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Efficient estimation of conditional covariance matrices for dimension reduction

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Abstract

We consider the problem of estimating a conditional covariance matrix in an inverse regression setting. We show that this estimation can be achieved by estimating a quadratic functional extending the results of Da Veiga & Gamboa (2008). We prove that this method provides a new efficient estimator whose asymptotic properties are studied.

1 Introduction

Consider the nonparametric regression

\[ Y = \varphi(X) + \epsilon, \]

where \( X \in \mathbb{R}^p, \ Y \in \mathbb{R} \) and \( \mathbb{E}[\epsilon] = 0 \). The main difficulty with any regression method is that, as the dimension of \( X \) becomes larger, the number of observations needed for a good estimator increases exponentially. This phenomena is usually called the curse of dimensionality. All the “classical” methods could break down, as the dimension \( p \) increases, unless we have at hand a very huge sample.
For this reason, there have been along the past decades a very large number of methods to cope with this issue. Their aim is to reduce the dimensionality of the problem, using just to name a few, the generalized linear model in Brillinger (1983), the additive models in Hastie & Tibshirani (1990), sparsity constraint models as Li (2007) and references therein.

Alternatively, Li (1991a) proposed the procedure of Sliced Inverse Regression (SIR) considering the following semiparametric model,

$$Y = \phi(v_1^\top X, \ldots, v_K^\top X, \epsilon)$$

where the $v_i$'s are unknown vectors in $\mathbb{R}^p$, $\epsilon$ is independent of $X$ and $\phi$ is an arbitrary function in $\mathbb{R}^{K+1}$. This model can gather all the relevant information about the variable $Y$, with only the projection of $X$ onto the $K \ll p$ dimensional subspace $(v_1^\top X, \ldots, v_K^\top X)$. In the case when $K$ is small, it is possible to reduce the dimension by estimating the $v_i$'s efficiently. This method is also used to search nonlinear structures in data and to estimate the projection directions $v_i$'s. For a review on SIR methods, we refer to Li (1991a,b); Duan & Li (1991); Hardle & Tsybakov (1991) and references therein. The $v_i$'s define the effective dimension reduction (e.d.r) direction and the eigenvectors of $\mathbb{E}[\text{Cov}(X|Y)]$ are the e.d.r. directions. Many estimators have been proposed in order to study the e.d.r directions in many different cases. For example, Zhu & Fang (1996) and Ferré & Yao (2005, 2003) use kernel estimators, Hsing (1999) combines nearest neighbor and SIR, Bura & Cook (2001) assume that $\mathbb{E}[X|Y]$ has some parametric form, Setodji & Cook (2004) use k-means and Cook & Ni (2005) transform SIR to least square form.

In this paper, we propose an alternate estimation of the matrix

$$\text{Cov}(\mathbb{E}[X|Y]) = \mathbb{E}[\mathbb{E}[X|Y] \mathbb{E}[X|Y]^\top] - \mathbb{E}[X] \mathbb{E}[X]^\top,$$

using ideas developed by Da Veiga & Gamboa (2008), inspired by the prior work of Laurent (1996). More precisely since $\mathbb{E}[X] \mathbb{E}[X]^\top$ can be easily estimated with many usual methods, we will focus on finding an estimator of $\mathbb{E}[\mathbb{E}[X|Y] \mathbb{E}[X|Y]^\top]$. For this we will show that this estimation implies an estimation of a quadratic functional rather than plugging non parametric estimate into this form as commonly used. This method has the advantage of getting an efficient estimator in a semi-parametric framework.

This paper is organized as follows. Section 2 is intended to motivate our investigation of $\text{Cov}(\mathbb{E}[X|Y])$ using a Taylor approximation. In Section 3.1 we set up notation and hypothesis. Section 3.2 is devoted to demonstrate that each coordinate of $\text{Cov}(\mathbb{E}[X|Y])$ converge efficiently. Also we find the normality asymptotic for the whole matrix. An asymptotic bound of the variance for the quadratic part for the Taylor’s expansion of $\text{Cov}(\mathbb{E}[X|Y])$ is found in Section 4. All technical Lemmas and their proofs are postponed to Sections 5 and 6 respectively.
2 Methodology

Our aim is to estimate $\text{Cov}(E[X|Y])$ efficiently when observing $X \in \mathbb{R}^p$, for $p \geq 1$, and $Y \in \mathbb{R}$. For this, write the matrix

$$\text{Cov}(E[X|Y]) = E[E[X|Y]E[X|Y]'] - E[X]E[X]'$$

where $A'$ means the transpose of $A$. If $E[X]$ can be easily estimated by classical methods, the remainder term

$$E[E[X|Y]E[X|Y]'] = (T^*_i)_{ij} \quad i, j = 1, \ldots, p;$$

is a non linear term whose estimation is the main topic of this paper. Each term of this matrix can be written as

$$T^*_i = \int \left( \frac{\int x_i f(x_i, x_j, y)dx_j}{\int f(x_i, x_j, y)dx_j} \right) \left( \frac{\int x_j f(x_i, x_j, y)dx_j}{\int f(x_i, x_j, y)dx_j} \right) f(x_i, x_j, y)dx_i dx_j dy,$$

where $f(x_i, x_j, y)$ for $i$ and $j$ fixed, is the joint density of $(X_i, X_j, Y)$ where $i, j = 1, \ldots, p$.

Hence, we focus on the efficient estimation of the corresponding non linear functional for $f \in L(dx_i, dx_j, dy)$

$$f \mapsto T_{ij}(f) = \int \left( \frac{\int x_i f(x_i, x_j, y)dx_j}{\int f(x_i, x_j, y)dx_j} \right) \left( \frac{\int x_j f(x_i, x_j, y)dx_j}{\int f(x_i, x_j, y)dx_j} \right) f(x_i, x_j, y)dx_i dx_j dy.$$

(1)

In the case $i = j$, this estimation has been considered in Da Veiga & Gamboa (2008); Laurent (1996). Here we extend their methodology to this case. Assume we have at hand an i.i.d sample $(X^{(k)}_i, X^{(k)}_j, Y^{(k)})$, $k = 1, \ldots, n$ such that it is possible to build a preliminary estimator $\hat{f}$ of $f$ with a subsample of size $n_1 < n$. Now, the main idea is to make a Taylor’s expansion of $T_{ij}(f)$ in a neighborhood of $\hat{f}$ which will play the role of a suitable approximation of $f$. More precisely, define an auxiliary function $F : [0, 1] \rightarrow \mathbb{R}$;

$$F(u) = T_{ij}(uf + (1 - u)\hat{f})$$

with $u \in [0, 1]$. The Taylor’s expansion of $F$ between 0 and 1 up to the third order is

$$F(1) = F(0) + F'(0) + \frac{1}{2} F''(0) + \frac{1}{6} F'''(\xi)(1 - \xi)^3$$

(3)

for some $\xi \in [0, 1]$. Moreover, we have

$F(1) = T_{ij}(f)$

$F(0) = T_{ij}(\hat{f}) = \int \left( \frac{\int x_i \hat{f}(x_i, x_j, y)dx_j}{\int \hat{f}(x_i, x_j, y)dx_j} \right) \left( \frac{\int x_j \hat{f}(x_i, x_j, y)dx_j}{\int \hat{f}(x_i, x_j, y)dx_j} \right) \hat{f}(x_i, x_j, y)dx_i dx_j dy.$

To simplify the notations, let

$$m_i(f_u, y) = \frac{\int x_i f_u(x_i, x_j, y)dx_j}{\int f_u(x_i, x_j, y)dx_j}$$

$$m_i(f_0, y) = m_i(\hat{f}, y) = \frac{\int x_i \hat{f}(x_i, x_j, y)dx_j}{\int \hat{f}(x_i, x_j, y)dx_j},$$
where \( f_u = uf + (1 - u)\hat{f}, \quad \forall u \in [0, 1] \). Then, we can rewrite \( F(u) \) as

\[
F(u) = \int m_i(f_u, y)m_j(f_u, y)f_u(x_i, x_j, y)dx_idx_jdy.
\]

The Taylor’s expansion of \( T_{ij}(f) \) is given in the next Proposition.

**Proposition 1** (Linearization of the operator \( T \)). For the functional \( T_{ij}(f) \) defined in (2), the following decomposition holds

\[
T_{ij}(f) = \int H_1(\hat{f}, x_i, x_j, y)f(x_i, x_j, y)dx_idx_jdy + \int H_2(\hat{f}, x_{i1}, x_{j2}, y)f(x_{i1}, x_{j1}, y)f(x_{i2}, x_{j2}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy + \Gamma_n \tag{4}
\]

where

\[
H_1(\hat{f}, x_i, x_j, y) = x_im_j(\hat{f}, y) + x_jm_i(\hat{f}, y) - m_i(\hat{f}, y)m_j(\hat{f}, y) \tag{5}
\]

\[
H_2(\hat{f}, x_{i1}, x_{j2}, y) = \frac{1}{\int f(x_i, x_j, y)dx_idx_j} (x_{i1} - m_i(\hat{f}, y))(x_{j2} - m_j(\hat{f}, y)) \tag{6}
\]

\[
\Gamma_n = \frac{1}{6} F_{\eta}''(\xi)(1 - \xi)^3, \tag{7}
\]

for some \( \xi \in ]0, 1[ \).

This decomposition has the main advantage of separating the terms to be estimated into a linear functional of \( f \), which can be easily estimated and a second part which is a quadratic functional of \( f \). In this case, Section 4 will be dedicated to estimate this kind of functionals and specifically to control its variance. This will enable to provide an efficient estimator of \( T_{ij}(f) \) using the decomposition of Proposition 1.

### 3 Main Results

In this section we build a procedure to estimate \( T_{ij}(f) \) efficiently. Since we used \( n_1 < n \) to build a preliminary approximation \( \hat{f} \), we will use a sample of size \( n_2 = n - n_1 \) to estimate (5) and (6). Since (5) is a linear functional of the density \( f \), it can be estimated by its empirical counterpart

\[
\frac{1}{n_2} \sum_{k=1}^{n_2} H_1(\hat{f}, X_i^{(k)}, X_j^{(k)}, Y^{(k)}). \tag{8}
\]

Since (6) is a nonlinear functional of \( f \), the estimation is harder. Its estimation will be a direct consequence of the technical results presented in Section 4 where we build an estimator for the general functional

\[
\theta(f) = \int \eta(x_{i1}, x_{j2}, y)f(x_{i1}, x_{j1}, y)f(x_{i2}, x_{j2}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy
\]

where \( \eta : \mathbb{R}^3 \rightarrow \mathbb{R} \) is a bounded function. The estimator \( \hat{\theta}_n \) of \( \theta(f) \) is an extension of the method developed in Da Veiga & Gamboa (2008).
3.1 Hypothesis and Assumptions

The following notations will be used throughout the paper. Let $d_s$ and $b_s$ for $s = 1, 2, 3$ be real numbers where $d_s < b_s$. Let, for $i$ and $j$ fixed, $L^2(dx, dx, dy)$ be the squared integrable functions in the cube $[d_1, b_1] \times [d_2, b_2] \times [d_3, b_3]$. Moreover, let $(p_i(x_i, x_j, y))_{i \in D}$ be an orthonormal basis of $L^2(dx, dx, dy)$, where $D$ is a countable set. Let $a_l = \int p_i f$ denote the scalar product of $f$ with $p_i$.

