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Efficient estimation of conditional covariance matrices for dimension reduction

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Abstract

We consider the problem of estimating a conditional covariance matrix in an inverse regression setting. We show that this estimation can be achieved by estimating a quadratic functional extending the results of Da Veiga & Gamboa (2008). We prove that this method provides a new efficient estimator whose asymptotic properties are studied.

1 Introduction

Consider the nonparametric regression

\[ Y = \varphi(X) + \epsilon, \]

where \( X \in \mathbb{R}^p, Y \in \mathbb{R} \) and \( \mathbb{E}[\epsilon] = 0 \). The main difficulty with any regression method is that, as the dimension of \( X \) becomes larger, the number of observations needed for a good estimator increases exponentially. This phenomena is usually called the \textit{curse of dimensionality}. All the “classical” methods could break down, as the dimension \( p \) increases, unless we have at hand a very huge sample.
For this reason, there have been along the past decades a very large number of methods to cope with this issue. Their aim is to reduce the dimensionality of the problem, using just to name a few, the generalized linear model in Brillinger (1983), the additive models in Hastie & Tibshirani (1990), sparsity constraint models as Li (2007) and references therein.

Alternatively, Li (1991a) proposed the procedure of Sliced Inverse Regression (SIR) considering the following semiparametric model,

$$Y = \phi(v_1^TX, \ldots, v_K^TX, \epsilon)$$

where the $v$'s are unknown vectors in $\mathbb{R}^p$, $\epsilon$ is independent of $X$ and $\phi$ is an arbitrary function in $\mathbb{R}^{K+1}$. This model can gather all the relevant information about the variable $Y$, with only the projection of $X$ onto the $K \ll p$ dimensional subspace $(v_1^TX, \ldots, v_K^TX)$. In the case when $K$ is small, it is possible to reduce the dimension by estimating the $v$'s efficiently. This method is also used to search nonlinear structures in data and to estimate the projection directions $v$'s. For a review on SIR methods, we refer to Li (1991a,b); Duan & Li (1991); Hardle & Tsybakov (1991) and references therein. The $v$'s define the effective dimension reduction (e.d.r) direction and the eigenvectors of $\mathbb{E}[\text{Cov}(X|Y)]$ are the e.d.r. directions. Many estimators have been proposed in order to study the e.d.r directions in many different cases. For example, Zhu & Fang (1996) and Ferré & Yao (2005, 2003) use kernel estimators, Hsing (1999) combines nearest neighbor and SIR, Bura & Cook (2001) assume that $\mathbb{E}[X|Y]$ has some parametric form, Setodji & Cook (2004) use k-means and Cook & Ni (2005) transform SIR to least square form.

In this paper, we propose an alternate estimation of the matrix

$$\text{Cov}(\mathbb{E}[X|Y]) = \mathbb{E}[\mathbb{E}[X|Y] \mathbb{E}[X|Y]^\top] - \mathbb{E}[X] \mathbb{E}[X]^\top,$$

using ideas developed by Da Veiga & Gamboa (2008), inspired by the prior work of Laurent (1996). More precisely since $\mathbb{E}[X] \mathbb{E}[X]^\top$ can be easily estimated with many usual methods, we will focus on finding an estimator of $\mathbb{E}[\mathbb{E}[X|Y] \mathbb{E}[X|Y]^\top]$. For this we will show that this estimation implies an estimation of a quadratic functional rather than plugging non parametric estimate into this form as commonly used. This method has the advantage of getting an efficient estimator in a semi-parametric framework.

This paper is organized as follows. Section 2 is intended to motivate our investigation of $\text{Cov}(\mathbb{E}[X|Y])$ using a Taylor approximation. In Section 3.1, we set up notation and hypothesis. Section 3.2 is devoted to demonstrate that each coordinate of $\text{Cov}(\mathbb{E}[X|Y])$ converge efficiently. Also we find the normality asymptotic for the whole matrix. An asymptotic bound of the variance for the quadratic part for the Taylor's expansion of $\text{Cov}(\mathbb{E}[X|Y])$ is found in Section 4. All technical Lemmas and their proofs are postponed to Sections 6 and 5 respectively.
2 Methodology

Our aim is to estimate $\text{Cov}(\mathbb{E}[X|Y])$ efficiently when observing $X \in \mathbb{R}^p$, for $p \geq 1$, and $Y \in \mathbb{R}$. For this, write the matrix

$$\text{Cov}(\mathbb{E}[X|Y]) = \mathbb{E}[X|Y] \mathbb{E}[X|Y]^\top - \mathbb{E}[X] \mathbb{E}[X]^\top,$$

where $A^\top$ means the transpose of $A$. If $\mathbb{E}[X]$ can be easily estimated by classical methods, the remainder term

$$\mathbb{E}[X|Y] \mathbb{E}[X|Y]^\top = (T^*_{ij})_{i,j} \ i, j = 1, \ldots, p;$$

is a non linear term whose estimation is the main topic of this paper. Each term of this matrix can be written as

$$T^*_{ij} = \int \left( \frac{\int x_i f(x_i, x_j, y)dx_i dx_j}{\int f(x_i, x_j, y)dx_i dx_j} \right) \left( \frac{\int x_j f(x_i, x_j, y)dx_i dx_j}{\int f(x_i, x_j, y)dx_i dx_j} \right) f(x_i, x_j, y)dx_i dx_j dy,$$

where $f(x_i, x_j, y)$ for $i$ and $j$ fixed, is the joint density of $(X_i, X_j, Y)$ $i, j = 1, \ldots, p$.

Hence, we focus on the efficient estimation of the corresponding non linear functional for $f \in L(dx_i, dx_j, dy)$

$$f \mapsto T_{ij}(f) = \int \left( \frac{\int x_i f(x_i, x_j, y)dx_i dx_j}{\int f(x_i, x_j, y)dx_i dx_j} \right) \left( \frac{\int x_j f(x_i, x_j, y)dx_i dx_j}{\int f(x_i, x_j, y)dx_i dx_j} \right) f(x_i, x_j, y)dx_i dx_j dy. \quad (2)$$

In the case $i = j$, this estimation has been considered in Da Veiga & Gamboa (2008); Laurent (1996). Here we extend their methodology to this case. Assume we have at hand an i.i.d sample $(X_i^{(k)}, X_j^{(k)}, Y^{(k)})$, $k = 1, \ldots, n$ such that it is possible to build a preliminary estimator $\hat{f}$ of $f$ with a subsample of size $n_1 < n$. Now, the main idea is to make a Taylor’s expansion of $T_{ij}(f)$ in a neighborhood of $\hat{f}$ which will play the role of a suitable approximation of $f$. More precisely, define an auxiliar function $F : [0, 1] \rightarrow \mathbb{R}$;

$$F(u) = T_{ij}(uf + (1 - u)\hat{f})$$

with $u \in [0, 1]$. The Taylor’s expansion of $F$ between 0 and 1 up to the third order is

$$F(1) = F(0) + F'(0) + \frac{1}{2} F''(0) + \frac{1}{6} F'''(\xi)(1 - \xi)^3 \quad (3)$$

for some $\xi \in [0, 1]$. Moreover, we have

$$F(1) = T_{ij}(f)$$

$$F(0) = T_{ij}(\hat{f}) = \int \left( \frac{\int x_i \hat{f}(x_i, x_j, y)dx_i dx_j}{\int \hat{f}(x_i, x_j, y)dx_i dx_j} \right) \left( \frac{\int x_j \hat{f}(x_i, x_j, y)dx_i dx_j}{\int \hat{f}(x_i, x_j, y)dx_i dx_j} \right) \hat{f}(x_i, x_j, y)dx_i dx_j dy.$$

To simplify the notations, let

$$m_i(f_u, y) = \frac{\int x_i f_u(x_i, x_j, y)dx_i dx_j}{\int f_u(x_i, x_j, y)dx_i dx_j}$$

$$m_i(f_0, y) = m_i(\hat{f}, y) = \frac{\int x_i \hat{f}(x_i, x_j, y)dx_i dx_j}{\int \hat{f}(x_i, x_j, y)dx_i dx_j},$$

$$m_i(f_u, y) - m_i(f_0, y) = \int \left( \frac{\int x_i f_u(x_i, x_j, y)dx_i dx_j}{\int f_u(x_i, x_j, y)dx_i dx_j} \right) - \frac{\int x_i \hat{f}(x_i, x_j, y)dx_i dx_j}{\int \hat{f}(x_i, x_j, y)dx_i dx_j}$$

$$dx_i dx_j dy.$$
where \( f_u = uf + (1 - u)\hat{f}, \quad \forall u \in [0, 1] \). Then, we can rewrite \( F(u) \) as
\[
F(u) = \int m_i(f_u, y)m_j(f_u, y)f_u(x_i, x_j, y)dx_idx_jdy.
\]
The Taylor’s expansion of \( T_{ij}(f) \) is given in the next Proposition.

**Proposition 1** (Linearization of the operator \( T \)). For the functional \( T_{ij}(f) \) defined in (2), the following decomposition holds
\[
T_{ij}(f) = \int H_1(\hat{f}, x_i, x_j, y)f(x_i, x_j, y)dx_idx_jdy

+ \int H_2(\hat{f}, x_{i1}, x_{j2}, y)f(x_{i1}, x_{j1}, y)f(x_{i2}, x_{j2}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy + \Gamma_n \tag{4}
\]
where
\[
H_1(\hat{f}, x_i, x_j, y) = x_im_j(\hat{f}, y) + x_jm_i(\hat{f}, y) - m_i(\hat{f}, y)m_j(\hat{f}, y) \tag{5}
\]
\[
H_2(\hat{f}, x_{i1}, x_{j2}, y) = \frac{1}{\int f(x_i, x_j, y)dx_i dx_j}(x_{i1} - m_i(\hat{f}, y))(x_{j2} - m_j(\hat{f}, y)) \tag{6}
\]
\[
\Gamma_n = \frac{1}{6}F''(\xi)(1 - \xi)^3, \tag{7}
\]
for some \( \xi \in ]0, 1[ \).

This decomposition has the main advantage of separating the terms to be estimated into a linear functional of \( f \), which can be easily estimated and a second part which is a quadratic functional of \( f \). In this case, Section 4 will be dedicated to estimate this kind of functionals and specifically to control its variance. This will enable to provide an efficient estimator of \( T_{ij}(f) \) using the decomposition of Proposition 1.

