tag{40}$$

From Eqs. (25) and (28) we get

$$T_{xz} = -2\Theta\frac{a^2\alpha y}{a^2+b^2}\exp\left[\frac{a^2b^2\beta}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right], \tag{41}$$

$$T_{yz} = 2\Theta\frac{b^2\alpha x}{a^2+b^2}\exp\left[\frac{a^2b^2\beta}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right], \tag{42}$$

$$\Phi(x,y) = \varphi(x,y) = \frac{b^2-a^2}{a^2+b^2}xy. \tag{43}$$

We note, the limit as β tends to zero in Eqs. (40), (41), (42) leads to the formulae of torsional rigidity and shearing stresses of twisted homogeneous elliptical cross-section.

4.2. *Solid equilateral triangle cross-section*

The boundary of the solid cross-section shown in Fig. 3 is an equilateral triangle of altitude h. The shear modulus of non-homogeneous bar is

$$G = f(u) = \alpha u^2 + \beta, \quad \alpha \geq 0, \quad \beta > 0. \tag{44}$$

It is clear that, in the present case (Lurje, 1970; Sokolnikoff, 1956)

$$u(x,y) = \frac{1}{2h}(3y^2-x^2)(x-h), \tag{45}$$

$$\varphi(x,y) = \frac{1}{2h}(y^3-3x^2y+4hxy), \tag{46}$$

and we have

$$F(u) = \frac{\alpha}{3}u^3 + \beta u. \tag{47}$$

Fig. 3.

Application of Eqs. (25) and (28) gives the next results

$$T_{xz} = \frac{3\alpha\Theta}{4h^3}(3y^2-x^2)^2(x-h)^3y+\frac{3\beta\Theta}{h}y(y-h), \tag{48}$$

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$$T_{yz} = \frac{\alpha\Theta}{8h^3}(3y^2 - x^2)(x-h)^2(3x^2 - 2xh - 3y^2) + \frac{\beta\Theta}{2h}(3x^2 - 2xh - 3y^2), \qquad (49)$$

$$\Phi = \frac{1}{2h}(y^3 - 3x^2y + 4hxy). \qquad (50)$$

Starting from formula, (27) by a detailed computation we get

$$K = 0.0003334\alpha h^8 + 0.03849\beta h^4. \qquad (51)$$

It is evident from Eqs. (48), (49), (50) the formulas of homogeneous cross-section is obtained as α tends to zero.

4.3. *Approximate solution for thin-walled tube*

Cross-section of thin-walled tube with smooth centerline a_m and uniform thickness H is shown in Fig. 4. The arc-length defined on the centerline a_m is denoted by s and the thickness coordinate is indicated by η. On the outer boundary curve $\eta = H/2$ and on the inner boundary curve $\eta = -H/2$. The shear modulus G changes continuously between the inner and outer walls, it does not depend on s, that is, $G = G(\eta)$. According to Bredt's formulation we can write (Kollbrunner and Basler, 1969; Murray, 1985)

$$u = u(\eta) = \frac{2A_m}{l}(\frac{H}{2} - \eta), \quad \nabla u = -\frac{2A_m}{l}\mathbf{n}, \qquad (52)$$

$$\varphi = \varphi(s) = \frac{2A_m}{l}s - \int_{\overparen{PP_0}} R_n ds, \qquad (53)$$

$$k = \frac{4A_m^2}{l}H, \qquad (54)$$

where A_m is the area enclosed by the centerline of walls, l is the length of a_m and $R_n = \mathbf{R} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal vector to the centerline (Fig. 4). Function $u = u(\eta)$ satisfies the boundary conditions (8), (9) and Eq. (10) but Eq. (7) is not satisfied by $u = u(\eta)$ (Kollbrunner and Basler, 1969). It is evident, in this case, G can be considered as a function of u, that is we have $G = f(u)$. From Eqs. (25) and (28) it follows that (Fig. 4)

$$\mathbf{T}_z = 2\Theta G(\eta)\frac{A_m}{l}\mathbf{e}, \quad \mathbf{e} = \frac{d\mathbf{R}}{ds}. \qquad (55)$$

$$\Phi(s) = \frac{2A_m}{l}s - \int_{\overparen{P_0P}} R_n ds. \qquad (56)$$

Fig. 4.