Furthermore, denote by $L^2(dx, dx, dy)$ (resp. $L^2(dy)$) the set of squared integrable functions in $[d_1, b_1] \times [d_2, b_2]$ (resp. $[d_3, b_3]$). If $(\alpha_t(x_i, x_j)_{t \in D_1})$ (resp. $(\beta_t(y)_{t \in D_2})$) is an orthonormal basis of $L^2(dx, dx, dy)$ (resp. $L^2(dy)$) then $p_i(x_i, x_j, y) = \alpha_t(x_i, x_j)\beta_t(y)$ with $l = (t, t) \in D_1 \times D_2$.

We also use the following subset of $L^2(dx, dx, dy)$

$$
E = \left\{ \sum_{l \in D} e_l p_i : (e_l)_{l \in D} \text{ is such that } \sum_{l \in D} \left| e_l \right|^2 < 1 \right\}
$$

where $(e_l)_{l \in D}$ is a given fixed sequence.

Moreover assume that $(X_i, X_j, Y)$ have a bounded joint density $f$ on $[d_1, b_1] \times [d_2, b_2] \times [d_3, b_3]$ which lies in the ellipsoid $E$.

In what follows, $X_n \xrightarrow{D} X$ (resp. $X_n \xrightarrow{P} X$) denotes the convergence in distribution or weak convergence (resp. convergence in probability) of $X_n$ to $X$. Additionally, the support of $f$ will be denoted by $\text{supp } f$.

Let $(M_n)_{n \geq 1}$ denote a sequence of subsets $D$. For each $n$ there exists $M_n$ such that $M_n \subset D$. Let us denote by $|M_n|$ the cardinal of $M_n$.

We shall make three main assumptions:

**Assumption 1.** For all $n \geq 1$ there is a subset $M_n \subset D$ such that $(\sup_{l \in M_n} |c_l|^2)^2 \approx |M_n| / n^2$ (for some positives constants $\lambda_1$ and $\lambda_2$). Moreover, $\forall f \in L^2(dx, dy)$, $\int (S_{M_n} f - f)^2 dx dy = 0$ when $n \to 0$, where $S_{M_n} f = \sum_{l \in M_n} a_l p_l$.

**Assumption 2.** $\text{supp } f \subset [d_1, b_1] \times [d_2, b_2] \times [d_3, b_3]$ and $\forall (x, y, z) \in \text{supp } f, 0 < \alpha \leq f(x, y, z) \leq \beta$ with $\alpha, \beta \in \mathbb{R}$.

**Assumption 3.** It is possible to find an estimator $\hat{f}$ of $f$ built with $n_1 \approx n / \log(n)$ observations, such that for $\epsilon > 0$,

$$
\forall (x, y, z) \in \text{supp } f, \quad 0 < \alpha - \epsilon \leq \hat{f}(x, y, z) \leq \beta + \epsilon
$$

and,

$$
\forall 2 \leq q \leq +\infty, \forall l \in \mathbb{N}^*, \quad E \| f - f \|_q^l \leq C(q, l) n^{-l} \lambda
$$

for some $\lambda > 1/6$ and some constant $C(q, l)$ not depending on $f$ belonging to the ellipsoid $E$.

Assumption 1 is necessary to bound the bias and variance of $\hat{\theta}_n$. Assumption 2 and 3 allow to establish that the remainder term in the Taylor expansion is negligible, i.e $\Gamma_n = O(1/n)$. Assumption 3 depends on the regularity of the density function. For instance for $x \in \mathbb{R}^p$, $s > 0$ and $L > 0$, consider the class
Theorem 1. Let Assumptions 1-3 hold and when compared to the other error terms.

Then, Assumption 3 is satisfied for \( f \in \mathcal{H}_q(s, L) \) with \( s > \frac{p}{q} \).

3.2 Efficient Estimation of \( T_{ij}(f) \)

As seen in Section 2, \( T_{ij}(f) \) can be decomposed as (4). Hence, using (8) and (14) we consider the following estimate

\[
\hat{T}_{ij}^{(n)} = \frac{1}{n_2} \sum_{k=1}^{n_2} H_1(\hat{f}, X_i^{(k)}, X_j^{(k)}, Y^{(k)}) \quad \text{and} \quad \hat{T}_{ij}^{(n)} - T_{ij}(f) \xrightarrow{D} N(0, C_{ij}(f)),
\]

and

\[
\lim_{n \to \infty} n\mathbb{E}\left[\hat{T}_{ij}^{(n)} - T_{ij}(f)\right]^2 = C_{ij}(f),
\]

where \( C_{ij}(f) = \text{Var}\left(H_1(f, X_i, X_j, Y)\right) \).

Note that, in Theorem 1 it appears that the asymptotic variance of \( T_{ij}(f) \) depends only on \( H_1(f, X_i, X_j, Y) \). Hence the asymptotic variance of \( \hat{T}_{ij}^{(n)} \) is explained only by the linear part of (4). This will entail that the estimator is naturally efficient as proved in the following.

Indeed, the semi-parametric Cramér-Rao bound is given in the next theorem.

Theorem 2 (Semi-parametric Cramér-Rao bound.). Consider the estimation of

\[
T_{ij}(f) = \int \left( \frac{\int x_i f(x_i, x_j, y)dx_i dx_j}{\int f(x_i, x_j, y)dx_i dx_j} \right) \left( \frac{\int x_j f(x_i, x_j, y)dx_i dx_j}{\int f(x_i, x_j, y)dx_i dx_j} \right) f(x_i, x_j, y)dx_i dx_j dy
\]
for a random vector \((X_i, X_j, Y)\) with joint density \(f \in \mathcal{E}\). Let \(f_0 \in \mathcal{E}\) be a density verifying the assumptions of Theorem 1. Then, for all estimator \(\widehat{T}^{(n)}_{ij}\) of \(T_{ij}(f)\) and every family \(\{\mathcal{V}_r(f_0)\}_{r > 0}\) of neighborhoods of \(f_0\) we have

\[
\inf_{\{\mathcal{V}_r(f_0)\}_{r > 0}} \liminf_{n \to \infty} \sup_{f \in \mathcal{V}_r(f_0)} n \mathbb{E} \left[ (\widehat{T}^{(n)}_{ij} - T_{ij}(f_0))^2 \right] \geq C_{ij}(f_0)
\]

where \(\mathcal{V}_r(f_0) = \{ f : \|f - f_0\|_2 < r \}\) for \(r > 0\).

Consequently, the estimator \(\widehat{T}^{(n)}_{ij}\) is efficient.

In the case of our estimate, its variance is \(C_{ij}(f)\), which proves its asymptotically efficiency.

Remark that Theorem 1 proves asymptotic normality entry by entry of the matrix \(T(f) = (T_{ij}(f))_{p \times p}\). To extend the result for the whole matrix it is necessary to introduce the half-vectorization operator \(\text{vech}\). This operator, stacks only the columns from the principal diagonal of a square matrix downwards in a column vector, that is, for an \(p \times p\) matrix \(A = (a_{ij})\),

\[\text{vech}(A) = [a_{11}, \ldots, a_{p1}, a_{p2}, \ldots, a_{33}, \ldots, a_{pp}]^\top.\]

Let define the estimator matrix \(\widehat{T}^{(n)} = (\widehat{T}^{(n)}_{ij})\) and \(H_1(f)\) denote the matrix with entries \((H_1(f, x_i, x_j, y))_{i,j}\). Now we are able to state the following

**Corollary 1.** Let Assumptions 1-3 hold and \(|M_n|/n \to 0\) when \(n \to \infty\). Then \(\widehat{T}^{(n)}\) has the following properties:

\[
\sqrt{n} \text{vech} \left( \widehat{T}^{(n)} - T(f) \right) \overset{D}{\to} \mathcal{N} \left( 0, C(f) \right), \tag{11}
\]

\[
\lim_{n \to \infty} n \mathbb{E} \left[ \text{vech} \left( \widehat{T}^{(n)} - T(f) \right) \text{vech} \left( \widehat{T}^{(n)} - T(f) \right)^\top \right] = C(f) \tag{12}
\]

where

\[C(f) = \text{Cov} \left( \text{vech}(H_1(f)) \right)\]

Previous results depend on the accurate estimation of the quadratic part of the estimator of \(T_{ij}^{(n)}\), which is the issue of the following section.

### 4 Estimation of quadratic functionals

As pointed out in Section 2 the decomposition (4) has a quadratic part (6) that we want to estimate. To achieve this we will construct a general estimator of the form:

\[
\theta = \int \eta(x_{i1}, x_{j2}, y)f(x_{i1}, x_{j1}, y)f(x_{i2}, x_{j2}, y)dx_{i1}dx_{j1}dx_{i2}dx_{j2}dy,
\]

for \(f \in \mathcal{E}\) and \(\eta : \mathbb{R}^3 \to \mathbb{R}\) a bounded function.
Given $M$, a subset of $D$, consider the estimator
\begin{align*}
\hat{\theta}_n &= \frac{1}{n(n-1)} \sum_{i \neq j, k \neq k'} \sum_{l \in M} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) \\
&\quad \int p_l(x_i, x_j, Y^{(k)}) \left( \eta(x_i, X_j^{(k')}, Y^{(k')}) + \eta(X_i^{(k')}, x_j, Y^{(k')}) \right) dx_i dx_j \\
&\quad - \frac{1}{n(n-1)} \sum_{i, j, k \neq k', l \in M} \sum_{l' \neq M} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k')})\psi_l(x_i, x_j, X_i^{(k')}, X_j^{(k')}, Y^{(k')}) dx_i dx_j \\
&\quad \int p_l(x_i, x_j, x_j, y)\psi_l(x_i, x_j, x_j, x_j, y) dx_i dx_j dx_2 dx_2 dy.
\end{align*}

In order to simplify the presentation of the main Theorem, let $\psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}; y) = \eta(x_{i1}, x_{j2}, y) + \eta(x_{i2}, x_{j1}, y)$ verifying
\begin{align*}
\int \psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy &= \int \psi(x_{i2}, x_{j2}, x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy.
\end{align*}

With this notation we can simplify (13) in
\begin{align*}
\hat{\theta}_n &= \frac{1}{n(n-1)} \sum_{i \neq j, k \neq k'} \sum_{l \in M} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k)}) \int p_l(x_i, x_j, Y^{(k)}) \psi(x_i, x_j, X_i^{(k')}, X_j^{(k')}, Y^{(k')}) dx_i dx_j \\
&\quad - \frac{1}{n(n-1)} \sum_{i, j, k \neq k', l \in M} \sum_{l' \neq M} p_l(X_i^{(k)}, X_j^{(k)}, Y^{(k')})\psi_l(x_i, x_j, X_i^{(k')}, X_j^{(k')}, Y^{(k')}) dx_i dx_j \\
&\quad \int p_l(x_i, x_{j1}, x_{j2}, y)\psi_l(x_i, x_{j1}, x_{j2}, x_{j1}, y) dx_i dx_{j1} dx_{j2} dx_{j2} dy.
\end{align*}

Using simple algebra, it is possible to prove that this estimator has bias equal to
\begin{align*}
- \int (S_M f(x_{i1}, x_{j1}, y) - f(x_{i1}, x_{j1}, y)) (S_M f(x_{i2}, x_{j2}, y) - f(x_{i2}, x_{j2}, y)) \\
\eta(x_{i1}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy
\end{align*}

(15)
The following Theorem gives an explicit bound for the variance of $\hat{\theta}_n$.