### 3 Main Results

In this section we build a procedure to estimate \( T_{ij}(f) \) efficiently. Since we used \( n_1 < n \) to build a preliminary approximation \( \hat{f} \), we will use a sample of size \( n_2 = n - n_1 \) to estimate (5) and (6). Since (5) is a linear functional of the density \( f \), it can be estimated by its empirical counterpart
\[
\frac{1}{n_2} \sum_{k=1}^{n_2} H_1(\hat{f}, X_i^{(k)}, X_j^{(k)}, Y^{(k)}). \tag{8}
\]
Since (6) is a nonlinear functional of \( f \), the estimation is harder. Its estimation will be a direct consequence of the technical results presented in Section 4 where we build an estimator for the general functional
\[
\theta(f) = \int \eta(x_{i1}, x_{j2}, y)f(x_{i1}, x_{j1}, y)f(x_{i2}, x_{j2}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy
\]
where \( \eta : \mathbb{R}^3 \to \mathbb{R} \) is a bounded function. The estimator \( \hat{\theta}_n \) of \( \theta(f) \) is an extension of the method developed in Da Veiga & Gamboa (2008).
3.1 Hypothesis and Assumptions

The following notations will be used throughout the paper. Let \( d_s \) and \( b_s \) for \( s = 1, 2, 3 \) be real numbers where \( d_s < b_s \). Let, for \( i \) and \( j \) fixed, \( \mathbb{L}^2(dx, dx, dy) \) be the squared integrable functions in the cube \([d_1, b_1] \times [d_2, b_2] \times [d_3, b_3] \). Moreover, let \((p_i(x_i, y_i, y_j))_{i \in D} \) be an orthonormal basis of \( \mathbb{L}^2(dx, dx, dy) \), where \( D \) is a countable set. Let \( a_l = \int p_i f \) denote the scalar product of \( f \) with \( p_i \).

Furthermore, denote by \( \mathbb{L}^2(dx, dx) \) (resp. \( \mathbb{L}^2(dy) \)) the set of squared integrable functions in \([d_1, b_1] \times [d_2, b_2] \) (resp. \([d_3, b_3] \)). If \((\alpha_{a}(x_i, x_j)_{i \in D_1}) \) (resp. \((\beta_{b}(y)_{i \in D_2}) \)) is an orthonormal basis of \( \mathbb{L}^2(dx, dx) \) (resp. \( \mathbb{L}^2(dy) \)) then \( p_l(x_i, x_j, y) = \alpha_{a}(x_i, x_j)\beta_{b}(y) \) with \( l = (l_{\alpha}, l_{\beta}) \in D_1 \times D_2 \).

We also use the following subset of \( \mathbb{L}^2(dx, dx, dy) \)

\[
\mathcal{E} = \left\{ \sum_{l \in D} e_l p_l : (e_l)_{l \in D} \text{ is such that } \sum_{l \in D} \left| e_l \right|^2 < 1 \right\}
\]

where \((e_l)_{l \in D} \) is a given fixed sequence.

Moreover assume that \((X_i, X_j, Y) \) have a bounded joint density \( f \) on \([d_1, b_1] \times [d_2, b_2] \times [d_3, b_3] \) which lies in the ellipsoid \( \mathcal{E} \).

In what follows, \( X_n \xrightarrow{D} X \) (resp. \( X_n \xrightarrow{P} X \)) denotes the convergence in distribution or weak convergence (resp. convergence in probability) of \( X_n \) to \( X \).

Additionally, the support of \( f \) will be denoted by \( \text{supp} f \).

Let \((M_n)_{n \geq 1} \) denote a sequence of subsets \( D \). For each \( n \) there exists \( M_n \) such that \( M_n \subseteq D \). Let us denote by \( |M_n| \) the cardinal of \( M_n \).

We shall make three main assumptions:

**Assumption 1.** For all \( n \geq 1 \) there is a subset \( M_n \subseteq D \) such that \( (\text{supp}_{l \in M_n} |c_l|^2)^2 \approx |M_n|/n^2 \) (\( A_n \approx B \) means \( \lambda_1 \leq A_n/B \leq \lambda_2 \) for some positives constants \( \lambda_1 \) and \( \lambda_2 \)). Moreover, \( \forall f \in \mathbb{L}^2(dx dy dz), \int (S_{M_n} f - f)^2 dx dy dz \to 0 \) when \( n \to 0 \), where \( S_{M_n} f = \sum_{l \in M_n} a_l p_l \)

**Assumption 2.** \( \text{supp} f \subseteq [d_1, b_1] \times [d_2, b_2] \times [d_3, b_3] \) and \( \forall (x, y, z) \in \text{supp} f, 0 < \alpha \leq f(x, y, z) \leq \beta \) with \( \alpha, \beta \in \mathbb{R} \).

**Assumption 3.** It is possible to find an estimator \( \hat{f} \) of \( f \) built with \( n_1 \approx n/\log(n) \) observations, such that for \( \epsilon > 0 \),

\[
\forall (x, y, z) \in \text{supp} f, 0 < \alpha - \epsilon \leq \hat{f}(x, y, z) \leq \beta + \epsilon
\]

and,

\[
\forall 2 \leq q \leq +\infty, \forall l \in \mathbb{N}^*, \mathbb{E}_f \| \hat{f} - f \|_q^l \leq C(q, l)n_1^{-l\lambda}
\]

for some \( \lambda > 1/6 \) and some constant \( C(q, l) \) not depending on \( f \) belonging to the ellipsoid \( \mathcal{E} \).

Assumption 1 is necessary to bound the bias and variance of \( \hat{\theta}_n \). Assumption 2 and 3 allow to establish that the remainder term in the Taylor expansion is negligible, i.e \( \Gamma_n = O(1/n) \). Assumption 3 depends on the regularity of the density function. For instance for \( x \in \mathbb{R}^p, s > 0 \) and \( L > 0 \), consider the class
\( \mathcal{H}_q(s, L) \) of Nikol’skii of functions \( f \in \mathbb{L}^q(dx) \) with partials derivatives up to order \( r = \lfloor s \rceil \) inclusive, and for each of these derivatives \( g^{(r)} \)

\[
\| f^{(r)}(\cdot + h) - f^{(r)}(\cdot) \|_q \leq L |h|^{s-r} \quad \forall h \in \mathbb{R}.
\]

Then, Assumption 3 is satisfied for \( f \in \mathcal{H}_q(s, L) \) with \( s > \frac{p}{4} \).

### 3.2 Efficient Estimation of \( T_{ij}(f) \)

As seen in Section 2, \( T_{ij}(f) \) can be decomposed as (4). Hence, using (8) and (14) we consider the following estimate

\[
\hat{T}_{ij}^{(n)} = \frac{1}{n} \sum_{k=1}^{n^2} H_1(\hat{f}, X_i^{(k)}, X_j^{(k)}, Y^{(k)}) + \frac{1}{n^2(n^2 - 1)} \sum_{l \in M, k \neq k'} \sum_{k=1}^{n^2} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) \int p_l(x_i, x_j, y^{(k')}) H_2(\hat{f}, x_i, x_j, X_i^{(k')}, X_j^{(k')}, Y^{(k')}) dx_i dx_j
\]

\[
- \frac{1}{n^2(n^2 - 1)} \sum_{l \in M, k \neq k'} \sum_{k=1}^{n^2} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) p_l(X_i^{(k')}, X_j^{(k')}, Y^{(k')}) \int p_l(x_i, x_j, y) p_l(x_i, x_j, y) H_2(\hat{f}, x_i, x_j, y) dx_i dx_j dy.
\]

where \( H_3(f, x_i, x_j, y) = H_2(f, x_i, x_j, y) + H_2(f, x_i, x_j, y) \) and \( n_2 = n - n_1 \). The remainder \( \Gamma_n \) does not appear because we will prove that it is negligible when compared to the other error terms.

The asymptotic behavior of \( \hat{T}_{ij}^{(n)} \) for \( i \) and \( j \) fixed is given in the next Theorem.

**Theorem 1.** Let Assumptions [15] hold and \( |M_n|/n \to 0 \) when \( n \to \infty \). Then:

\[
\sqrt{n}(\hat{T}_{ij}^{(n)} - T_{ij}(f)) \xrightarrow{p} \mathcal{N}(0, C_{ij}(f)), \quad (9)
\]

and

\[
\lim_{n \to \infty} n \mathbb{E} \left[ \hat{T}_{ij}^{(n)} - T_{ij}(f) \right]^2 = C_{ij}(f), \quad (10)
\]

where

\[
C_{ij}(f) = \text{Var}(H_1(f, x_i, x_j, Y)).
\]

Note that, in Theorem 1 it appears that the asymptotic variance of \( T_{ij}(f) \) depends only on \( H_1(f, x_i, x_j, Y) \). Hence the asymptotic variance of \( \hat{T}_{ij}^{(n)} \) is explained only by the linear part of (4). This will entail that the estimator is naturally efficient as proved in the following.

Indeed, the semi-parametric Cramér-Rao bound is given in the next theorem.

**Theorem 2** (Semi-parametric Cramér-Rao bound.). Consider the estimation of

\[
T_{ij}(f) = \int \left( \frac{\int x_i f(x_i, x_j, y) dx_i dx_j}{\int f(x_i, x_j, y) dx_i dx_j} \right) \left( \frac{\int x_j f(x_i, x_j, y) dx_i dx_j}{\int f(x_i, x_j, y) dx_i dx_j} \right) f(x_i, x_j, y) dx_i dx_j dy.
\]
for a random vector \((X_i, X_j, Y)\) with joint density \(f \in \mathcal{E}\). Let \(f_0 \in \mathcal{E}\) be a density verifying the assumptions of Theorem 1. Then, for all estimator \(\hat{T}^{(n)}_{ij}\) of \(T_{ij}(f)\) and every family \(\{\mathcal{V}_r(f_0)\}_{r > 0}\) of neighborhoods of \(f_0\) we have

\[
\inf_{\{\mathcal{V}_r(f_0)\}_{r > 0}} \liminf_{n \to \infty} \sup_{f \in \mathcal{V}_r(f_0)} n \mathbb{E} \left[ \left( \hat{T}^{(n)}_{ij} - T_{ij}(f_0) \right)^2 \right] \geq C_{ij}(f_0)
\]

where \(\mathcal{V}_r(f_0) = \{ f : \| f - f_0 \|_2 < r \}\) for \(r > 0\).

Consequently, the estimator \(\hat{T}^{(n)}_{ij}\) is efficient.

In the case of our estimate, its variance is \(C_{ij}(f)\), which proves its asymptotically efficiency.

Remark that Theorem 1 proves asymptotic normality entry by entry of the matrix \(T(f) = (T_{ij}(f))_{p \times p}\). To extend the result for the whole matrix it is necessary to introduce the half-vectorization operator \(\text{vech}\). This operator, stacks only the columns from the principal diagonal of a square matrix downwards in a column vector, that is, for an \(p \times p\) matrix \(A = (a_{ij})\),

\[
\text{vech}(A) = [a_{11}, \cdots, a_{p1}, a_{22}, \cdots, a_{p2}, \cdots, a_{33}, \cdots, a_{pp}]^\top.
\]

Let define the estimator matrix \(\hat{T}^{(n)} = (\hat{T}^{(n)}_{ij})\) and \(H_1(f)\) denote the matrix with entries \((H_1(f, x_i, x_j, y))_{i,j}\). Now we are able to state the following

**Corollary 1.** Let Assumptions 1-3 hold and \(|M_n|/n \to 0\) when \(n \to \infty\). Then \(\hat{T}^{(n)}\) has the following properties:

\[
\sqrt{n} \text{vech}\left(\hat{T}^{(n)} - T(f)\right) \xrightarrow{D} \mathcal{N}(0, C(f)),
\]

\[
\lim_{n \to \infty} n \mathbb{E} \left[ \text{vech}\left(\hat{T}^{(n)} - T(f)\right) \text{vech}(\hat{T}^{(n)} - T(f))^\top \right] = C(f)
\]

where

\[
C(f) = \text{Cov}\left(\text{vech}(H_1(f))\right)
\]

Previous results depend on the accurate estimation of the quadratic part of the estimator of \(T^{(n)}_{ij}\), which is the issue of the following section.