Application of formula (27) gives

$$K = \int_A f(u(x,y))|\nabla u|^2 dA = \frac{4A_m^2}{l^2}\oiint_{a_m}\int_{-H/2}^{H/2} G(\eta)\left(1 + \frac{\eta}{\rho}\right)d\eta ds = \frac{4A_m^2}{l}\left(\Gamma_0 + \frac{2\pi}{l}\Gamma_1\right), \qquad (57)$$

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where

$$\Gamma_i = \int_{-H/2}^{H/2} \eta^i G(\eta) d\eta \qquad (i = 0, 1) \tag{58}$$

and we have used

$$dA = \left(1 + \frac{\eta}{\rho}\right) d\eta ds, \quad \oiint_{a_m} \frac{ds}{\rho} = 2\pi. \tag{59}$$

Here, ρ is the radius of curvature of smooth centerline a_m.

For thin-walled non-homogeneous circular tube, whose centerline is a circle of radius b, (Fig. 5), the exact solution of the torsional problem is known (Horgan and Chan, 1999) which is as follows

$$\mathbf{T}_z^{(e)} = \Theta G(\eta)(b + \eta)\mathbf{e}, \quad K^{(e)} = 2\pi(b^3\Gamma_0 + 3b^2\Gamma_1 + 3b\Gamma_2 + \Gamma_3), \tag{60}$$

where

$$\Gamma_i = \int_{-H/2}^{H/2} \eta^i G(\eta) d\eta \qquad (i = 0, 1, 2, 3). \tag{61}$$

The present formulation gives

$$\mathbf{T}_z = \Theta G(\eta)b\mathbf{e}, \quad K = 2\pi(b^3\Gamma_0 + b^2\Gamma_1). \tag{62}$$

Assuming that Θ is specified, from Eqs. $(60)_1$ and $(62)_1$ it follows that the relative error in stresses is smaller than 5% if $H/b \le 0.1$ independetly of material properties.
The error of torsional rigidity of approximate solution is analysed when the shear modulus is a linear function of thickness coordinate η as

$$G = G_0(1 + 12\frac{\eta}{H}) \tag{63}$$

and $b = \lambda H$. The exact and approximate values of torsional rigidity by these data are

$$K^{(e)} = 2\pi G_0 H^4 (\lambda^3 + 3\lambda^2 + 0.25\lambda + 0.15), \quad K = 2\pi G_0 H^4 (\lambda^3 + \lambda^2). \tag{64}$$

The relative error ε for the torsional rigidity is as follows

$$\varepsilon = \frac{K^{(e)} - K}{K^{(e)}} = \frac{2\lambda^2 + 0,25\lambda + 0.15}{\lambda^3 + 3\lambda^2 + 0,25\lambda + 0.15} \tag{65}$$

and ε as function of λ is shown in Fig. 6. In the present problem the relative error is smaller than 6% if $H/b \le 1/30$.

Fig. 5, Fig. 6.

An approximate analytical formulation is presented by Arghavan and Hematiyan (2008) for the torsional analysis of functionally graded hollow tubes with uniform thickness, whose

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centerlines consist of straight lines and circular arcs. Solution obtained by Arghavan and Hematiyan (2008) is more accurate as that given by formulas (55) and (57) since in their analitical formulation Eq. (14), which is the local condition of compatibility for torsion problem, is satisfied. Their approximate solution can be used for thin and moderate thick walled functionally graded hollow tubes. Paper by Arghavan and Hematiyan (2008) uses the Prandtl's stress function formulation, it does not deal with the determination of the warping of hollow tubes.