**Theorem 3.** Let Assumption 1 hold. Then if $|M_n| / n \to 0$ when $n \to 0$, then $\hat{\theta}_n$ has the following property
\begin{align*}
\mathbb{E}[(\hat{\theta}_n - \theta)^2] - \Lambda(f, \eta) \leq \gamma \left[ \frac{|M_n|}{n} + \|S_M f - f\|_2 + \|S_n g - g\|_2 \right],
\end{align*}
where $g(x_i, x_j, y) = \int f(x_{i2}, x_{j2}, y)\psi(x_i, x_j, x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}$ and
\begin{align*}
\Lambda(f, \eta) &= \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2,
\end{align*}
where $\gamma$ is constant depending only on $\|f\|_{\infty}$, $\|\eta\|_{\infty}$, and $\Delta_{x,i,j} = (b_1 - a_1) \times (b_2 - a_2)$. Moreover, this constant is an increasing function of these quantities.
Now, using (17) we first compute $F_\ast$, which has the form of the quadratic functional $\theta$ with the particular choice $\eta(x_{i1}, x_{j2}, y) = H_2(\hat{f}, x_{i1}, x_{j2}, y)$. We point out that we also show that in this particular frame, we get $\Lambda(f, \eta) = 0$. This is the reason why the asymptotic variance of the estimate $\hat{F}^{(n)}_{ij}$ built in the previous section, is only governed by its linear part, yielding asymptotic efficiency.

5 Proofs

Proof of Proposition 1

We need to calculate the three first derivatives of $F(u)$. In order to facilitate the calculation, we are going to differentiate $m_i(f_u, y)$:

$$\frac{d}{du}(m_i(f_u, y)) = \frac{d}{du} \left( \frac{\int x_i f_u(x_i, x_j, y) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j} \right)
= \frac{\int x_i (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j}
- \frac{\int x_i f_u(x_i, x_j, y) dx_i dx_j \int f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) dx_i dx_j}{\left( \int f_u(x_i, x_j, y) dx_i dx_j \right)^2},$$
$$\frac{m_i(f_u, y) \int f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j}$$
$$- \frac{\int (x_i - m_i(f_u, y)) (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j}{\int f_u(x_i, x_j, y) dx_i dx_j}. \quad (17)$$

Now, using (17) we first compute $F'(u)$,

$$\int \frac{d}{du}(m_i(f_u, y)) m_j(f_u, y) f_u(x_i, x_j, y) + m_i(f_u, y) \frac{d}{du}(m_j(f_u, y)) f_u(x_i, x_j, y)$$
$$+ m_i(f_u, y) m_j(f_u, y) \frac{d}{du} (f_u(x_i, x_j, y)) dx_i dx_j dy,$n
$$= \int \left[ x_i m_j(f_u, y) + x_j m_i(f_u, y) - m_i(f_u, y) m_j(f_u, y) \right] \left( f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) \right) dx_i dx_j dy.$$

Taking $u = 0$ we have

$$F'(0) = \int \left[ x_i m_j(\hat{f}, y) + x_j m_i(\hat{f}, y) - m_i(\hat{f}, y) m_j(\hat{f}, y) \right] \left( f(x_i, x_j, y) - \hat{f}(x_i, x_j, y) \right) dx_i dx_j dy. \quad (18)$$
We derive now \( m_i(f_u, y)m_j(f_u, y) \) to obtain

\[
\frac{d}{du} (m_i(f_u, y)m_j(f_u, y)) = \frac{d}{du} (m_i(f_u, y)) m_j(f_u, y) + m_i(f_u, y) \frac{d}{du} (m_j(f_u, y)) \\
= m_j(f_u, y) \int \left( x_i - m_i(f_u, y) \right) (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j \\
+ m_i(f_u, y) \int \left( x_j - m_j(f_u, y) \right) (f(x_i, x_j, y) - \hat{f}(x_i, x_j, y)) dx_i dx_j.
\]

(19)

Following with \( F''(u) \) and using (17) and (19) we get,

\[
F''(u) = \int \left[ x_{i1} \int \left( (x_{j2} - m_j(f_u, y))(f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i2} dx_{j2} \right) \\
+ x_{j1} \int \left( (x_{i2} - m_i(f_u, y))(f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i2} dx_{j2} \right) \\
- m_j(f_u, y) \int \left( (x_{i2} - m_i(f_u, y))(f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i2} dx_{j2} \right) \\
- m_i(f_u, y) \int \left( (x_{j2} - m_j(f_u, y))(f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i2} dx_{j2} \right) \\
( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) ) dx_{i1} dx_{j1} dy. \right]
\]

Simplifying the last expression we obtain

\[
F''(u) = \int \frac{1}{f_u(x_i, x_j, y) dx_i dx_j} \left\{ (x_{i1} - m_i(f_u, y))(x_{j2} - m_j(f_u, y)) + (x_{i2} - m_i(f_u, y))(x_{j1} - m_j(f_u, y)) \right\} \\
( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) ) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy.
\]

Besides, when \( u = 0 \)

\[
F''(0) = \int \frac{1}{f(x_i, x_j, y) dx_i dx_j} \left\{ (x_{i1} - \hat{m}(\hat{f}, y))(x_{j2} - m_j(\hat{f}, y)) + (x_{i2} - m_i(\hat{f}, y))(x_{j1} - m_j(\hat{f}, y)) \right\} \\
( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) ) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy \\
= \int \frac{2}{f(x_i, x_j, y) dx_i dx_j} (x_{i1} - m_i(\hat{f}, y))(x_{j2} - m_j(\hat{f}, y)) \\
( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) ) (f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y)) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy.
\]

(21)
Using the previous arguments we can finally find $F'''(u)$:

$$
F'''(u) = \int \frac{-6}{f_u(x, x, y)dx_i dx_j} (x_{i1} - m_j(f_u, y))(x_{j2} - m_j(f_u, y))
$$

\begin{align*}
&\left( f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right) \left( f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y) \right) \\
&\left( f(x_{i3}, x_{j3}, y) - \hat{f}(x_{i3}, x_{j3}, y) \right)
\end{align*}

(22)

Replacing (18), (21) and (22) into (3) we get the desired decomposition.

**Proof of Theorem 1**

We will first control the remaining term (7),

$$
\Gamma_n = \frac{1}{6} F'''(\xi)(1 - \xi)^3.
$$

Remember that

$$
F'''(\xi) = -6 \int \frac{(x_{i1} - m_1(f_\xi, y))(x_{j2} - m_2(f_\xi, y))}{(\int f_\xi(x, y) dx_i dx_j)^2} 
$$

\begin{align*}
&\left( f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y) \right) \\
&\left( f(x_{i3}, x_{j3}, y) - \hat{f}(x_{i3}, x_{j3}, y) \right)
\end{align*}

Assumptions 1 and 2 ensure that the first part of the integrand is bounded by a constant $\mu$. Furthermore,

\begin{align*}
\left| \Gamma_n \right| &\leq \mu \int \left| f(x_{i1}, x_{j1}, y) - \hat{f}(x_{i1}, x_{j1}, y) \right| \left| f(x_{i2}, x_{j2}, y) - \hat{f}(x_{i2}, x_{j2}, y) \right| \\
&\leq \mu \int \left| f(x_{i3}, x_{j3}, y) - \hat{f}(x_{i3}, x_{j3}, y) \right| dx_i dx_j
\end{align*}

by the Hölder inequality. Then $\mathbb{E}[\Gamma_n^2] = O(\mathbb{E}(\int |f - \hat{f}|^3)^2) = O(\mathbb{E}(\|f - \hat{f}\|_6^2))$. Since $\hat{f}$ verifies Assumption 3, this quantity is of order $O(n_1^{-6\lambda})$. Since we also assume $n_1 \approx n/\log(n)$ and $\lambda > 1/6$, then $n_1^{-6\lambda} = o\left(\frac{1}{n}\right)$. Therefore, we get $\mathbb{E}[\Gamma_n^2] = o(1/n)$ which implies that the remaining term $\Gamma_n$ is negligible.

To prove the asymptotic normality of $\hat{T}^{(n)}_{ij}$, we shall show that $\sqrt{n} \left( \hat{T}^{(n)}_{ij} - T_{ij}(f) \right)$ and define

$$
Z_{ij}^{(n)} = \frac{1}{n_2} \sum_{k=1}^{n_2} H_1(f, X_{i}^{(k)}, X_{j}^{(k)}, Y^{(k)}) - \int H_1(f, x_i, x_j, y) f(x_i, x_j, y))dx_i dx_j dy
$$

(23)

have the same asymptotic behavior. We can get for $Z_{ij}^{(n)}$ a classic central limit theorem with variance

$$
C_{ij}(f) = \text{Var}(H_1(f, x_i, x_j, y))
$$

$$
= \int H_1(f, x_i, x_j, y)^2 f(x_i, x_j, y))dx_i dx_j dy - \left( \int H_1(f, x_i, x_j, y) f(x_i, x_j, y))dx_i dx_j dy \right)^2
$$

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which implies (9) and (10). In order to establish our claim, we will show that

\[ R_{ij}^{(n)} = \sqrt{n} \left[ \hat{T}_{ij}^{(n)} - T_{ij}(f) - Z_{ij}^{(n)} \right] \]

(24)

has second-order moment converging to 0.