**4 Estimation of quadratic functionals**

As pointed out in Section 2 the decomposition \(4\) has a quadratic part \(6\) that we want to estimate. To achieve this we will construct a general estimator of the form:

\[
\theta = \int \eta(x_{i1}, x_{j2}, y) f(x_{i1}, x_{j1}, y) f(x_{i2}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy,
\]

for \(f \in \mathcal{E}\) and \(\eta : \mathbb{R}^3 \to \mathbb{R}\) a bounded function.
Given $M$, a subset of $D$, consider the estimator
\[
\hat{\theta}_n = \frac{1}{n(n-1)} \sum_{i,M,k \neq k'} \sum_{l,M,k \neq k'} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) 
\int p_l(x_i, x_j, Y^{(k)}) \left( \eta(x_i, X_j^{(k)}, Y^{(k)}) + \eta(X_i^{(k)}, x_j, Y^{(k)}) \right) dx_i dx_j 
- \frac{1}{n(n-1)} \sum_{l,l',M,k \neq k'} \sum_{l,l',M,k \neq k'} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) p_{l'}(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) 
\int p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i2}, x_{j2}, y)\eta(x_{i1}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy. \tag{13}
\]
In order to simplify the presentation of the main Theorem, let $\psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}, y) = \eta(x_{i1}, x_{j2}, y) + \eta(x_{i2}, x_{j1}, y)$ verifying
\[
\int \psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy = \int \psi(x_{i2}, x_{j2}, x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy. \tag{14}
\]
With this notation we can simplify (13) in
\[
\hat{\theta}_n = \frac{1}{n(n-1)} \sum_{i,M,k \neq k'} \sum_{l,M,k \neq k'} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) \int p_l(x_i, x_j, Y^{(k)}) \psi(x_i, x_j, X_i^{(k)}, X_j^{(k)}, Y^{(k)}) dx_i dx_j 
- \frac{1}{n(n-1)} \sum_{l,l',M,k \neq k'} \sum_{l,l',M,k \neq k'} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) p_{l'}(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) 
\int p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i2}, x_{j2}, y)\eta(x_{i1}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy. \tag{15}
\]
Using simple algebra, it is possible to prove that this estimator has bias equal to
\[
- \int (S_M f(x_{i1}, x_{j1}, y) - f(x_{i1}, x_{j1}, y))(S_M f(x_{i2}, x_{j2}, y) - f(x_{i2}, x_{j2}, y)) 
\eta(x_{i1}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy \tag{16}
\]
The following Theorem gives an explicit bound for the variance of $\hat{\theta}_n$.

**Theorem 3.** Let Assumption (1) hold. Then if $|M_n|/n \to 0$ when $n \to 0$, then $\hat{\theta}_n$ has the following property
\[
\left| n \mathbb{E}[\left( \hat{\theta}_n - \theta \right)^2] - \Lambda(f, \eta) \right| \leq \gamma \left[ \frac{|M_n|}{n} + \| S_M f - f \|_2 + \| S_M g - g \|_2 \right],
\]
where $g(x_i, x_j, y) = \int f(x_{i2}, x_{j2}, y) \psi(x_i, x_j, x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}$ and
\[
\Lambda(f, \eta) = \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2,
\]
where $\gamma$ is constant depending only on $\| f \|_{\infty}$, $\| \eta \|_{\infty}$, and $\Delta_{x_i x_j} = (b_1 - a_1)(b_2 - a_2)$. Moreover, this constant is an increasing function of these quantities.
Note that equation (16) implies that
\[ \lim_{n \to \infty} n \mathbb{E} \left[ \hat{\theta}_n - \theta \right]^2 = \Lambda(f, \eta). \]
These results will be stated in order to control the term
\[ Q = \int H_2(\hat{f}, x_{i1}, x_{j2}, y) f(x_{i1}, x_{j1}, y) f(x_{i2}, x_{j2}, y) dx_{i1} dx_{j1} dx_{j2} dy \]
which has the form of the quadratic functional \( \eta \) with the particular choice \( \eta(x_{i1}, x_{j2}, y) = H_2(\hat{f}, x_{i1}, x_{j2}, y) \). We point out that we also show that in this particular frame, we get \( \Lambda(f, \eta) = 0 \). This is the reason why the asymptotic variance of the estimate \( \hat{\theta}^{(n)} \) built in the previous section, is only governed by its linear part, yielding asymptotic efficiency.

5 Proofs

Proof of Proposition 1

We need to calculate the three first derivatives of \( F(u) \). In order to facilitate the calculation, we are going to differentiate \( m_i(f, y) \):
\[
\frac{d}{du} (m_i(f, y)) = \frac{d}{du} \left( \frac{\int x_i f_u(x_i, x_j, y) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j} \right)
= \frac{\int x_i (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j}
- \frac{\int x_i f_u(x_i, x_j, y) dx_i dx_j \int f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) dx_i dx_j}{\left( \int f_u(x_i, x_j, y) dx_i dx_j \right)^2},
\]
\[
= \frac{\int x_i (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j}
- \frac{m_i(f, y) \int f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j},
\]
\[
= \frac{\int (x_i - m_i(f, y)) f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j}. \tag{17}
\]

Now, using (17) we first compute \( F'(u) \):
\[
\int \frac{d}{du} (m_i(f, y)) m_j(f, y) f_u(x_i, x_j, y) + m_i(f, y) \frac{d}{du} (m_j(f, y)) f_u(x_i, x_j, y)
+ m_i(f, y) m_j(f, y) \frac{d}{du} (f_u(x_i, x_j, y)) dx_i dx_j dy,
= \int [x_i m_j(f, y) + x_j m_i(f, y) - m_i(f, y) m_j(f, y)] (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j dy.
\]
Taking \( u = 0 \) we have
\[
F'(0) = \int [x_i m_j(f, y) + x_j m_i(f, y) - m_i(f, y) m_j(f, y)] (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j dy. \tag{18}
\]
We derive now \( m_i(f_u, y)m_j(f_u, y) \) to obtain
\[
\frac{d}{du} (m_i(f_u, y)m_j(f_u, y)) = \frac{d}{du} (m_i(f_u, y)m_j(f_u, y)) + m_i(f_u, y) \frac{d}{du} (m_j(f_u, y)) \\
= m_j(f_u, y) \int \left( x_i - m_i(f_u, y) \right) \left( f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) \right) dx_i dx_j \\
+ m_i(f_u, y) \int \left( x_j - m_j(f_u, y) \right) \left( f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) \right) dx_i dx_j.
\]

Following with \( F''(u) \) and using (17) and (19) we get,
\[
F''(u) = \int \left[ x_1 \int \left( x_j - m_j(f_u, y) \right) \left( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right) dx_i dx_j \\
+ x_1 \int \left( x_i - m_i(f_u, y) \right) \left( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right) dx_j dx_i \\
- m_j(f_u, y) \int \left( x_i - m_i(f_u, y) \right) \left( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right) dx_i dx_j \\
- m_i(f_u, y) \int \left( x_j - m_j(f_u, y) \right) \left( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right) dx_i dx_j \\
\right] \right] dx_i dx_j dx_{i1} dx_{j1} dy.
\]

Simplifying the last expression we obtain
\[
F''(u) = \int \frac{1}{f_u(x_i, x_j, y) dx_i dx_j} \left\{ (x_{i1} - m_i(f_u, y)) (x_{j2} - m_j(f_u, y)) + (x_{i2} - m_i(f_u, y)) (x_{j1} - m_j(f_u, y)) \right\} \\
(f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y)) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_i dx_j dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy.
\]

Besides, when \( u = 0 \)
\[
F''(0) = \int \frac{1}{\hat{f}(x_i, x_j, y) dx_i dx_j} \left\{ (x_{i1} - m_i(\hat{f}, y)) (x_{j2} - m_j(\hat{f}, y)) + (x_{i2} - m_i(\hat{f}, y)) (x_{j1} - m_j(\hat{f}, y)) \right\} \\
(f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y)) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_i dx_j dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy = \int \frac{2}{\hat{f}(x_i, x_j, y) dx_i dx_j} (x_{i1} - m_i(\hat{f}, y)) (x_{j2} - m_j(\hat{f}, y)) \\
(f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y)) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_i dx_j dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy.
\]

(20)

(21)
Using the previous arguments we can finally find $F'''(u)$:

$$F'''(u) = \int \frac{-6}{f_u(x_i, x_j, y) dx_i dx_j} (x_{i1} - m_j(f_u, y)) (x_{j2} - m_j(f_u, y)) (f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y)) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y))$$

$$\left( f(x_{i3}, x_{j3}, y) - \hat{f}(x_{i3}, x_{j3}, y) \right) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy \quad (22)$$

Replacing (18), (21) and (22) into (3) we get the desired decomposition. □

**Proof of Theorem 1**

We will first control the remaining term (7),

$$\Gamma_n = \frac{1}{6} F'''(\xi)(1 - \xi)^3.$$

Remember that

$$F'''(\xi) = -6 \int \frac{(x_{i1} - m_j(f_\xi, y)) (x_{j2} - m_j(f_\xi, y))}{(f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y))} \left( f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y) \right) \left( f(x_{i3}, x_{j3}, y) - \hat{f}(x_{i3}, x_{j3}, y) \right) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy,$$

Assumptions 1 and 2 ensure that the first part of the integrand is bounded by a constant $\mu$. Furthermore,

$$|\Gamma_n| \leq \mu \int \left| f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right| \left| f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y) \right|$$

$$\left| f(x_{i3}, x_{j3}, y) - \hat{f}(x_{i3}, x_{j3}, y) \right| dx_{i1} dx_{j1} dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy$$

$$= \mu \int \left( \int \left| f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) \right| dx_i dx_j \right)^3 dy$$

$$\leq \mu \Delta_{x,y}^3 \int \left| f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right|^3 dx_i dx_j dy$$

by the Hölder inequality. Then $\mathbb{E}[\Gamma_n^2] = O(\mathbb{E}(\int |f - \hat{f}|^3)^2) = O(\mathbb{E}||f - \hat{f}||_q^3)$. Since $\hat{f}$ verifies Assumption 3, this quantity is of order $O(n^{-6\lambda})$. Since we also assume $n_1 \approx n / \log(n)$ and $\lambda > 1/6$, then $n^{-6\lambda} = o(1/n)$. Therefore, we get $\mathbb{E}[\Gamma_n^2] = o(1/n)$ which implies that the remaining term $\Gamma_n$ is negligible.