4.4 *Rectangular cross-section*

The bar with rectangular cross-section (Fig. 7) is an important structural component. The Prandtl's stress function of homogeneous rectangular cross section is known in infinite series form as (Lurje, 1970)

$$u(x,y) = b^2 - y^2 + \frac{32b^2}{\pi^3} \sum_{i=0}^{\infty} \frac{(-1)^{i+1}}{(2i+1)^3} \frac{\cosh\dfrac{(2i+1)\pi x}{2b}}{\cosh\dfrac{(2i+1)\pi a}{2b}} \cos\frac{(2i+1)\pi y}{2b}. \tag{66}$$

In this case, the application of the presented formulation gives only approximate solution for the stresses and torsional rigidity. The shear stresses can be obtained point-by-point as

$$T_{xz}(x_m, y_n) = \Theta f(x_m, y_n)\tilde{t}_{xz}(x_m, y_n), \qquad \tilde{t}_{xz}(x,y) = \frac{\partial u}{\partial y}, \tag{67}$$

$$T_{yz}(x_m, y_n) = \Theta f(x_m, y_n)\tilde{t}_{yz}(x_m, y_n), \qquad \tilde{t}_{xz}(x,y) = -\frac{\partial u}{\partial x}, \tag{68}$$

where (x_m, y_n) is an arbitrary point of the considred rectangular cross-section (Fig. 7). The approximate value of the torsional rigidity can be computed by a numerical integration which is based on Eq. (26). By the use of a suitable quadratura formula we can write

$$K = 2\sum_{m,n} p_{mn} F\left(u(x_m, y_n)\right), \tag{69}$$

where p_{mn} $(m,n = 0,1,2,...)$ are the weight coefficients of the quadratura formula used (Kopal, 1961). According to Eq. (28) independently of the shape of function $f = f(\xi)$ we have for non-homogeneous rectangular cross-section (Slaugter, 2001)

$$\Phi = xy - \frac{32a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n \sin\dfrac{(2n+1)\pi}{2a}x \sinh\dfrac{(2n+1)\pi}{2a}y}{(2n+1)^3 \cosh\dfrac{(2n+1)\pi}{2a}b}. \tag{70}$$

Fig. 7

4.5. *Narrow rectangular cross-section*

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We consider a non-homogeneous narrow rectangular cross-section when $a/b \to \infty$ or $b \to 0$ (Fig. 7). The shear modulus of the non-homogeneous cross-section is an even function of ''thickness'' coordinate y. The approximate solution of the corresponding homogeneous torsion problem (Lurje, 1970) is

$$u = b^2 - y^2, \qquad |y| \le b. \tag{71}$$

This function does not satisfy all the boundary conditions since

$$u(y, -a) \ne 0 \qquad \text{and} \qquad u(y, a) \ne 0 \qquad |y| \le b. \tag{72}$$

It is evident, in this case the shear modulus can be considered as a function of u that is, we have

$$G(y) = f(u) = f(b^2 - y^2). \tag{73}$$

Application of Eqs. (25) and (27) for the considered non-homogeneous narrow rectangular cross-section yields

$$T_{xz} = -2\Theta y G(y), \qquad K = 8a \int_{-b}^{b} y^2 G(y) dy. \tag{74}$$

These formulas for non-homogeneous narrow rectangular cross-section were derived by Kolchin (1971). Kolchin's approach is based on the solution of the boundary value problem

$$\frac{d}{dy}\left(\frac{1}{G(y)} \frac{du}{dy} \right) = -2, \qquad u(-b) = u(b) = 0. \tag{75}$$

5. Conclusions

The Saint-Venant's torsional problem of linearly elastic, isotropic, non-homogeneous cylindrical bars is treated. The shear modulus of the investigated non-homogeneous bar is a given function of the Prandtl's stress function of a homogeneous bar, which has the same cross-section as the considered non-homogeneous bar. The main result of the present paper is a contribution to the existing exact benchmark solutions for functionally graded twisted elastic cylinders. Five examples illustrate the applications of presented formulation. An extension of Bredt's theory for non-homogeneous thin-walled tubes is also given.

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Captions of Figures

Fig. 1. Non-homogeneous cross-section.

Fig. 2. Hollow elliptical cross-section.

Fig. 3. Solid equilateral triangle cross-section.

Fig. 4. Hollow thin-walled tube cross-section.

Fig. 5. Cross-section of circular tube.

Fig. 6 Relative error of torsional rigidity as a function of b/H.

Fig. 7. Rectangular cross-section.

$\vec{e}_y$

y

$2\alpha = \dfrac{\pi}{3}$

α

$\vec{e}_x$

α

x

h