Define \( \hat{Z}_{ij}^{(n)} \) as \( Z_{ij}^{(n)} \) with \( f \) replaced by \( \hat{f} \). Let us note that \( R_{ij}^{(n)} = R_1 + R_2 \) where

\[ R_1 = \sqrt{n} \left[ \hat{T}_{ij}^{(n)} - T_{ij}(f) - \hat{Z}_{ij}^{(n)} \right] \]
\[ R_2 = \sqrt{n} \left[ \hat{Z}_{ij}^{(n)} - Z_{ij}^{(n)} \right]. \]

It only remains to state that \( \mathbb{E}[R_1^2] \) and \( \mathbb{E}[R_2^2] \) converges to 0. We can rewrite \( R_1 \) as

\[ R_1 = -\sqrt{n} \left[ \hat{Q} - Q + \Gamma_n \right] \]

where we note that

\[ Q = \int H_2(\hat{f}, x_{11}, x_{22}, y) f(x_{11}, x_{21}, y) f(x_{12}, x_{22}, y) dx_{11} dx_{21} dx_{12} dx_{22} dy \]
\[ H_2(\hat{f}, x_{11}, x_{22}, y) = \frac{1}{\int \hat{f}(x_{11}, x_{22}, y) dx_{12} dx_{22}} \left( x_{11} - m_1(\hat{f}, y) \right) \left( x_{22} - m_2(\hat{f}, y) \right) \]

has the form of a quadratic functional studied in Section 4 with \( \eta(x_{11}, x_{22}, y) = H_2(\hat{f}, x_{11}, x_{22}, y) \). Hence such functional can be estimated as done in Section 4 and let \( \hat{Q} \) be its corresponding estimator. Since \( \mathbb{E}[\Gamma_n^2] = o(1/n) \), we only have to control the term \( \sqrt{n}(\hat{Q} - Q) \) which is such that \( \lim_{n \to \infty} n\mathbb{E}[\hat{Q} - Q]^2 = 0 \) by Lemma 7. This Lemma implies that \( \mathbb{E}[R_1^2] \to 0 \) as \( n \to \infty \). For \( R_2 \) we have

\[ \mathbb{E}[R_2^2] = \frac{n}{n^2} \left[ \int \left( H_1(f, x_{11}, x_{21}, y) - H_1(\hat{f}, x_{11}, x_{21}, y) \right)^2 f(x_{11}, x_{21}, y) dx_{11} dx_{21} dx_{12} dx_{21} \right] 
- \frac{n}{n^2} \left[ \int H_1(f, x_{11}, x_{21}, y) f(x_{11}, x_{21}, y) dx_{11} dx_{21} dx_{12} dx_{21} \right] \]

(25)

The same arguments as the ones of Lemma 7 (mean value and Assumptions 2 and 3) show that \( \mathbb{E}[R_2^2] \to 0 \).

\[ \square \]

**Proof of Theorem 2.** To prove the inequality we will use the usual framework described in [Ibragimov & Khas’minskii (1991)]. The first step is to calculate the Fréchet derivative of \( T_{ij}(f) \) at some point \( f_0 \in \mathcal{E} \). Assumptions 2 and 3 and equation (4), imply that

\[ T_{ij}(f) - T_{ij}(f_0) = \int \left( x_i m_j(f_0, y) + x_j m_i(f_0, y) - m_i(f_0, y)m_j(f_0, y) \right) \]
\[ \left( f(x_i, x_j, y) - f_0(x_i, x_j, y) \right) dx_i dx_j dy + O \left( \int (f - f_0)^2 \right) \]

\[ \square \]
where \( m_i(f_0, y) = \int x_i f_0(x_i, x_j, y) dx_i dx_j dy \) for any \( u \) and \( \Rightarrow \) equation \( (24) \) we have

\[
H_1(f_0, x_i, x_j, y) = x_i m_j(f_0, y) + x_j m_i(f_0, y) - m_i(f_0, y)m_j(f_0, y).
\]

Using the results of Ibragimov & Khas’minskii (1991), denote \( H(f_0) = \{ u \in \mathbb{L}^2(dx_i dx_j dy) : \int u(x_i, x_j, y) \sqrt{f_0(x_i, x_j, y)} dx_i dx_j dy = 0 \} \) the set of functions in \( \mathbb{L}^2(dx_i dx_j dy) \) orthogonal to \( \sqrt{f_0} \), \( \Pr_{H(f_0)} \) the projection onto \( H(f_0) \), \( A_n(t) = (\sqrt{f_0})/\sqrt{n} \) and \( P_f^{(n)} \) the joint distribution of \( (X_i^{(k)}, X_j^{(k)}) \) \( k = 1, \ldots, n \) under \( f_0 \). Since \( (X_i^{(k)}, X_j^{(k)}) \) \( k = 1, \ldots, n \) are i.i.d., the family \( \{ P_f^{(n)}, f \in \mathcal{E} \} \) is differentiable in quadratic mean at \( f_0 \) and therefore locally asymptotically normal at all points \( f_0 \in \mathcal{E} \) in the direction \( H(f_0) \) with normalizing factor \( A_n(f_0) \) (see the details in Van der Vaart (2000)). Then, by the results of Ibragimov & Khas’minskii (1991) say that under these conditions, denoting \( K_n = B_n \theta'(f_0) A_n \Pr_{H(f_0)} \) with \( B_n = \sqrt{n} u \) if \( K_n \xrightarrow{D} K \) and if \( K(u) = \langle t, u \rangle \), then for every estimator \( \hat{T}^{(n)}_{ij} \) of \( T_{ij} \) and every family \( \mathcal{V} \) of vicinities of \( f_0 \), we have

\[
\inf_{\mathcal{V}(f_0)} \liminf_{n \to \infty} \sup_{f \in \mathcal{V}(f_0)} n \mathbb{E} \left[ \left( \hat{T}^{(n)}_{ij} - T_{ij}(f_0) \right)^2 \right] \geq \| t \|_{\mathbb{L}^2(dx_i dx_j dy)}^2.
\]

Here,

\[
K_n(u) = \sqrt{n T'(f_0) \cdot \frac{\sqrt{f_0}}{\sqrt{n}} \Pr_{H(f_0)}(u) = T'(f_0) \left( \sqrt{f_0} \left( u - \sqrt{f_0} \int u \sqrt{f_0} \right) \right)}.
\]

since for any \( u \in \mathbb{L}^2(dx_i dx_j dy) \) we can write it as \( u = \sqrt{f_0} (\sqrt{f_0} u) + \Pr_{H(f_0)}(u) \). In this case \( K_n(u) \) does not depend on \( n \) and

\[
K(u) = T'(f_0) \cdot \left( \sqrt{f_0} \left( u - \sqrt{f_0} \int u \sqrt{f_0} \right) \right)
= \int H_1(f_0, \cdot) \sqrt{f_0} u - \int H_1(f_0, \cdot) \sqrt{f_0} \int u \sqrt{f_0}
= \langle t, u \rangle
\]

with

\[
t(x_i, x_j, y) = H_1(f_0, x_i, x_j, y) \sqrt{f_0} - \left( \int H_1(f_0, x_i, x_j, y) f_0 \right) \sqrt{f_0}.
\]

The semi-parametric Cramér-Rao bound for this problem is thus

\[
\| t \|_{\mathbb{L}^2(dx_i dx_j dy)} = \int H_1(f_0, x_i, x_j, y)^2 f_0 dx_i dx_j dy - \left( \int H_1(f_0, x_i, x_j, y) f_0 dx_i dx_j dy \right)^2 = C_{ij}(f_0)
\]

and we recognize the expression \( C_{ij}(f_0) \) found in Theorem 1.

**Proof of Corollary** The proof is based in the following observation. Employing equation \( (24) \) we have

\[
\hat{T}^{(n)} - T(f) = Z^{(n)}(f) + \frac{R^{(n)}}{\sqrt{n}}
\]

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where $Z^{(n)}(f)$ and $R^{(n)}$ are matrices with elements $Z_{ij}^{(n)}$ and $R_{ij}^{(n)}$, defined in (23) and (24), respectively.

Hence we have,

$$n\mathbb{E}[\| \text{vech} \left( \hat{T}^{(n)} - T(f) - Z^{(n)}(f) \right) \|^2] = \mathbb{E}[\| \text{vech} \left( R^{(n)} \right) \|^2] = \sum_{i\leq j} \mathbb{E}[\left( R_{ij}^{(n)} \right)^2].$$

We see by Lemma 7 that $\mathbb{E}[R_{ii}^{2}] \to 0$ as $n \to 0$. It follows that

$$n\mathbb{E}[\| \text{vech} \left( \hat{T}^{(n)} - T(f) - Z^{(n)}(f) \right) \|^2] \to 0 \text{ as } n \to 0.$$

We know that if $X_n$, $X$ and $Y_n$ are random variables, then if $X_n \overset{D}{\rightarrow} X$ and $(X_n - Y_n) \overset{D}{\rightarrow} 0$, follows that $Y_n \overset{D}{\rightarrow} X$.

Remember also that convergence in $\mathbb{L}^2$ implies convergence in probability, therefore

$$\sqrt{n} \text{vech} \left( \hat{T}^{(n)} - T(f) - Z^{(n)}(f) \right) \overset{D}{\rightarrow} 0.$$

By the multivariate central limit theorem we have that $\sqrt{n} \text{vech} \left( Z^{(n)}(f) \right) \overset{D}{\rightarrow} N(0, C(f))$. Therefore, $\sqrt{n} \text{vech} \left( T^{(n)} - T(f) \right) \overset{D}{\rightarrow} N(0, C(f))$.

**Proof of Theorem 3** For abbreviation, we write $M$ instead of $M_n$ and set $m = |M_n|$. We first compute the mean squared error of $\hat{\theta}_n$ as

$$\mathbb{E}[\hat{\theta}_n - \theta]^2 = \text{Bias}^2(\hat{\theta}_n) + \text{Var}(\hat{\theta}_n)$$

where $\text{Bias}(\hat{\theta}_n) = \mathbb{E}[\hat{\theta}_n] - \theta$.

We begin the proof by bounding $\text{Var}(\hat{\theta}_n)$. Let $A$ and $B$ be $m \times 1$ vectors with components

$$\begin{align*}
a_l &= \int p_l(x_i, x_j, y)f(x_i, x_j, y)dx_idx_jdy \quad l = 1, \ldots, m, \\
b_l &= \int p_l(x_i, x_j, y)f(x_i, x_j, y)\psi(x_i, x_j, x_i, x_j, x_j, y)dx_idx_jdx_jdy \\
&= \int p_l(x_i, x_j, y)g(x_i, x_j, y)dx_idx_jdy \quad l = 1, \ldots, m
\end{align*}$$

where $g(x_i, x_j, y) = \int f(x_i, x_j, y)\psi(x_i, x_j, x_i, x_j, x_j, y)dx_i2dx_j2$. Let $Q$ and $R$ be $m \times 1$ vectors of centered functions

$$\begin{align*}
q_l(x_i, x_j, y) &= p_l(x_i, x_j, y) - a_l \\
r_l(x_i, x_j, y) &= \int p_l(x_i, x_j, y)\psi(x_i, x_j, x_i, x_j, x_j, y)dx_i2dx_j2 - b_l
\end{align*}$$

for $l = 1, \ldots, m$. Let $C$ a $m \times m$ matrix of constants

$$c_{ll'} = \int p_l(x_i, x_j, y)p_{l'}(x_i, x_j, y)\eta(x_i, x_j, y)dx_idx_jdx_i2dx_j2dy \quad l, l' = 1, \ldots, m.$$
Let us denote by $U_n$ the process

$$U_n h = \frac{1}{n(n - 1)} \sum_{k \neq k' = 1}^{n} h(X_i^{(k)}, X_j^{(k')}, Y^{(k)}, X_i^{(k')}, Y^{(k')})$$

and $P_n$ the empirical measure

$$P_n h = \frac{1}{n} \sum_{k = 1}^{n} h(X_i^{(k)}, X_j^{(k)}, Y^{(k)})$$

for some $h$ in $L^2(dx_i, dx_j, dy)$. With these notations, $\hat{\theta}_n$ has the Hoeffding’s decomposition

$$\hat{\theta}_n = \frac{1}{n(n - 1)} \sum_{l \in M} \sum_{k \neq k' = 1}^{n} (q_l(X_i^{(k)}, X_j^{(k')}, Y^{(k')}) + a_l) (r_l(X_i^{(k')}, X_j^{(k')}, Y^{(k)}) + b_l)$$

$$\quad - \frac{1}{n(n - 1)} \sum_{l \in M} \sum_{k \neq k' = 1}^{n} (q_l(X_i^{(k)}, X_j^{(k')}, Y^{(k')}) + a_l) (q_{l'}(X_i^{(k')}, X_j^{(k')}, Y^{(k')}) + a_{l'}) \epsilon_{l,l'}$$

$$\quad = U_n K + P_n L + A^\top B - A^\top CA$$

where

$$K(x_1, x_2, x_1, x_2, y_1, y_2) = Q^\top(x_1, x_1, y_1) R(x_2, x_2, y_2) - Q^\top(x_1, x_1, y_1) C Q(x_2, x_2, y_2)$$

$$L(x_1, x_2, y) = A^\top R(x_1, x_2, y) + B Q(x_1, x_2, y) - 2 A^\top C Q(x_1, x_2, y).$$