To prove the asymptotic normality of $\hat{T}_{ij}^{(n)}$, we shall show that $\sqrt{n} \left( \hat{T}_{ij}^{(n)} - T_{ij}(f) \right)$ and define

$$Z_{ij}^{(n)} = \frac{1}{n_2} \sum_{k=1}^{n_2} H_1(f, x_i^{(k)}, x_j^{(k)}, y^{(k)}) - \int H_1(f, x_i, x_j) f(x_i, x_j, y) dx_i dx_j dy \quad (23)$$

have the same asymptotic behavior. We can get for $Z_{ij}^{(n)}$ a classic central limit theorem with variance

$$C_{ij}(f) = \text{Var}(H_1(f, x_i, x_j))$$

$$= \int H_1(f, x_i, x_j)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int H_1(f, x_i, x_j) f(x_i, x_j, y) dx_i dx_j dy \right)^2$$

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which implies (9) and (10). In order to establish our claim, we will show that

$$R_{ij}^{(n)} = \sqrt{n} \left[ \hat{T}_{ij}^{(n)} - T_{ij}(f) - Z_{ij}^{(n)} \right]$$

(24)

has second-order moment converging to 0.

Define \( \hat{Z}_{ij}^{(n)} \) as \( Z_{ij}^{(n)} \) with \( f \) replaced by \( \hat{f} \). Let us note that \( R_{ij}^{(n)} = R_1 + R_2 \) where

$$R_1 = \sqrt{n} \left[ \hat{T}_{ij}^{(n)} - T_{ij}(f) - \hat{Z}_{ij}^{(n)} \right]$$

$$R_2 = \sqrt{n} \left[ \hat{Z}_{ij}^{(n)} - Z_{ij}^{(n)} \right].$$

It only remains to state that \( \mathbb{E}[R_1^2] \) and \( \mathbb{E}[R_2^2] \) converges to 0. We can rewrite \( R_1 \) as

$$R_1 = -\sqrt{n} \left[ \hat{Q} - Q + \Gamma_n \right]$$

where we note that

$$Q = \int H_2(\hat{f}, x_{i1}, x_{j2}, y) f(x_{i1}, x_{j1}, y) f(x_{i2}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy$$

$$H_2(\hat{f}, x_{i1}, x_{j2}, y) = \frac{1}{\int \hat{f}(x_{i1}, x_{j2}, y) dx_{i1} dx_{j2}} \left( x_{i1} - m_i(\hat{f}, y) \right) \left( x_{j2} - m_j(\hat{f}, y) \right)$$

has the form of a quadratic functional studied in Section 4 with \( \eta(x_{i1}, x_{j2}, y) = H_2(\hat{f}, x_{i1}, x_{j2}, y) \). Hence such functional can be estimated as done in Section 4 and let \( \hat{Q} \) be its corresponding estimator. Since \( \mathbb{E}[\Gamma_n^2] = o(1/n) \), we only have to control the term \( \sqrt{n}(\hat{Q} - Q) \) which is such that \( \lim_{n \to \infty} n \mathbb{E}[(\hat{Q} - Q)^2] = 0 \) by Lemma 7. This Lemma implies that \( \mathbb{E}[R_1^2] \to 0 \) as \( n \to \infty \). For \( R_2 \) we have

$$\mathbb{E}[R_2^2] = \frac{n}{n_2} \left[ \int \left( H_1(f, x_{i1}, x_{j1}, y) - H_1(\hat{f}, x_{i1}, x_{j1}, y) \right)^2 f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy \right]$$

$$- \frac{n}{n_2} \left[ \int H_1(f, x_{i1}, x_{j1}, y) f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy - \int H_1(\hat{f}, x_{i1}, x_{j1}, y)^2 f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy \right].$$

The same arguments as the ones of Lemma 7 (mean value and Assumptions 2 and 3) show that \( \mathbb{E}[R_2^2] \to 0 \).

**Proof of Theorem 2.** To prove the inequality we will use the usual framework described in Ibragimov & Khas’minskiĭ (1991). The first step is to calculate the Fréchet derivative of \( T_{ij}(f) \) at some point \( f_0 \in \mathcal{E} \). Assumptions 2 and 3 and equation (4), imply that

$$T_{ij}(f) - T_{ij}(f_0) = \int \left( x_{i1} m_i(f_0, y) + x_{j1} m_j(f_0, y) - m_i(f_0, y) m_j(f_0, y) \right)$$

$$\left( f(x_{i1}, x_{j1}, y) - f_0(x_{i1}, x_{j1}, y) \right) dx_{i1} dx_{j1} dy + O \left( \int (f - f_0)^2 \right)$$
where \( m_i(f_0, y) = \int x_i f_0(x_i, x_j, y) dx_i dx_j dy / \int f_0(x_i, x_j, y) dx_i dx_j dy \). Therefore, the Fréchet derivative of \( T_{ij}(f) \) at \( f_0 \) is \( T_{ij}(f_0) \cdot h = \langle H_1(f_0, \cdot), h \rangle \) with

\[
H_1(f_0, x_i, x_j, y) = x_i m_j(f_0, y) + x_j m_i(f_0, y) - m_i(f_0, y) m_j(f_0, y).
\]

Using the results of Ibragimov & Khas’minskii (1991), denote \( H(f_0) = \{ u \in L^2(dx_i dx_j dy), \int u(x_i, x_j, y) \sqrt{f_0(x_i, x_j, y)} dx_i dx_j dy = 0 \} \) the set of functions in \( L^2(dx_i dx_j dy) \) orthogonal to \( \sqrt{f_0} \), \( Pr_{H(f_0)} \) the projection onto \( H(f_0) \),

\[
A_n(t) = (\sqrt{f_0} t / \sqrt{n} \text{ and } P^{(n)}_f \text{ the joint distribution of } (X^{(k)}_i, X^{(k)}_j) k = 1, \ldots, n \text{ under } f_0. \]

Since \( (X^{(k)}_i, X^{(k)}_j) k = 1, \ldots, n \) are i.i.d., the family \( \{ P^{(n)}_f, f \in \mathcal{E} \} \) is differentiable in quadratic mean at \( f_0 \) and therefore locally asymptotically normal at all points \( f_0 \in \mathcal{E} \) in the direction \( H(f_0) \) with normalizing factor \( A_n(f_0) \) (see the details in Van der Vaart (2000)). Then, by the results of Ibragimov & Khas’minskii (1991) say that under these conditions, denoting \( K_n = B_n \theta'(f_0) A_n Pr_{H(f_0)} \) with \( B_n = \sqrt{n} u \), if \( K_n \xrightarrow{D} K \) and if \( K(u) = \langle t, u \rangle \), then for every estimator \( \hat{T}^{(n)}_{ij} \) of \( T_{ij}(f) \) and every family \( \mathcal{V}(f_0) \) of vicinities of \( f_0 \), we have

\[
\inf_{\mathcal{V}(f_0)} \liminf_{n \to \infty} \sup_{f \in \mathcal{V}(f_0)} n \mathbb{E} [\hat{T}^{(n)}_{ij} - T_{ij}(f_0)]^2 \geq \| t \|^2_{L^2(dx_i dx_j dy)}.
\]

Here,

\[
K_n(u) = \sqrt{n} T'(f_0) \cdot \sqrt{f_0} Pr_{H(f_0)}(u) = T'(f_0) \left( \sqrt{f_0} \left( u - \sqrt{f_0} \int u \sqrt{f_0} dx \right) \right),
\]

since for any \( u \in L^2(dx_i dx_j dy) \) we can write it as \( u = \sqrt{f_0} \sqrt{f_0} + Pr_{H(f_0)}(u) \). In this case \( K_n(u) \) does not depend on \( n \) and

\[
K(h) = T'(f_0) \cdot \sqrt{f_0} \left( u - \sqrt{f_0} \int h \sqrt{f_0} dx \right) = \int H_1(f_0, \cdot) \sqrt{f_0} u - \int H_1(f_0, \cdot) \sqrt{f_0} \int u \sqrt{f_0} \ dx = \langle t, u \rangle
\]

with

\[
t(x_i, x_j, y) = H_1(f_0, x_i, x_j, y) \sqrt{f_0} - \left( \int H_1(f_0, x_i, x_j, y) f_0 dx \right) \sqrt{f_0}.
\]

The semi-parametric Cramér-Rao bound for this problem is thus

\[
\| t \|^2_{L^2(dx_i dx_j dy)} = \int H_1(f_0, x_i, x_j, y)^2 f_0 dx_i dx_j dy - \left( \int H_1(f_0, x_i, x_j, y) f_0 dx_i dx_j dy \right)^2 = C_{ij}(f_0)
\]

and we recognize the expression \( C_{ij}(f_0) \) found in Theorem 1.

Proof of Corollary 1 The proof is based in the following observation. Employing equation (24) we have

\[
\hat{T}^{(n)}(f) - T(f) = Z^{(n)}(f) + \frac{R^{(n)}}{\sqrt{n}}
\]
where $Z^{(n)}(f)$ and $R^{(n)}$ are matrices with elements $Z_{ij}^{(n)}$ and $R_{ij}^{(n)}$, defined in (23) and (24), respectively.

Hence we have,

$$nE[\|\text{vech} \left( \hat{T}^{(n)} - T(f) - Z^{(n)}(f) \right) \|^2] = E[\|\text{vech} \left( R^{(n)} \right) \|^2] = \sum_{i \leq j} E \left[ \left( R_{ij}^{(n)} \right)^2 \right].$$

We see by Lemma 7 that $E[R_{ij}^{(n)}] \to 0$ as $n \to 0$. It follows that

$$nE[\|\text{vech} \left( \hat{T}^{(n)} - T(f) - Z^{(n)}(f) \right) \|^2] \to 0 \text{ as } n \to 0.$$  

We know that if $X_n$, $X$ and $Y_n$ are random variables, then if $X_n \overset{D}{\to} X$ and $(X_n - Y_n) \overset{D}{\to} 0$, follows that $Y_n \overset{D}{\to} X$.

Remember also that convergence in $\mathbb{L}^2$ implies convergence in probability, therefore

$$\sqrt{n} \text{vech} \left( \hat{T}^{(n)} - T(f) - Z^{(n)}(f) \right) \overset{D}{\to} 0.$$  

By the multivariate central limit theorem we have that $\sqrt{n} \text{vech} \left( Z^{(n)}(f) \right) \overset{D}{\to} \mathcal{N}(0, C(f))$. Therefore, $\sqrt{n} \text{vech} \left( \hat{T}^{(n)} - T(f) \right) \overset{D}{\to} \mathcal{N}(0, C(f)). \quad \Box$

**Proof of Theorem 3** For abbreviation, we write $M$ instead of $M_n$ and set $m = |M_n|$. We first compute the mean squared error of $\hat{\theta}_n$ as

$$E[\hat{\theta}_n - \theta]^2 = \text{Bias}^2(\hat{\theta}_n) + \text{Var}(\hat{\theta}_n)$$

where $\text{Bias}(\hat{\theta}_n) = E[\hat{\theta}_n] - \theta$.