Therefore $\text{Var}(\hat{\theta}_n) = \text{Var}(U_n K) + \text{Var}(P_n L) - 2 \text{Cov}(U_n K, P_n L)$. These three terms are bounded in Lemmas 2–4 which gives

$$\text{Var}(\hat{\theta}_n) \leq \frac{20}{n(n - 1)} \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta_{x,x_j}^2 (m + 1) + \frac{12}{n} \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta_{x,x_j}^2.$$ 

For $n$ enough large and a constant $\gamma \in \mathbb{R}$,

$$\text{Var}(\hat{\theta}_n) \leq \gamma \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta_{x,x_j}^2 \left( \frac{m}{n^2} + \frac{1}{n} \right).$$

The term $\text{Bias}(\hat{\theta}_n)$ is easily computed, as proven in Lemma 5, is equal to

$$- \int (S_M f(x_{i_1}, x_{j_1}, y) - f(x_{i_1}, x_{j_1}, y)) (S_M f(x_{i_2}, x_{j_2}, y) - f(x_{i_2}, x_{j_2}, y))$$

$$\eta(x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dy.$$

From Lemma 5 the bias of $\hat{\theta}_n$ is bounded by

$$\left| \text{Bias}(\hat{\theta}_n) \right| \leq \Delta_{x,x_j} \| \eta \|_\infty \sup_{l \in M} \| c_l \|^2.$$

The assumption of $\left( \sup_{l \in M} \| c_l \|^2 \right)^2 \approx m/n^2$ and since $m/n \to 0$, we deduce that $\mathbb{E} [\hat{\theta}_n - \theta]^2$ has a parametric rate of convergence $O(1/n)$.
Finally to prove (16), note that
\[ nE[\hat{\theta}_n - \theta]^2 = n \text{Bias}^2(\hat{\theta}_n) + n \text{Var}(\hat{\theta}_n) = n \text{Bias}^2(\hat{\theta}_n) + n \text{Var}(U_nK) + n \text{Var}(P_nL). \]

We previously proved that for some \( \lambda_1, \lambda_2 \in \mathbb{R} \)
\[ n \text{Bias}^2(\hat{\theta}_n) \leq \lambda_1 \Delta^2_{x_i x_j} ||\eta||^2 \frac{m}{n}, \]
\[ n \text{Var}(U_nK) \leq \lambda_2 \Delta^2_{x_i x_j} ||f||^2 \frac{m}{n}. \]

Thus, Lemma 5 implies
\[ |n \text{Var}(P_nL) - \Lambda(f, \eta)| \leq \lambda [||S_M f - f||_2 + ||S_M g - g||_2], \]
where \( \lambda \) is a increasing function of \( ||f||^2_\infty, ||\eta||^2_\infty \) and \( \Delta^2_{x_i x_j} \). From all this we deduce (16) which ends the proof of Theorem 3.

6 Technical Results

Lemma 1 (Bias of \( \hat{\theta}_n \)). The estimator \( \hat{\theta}_n \) defined in (14) estimates \( \theta \) with bias equal to
\[ -\int (S_M f(x_{i1}, x_{j1}, y) - f(x_{i1}, x_{j1}, y)) (S_M f(x_{i2}, x_{j2}, y) - f(x_{i2}, x_{j2}, y)) \eta(x_{i1}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy. \]

Proof. Let \( \hat{\theta}_n = \hat{\theta}_n^1 - \hat{\theta}_n^2 \) where
\[ \hat{\theta}_n^1 = \frac{1}{n(n-1)} \sum_{l \in M} \sum_{k \neq k'} p_l(x_i^{(k)}, x_j^{(k)}, Y^{(k)}) \int p_l(x_i, x_j, Y^{(k)}) \psi(x_i, x_j, x_i^{(k)}, x_j^{(k)}, Y^{(k)}) dx_idx_j \]
\[ \hat{\theta}_n^2 = -\frac{1}{n(n-1)} \sum_{l \in M} \sum_{k \neq k'} p_l(x_i^{(k)}, x_j^{(k)}, Y^{(k)}) p_l(x_i^{(k')}, x_j^{(k')}, Y^{(k')}) \]
\[ \int p_l(x_i, x_j, y) p_l(x_i^{(k')}, x_j^{(k')}, Y^{(k')}) \eta(x_{i1}, x_{j2}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy. \]
Let us first compute $\mathbb{E}[\hat{\theta}_n^1]$.

$$
\mathbb{E}[\hat{\theta}_n^1] = \sum_{l \in M} \int p_l(x_{i1}, x_{j1}, y) f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy
$$

$$
= \sum_{l \in M} \int p_l(x_{i1}, x_{j1}, y) \psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}) f(x_{i2}, x_{j2}, y) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy
$$

$$
= \sum_{l \in M} a_l \int p_l(x_{i1}, x_{j1}, y) \psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}) f(x_{i2}, x_{j2}, y) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy
$$

$$
= \int \left( \sum_{l \in M} a_l p_l(x_{i2}, x_{j2}, y) \right) \psi(x_{i1}, x_{j1}, x_{i2}, x_{j2}) f(x_{i2}, x_{j2}, y) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy
$$

$$
= \int S_M f(x_{i1}, x_{j1}, y) f(x_{i2}, x_{j2}, y) \eta(x_{i1}, x_{j1}, x_{i2}, x_{j2}) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy
$$

Now for $\hat{\theta}_n^2$, we get

$$
\mathbb{E}[\hat{\theta}_n^2] = \sum_{l, l' \in M} \int p_l(x_{i1}, x_{j1}, y) f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy \int p_{l'}(x_{i1}, x_{j1}, y) f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy
$$

$$
= \sum_{l, l' \in M} \int p_l(x_{i1}, x_{j1}, y) p_{l'}(x_{i2}, x_{j2}, y) \eta(x_{i1}, x_{j1}, x_{i2}, x_{j2}) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy
$$

$$
= \int \left( \sum_{l \in M} a_l p_l(x_{i1}, x_{j1}, y) \right) \left( \sum_{l' \in M} a_{l'} p_{l'}(x_{i2}, x_{j2}, y) \right) \eta(x_{i1}, x_{j1}, x_{i2}, x_{j2}) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy
$$

$$
= \int S_M f(x_{i1}, x_{j1}, y) S_M f(x_{i2}, x_{j2}, y) \eta(x_{i1}, x_{j1}, x_{i2}, x_{j2}) dx_{i1} dx_{i2} dx_{j1} dx_{j2} dy.
$$

Arranging these terms and using

$$
\text{Bias} (\hat{\theta}_n) = \mathbb{E}[\hat{\theta}_n] - \theta = \mathbb{E}[\hat{\theta}_n^1] - \mathbb{E}[\hat{\theta}_n^2] - \theta
$$

we obtain the desire bias.

\[ \square \]

**Lemma 2 (Bound of $\text{Var}(U_n K)$).** Under the assumptions of Theorem 3, we have

$$
\text{Var}(U_n K) \leq \frac{20}{n(n-1)} \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta^2_{x_i x_j} (m + 1)
$$

\[ \text{Proof.} \text{ Note that } U_n K \text{ is centered because } Q \text{ and } R \text{ are centered and } (X^{(k)}_i, X^{(k)}_j, Y^{(k)}), k = .... \]
1, ..., n is an independent sample. So \( \text{Var}(U_n K) \) is equal to

\[
\mathbb{E} [U_n K]^2 = \mathbb{E} \left( \frac{1}{(n(n-1))^2} \sum_{k_1 \neq k_2} \sum_{k_1 \neq k_1'} \sum_{k_2 \neq k_2'} K(X_i^{(k_1)}, X_j^{(k_2)}, Y^{(k_1)}, X_i^{(k_1')}, X_j^{(k_2')}, Y^{(k_1')}) \right)
\]

By the Cauchy-Schwarz inequality, we get

\[
\text{Var}(U_n K) \leq \frac{2}{n(n-1)} \mathbb{E} \left[ K^2 \left( X_i^{(1)}, X_j^{(1)}, Y^{(1)}, X_i^{(2)}, X_j^{(2)}, Y^{(2)} \right) \right].
\]

Moreover, using the fact that \( 2 \mathbb{E}[XY] \leq \mathbb{E}[X^2] + \mathbb{E}[Y^2] \), we obtain

\[
\mathbb{E} \left[ K^2 \left( X_i^{(1)}, X_j^{(1)}, Y^{(1)}, X_i^{(2)}, X_j^{(2)}, Y^{(2)} \right) \right] \leq 2 \left[ \mathbb{E} \left[ (Q^T X_i^{(1)}, X_j^{(1)}, Y^{(1)}) R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) \right]^2 \right] + \mathbb{E} \left[ (Q^T X_i^{(1)}, X_j^{(1)}, Y^{(1)}) C Q(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) \right]^2.
\]

We will bound these two terms. The first one is

\[
\mathbb{E} \left[ (Q^T X_i^{(1)}, X_j^{(1)}, Y^{(1)}) R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) \right]^2
\]

\[
= \sum_{l,j \in M} \left( \int p_l(x_i, x_j, y) p_l(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - a_l a_{l'} \right)
\]

\[
\left( \int p_l(x_i, x_j, y) p_l(x_i, x_j, y) \psi(x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, y) \right)
\]

\[
\psi(x_{i_1}, x_{j_1}, x_{i_2}, x_{j_2}, y) f(x_{i_1}, x_{j_1}, y) dx_{i_1} dx_{j_1} dx_{i_2} dx_{j_2} dx_{i_3} dx_{j_3} dy - b_l b_{l'} \right)
\]

\[
=W_1 - W_2 - W_3 + W_4
\]
where

\[ W_1 = \int \sum_{l,l'\in M} p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)p_l(x_{i2}, x_{j2}, y')p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y') \]

\[ \psi(x_{i4}, x_{j4}, x_{i3}, x_{j3}, y')f(x_{i1}, x_{j1}, y)f(x_{i4}, x_{j4}, y')dx_{i1}dx_{j1}dx_{i2}dx_{j2}dx_{i3}dx_{j3}dx_{i4}dx_{j4}dydy' \]

\[ W_2 = \int \sum_{l,l'\in M} b_lb_{l'}p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)f(x_{i1}, x_{j1}, y)dx_{i1}dx_{j1}dy \]

\[ W_3 = \int \sum_{l,l'\in M} a_la_{l'}p_l(x_{i2}, x_{j2}, y')p_{l'}(x_{i3}, x_{j3}, y') \]

\[ \psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{j4}, x_{i3}, x_{j3}, y')f(x_{i4}, x_{j4}, y')dx_{i2}dx_{j2}dx_{i3}dx_{j3}dx_{i4}dx_{j4}dy' \]

\[ W_4 = \sum_{l,l'\in M} a_la_{l'}b_lb_{l'} \]

\[ W_2 \text{ and } W_3 \text{ are positive, hence} \]

\[ \mathbb{E}\left[ \left( 2Q^\top (X_i^{(1)}, X_j^{(1)}, Y^{(1)})R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) \right)^2 \right] \leq W_1 + W_4. \]

\[ W_1 = \int \sum_{l,l'\in M} p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)\left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right) \]

\[ \left( \int p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{j4}, x_{i3}, x_{j3}, y')dx_{i3}dx_{j3} \right) f(x_{i1}, x_{j1}, y)f(x_{i4}, x_{j4}, y')dx_{i1}dx_{j1}dx_{i4}dx_{j4}dydy' \]

\[ \leq \| f \|^2 \sum_{l,l'\in M} \int p_l(x_{i1}, x_{j1}, y)p_{l'}(x_{i1}, x_{j1}, y)dx_{i1}dx_{j1}dy \]

\[ \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right) \]

\[ \left( \int p_{l'}(x_{i3}, x_{j3}, y')\psi(x_{i4}, x_{j4}, x_{i3}, x_{j3}, y')dx_{i3}dx_{j3} \right) dx_{i2}dx_{j2}dx_{i4}dx_{j4}dy' \]

Since \( p_l \)'s are orthonormal we have

\[ W_1 \leq \| f \|^2 \sum_{l\in M} \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right)^2 dx_{i4}dx_{j4}dy'. \]

Moreover by the Cauchy-Schwarz inequality and \( \| \psi \|_\infty \leq 2\| \eta \|_\infty \)

\[ \left( \int p_l(x_{i2}, x_{j2}, y')\psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y')dx_{i2}dx_{j2} \right)^2 \leq \int p_l(x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2} \]

\[ \int \psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2} \]

\[ \leq \| \psi \|^2_\infty \Delta_{x_{i4},x_{j4}} \int p_l(x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2} \]

\[ \leq 4\| \eta \|^2_\infty \Delta_{x_{i4},x_{j4}} \int p_l(x_{i2}, x_{j2}, y')^2dx_{i2}dx_{j2}, \]
and then
\[
\int \left( \int p_l(x_{i2}, x_{j2}, y') \psi(x_{i4}, x_{j4}, x_{i2}, x_{j2}, y') dx_{i2} dx_{j2} \right)^2 dx_{i4} dx_{j4} dy' \\
\leq 4 \| \eta \|_\infty^2 \Delta_{x, x_j}^2 \int p_l(x_{i2}, x_{j2}, y')^2 dx_{i2} dx_{j2} dy' \\
= 4 \| \eta \|_\infty^2 \Delta_{x, x_j}^2.
\]

Finally,
\[
W_1 \leq 4 \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta_{x, x_j}^2, m.
\]

For the term \(W_4\) using the facts that \(S_M f\) and \(S_M g\) are projection and that \(\int f = 1\), we have
\[
W_4 = \left( \sum_{l \in M} a_l b_l \right)^2 \leq \sum_{l \in M} a_l^2 \sum_{l' \in M} b_{l'}^2 \leq \| f \|_2^2 \| g \|_2^2 \leq \| f \|_\infty \| g \|_2^2.
\]

By the Cauchy-Schwartz inequality we have \(\| g \|_2^2 \leq 4 \| \eta \|_\infty^2 \| f \|_\infty \Delta_{x, x_j}^2\) and then
\[
W_4 \leq 4 \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta_{x, x_j}^2
\]

which leads to
\[
\mathbb{E} \left[ (Q^T (X_{i}^{(1)}, X_j^{(1)}, Y^{(1)}) R(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) )^2 \right] \leq 4 \| \eta \|_\infty^2 \| f \|_\infty^2 \Delta_{x, x_j}^2 (m + 1). \tag{25}
\]

The second term \(\mathbb{E} \left[ (Q^T (X_{i}^{(1)}, X_j^{(1)}, Y^{(1)}) C Q(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) ) \right] = W_5 - 2W_6 + W_7\) where

\[
W_5 = \int \sum_{l_1, l_2} \sum_{j_1, j_2} c_{l_1} \epsilon_{l_1} c_{l_2} \epsilon_{l_2} p_{l_1}(x_{i1}, x_{j1}, y) p_{l_2}(x_{i1}, x_{j1}, y) \eta_{l_1}(x_{i2}, x_{j2}, y') \eta_{l_2}(x_{i2}, x_{j2}, y') f(x_{i1}, x_{j1}, y) f(x_{i2}, x_{j2}, y') dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy' dy
\]

\[
W_6 = \int \sum_{l_1, l_2} \sum_{j_1, j_2} c_{l_1} \epsilon_{l_1} c_{l_2} \epsilon_{l_2} a_{l_1} a_{l_2} \eta_{l_1}(x_{i1}, x_{j1}, y) \eta_{l_2}(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy
\]

\[
W_7 = \sum_{l_1, l_2} \sum_{j_1, j_2} c_{l_1} \epsilon_{l_1} c_{l_2} \epsilon_{l_2} a_{l_1} a_{l_2} \eta_{l_1} \eta_{l_2}.
\]

Using the previous manipulation, we show that \(W_6 \geq 0\). Thus
\[
\mathbb{E} \left[ (Q^T (X_{i}^{(1)}, X_j^{(1)}, Y^{(1)}) C Q(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) ) \right] \leq W_5 + W_7.
\]
First, observe that

\[ W_5 = \sum_{l_i,l'_i} \sum_{l_2,l'_2} c_{l_i}c_{l'_i} \left( \int p_{l_i}(x_{i_1}, x_{j_1}, y)p_{l'_i}(x_{i_1}, x_{j_1}, y)f(x_{i_1}, x_{j_1}, y)dx_{i_1}dx_{j_1}dy \right) \]

\[ \leq \|f\|_\infty^2 \sum_{l_i} c_{l_i} \left( \int p_{l_i}(x_{i_1}, x_{j_1}, y)p_{l_i}(x_{i_1}, x_{j_1}, y)dx_{i_1}dx_{j_1}dy \right) \]

\[ = \|f\|_\infty^2 \sum_{l_i} c_{l_i}^2 \]

again using the orthonormality of the the \( p_{l_i} \)'s. Besides given the decomposition \( p_{l_i}(x_{i_1}, x_{j_1}, y) = \alpha_{l_i}(x_{i_1}, x_{j_1})\beta_{l_i}(y) \),

\[ \sum_{l_i,l'_i} c_{l_i}^2 = \sum_{l_i,l'_i} \int \int \beta_{l_i}(y')\beta_{l'_i}(y')dydy' \]

\[ \sum_{l_i} \left( \int \alpha_{l_i}(x_{i_1}, x_{j_1})\alpha_{l_i}(x_{i_2}, x_{j_2})\eta(x_{i_1}, x_{j_2}, y)dx_{i_1}dx_{j_1}dx_{i_2}dx_{j_2} \right) \]

\[ \leq \sum_{l_i} \left( \int \alpha_{l_i}(x_{i_1}, x_{j_1})\eta(x_{i_1}, x_{j_1}, y)\alpha_{l_i}(x_{i_2}, x_{j_2})dx_{i_1}dx_{j_1}dx_{i_2}dx_{j_2} \right) \]

\[ \leq \Delta_{x_{i_1},x_{j_1}}^2 \|\eta\|_\infty^2 \]

\[ \]
using the orthonormality of the basis $\alpha_i$. Then we get
\[
\sum_{l,l'} c_{ll'}^2 \leq \Delta_{x,x_j}^2 \|\eta\|^2_\infty \left( \int \sum_{l,l'} \beta_{l,j}(y) \beta_{l',j}(y') \beta_{l,j}(y) \beta_{l',j}(y') dy dy' \right)
\]
\[
= \Delta_{x,x_j}^2 \|\eta\|^2_\infty \sum_{l,l'} \left( \int \beta_{l,j}(y) \beta_{l',j}(y) dy \right)^2
\]
\[
\leq \Delta_{x,x_j}^2 \|\eta\|^2_\infty \sum_{l,j} \left( \int \beta_{l,j}^2(y) dy \right)^2
\]
\[
\leq \Delta_{x,x_j}^2 \|\eta\|^2_\infty \Delta_{x,x_j} m
\]
since the $\beta_{l,j}$ are orthonormal. Finally
\[
W_5 \leq \|f\|_\infty^2 \|\eta\|_\infty^2 \Delta_{x,x_j}^2 m.
\]

Now for $W_7$ we first will bound,
\[
\sum_{l,l'} c_{ll'} a_{ll'} = \left| \int \sum_{l,l'\in\mathcal{M}} a_{ll'} p_l(x_1, x_j, y) p_{l'}(x_2, x_j, y) \eta(x_1, x_2, y) dx_1 dx_2 dx_1 dx_2 dy dy' \right|
\]
\[
\leq \int |S_M(x_1, x_1, y) S_M(x_2, x_2, y) \eta(x_1, x_2, y)| dx_1 dx_2 dx_1 dx_2 dy dy'
\]
\[
\leq \|\eta\|_\infty \int \left( \int |S_M(x_1, x_1, y) S_M(x_2, x_2, y)| dy \right) dx_1 dx_2 dx_1 dx_2 dy dy'
\]
Taking squares in both sides and using the Cauchy-Schwartz inequality twice, we get
\[
\left( \sum_{l,l'} c_{ll'} a_{ll'} \right)^2 = \|\eta\|^2_\infty \left( \int \left( \int |S_M(x_1, x_1, y) S_M(x_2, x_2, y)| dy \right) dx_1 dx_2 dx_1 dx_2 dy dy' \right)^2
\]
\[
\leq \|\eta\|^2_\infty \Delta_{x,x_j}^2 \left( \int \left( \int |S_M(x_1, x_1, y) S_M(x_2, x_2, y)| dy \right)^2 dx_1 dx_2 dx_1 dx_2 dy dy' \right)
\]
\[
\leq \|\eta\|^2_\infty \Delta_{x,x_j}^2 \left( \int S_M(x_1, x_1, y)^2 dy \right) \left( \int S_M(x_2, x_2, y)^2 dy \right) dx_1 dx_2 dx_1 dx_2 dy dy'
\]
\[
= \|\eta\|^2_\infty \Delta_{x,x_j}^2 \left( \int S_M(x_1, x_1, y)^2 dy \right) \left( \int S_M(x_2, x_2, y)^2 dy \right) dx_1 dx_2 dx_1 dx_2 dy dy'
\]
\[
= \|\eta\|^2_\infty \Delta_{x,x_j}^2 \left( \int S_M(x_1, x_1, y)^2 dx_1 dx_2 dy \right)
\]
\[
\leq \|\eta\|^2_\infty \Delta_{x,x_j}^2 \|f\|_\infty^2.
\]
Finally,
\[
\mathbb{E} \left[ (Q^\top (X_i^{(1)}, X_j^{(1)}, Y^{(1)}) C Q(X_i^{(2)}, X_j^{(2)}, Y^{(2)}))^2 \right] \leq \|\eta\|^2_\infty \|f\|^2_\infty \Delta_{x,x_j}^2 (m + 1). \quad (26)
\]
Collecting (25) and (26), we obtain

$$\text{Var}(U_nK) \leq \frac{20}{n(n-1)} \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta^2_{x,x}, (m+1)$$

which concludes the proof of Lemma 2.