We begin the proof by bounding $\text{Var}(\hat{\theta}_n)$. Let $A$ and $B$ be $m \times 1$ vectors with components

$$a_l = \int p_l(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \quad l = 1, \ldots, m,$$

$$b_l = \int p_l(x_i, x_j, y) f(x_i, x_j, y) \psi(x_i, x_j, x_i, x_j, x_i, x_j, x_i, x_j, y) dx_i dx_j dx_i dx_j dy$$

$$= \int p_l(x_i, x_j, y) g(x_i, x_j, y) dx_i dx_j dy \quad l = 1, \ldots, m$$

where $g(x_i, x_j, y) = \int f(x_i, x_j, y) \psi(x_i, x_j, x_i, x_j, x_i, x_j, x_i, x_j, y) dx_i dx_j dy$. Let $Q$ and $R$ be $m \times 1$ vectors of centered functions

$$q_l(x_i, x_j, y) = p_l(x_i, x_j, y) - a_l$$

$$r_l(x_i, x_j, y) = \int p_l(x_i, x_j, y) \psi(x_i, x_j, x_i, x_j, x_i, x_j, x_i, x_j, y) dx_i dx_j dy - b_l$$

for $l = 1, \ldots, m$. Let $C$ a $m \times m$ matrix of constants

$$c_{ll'} = \int p_l(x_i, x_j, y) p_{l'}(x_i, x_j, y) \eta(x_i, x_j, y) dx_i dx_j dx_i dx_j dy \quad l, l' = 1, \ldots, m.$$
Let us denote by $U_n$ the process
\[
U_n h = \frac{1}{n(n-1)} \sum_{k \neq k'=1}^{n} h(X_i^{(k)}, X_j^{(k)}, Y^{(k)}, X_i^{(k')}, X_j^{(k')}, Y^{(k')})
\]
and $P_n$ the empirical measure
\[
P_n h = \frac{1}{n} \sum_{k=1}^{n} h(X_i^{(k)}, X_j^{(k)}, Y^{(k)})
\]
for some $h$ in $L^2(dx_i, dx_j, dy)$. With these notations, $\hat{\theta}_n$ has the Hoeffding’s decomposition
\[
\hat{\theta}_n = \frac{1}{n(n-1)} \sum_{l \in M, k \neq k'=1}^{n} (q_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) + a_l) (r_{l'}(X_i^{(k')}, X_j^{(k')}, Y^{(k')}) + b_l)
\]
\[
- \frac{1}{n(n-1)} \sum_{l \in M} \sum_{k \neq k'=1}^{n} (q_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) + a_l) (q_{l'}(X_i^{(k')}, X_j^{(k')}, Y^{(k')}) + a_{l'}) c_{ll'}
\]
\[
= U_n K + P_n L + A^\top B - A^\top CA
\]
where
\[
K (x_1, x_2, x_3, x_4, x_5, x_6, x_7, y) = Q(x_1, x_2, x_3, x_4, x_5, x_6, x_7, y) R(x_1, x_2, x_3, x_4, x_5, x_6, x_7, y)
\]
\[
L(x_1, x_2, y) = A^\top R(x_1, x_2, y) + BQ(x_1, x_2, y) - 2A^\top C Q(x_1, x_2, y).
\]
Therefore $\text{Var}(\hat{\theta}_n) = \text{Var}(U_n K) + \text{Var}(P_n L) - 2 \text{Cov}(U_n K, P_n L)$. These three terms are bounded in Lemmas 2-4 which gives
\[
\text{Var}(\hat{\theta}_n) \leq \frac{20}{n(n-1)} \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta_{x, x_j}^2 (m+1) + \frac{12}{n} \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta_{x, x_j}^2.
\]
For $n$ enough large and a constant $\gamma \in \mathbb{R}$,
\[
\text{Var}(\hat{\theta}_n) \leq \gamma \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta_{x, x_j}^2 \left(\frac{m}{n^2} + \frac{1}{n}\right).
\]
The term Bias ($\hat{\theta}_n$) is easily computed, as proven in Lemma 5, is equal to
\[
- \int (S_M f(x_1, x_2, y) - f(x_1, x_2, y)) (S_M f(x_1, x_2, y) - f(x_1, x_2, y))
\]
\[
\eta(x_1, x_2, x_1, x_2, y) dx_1 dx_2 dy.
\]
From Lemma 5 the bias of $\hat{\theta}_n$ is bounded by
\[
|\text{Bias}(\hat{\theta}_n)| \leq \Delta_{x, x_j} \|\eta\|_\infty \sup_{l \in M} |c_l|^2.
\]
The assumption of $(\sup_{l \in M} |c_l|^2)^2 \approx m/n^2$ and since $m/n \to 0$, we deduce that $\mathbb{E}[\hat{\theta}_n - \theta]^2$ has a parametric rate of convergence $O(1/n)$.
Finally to prove (16), note that
\[
 n \mathbb{E} \left[ \hat{\theta}_n - \hat{\theta} \right]^2 = n \text{Bias}^2 \left( \hat{\theta}_n \right) + n \text{Var} \left( \hat{\theta}_n \right)
 = n \text{Bias}^2 \left( \hat{\theta}_n \right) + n \text{Var}(U_nK) + n \text{Var}(P_nL).
\]

We previously proved that for some \( \lambda_1, \lambda_2 \in \mathbb{R} \)
\[
 n \text{Bias}^2 \left( \hat{\theta}_n \right) \leq \lambda_1 \Delta^2_{x_i, x_j, \| \eta \|_\infty} \frac{m}{n}
 n \text{Var}(U_nK) \leq \lambda_2 \Delta^2_{x_i, x_j, \| f \|_\infty, \| \eta \|_\infty} \frac{m}{n}.
\]

Thus, Lemma 5 implies
\[
 \left| n \text{Var}(P_nL) - \Lambda(f, \eta) \right| \leq \lambda \left[ \| S_M f - f \|_2 + \| S_M g - g \|_2 \right],
\]
where \( \lambda \) is an increasing function of \( \| f \|_\infty, \| \eta \|_\infty \) and \( \Delta_{x_i, x_j} \). From all this we deduce (16) which ends the proof of Theorem 3. \( \square \)

6 Technical Results

Lemma 1 (Bias of \( \hat{\theta}_n \)). The estimator \( \hat{\theta}_n \) defined in (14) estimates \( \theta \) with bias equal to
\[
 - \int (S_M f(x_{i1}, x_{j1}, y) - f(x_{i1}, x_{j1}, y)) (S_M f(x_{i2}, x_{j2}, y) - f(x_{i2}, x_{j2}, y))
 \eta(x_{i1}, x_{j1}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy.
\]

Proof. Let \( \hat{\theta}_n = \hat{\theta}_n^1 - \hat{\theta}_n^2 \) where
\[
 \hat{\theta}_n^1 = \frac{1}{n(n-1)} \sum_{l \in M} \sum_{k \neq k'} p_l(X_i^{(k)}, X_j^{(k')}, Y^{(k')}) \int p_l(x_i, x_j, Y^{(k')}) \psi(x_i, x_j, X_i^{(k')}, X_j^{(k')}, Y^{(k')})dx_i dx_j
\]
\[
 \hat{\theta}_n^2 = - \frac{1}{n(n-1)} \sum_{l, l' \in M} \sum_{k \neq k'} p_l(X_i^{(k)}, X_j^{(k')}, Y^{(k')}) p_{l'}(X_i^{(k')}, X_j^{(k')}, Y^{(k')})
 \int p_l(x_i, x_j, y)p_{l'}(x_{i2}, x_{j2}, y) \eta(x_{i1}, x_{j1}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy.
\]
Let us first compute $E[\hat{\theta}_n^1]$.

$$
E[\hat{\theta}_n^1] = \sum_{l \in M} \int p_l(x_{i_1}, x_{j_1}, y) f(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dy
$$

$$
\quad \quad \quad = \sum_{l \in M} a_l \int p_l(x_{i_1}, x_{j_1}, y) \psi(x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, y) f(x_{i_2}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy
$$

$$
\quad \quad \quad = \int \left( \sum_{l \in M} a_l p_l(x_{i_2}, x_{j_2}, y) \right) \psi(x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, y) f(x_{i_2}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy
$$

$$
\quad \quad \quad = \int S_M f(x_{i_1}, x_{j_1}, y) f(x_{i_2}, x_{j_2}, y) \eta(x_{i_1}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy
$$

Now for $\hat{\theta}_n^2$, we get

$$
E[\hat{\theta}_n^2] = \sum_{l, l' \in M} \int p_l(x_{i_1}, x_{j_1}, y) f(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dy \int p_{l'}(x_{i_1}, x_{j_1}, y) f(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dy
$$

$$
\quad \quad \quad \quad = \sum_{l, l' \in M} a_l a_{l'} \int p_l(x_{i_1}, x_{j_1}, y) p_{l'}(x_{i_2}, x_{j_2}, y) \eta(x_{i_1}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy
$$

$$
\quad \quad \quad \quad = \int \left( \sum_{l \in M} a_l p_l(x_{i_1}, x_{j_1}, y) \right) \left( \sum_{l' \in M} a_{l'} p_{l'}(x_{i_2}, x_{j_2}, y) \right) \eta(x_{i_1}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy
$$

$$
\quad \quad \quad \quad = \int S_M f(x_{i_1}, x_{j_1}, y) S_M f(x_{i_2}, x_{j_2}, y) \eta(x_{i_1}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy.
$$

Arranging these terms and using

$$
\text{Bias} (\hat{\theta}_n) = E[\hat{\theta}_n] - \theta = E[\hat{\theta}_n^1] - E[\hat{\theta}_n^2] - \theta
$$

we obtain the desire bias.

\[ \square \]

**Lemma 2 (Bound of $\text{Var}(U_n K)$).** Under the assumptions of Theorem 3 we have

$$
\text{Var}(U_n K) \leq \frac{20}{n(n-1)} \| \eta \|_{\infty}^2 \| f \|_{\infty}^2 \Delta_{x_1 x_2} (m + 1)
$$

**Proof.** Note that $U_n K$ is centered because $Q$ and $R$ are centered and $(X^{(k)}_i, X^{(k)}_j, Y^{(k)})$, $k =
1, \ldots, n is an independent sample. So $\text{Var}(U_n K)$ is equal to

$$
\mathbb{E}[U_n K]^2 = \mathbb{E} \left( \frac{1}{n(n-1)} \sum_{k_1 \neq k_2=1}^{n} \sum_{k_2 \neq k_2' \neq 1}^{n} K(X_i^{(k_1)}, X_j^{(k_1)}, Y^{(k_1)}, X_i^{(k_2)}, X_j^{(k_2)}, Y^{(k_2)}) \right)
$$

$$
= \frac{1}{n(n-1)} \mathbb{E} \left( K^2(X_i^{(1)}, X_j^{(1)}, Y^{(1)}, X_i^{(2)}, X_j^{(2)}, Y^{(2)}) \right)
$$

Moreover, using the fact that $2 \mathbb{E}[XY] \leq \mathbb{E}[X^2] + \mathbb{E}[Y^2]$, we obtain

$$
\mathbb{E}[K^2(X_i^{(1)}, X_j^{(1)}, Y^{(1)}, X_i^{(2)}, X_j^{(2)}, Y^{(2)})] \leq 2 \left[ \mathbb{E}[(Q^T(X_i^{(1)}, X_j^{(1)}, Y^{(1)})R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}))^2] \right]
$$

$$
+ \mathbb{E}[(Q^T(X_i^{(1)}, X_j^{(1)}, Y^{(1)})CQ(X_i^{(2)}, X_j^{(2)}, Y^{(2)}))^2] .
$$

We will bound these two terms. The first one is

$$
\mathbb{E}[(Q^T(X_i^{(1)}, X_j^{(1)}, Y^{(1)})R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}))^2]
$$

$$
= \sum_{l, l' \in M} \left( \int p_l(x_i, x_j, y) p_{l'}(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - a_l a_{l'} \right)
$$

$$
\int p_l(x_i, x_j, y) p_{l'}(x_i, x_j, y) \psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}) dy
$$

$$
\psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}) f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy - b_l b_{l'} \right)
$$

$$
=W_1 - W_2 - W_3 + W_4
$$
where

\[ W_1 = \int \sum_{l,l' \in M} p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)p_l(x_{i2}, x_{j2}, y')p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{i4}, x_{j4}, x_{j2}, y')dx_{i1}dx_{i2}dx_{i3}dx_{j3}dx_{i4}dx_{j4}dy'dy' \]

\[ W_2 = \int \sum_{l,l' \in M} b_lb_{l'}p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)f(x_{i1}, x_{j1}, y)dx_{i1}dx_{j1}dy \]

\[ W_3 = \int \sum_{l,l' \in M} a_la_{l'}p_l(x_{i2}, x_{j2}, y')p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{i4}, x_{j4}, x_{j2}, y')f(x_{i4}, x_{j4}, y')dx_{i2}dx_{j2}dx_{i3}dx_{j3}dx_{i4}dx_{j4}dy'dy' \]

\[ W_4 = \sum_{l,l' \in M} a_la_{l'}b_lb_{l'} \]

W_2 and W_3 are positive, hence

\[ \mathbb{E} \left[ \left( 2Q^\top (X_{i1}^{(1)}, X_{j1}^{(1)}, Y^{(1)})R(X_{i1}^{(2)}, X_{j1}^{(2)}, Y^{(2)}) \right)^2 \right] \leq W_1 + W_4. \]

\[ W_1 = \int \sum_{l,l' \in M} p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y) \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{i4}, x_{j4}, x_{j2}, y')dx_{i2}dx_{j2} \right) \]

\[ \left( \int p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{i4}, x_{i3}, x_{j3}, y')dx_{i3}dx_{j3} \right) f(x_{i1}, x_{j1}, y)f(x_{i4}, x_{j4}, y')dx_{i1}dx_{j1}dx_{i4}dx_{j4}dy'dy' \]

\[ \leq \| f \|^2_\infty \sum_{l,l' \in M} \int p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)dx_{i1}dx_{j1}dy \]

\[ \int \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{i4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right) \]

\[ \left( \int p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{i4}, x_{i3}, x_{j3}, y')dx_{i3}dx_{j3} \right) dx_{i2}dx_{j2}dx_{i4}dx_{j4}dy'dy' \]

Since \( p_l \)'s are orthonormal we have

\[ W_1 \leq \| f \|^2_\infty \sum_{l \in M} \int \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{i4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right)^2 dx_{i4}dx_{j4}dy'dy'. \]

Moreover by the Cauchy-Schwarz inequality and \( \| \psi \|_\infty \leq 2\| \eta \|_\infty \)

\[ \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{i4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right)^2 \leq \int p_l(x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2} \]

\[ \int \psi(x_{i4}, x_{i4}, x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2} \]

\[ \leq \| \psi \|^2_\infty \Delta_{x_{i4}} \int p_l(x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2} \]

\[ \leq 4\| \eta \|^2_\infty \Delta_{x_{i4}} \int p_l(x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2}, \]

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Using the previous manipulation, we show that

\[ W = \frac{1}{4} \Delta_{x, x_0} \int p(x_{i_2}, x_{j_2}, y') dx_{i_2} dx_{j_2} dy' \]

which leads to

\[ W = 4\| \eta \|^2 \Delta_{x, x_0}^2. \]

Finally,

\[ W_1 = 4\| \eta \|^2 \Delta_{x, x_0}^2, m. \]

For the term \( W_4 \) using the facts that \( S_M f \) and \( S_M g \) are projection and that \( \int f = 1 \), we have

\[ W_4 = \left( \sum_{i \in M} a_i b_i \right)^2 \leq \sum_{i \in M} a_i^2 \sum_{i \in M} b_i^2 \leq \| f \|_2^2 \| g \|_2^2 \leq \| f \|_\infty \| g \|_\infty^2. \]

By the Cauchy-Schwartz inequality we have \( \| g \|_2^2 \leq 4\| \eta \|^2 \| f \|_\infty \Delta_{x, x_0}^2 \) and then

\[ W_4 \leq 4\| \eta \|^2 \| f \|_\infty \Delta_{x, x_0}^2, \]

which leads to

\[ \mathbb{E}\left[ (Q^\top (X_i^{(1)}, X_j^{(1)}, Y^{(1)}) R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}))^2 \right] \leq 4\| \eta \|^2 \| f \|_\infty \Delta_{x, x_0}^2 (m + 1). \quad (25) \]

The second term \( \mathbb{E}\left[ (Q^\top (X_i^{(1)}, X_j^{(1)}, Y^{(1)}) CQ(X_i^{(2)}, X_j^{(2)}, Y^{(2})) \right] = W_5 - 2W_6 + W_7 \)

where

\[ W_5 = \int \sum_{i_1} \sum_{i_2} c_{l_1} c_{l_2} p_{l_1}(x_{i_1}, x_{j_1}, y) p_{l_2}(x_{i_2}, x_{j_2}, y) p_{l_1}(x_{i_2}, x_{j_2}, y') p_{l_2}(x_{i_2}, x_{j_2}, y') f(x_{i_1}, x_{j_1}, y) f(x_{i_2}, x_{j_2}, y') dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy dy' \]

\[ W_6 = \int \sum_{i_1} \sum_{i_2} c_{l_1} c_{l_2} a_{l_1} a_{l_2} p_{l_1}(x_{i_1}, x_{j_1}, y) p_{l_2}(x_{i_2}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy \]

\[ W_7 = \sum_{i_1} \sum_{i_2} c_{l_1} c_{l_2} a_{l_1} a_{l_2} a_{l_1} a_{l_2} a_{l_1} a_{l_2}. \]

Using the previous manipulation, we show that \( W_6 \geq 0 \). Thus

\[ \mathbb{E}\left[ (Q^\top (X_i^{(1)}, X_j^{(1)}, Y^{(1)}) CQ(X_i^{(2)}, X_j^{(2)}, Y^{(2})) \right] \leq W_5 + W_7. \]
First, observe that

\[
W_5 = \sum_{l_1, l_1', l_2, l_2'} c_{l_1, l_1'} c_{l_2, l_2'} \left( \int p_{l_1}(x_{i_1}, x_{j_1}, y) p_{l_2}(x_{i_1}, x_{j_1}, y) f(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dy \right) \\
\left( \int p_{l_1'}(x_{i_2}, x_{j_2}, y') p_{l_2'}(x_{i_2}, x_{j_2}, y') f(x_{i_2}, x_{j_2}, y') dx_{i_2} dx_{j_2} dy' \right) \\
\leq \|f\|^2 \sum_{l_1, l_1', l_2, l_2'} c_{l_1, l_1'} c_{l_2, l_2'} \left( \int p_{l_1}(x_{i_1}, x_{j_1}, y) p_{l_2}(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dy \right) \\
\left( \int p_{l_1'}(x_{i_2}, x_{j_2}, y') p_{l_2'}(x_{i_2}, x_{j_2}, y') dx_{i_2} dx_{j_2} dy' \right) \\
= \|f\|^2 \sum_{l, l'} c_{l, l'}^2
\]

again using the orthonormality of the the \(p_l\)’s. Besides given the decomposition \(p_l(x_i, x_j, y) = \alpha_{l_n}(x_i, x_j) \beta_{l_n}(y)\),

\[
\sum_{l, l'} c_{l, l'}^2 = \int \left( \sum_{l_n, l_n'} \beta_{l_n}(y) \beta_{l_n'}(y) \beta_{l_n}(y') \beta_{l_n'}(y') \right) \\
\left( \int \alpha_{l_n}(x_{i_1}, x_{j_1}) \alpha_{l_n'}(x_{i_2}, x_{j_2}) \eta(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} \right) \\
\left( \int \alpha_{l_n}(x_{i_3}, x_{j_3}) \alpha_{l_n'}(x_{i_4}, x_{j_4}) \eta(x_{i_3}, x_{j_3}, y') dx_{i_3} dx_{j_3} dx_{i_4} dx_{j_4} \right) dy dy'
\]

But

\[
\sum_{l_n, l_n'} \left( \int \alpha_{l_n}(x_{i_1}, x_{j_1}) \alpha_{l_n'}(x_{i_2}, x_{j_2}) \eta(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} \right) \\
\left( \int \alpha_{l_n}(x_{i_3}, x_{j_3}) \alpha_{l_n'}(x_{i_4}, x_{j_4}) \eta(x_{i_3}, x_{j_3}, y') dx_{i_3} dx_{j_3} dx_{i_4} dx_{j_4} \right) \\
= \sum_{l_n, l_n'} \int \alpha_{l_n}(x_{i_1}, x_{j_1}) \alpha_{l_n'}(x_{i_2}, x_{j_2}) \eta(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} \eta(x_{i_3}, x_{j_3}, y') dx_{i_3} dx_{j_3} dx_{i_4} dx_{j_4} \\
= \int \sum_{l_n} \left( \int \alpha_{l_n}(x_{i_1}, x_{j_1}) \eta(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} \right) \alpha_{l_n}(x_{i_3}, x_{j_3}) \\
\sum_{l_n} \left( \int \alpha_{l_n}(x_{i_4}, x_{j_4}) \eta(x_{i_3}, x_{j_4}, y') dx_{i_4} dx_{j_4} \right) \alpha_{l_n'}(x_{i_2}, x_{j_2}) dx_{i_2} dx_{j_2} dx_{i_3} dx_{j_3} \\
\leq \int \eta(x_{i_3}, x_{j_3}, x_{i_2}, x_{j_2}, y) \eta(x_{i_3}, x_{j_2}, y') dx_{i_2} dx_{j_2} dx_{i_3} dx_{j_3} \\
\leq \Delta^2_{x, x_j} \|\eta\|_\infty^2
\]
using the orthonormality of the basis $\alpha_l$. Then we get

$$\sum_{l,l'} c_{ll'}^2 \leq \Delta_{x,x_j}^2 \|\eta\|_\infty^2 \left( \int \sum_{l,l',l''} \beta_{l,y}(y) \beta_{l',y}(y') dy dy' \right)$$

$$= \Delta_{x,x_j}^2 \|\eta\|_\infty^2 \sum_{l,l'} \left( \int \beta_{l,y}(y) \beta_{l',y}(y) dy \right)^2$$

$$\leq \Delta_{x,x_j}^2 \|\eta\|_\infty^2 \sum_{l,l'} \left( \int \beta_{l,y}^2(y) dy \right)^2$$

$$\leq \Delta_{x,x_j}^2 \|\eta\|_\infty^2 m$$

since the $\beta_{l,y}$ are orthonormal. Finally

$$W_5 \leq \|f\|_\infty^2 \|\eta\|_\infty^2 \Delta_{x,x_j}^2 m.$$  

Now for $W_7$ we first will bound,

$$\left| \sum_{l,l'} c_{ll'} a_{l}a_{l'} \right| = \left| \int \sum_{l,l',l''} a_{l}a_{l'} p_{l}(x_{i1},x_{j1},y)p_{l'}(x_{i2},x_{j2},y) \eta(x_{i1},x_{j2},y) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy \right|$$

$$\leq \int |S_M(x_{i1},x_{j1},y)S_M(x_{i2},x_{j2},y) \eta(x_{i1},x_{j2},y)| dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy$$

$$\leq \|\eta\|_\infty \int \left( \int |S_M(x_{i1},x_{j1},y)S_M(x_{i2},x_{j2},y)| dy \right) dx_{i1} dx_{i2} dx_{j1} dx_{j2}.$$