Lemma 3 (Bound for \(\text{Var}(P_nL)\)). Under the assumptions of Theorem 3 we have

$$\text{Var}(P_nL) \leq \frac{12}{n} \|\eta\|_\infty^2 \|f\|_\infty^2 \Delta^2_{x,x}.$$  

Proof. First note that given the independence of \((X_i^{(k)}, X_j^{(k)}, Y^{(k)})\) \(k = 1, \ldots, n\) we have

$$\text{Var}(P_nL) = \frac{1}{n} \text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))$$

we can write \(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})\) as

$$A^\top R \left( X_i^{(1)}, X_j^{(1)}, Y^{(1)} \right) + B^\top Q \left( X_i^{(1)}, X_j^{(1)}, Y^{(1)} \right) - 2A^\top C Q \left( X_i^{(1)}, X_j^{(1)}, Y^{(1)} \right)$$

$$= \sum_{l \in M} a_l \left( \int p_l(x_i, x_j, Y^{(1)}) \psi(x_i, x_j, X_i^{(1)}, X_j^{(1)}, Y^{(1)}) dx_i dx_j - b_l \right)$$

$$+ \sum_{l \in M} b_l \left( p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - a_l \right) - 2 \sum_{l, l' \in M} c_{ll'} a_{ll'} \left( p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - a_l \right)$$

$$= \int \sum_{l \in M} a_l p_l(x_i, x_j, Y^{(1)}) \psi(x_i, x_j, X_i^{(1)}, X_j^{(1)}, Y^{(1)}) dx_i dx_j$$

$$+ \sum_{l \in M} b_l p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2 \sum_{l, l' \in M} c_{ll'} a_{ll'} p_l(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^\top B - 2A^\top CA.$$  

Let \(h(x_i, x_j, y) = \int S_M f(x_i, x_j, x_{i2}, x_{j2}, y) \psi(x_i, x_j, x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}\), we have

$$S_M h(x_i, x_j, y)$$

$$= \sum_{l \in M} \left( \int h(x_{i2}, x_{j2}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dy \right) p_l(x_i, x_j, y)$$

$$= \sum_{l \in M} \left( \int S_M f(x_{i3}, x_{j3}, y) \psi(x_{i2}, x_{j2}, x_{i3}, x_{j3}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dx_{i3} dx_{j3} dy \right) p_l(x_i, x_j, y)$$

$$= \sum_{l, l' \in M} \left( \int a_{ll'} p_{ll'}(x_{i3}, x_{j3}, y) \psi(x_{i2}, x_{j2}, x_{i3}, x_{j3}, y) p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dx_{i3} dx_{j3} dx_{i3} dx_{j3} dy \right) p_l(x_i, x_j, y)$$

$$= 2 \sum_{l, l' \in M} \left( \int a_{ll'} c_{ll'} p_l(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} dx_{i3} dx_{j3} dx_{i3} dx_{j3} dy \right) p_l(x_i, x_j, y)$$

$$= 2 \sum_{l, l' \in M} a_{ll'} c_{ll'} p_l(x_i, x_j, y)$$
and we can write
\[
L(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) = h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^\top B - 2A^\top C A.
\]

Thus,
\[
\text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) \\
= \text{Var}(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) \\
\leq \mathbb{E}[(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \\
\leq \mathbb{E}[(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2 + (S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2 + (S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2].
\]

Each of these terms can be bounded
\[
\mathbb{E}[(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] = \int \left( \int S_M f(x_{i1}, x_{j2}, y) \psi(x_{i1}x_{j2}, x_{i1}, x_{j2}, y) dx_{i1} dx_{j2} \right)^2 f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dy \\
\leq \Delta_{x_{i}x_{j}} \int S_M f(x_{i2}, x_{j2}, y)^2 \psi(x_{i1}x_{j2}, x_{i2}, x_{j2}, y)^2 f(x_{i1}, x_{j1}, y) dx_{i1} dx_{j1} dx_{i2} dx_{j2} dy \\
\leq 4\Delta^2_{x_{i}x_{j}} f_\infty^2 \|\eta\|^2 \int S_M f(x_{i}, x_{j}, y)^2 dx_i dx_j dy \\
= 4\Delta^2_{x_{i}x_{j}} f_\infty^2 \|\eta\|^2 \|S_M f\|^2_2 \\
\leq 4\Delta^2_{x_{i}x_{j}} f_\infty^2 \|\eta\|^2 \|f\|^2_2 \\
\leq 4\Delta^2_{x_{i}x_{j}} f_\infty^2 \|\eta\|^2_\infty
\]

and similar calculations are valid for the others two terms,
\[
\mathbb{E}[(S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \leq \|f\|_\infty^2 \|S_M g\|^2_2 \leq \|f\|_\infty^2 \|g\|^2_2 \leq 4\Delta^2_{x_{i}x_{j}} f_\infty^2 \|\eta\|^2_\infty \\
\mathbb{E}[(S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))^2] \leq \|f\|_\infty^2 \|S_M h\|^2_2 \leq \|f\|_\infty^2 \|h\|^2_2 \leq 4\Delta^2_{x_{i}x_{j}} f_\infty^2 \|\eta\|^2_\infty.
\]

Finally we get,
\[
\text{Var}(P_n L) \leq \frac{12}{n} \|\eta\|^2_\infty \|f\|^2_\infty \Delta^2_{x_{i}x_{j}}.
\]

\textbf{Lemma 4 (Computation of Cov($U_nK, P_nL$)). Under the assumptions of Theorem 3, we have}
\[
\text{Cov}(U_nK, P_nL) = 0.
\]
Proof of Lemma 4 Since $U_nK$ and $P_nL$ are centered, we have

$$
\text{Cov}(U_nK, P_nL) = \mathbb{E}[U_nKP_nL]
$$

$$
= \mathbb{E}\left[\frac{1}{n^2(n-1)} \sum_{k \neq k' = 1}^{n} K(X_i^{(k)}, X_j^{(k)}, Y^{(k)}, X_i^{(k')}, X_j^{(k')}, Y^{(k')}) \sum_{k=1}^{n} L(X_i^{(k)}, X_j^{(k)}, Y^{(k)})\right]
$$

$$
= \frac{1}{n} \mathbb{E}\left[ K(X_i^{(1)}, X_j^{(1)}, Y^{(1)}, X_i^{(2)}, X_j^{(2)}, Y^{(2)}) L(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + L(X_i^{(2)}, X_j^{(2)}, Y^{(2)})\right]
$$

$$
= \frac{1}{n} \mathbb{E}\left[ (Q^T (X_i^{(1)}, X_j^{(1)}, Y^{(1)}) R(X_i^{(1)}, X_j^{(2)}, Y^{(2)}) - Q^T (X_i^{(1)}, X_j^{(1)}, Y^{(1)}) CQ(X_i^{(2)}, X_j^{(2)}, Y^{(2)}))
\right.

$$

(A^T R(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + B^T Q(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) - 2A^T CQ(X_i^{(1)}, X_j^{(1)}, Y^{(1)})

(A^T R(X_i^{(1)}, X_j^{(2)}, Y^{(2)}) + B^T Q(X_i^{(2)}, X_j^{(2)}, Y^{(2)}) - 2A^T CQ(X_i^{(2)}, X_j^{(2)}, Y^{(2)})]
$$

$$= 0.
$$

Since $K, L, Q$ and $R$ are centered.

Lemma 5 (Bound of Bias ($\hat{\theta}_n$)). Under the assumptions of Theorem 3 we have

$$
|\text{Bias}(\hat{\theta}_n)| \leq \Delta_{x_i x_j} \|\eta\|_\infty \sup_{l \notin \mathcal{M}} |c_l|^2.
$$

Proof.

$$
|\text{Bias}(\hat{\theta}_n)| \leq \|\eta\|_\infty \int \left( \int |S_M f(x_{i1}, x_{j1}, y) - f(x_{i1}, x_{j1}, y)| \, dx_{i1} \, dx_{j1} \right)
$$

$$
\left( \int |S_M f(x_{i2}, x_{j2}, y) - f(x_{i2}, x_{j2}, y)| \, dx_{i2} \, dx_{j2} \right) \, dy
$$

$$
= \|\eta\|_\infty \int \left( \int |S_M f(x_i, x_j, y) - f(x_i, x_j, y)| \, dx_i \, dx_j \right)^2 \, dy
$$

$$
\leq \Delta_{x_i x_j} \|\eta\|_\infty \int (S_M f(x_i, x_j, y) - f(x_i, x_j, y))^2 \, dx_i \, dx_j \, dy
$$

$$
= \Delta_{x_i x_j} \|\eta\|_\infty \sum_{l \notin \mathcal{M}} a_l a_l \int p_l(x_i, x_j, y) p_l(x_i, x_j, y) \, dx_i \, dx_j \, dy
$$

$$
= \Delta_{x_i x_j} \|\eta\|_\infty \sum_{l \notin \mathcal{M}} |a_l|^2 \leq \Delta_{x_i x_j} \|\eta\|_\infty \sup_{l \notin \mathcal{M}} |c_l|^2.
$$

We use the Hölder's inequality and the fact that $f \in \mathcal{E}$ then $\sum_{l \notin \mathcal{M}} |a_l|^2 \leq \sup_{l \notin \mathcal{M}} |c_l|^2$.