Taking squares in both sides and using the Cauchy-Schwartz inequality twice, we get

$$\left( \sum_{l,l'} c_{ll'} a_{l}a_{l'} \right)^2 = \|\eta\|_\infty^2 \left( \int \left( \int |S_M(x_{i1},x_{j1},y)S_M(x_{i2},x_{j2},y)| dy \right) dx_{i1} dx_{i2} dx_{j1} dx_{j2} \right)^2$$

$$\leq \|\eta\|_\infty^2 \Delta_{x,x_j}^2 \left( \int \left( \int |S_M(x_{i1},x_{j1},y)S_M(x_{i2},x_{j2},y)| dy \right)^2 dx_{i1} dx_{i2} dx_{j1} dx_{j2} \right)$$

$$\leq \|\eta\|_\infty^2 \Delta_{x,x_j}^2 \left( \int \left( \int |S_M(x_{i1},x_{j1},y)^2 dy \right) \left( \int |S_M(x_{i2},x_{j2},y')^2 dy' \right) dx_{i1} dx_{i2} dx_{j1} dx_{j2} \right)$$

$$\leq \|\eta\|_\infty^2 \Delta_{x,x_j}^2 \left( \int S_M(x_{i1},x_{j1},y)^2 dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy dy' \right)$$

$$= \|\eta\|_\infty^2 \Delta_{x,x_j}^2 \left( \int S_M(x_{i1},x_{j1},y)^2 dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy' \right)$$

$$\leq \|\eta\|_\infty^2 \Delta_{x,x_j}^2 \|f\|_\infty^2.$$  

Finally,

$$E \left[ (Q^\top (X^{(1)}_i, X^{(1)}_j, Y^{(1)}_j) C Q (X^{(2)}_i, X^{(2)}_j, Y^{(2)}_j))^2 \right] \leq \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta_{x,x_j}^2 (m + 1).$$  (26)
Collecting (25) and (26), we obtain

\[ \text{Var}(U_n K) \leq \frac{20}{n(n-1)} \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta^2_{x_i x_j}(m + 1) \]

which concludes the proof of Lemma 2. \qed

**Lemma 3** (Bound for \( \text{Var}(P_n L) \)). **Under the assumptions of Theorem 3**, we have

\[ \text{Var}(P_n L) \leq \frac{12}{n} \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta^2_{x_i x_j}. \]

**Proof.** First note that given the independence of \((X_i^{(k)}, X_j^{(k)}, Y^{(k)})\) \(k = 1, \ldots, n\) we have

\[ \text{Var}(P_n L) = \frac{1}{n} \text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) \]

we can write \(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})\) as

\[
A^\top R \left(X_i^{(1)}, X_j^{(1)}, Y^{(1)}\right) + B^\top Q \left(X_i^{(1)}, X_j^{(1)}, Y^{(1)}\right) - 2A^\top CQ \left(X_i^{(1)}, X_j^{(1)}, Y^{(1)}\right)
\]

\[= \sum_{l \in M} a_l \left( \int p_l(x_i, x_j, Y^{(1)}) \psi(x_i, x_j, X_i^{(1)}, X_j^{(1)}, Y^{(1)}) dx_i dx_j - b_l \right)
+ \sum_{l \in M} b_l \left( p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - a_l \right) - 2 \sum_{l, l' \in M} c_{ll'} a_{ll'} \left( p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - a_l \right)
\]

\[= \int \sum_{l \in M} a_l p_l(x_i, x_j, Y^{(1)}) \psi(x_i, x_j, X_i^{(1)}, X_j^{(1)}, Y^{(1)}) dx_i dx_j
+ \sum_{l \in M} b_l p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)} - 2 \sum_{l, l' \in M} c_{ll'} a_{ll'} p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^\top B - 2A^\top CA.
\]

\[= \int S_M f(x_i, x_j, Y^{(1)}) \psi(x_i, x_j, X_i^{(1)}, X_j^{(1)}, Y^{(1)}) dx_i dx_j + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)})
- 2 \sum_{l, l' \in M} c_{ll'} a_{ll'} p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^\top B - 2A^\top CA.
\]

Let \(h(x_i, x_j, y) = \int S_M f(x_i, x_j, y) \psi(x_i, x_j, x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}, we have

\[
S_M h(x_i, x_j, y)
= \sum_{l \in M} \left( \int h(x_{i2}, x_{j2}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dy \right) p_l(x_i, x_j, y)
= \sum_{l \in M} \left( \int S_M f(x_{i3}, x_{j3}, y) \psi(x_{i2}, x_{j2}, x_{i3}, x_{j3}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy \right) p_l(x_i, x_j, y)
= \sum_{l, l' \in M} \left( \int a_{ll'} p_{ll'}(x_{i3}, x_{j3}, y) \psi(x_{i2}, x_{j2}, x_{i3}, x_{j3}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy \right) p_l(x_i, x_j, y)
= 2 \sum_{l, l' \in M} \left( \int a_{ll'} p_{ll'}(x_{i3}, x_{j3}, y) \eta(x_{i2}, x_{j2}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy \right) p_l(x_i, x_j, y)
= 2 \sum_{l, l' \in M} a_{ll'} p_{ll'}(x_i, x_j, y)
\]
and we can write

\[ L(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) = h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^\top B - 2A^\top CA. \]

Thus,

\[
\text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) \\
= \text{Var}(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) \\
\leq \mathbb{E}[(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \\
\leq \mathbb{E}[(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2 + (S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2 + (S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2].
\]

Each of these terms can be bounded

\[
\mathbb{E}[(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \\
= \int \left( \int S_M f(x_{i1}, x_{j2}, y) \psi(x_{i1}x_{j2}, x_{i1}, x_{j2}, y) dx_{i1}dx_{j2} \right)^2 f(x_{i1}, x_{j2}, y) dx_{i1}dx_{j2}dy \\
\leq \Delta_{x_{i1},x_{j2}} \int S_M f(x_{i1}, x_{j2}, y)^2 \psi(x_{i1}x_{j2}, x_{i1}, x_{j2}, y)^2 f(x_{i1}, x_{j2}, y) dx_{i1}dx_{j2}dy \\
\leq 4\Delta^2_{x_{i1},x_{j2}} \|f\|_\infty \|\eta\|_\infty^2 \int S_M f(x_{i1}, x_{j2}, y)^2 dx_{i1}dx_{j2}dy \\
= 4\Delta^2_{x_{i1},x_{j2}} \|f\|_\infty \|\eta\|_\infty^2 \|S_M f\|_2^2 \\
\leq 4\Delta^2_{x_{i1},x_{j2}} \|f\|_\infty \|\eta\|_\infty^2 \|f\|_2^2 \\
\leq 4\Delta^2_{x_{i1},x_{j2}} \|f\|_\infty \|\eta\|_\infty^2
\]

and similar calculations are valid for the others two terms,

\[
\mathbb{E}[(S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \leq \|f\|_\infty \|S_M g\|_2^2 \leq \|f\|_\infty \|g\|_2^2 \leq 4\Delta^2_{x_{i1},x_{j2}} \|f\|_\infty \|\eta\|_\infty^2 \|g\|_2^2 \\
\mathbb{E}[(S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \leq \|f\|_\infty \|S_M h\|_2^2 \leq \|f\|_\infty \|h\|_2^2 \leq 4\Delta^2_{x_{i1},x_{j2}} \|f\|_\infty \|\eta\|_\infty^2 \|h\|_2^2.
\]

Finally we get,

\[
\text{Var}(P_n L) \leq \frac{12}{n} \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta^2_{x_{i1},x_{j2}}.
\]

\[\square\]

**Lemma 4** (Computation of Cov$(U_n K, P_n L)$). Under the assumptions of Theorem \[\text{we have} \]

\[
\text{Cov}(U_n K, P_n L) = 0.
\]
Proof of Lemma 4 Since \( U_nK \) and \( P_nL \) are centered, we have

\[
\text{Cov}(U_nK, P_nL) = \mathbb{E}[U_nK P_nL] = \mathbb{E}
\left[ \frac{1}{n^2(n-1)} \sum_{k \neq k'}^n K(X_i^{(k)}, X_j^{(k)}, Y^{(k)}, X_i^{(k')}, X_j^{(k')}, Y^{(k')}) \sum_{k=1}^n L(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) \right
\]

\[
= \frac{1}{n} \mathbb{E} \left[ K(X_i^{(1)}, X_j^{(1)}, Y^{(1)}, X_i^{(2)}, X_j^{(2)}, Y^{(2)}) (L(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + L(X_i^{(2)}, X_j^{(2)}, Y^{(2)})) \right
\]

\[
= \frac{1}{n} \mathbb{E} \left[ (Q^T(X_i^{(1)}, X_j^{(1)}, Y^{(1)})R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) - Q^T(X_i^{(1)}, X_j^{(1)}, Y^{(1)})CQ(X_i^{(2)}, X_j^{(2)}, Y^{(2)})) \right
\]

\[
+ A^T R(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + B^T Q(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^T CQ(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) \right
\]

\[= 0. \]

Since \( K, L, Q \) and \( R \) are centered. □

Lemma 5 (Bound of Bias \((\hat{\theta}_n)\)). Under the assumptions of Theorem 3 we have

\[
\left| \text{Bias}(\hat{\theta}_n) \right| \leq \Delta_{x_i,x_j} \|\eta\|_{\infty} \sup_{l \notin M} |c_l|^2. \]

Proof.

\[
\left| \text{Bias}(\hat{\theta}_n) \right| \leq \|\eta\|_{\infty} \int \left( \int |S_M f(x_{i1}, x_{j1}, y)| \, dx_{i1} \, dx_{j1} \right)
\]

\[
\left( \int |S_M f(x_{i2}, x_{j2}, y)| \, dx_{i2} \, dx_{j2} \right) \, dy
\]

\[= \|\eta\|_{\infty} \int \left( \int |S_M f(x_i, x_j, y) - f(x_i, x_j, y)| \, dx_i \, dx_j \right)^2 \, dy
\]

\[\leq \Delta_{x_i,x_j} \|\eta\|_{\infty} \int (S_M f(x_i, x_j, y) - f(x_i, x_j, y))^2 \, dx_i \, dx_j \, dy
\]

\[= \Delta_{x_i,x_j} \|\eta\|_{\infty} \sum_{l \notin M} a_l a_l' \int p_l(x_i, x_j, y) p_l'(x_i, x_j, y) \, dx_i \, dx_j \, dy
\]

\[= \Delta_{x_i,x_j} \|\eta\|_{\infty} \sum_{l \notin M} |a_l|^2 \leq \Delta_{x_i,x_j} \|\eta\|_{\infty} \sup_{l \notin M} |c_l|^2. \]

We use the Hölder's inequality and the fact that \( f \in \mathcal{E} \) then \( \sum_{l \notin M} |a_l|^2 \leq \sup_{l \notin M} |c_l|^2. \) □

Lemma 6 (Asymptotic variance of \( \sqrt{n}(P_nL) \)). Under the assumptions of Theorem 3 we have

\[
n \text{Var}(P_nL) \to \Lambda(f, \eta)
\]

where

\[
\Lambda(f, \eta) = \int g(x_i, x_j, y)^2 f(x_i, x_j, y) \, dx_i \, dx_j \, dy - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) \, dx_i \, dx_j \, dy \right)^2.
\]

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Proof. We proved in Lemma 3 that
\[ \text{Var}(L(X_1^{(1)}, X_j^{(1)}, Y^{(1)})) = \text{Var}(h(X_1^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_1^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M h(X_1^{(1)}, X_j^{(1)}, Y^{(1)})) = \sum_{k,l=1}^{3} \text{Cov}(A_k, A_l). \]

We claim that for all \( k, l \in \{1, 2, 3\}^2 \), we have
\[ \left| \text{Cov}(A_k, A_l) - \epsilon_{kl} \right| \left( \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right) \leq \lambda \left[ \| S_M f - f \|_2 + \| S_M g - g \|_2 \right] \tag{27} \]
where
\[ \epsilon_{kl} = \begin{cases} -1 & \text{if } k = 3 \text{ or } l = 3 \text{ and } k \neq l \\ 1 & \text{otherwise} \end{cases}, \]
and where \( \lambda \) depends only on \( \| f \|_\infty, \| \eta \|_\infty \) and \( \Delta_{x,x_j} \). We will do the details only for the case \( k = l = 3 \) since the calculations are similar for other configurations.

\begin{align*}
\text{Var}(A_3) &= \int S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \left( \int S_M h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2.
\end{align*}

The computation will be done in two steps. We first bound the quantity by the Cauchy-Schwartz inequality
\[ \left| \int S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy \right| \leq \int |S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) - S_M g(x_i, x_j, y) f(x_i, x_j, y)| dx_i dx_j dy \\
&\quad + \int |S_M h(x_i, x_j, y) f(x_i, x_j, y) - g(x_i, x_j, y)^2 f(x_i, x_j, y)| dx_i dx_j dy \\
&\leq \| f \|_\infty \| S_M h - S_M g \|_2 \| S_M h \|_2 + \| f \|_\infty \| S_M g + g \|_2 \| S_M g - g \|_2.
\]
Using several times the fact that since \( S_M \) is a projection, \( \| S_M g \|_2 \leq \| g \|_2 \), the sum is bounded by
\[ \| f \|_\infty \| h + g \|_2 \| h - g \|_2 + 2 \| f \|_\infty \| g \|_2 \| S_M g - g \|_2 \\
\leq \| f \|_\infty (\| h \|_2 + \| g \|_2) \| h - g \|_2 + 2 \| f \|_\infty \| g \|_2 \| S_M g - g \|_2.
\]
We saw previously that \( \| g \|_2 \leq 2\Delta_{x,x_j} \| f \|_\infty^{1/2} \| \eta \|_\infty \) and \( \| h \|_2 \leq 2\Delta_{x,x_j} \| f \|_\infty^{1/2} \| \eta \|_\infty \). The sum is then bound by
\[ 4\Delta_{x,x_j} \| f \|_\infty^{3/2} \| \eta \|_\infty \| h - g \|_2 + 4\Delta_{x,x_j} \| f \|_\infty^{3/2} \| \eta \|_\infty \| S_M g - g \|_2. \]
We now have to deal with \( \| h - g \|_2^2 \):
\[
\| h - g \|_2^2 = \int \left( \int (S_M f(x_i, x_j, y) - f(x_i, x_j, y)) \psi(x_i, x_j, x_i, x_j, y) dx_i dx_j dy \right)^2 dx_i dx_j dy
\leq \int \left( \int (S_M f(x_i, x_j, y) - f(x_i, x_j, y))^2 dx_i dx_j dy \right) \left( \int \psi^2(x_i, x_j, x_i, x_j, y) dx_i dx_j dy \right) dx_i dx_j dy
\leq 4\Delta^2 \| \eta \|_\infty^2 \| S_M f - f \|_2^2.
\]
Finally this first part is bounded by
\[
\left| \int S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy \right|
\leq 4\Delta \| f \|_\infty^{3/2} \| \eta \|_\infty \left( 2\Delta \| \eta \|_\infty \| S_M f - f \|_2 + \| S_M g - g \|_2 \right).
\]
Following with the second quantity
\[
\left| \left( \int S_M h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right|
= \left| \left( \int (S_M h(x_i, x_j, y) - g(x_i, x_j, y)) f(x_i, x_j, y) dx_i dx_j dy \right) \left( \int (S_M h(x_i, x_j, y) + g(x_i, x_j, y)) f(x_i, x_j, y) dx_i dx_j dy \right) \right|
\]
\[
\leq 4\Delta \| f \|_\infty^{3/2} \| \eta \|_\infty \left( 2\Delta \| \eta \|_\infty \| S_M f - f \|_2 + \| S_M g - g \|_2 \right)
\]
using the previous calculations. Collecting the two inequalities gives (27) for \( k = l = 3 \). Finally, since by assumption \( \forall t \in \mathbb{L}^2(d\mu), \| S_M t - t \|_2 \to 0 \) when \( n \to \infty \) a direct consequence of (27) is
\[
\lim_{n \to \infty} \text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) = g^2(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 = \Lambda(f, \eta).
\]
We conclude by noting that \( \text{Var}(\sqrt{n}(P_n L)) = \text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) \).
Lemma 7 (Asymptotics for $\sqrt{n}(\hat{Q} - Q)$). Under the assumptions of Theorem 7 we have

$$\lim_{n \to \infty} n \mathbb{E} [\hat{Q} - Q]^2 = 0.$$ 

Proof. The bound given in (16) states that if $|M_n|/n \to 0$ we have

$$\left| \frac{n \mathbb{E} [(\hat{Q} - Q)^2]}{\mathbb{E} [(\hat{Q} - Q)^2]} \right| \leq \gamma \left( \|f\|_\infty, \|\eta\|_\infty, \Delta_{x,y} \right) \left[ \frac{|M_n|}{n} + \|S_M f - f\|_2 + \|S_M \hat{g} - \hat{g}\|_2 \right]$$

where $\hat{g}(x, y) = \int H_3(f, x_i, x_j, x_{i2}, y) \cdot S_{xy} \cdot dy \cdot dx_{i2} \cdot dx_{j2}$, where we recall that $H_3(f, x_i, x_j, x_{i2}, x_{j2}) = H_2(f, x_i, x_j, y) + H_2(f, x_{i2}, x_{j2}, y)$ with $H_2(f, x_i, x_j, y) = \int f(x_i, x_j, x_{i2}, x_{j2}, y) \cdot S_{xy} \cdot dy \cdot dx_{i2} \cdot dx_{j2}$. By deconditioning we get

$$\left| \frac{n \mathbb{E} [(\hat{Q} - Q)^2]}{\mathbb{E} [(\hat{Q} - Q)^2]} \right| \leq \gamma \left( \|f\|_\infty, \|\eta\|_\infty, \Delta_{x,y} \right) \left[ \frac{|M_n|}{n} + \|S_M f - f\|_2 + \mathbb{E} [\|S_M \hat{g} - \hat{g}\|_2] \right]$$

Note that

$$\mathbb{E} [\|S_{M_n} \hat{g} - \hat{g}\|_2^2] \leq \mathbb{E} [\|S_M \hat{g} - S_M g\|_2^2] + \mathbb{E} [\|\hat{g} - g\|_2^2] + \mathbb{E} [\|S_{M_n} g - g\|_2^2]$$

where $g(x_i, x_j, y) = \int H_3(f, x_i, x_j, x_{i2}, x_{j2}) \cdot S_{xy} \cdot dy \cdot dx_{i2} \cdot dx_{j2}$. The second term converges to 0 since $g \in \mathbb{L}^2(dx_{i2} dx_{j2})$ and $\forall t \in \mathbb{L}^2(dx_{i2} dx_{j2})$, $\int (S_{M_n} t - t)^2 dx_{i2} dx_{j2} \to 0$ . Moreover

$$\|\hat{g} - g\|_2^2 = \int (\hat{g}(x_i, x_j, y) - g(x_i, x_j, y))^2 dx_{i2} dx_{j2}$$

$$= \int \left[ \int (H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y)) \cdot S_{xy} \cdot dy \cdot dx_{i2} \cdot dx_{j2} \right]^2 dx_{i2} dx_{j2}$$

$$\leq \int \left[ \int (H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y)) \cdot S_{xy} \cdot dy \cdot dx_{i2} \cdot dx_{j2} \right]$$

$$\leq \Delta_{x_i x_j} \|f\|_\infty^2 \cdot \int (H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y))^2 dx_{i2} dx_{j2}$$

$$\leq \delta \Delta_{x_i x_j} \|f\|_\infty^2 \cdot \int (\hat{f}(x_i, x_j, y) - f(x_i, x_j, y))^2 dx_{i2} dx_{j2}$$

for some constant $\delta$ that comes out of applying the mean value theorem to $H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y)$. The constant $\delta$ was taken under Assumptions 7. Since

$$\mathbb{E} [\|f - \hat{f}\|_2] \to 0$$

then $\mathbb{E} [\|g - \hat{g}\|_2] \to 0$. Now show that the expectation of

$$\int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_{i2} dx_{j2} dy - \left( \int \hat{g}(x_i, x_j, y) f(x_i, x_j, y) dx_{i2} dx_{j2} dy \right)^2$$

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converges to 0. We develop the proof for only the first term. We get
\[
\left| \int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy \right|
\leq \int |\hat{g}(x_i, x_j, y)^2 - g(x_i, x_j, y)^2| f(x_i, x_j, y) dx_i dx_j dy
\leq \lambda \int (\hat{g}(x_i, x_j, y) - g(x_i, x_j, y))^2 dx_i dx_j dy
= \lambda \| \hat{g} - g \|_2^2
\]
for some constant $\lambda$. By taking the expectation of both sides, we see it is enough to show that $E[\| \hat{g} - g \|_2^2] \to 0$. Besides, we can verify
\[
\hat{g}(x_i, x_j, y) = \int H_3(f, x_i, x_j, x_{i2}, x_{j2}, y) f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}
= \frac{2}{\int f(x_i, x_j, y) dx_i dx_j} \left( \int x_{j2} f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} - \hat{m}_j(y) \int f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} \right)
= 0
\]
which proves that the expectation of $\int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j$ converges to 0. Similar computations shows that the expectation of $\left( \int \hat{g}(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j \right)^2$ also converges to 0. Finally we have
\[
\lim_{n \to \infty} nE[\hat{Q} - Q]^2 = 0.
\]

References