Lemma 6 (Asymptotic variance of $\sqrt{n}(P_nL)$). Under the assumptions of Theorem 3 we have

$$
n \text{Var}(P_nL) \to \Lambda(f, \eta)
$$

where

$$
\Lambda(f, \eta) = \int g(x_i, x_j, y)^2 f(x_i, x_j, y) \, dx_i \, dx_j \, dy - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) \, dx_i \, dx_j \, dy \right)^2.
$$
We proved in Lemma 3 that
\[
\text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) = \text{Var}(h(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M g(X_i^{(1)}, X_j^{(1)}, Y^{(1)}) + S_M h(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) = \frac{3}{2} \sum_{k,l=1}^{3} \text{Cov}(A_k, A_l).
\]

We claim that \( \forall k, l \in \{1, 2, 3\}^2 \), we have
\[
\left| \text{Cov}(A_k, A_l) - \epsilon_{kl} \left[ \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right] \right| \leq \lambda \left[ \|S_M f - f\|_2 + \|S_M g - g\|_2 \right] \tag{27}
\]
where
\[
\epsilon_{kl} = \begin{cases} 
-1 & \text{if } k = 3 \text{ or } l = 3 \text{ and } k \neq l \\
1 & \text{otherwise}
\end{cases}
\]
and where \( \lambda \) depends only on \( \|f\|_\infty, \|\eta\|_\infty \) and \( \Delta_{x_i, x_j} \). We will do the details only for the case \( k = l = 3 \) since the calculations are similar for other configurations.

\[
\text{Var}(A_3) = \int S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \left( \int S_M h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2.
\]

The computation will be done in two steps. We first bound the quantity by the Cauchy-Schwartz inequality
\[
\left| \int S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \left( \int S_M h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right| \leq \left\| S_M h - S_M g \right\|_2 \left\| S_M h - S_M g \right\|_2 + \left\| f \right\|_\infty \left\| S_M g - g \right\|_2 \left\| S_M g - g \right\|_2.
\]

Using several times the fact that since \( S_M \) is a projection, \( \left\| S_M g \right\|_2 \leq \left\| g \right\|_2 \), the sum is bounded by
\[
\left\| f \right\|_\infty \left\| h + g \right\|_2 \left\| h - g \right\|_2 + 2 \left\| f \right\|_\infty \left\| g \right\|_2 \left\| S_M g - g \right\|_2 \leq \left\| f \right\|_\infty \left( \left\| h \right\|_2 + \left\| g \right\|_2 \right) \left\| h - g \right\|_2 + 2 \left\| f \right\|_\infty \left\| g \right\|_2 \left\| S_M g - g \right\|_2.
\]

We saw previously that \( \left\| g \right\|_2 \leq 2 \Delta_{x_i, x_j} \left\| f \right\|_\infty^{1/2} \left\| \eta \right\|_\infty \) and \( \left\| h \right\|_2 \leq 2 \Delta_{x_i, x_j} \left\| f \right\|_\infty^{1/2} \left\| \eta \right\|_\infty \).

The sum is then bound by
\[
4 \Delta_{x_i, x_j} \left\| f \right\|_\infty^{3/2} \left\| \eta \right\|_\infty \left\| h - g \right\|_2 + 4 \Delta_{x_i, x_j} \left\| f \right\|_\infty^{3/2} \left\| \eta \right\|_\infty \left\| S_M g - g \right\|_2.
\]
We now have to deal with $\| h - g \|_2^2$:

\[
\begin{align*}
\| h - g \|_2^2 &= \int \left( \int (S_M f(x_i, x_j, y) - f(x_i, x_j, y)) \psi(x_i, x_j, x_k, x_l, y) dx_i dx_j dy \right)^2 dx_i dx_j dy \\
&\leq \int \left( \int (S_M f(x_i, x_j, y) - f(x_i, x_j, y))^2 dx_i dx_j dy \right) \left( \int \psi^2(x_i, x_j, x_k, x_l, y) dx_i dx_j dy \right) dx_i dx_j dy \\
&\leq 4 \Delta_{x,x}^2 \| \eta \|_\infty^2 \| S_M f - f \|_2^2.
\end{align*}
\]

Finally this first part is bounded by

\[
\begin{align*}
&\left| \int S_M^2 h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy - \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy \right| \\
&\leq 4 \Delta_{x,x} \| f \|_\infty^{3/2} \| \eta \|_\infty \left( 2 \Delta_{x,x} \| \eta \|_\infty \| S_M f - f \|_2 + \| S_M g - g \|_2 \right)
\end{align*}
\]

Following with the second quantity

\[
\begin{align*}
&\left| \left( \int S_M h(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 - \left( \int g(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right| \\
&= \left| \left( \int (S_M h(x_i, x_j, y) - g(x_i, x_j, y)) f(x_i, x_j, y) dx_i dx_j dy \right) - \left( \int (S_M h(x_i, x_j, y) + g(x_i, x_j, y)) f(x_i, x_j, y) dx_i dx_j dy \right) \right|
\end{align*}
\]

By using the Cauchy-Schwartz inequality, it is bounded by

\[
\begin{align*}
&\left| \| f \|_2 \| S_M h - g \|_2 \| f \|_2 \| S_M h + g \|_2 \\
&\leq \| f \|_2 \left( \| h \|_2 + \| g \|_2 \right) \left( \| S_M h - S_M g \|_2 + \| S_M g - g \|_2 \right) \\
&\leq 4 \Delta_{x,x} \| f \|_\infty^{3/2} \| \eta \|_\infty \left( \| h - g \|_2 + \| S_M g - g \|_2 \right) \\
&\leq 4 \Delta_{x,x} \| f \|_\infty^{3/2} \| \eta \|_\infty \left( 2 \Delta_{x,x} \| \eta \|_\infty \| S_M f - f \|_2 + \| S_M g - g \|_2 \right)
\end{align*}
\]

using the previous calculations. Collecting the two inequalities gives (27) for $k = l = 3$. Finally, since by assumption $\forall t \in \L^2(d\mu)$, $\| S_M t - t \|_2 \to 0$ when $n \to \infty$ a direct consequence of (27) is

\[
\lim_{n \to \infty} \text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)})) = \int g^2(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy = \frac{1}{2} \Lambda(f, \eta).
\]

We conclude by noting that $\text{Var}(\sqrt{n}(P_n L)) = \text{Var}(L(X_i^{(1)}, X_j^{(1)}, Y^{(1)}))$. 

\[\square\]
Lemma 7 (Asymptotics for \(\sqrt{n}(\hat{Q} - Q)\)). Under the assumptions of Theorem 1 we have
\[
\lim_{n \to \infty} n\mathbb{E}[(\hat{Q} - Q)^2] = 0.
\]

Proof. The bound given in (16) states that if \(|M_n|/n \to 0\) we have
\[
n\mathbb{E}[(\hat{Q} - Q)^2] - \mathbb{E} \left[ \int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int \hat{g}(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right] \\
\leq \gamma(f, \|\eta\|_{\infty}, \Delta_{x,x}) \left[ \frac{|M_n|}{n} + \|S_M f - f\|_2 + \mathbb{E}[\|S_M \hat{g} - \hat{g}\|_2] \right]
\]
where \(\hat{g}(x_i, x_j, y) = \int H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}\), where we recall that \(H_3(f, x_{i1}, x_{j1}, x_{i2}, y) = H_2(f, x_{i1}, x_{j2}, y) + H_2(f, x_{i2}, x_{j1}, y)\) with \(H_2(\hat{f}, x_{i1}, x_{j2}, y) = \frac{1}{f(f(x_{i1}, x_{j2}, y) dx_{i2} dx_{j2})} (x_{i1} - m_i(\hat{f}, y))(x_{j2} - m_j(\hat{f}, y))\). By deconditioning we get
\[
n\mathbb{E}[(\hat{Q} - Q)^2] - \mathbb{E} \left[ \int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int \hat{g}(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2 \right] \\
\leq \gamma(f, \|\eta\|_{\infty}, \Delta_{x,x}) \left[ \frac{|M_n|}{n} + \|S_M f - f\|_2 + \mathbb{E}[\|S_M \hat{g} - \hat{g}\|_2] \right]
\]
Note that
\[
\mathbb{E}[\|S_M \hat{g} - \hat{g}\|_2] \leq \mathbb{E}[\|S_M \hat{g} - S_M g\|_2] + \mathbb{E}[\|\hat{g} - g\|_2] + \mathbb{E}[\|S_M g - g\|_2] \\
\leq 2\mathbb{E}[\|\hat{g} - g\|_2] + \mathbb{E}[\|S_M g - g\|_2]
\]
where \(g(x_i, x_j, y) = \int H_3(f, x_i, x_j, x_{i2}, x_{j2}, y) f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}\). The second term converges to 0 since \(g \in L^2(dx dy dz)\) and \(\forall t \in L^2(dx dy dz), \int (S_M t - t)^2 dx dy dz \to 0\). Moreover
\[
\|\hat{g} - g\|_2^2 = \int [\hat{g}(x_i, x_j, y) - g(x_i, x_j, y)]^2 dx_i dx_j dy \\
= \int \left[ \int (H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y))^2 f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} \right] dx_i dx_j dy \\
\leq \int \left[ \int (H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y))^2 dx_{i2} dx_{j2} \right] dx_i dx_j dy \\
\leq \Delta_{x,x} \|f\|_\infty^2 \int (H_2(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_2(f, x_i, x_j, x_{i2}, x_{j2}, y))^2 dx_i dx_j dx_{i2} dx_{j2} dy \\
\leq \delta \Delta_{x,x} \|f\|_\infty^2 \int (\hat{f}(x_i, x_j, y) - f(x_i, x_j, y))^2 dx_i dx_j dy
\]
for some constant \(\delta\) that comes out of applying the mean value theorem to \(H_3(\hat{f}, x_i, x_j, x_{i2}, x_{j2}, y) - H_3(f, x_i, x_j, x_{i2}, x_{j2}, y)\). The constant \(\delta\) was taken under Assumptions 13. Since \(\mathbb{E}[\|f - \hat{f}\|_2] \to 0\) then \(\mathbb{E}[\|g - \hat{g}\|_2] \to 0\). Now show that the expectation of
\[
\int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \left( \int \hat{g}(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j dy \right)^2
\]
converges to 0. We develop the proof for only the first term. We get
\[
\left| \int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy - \int g(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j dy \right|
\leq \int [\hat{g}(x_i, x_j, y)^2 - g(x_i, x_j, y)^2] f(x_i, x_j, y) dx_i dx_j dy
\leq \lambda \int (\hat{g}(x_i, x_j, y) - g(x_i, x_j, y))^2 dx_i dx_j dy
= \lambda \| \hat{g} - g \|_2^2
\]
for some constant \( \lambda \). By taking the expectation of both sides, we see it is enough
to show that \( E [\| \hat{g} - g \|_2^2] \to 0 \). Besides, we can verify
\[
g(x_i, x_j, y) = \int H_3(f, x_i, x_j, x_{i2}, x_{j2}, y) f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2}
= \frac{2}{\int f(x_i, x_j, y) dx_i dx_j} \left( x_i - \hat{m}_i(y) \right) \left( \int x_{j2} f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} - \hat{m}_j(y) \int f(x_{i2}, x_{j2}, y) dx_{i2} dx_{j2} \right)
= 0
\]
which proves that the expectation of \( \int \hat{g}(x_i, x_j, y)^2 f(x_i, x_j, y) dx_i dx_j \) converges to 0. Similar computations shows that the expectation of \( (\int \hat{g}(x_i, x_j, y) f(x_i, x_j, y) dx_i dx_j)^2 \) also converges to 0. Finally we have
\[
\lim_{n \to \infty} n E [\hat{Q} - Q]^2 = 0.
\]
\[\square\]

References

variables. *A Festschrift for Erich L. Lehmann in Honor of His Sixty-Fifth Birthday*,
(pp. 97–114).

B (Statistical Methodology)*, 63(2), 393–410.

*Journal of the American Statistical Association*, 100(470), 410–428.


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