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Adaptation of Preissmann's scheme for transcritical open channel flows

Adaptation du schéma de Preissmann pour les écoulements transcritiques à surface libre

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Abstract

Despite widely used for the solution of one-dimensional subcritical flows governed by Saint-Venant's equations, the Preissmann's scheme cannot solve transcritical flows. This inability is due only to the solution methods created for non-transcritical flows. Transcritical transitions present specific properties which have to be adequately represented in the numerical method. A modified version of Preissmann's method is presented herein which changes the formulation only in transcritical zones, while keeping its conservative property and shock capturing form otherwise. A solution method is proposed for the implicit system, through storing the transcritical positions. This enables to solve the system with simple and double sweep methods. The transcritical transition problem is solved locally, either by associating the cells involved in a bore and adding an equation to characterize the information transferred in the subcritical domain, or by the addition of an internal boundary condition to characterize the expansion fan at the critical point.

RÉSUMÉ

Bien qu'il soit fréquemment utilisé pour la résolution des écoulements 1D fluviaux régis par les équations de Saint-Venant, le schéma de Preissmann ne peut pas traiter les écoulements transcritiques. Cette incapacité est uniquement due aux méthodes de résolution généralement utilisées, car elles ont été mises au point pour les écoulements non transcritiques. Ces transitions présentent des caractéristiques particulières qui doivent être correctement gérées par la méthode numérique employée. Une version modifiée du schéma de Preissmann est présentée, adaptant la formulation dans ces zones, tout en gardant par ailleurs ses propriétés conservatives et de capture des chocs. Une méthode de résolution est proposée pour le système implicite, à travers le stockage des positions transcritiques. Cela permet de résoudre le système avec des techniques simples et la méthode du double balayage. Le problème de la transition transcritique est résolu localement, soit par l'association des cellules impliquées dans un ressaut en y ajoutant une équation pour caractériser les informations transmises dans le domaine fluvial, soit par l'ajout d'une condition à la limite interne pour caractériser le point critique.

Keywords: Fluvial hydraulics, hydraulic jump, numerical model, numerical scheme, open channel flow, transcritical flow, transient flow

1 Introduction

An essential practical aim in river hydraulics is to model hydraulic networks, whether for irrigation enhancement, flood planning or management of urban-drainage. This implies simulations of lengthy events and inclusion of hydraulic structures (Baume *et al.* 2005). Even if the flow is generally subcritical, supercritical regions may appear. The different regimes as well as the transitions between these – bores and critical points – must be numerically represented. Nonetheless, there is no need for a precise localisation of a sharp front. In other words, the expense of sophisticate techniques created for the solution of dam-break problems would be neither justified nor appropriate for our above applications.

For subcritical flows, Preissmann's scheme (Cunge *et al.* 1980) is one of the most widely used in industrial models. This implicit finite-difference method is unconditionally stable and extremely robust. Moreover, each discrete equation implying only two discrete spatial positions makes easy the inclusion of hydraulic structures that can be represented between two sections, without perturbing the modelling of neighbouring fluid zones. Unfortunately, while also well suited for fully supercritical flow, it can be demonstrated that in its original usage, the scheme is invalid for transcritical flows, *i.e.* for flows in which the two regimes coexist (Meselhe and Holly 1997).

A classical procedure allows the treatment of transcritical flows with a numerical method created for subcritical flows (Kutija 1994). The solution is to apply the method to a degenerated system of equations, reducing the influence of the convective momentum term as the flow becomes supercritical. But this trick does not conserve volumes.

Noticeably, circumventing Meselhe and Holly's invalidity conditions, Johnson *et al.* (2002) have proposed a method for transcritical flow based on Preissmann's scheme involving the complete Saint-Venant equations. Using the idea of splitting methods (Roe 1981), Preissmann's equations from two adjacent cells are combined. The matrix of the resulting scheme loses the bi-diagonal structure associated with Preissmann's scheme. This structure is highly desirable as it permits solving algebraically the implicit system.

This research focuses on the direct adaptation of Preissmann's scheme for transcritical flows. First, the observations of Meselhe and Holly (1997) concerning the invalidity of direct use of Preissmann's method for the solution of transcritical flows using the full Saint-Venant's equations are discussed. Then the interesting features proposed by Johnson *et al.* (2002) to describe transcritical transitions with a through method (i.e. a method whereby all possible wave configurations are handled within one single mathematical formula, without the need of specific or *ad hoc* treatments) compatible with Preissmann's scheme are highlighted. The resulting method uses Preissmann's scheme in non-transcritical domains, which are linked by internal boundary conditions characterizing the physics of transcritical transitions. Furthermore, an automatic procedure based on discrete Froude numbers is used to solve the transcritical implicit system. It corresponds to an adaptation of the simple- and double-sweep methods which only require localising the transcritical transitions, thus keeping one of the major advantages of Preissmann's scheme. The method is presented on a transient problem involving appearance and disappearance of a supercritical region between a critical point and a bore. Finally, conclusions on this research are drawn.

2 Generalities on numerical methods for free surface hydraulics

2.1 *Saint-Venant's equations*

- *Governing equations in conservation form*

To model one dimensional (1D) open channel flows (Cunge *et al.* 1980), the depth-integrated equations for conservation of mass and momentum can be written for the variables wetted cross-sectional area A and discharge Q as

$$\begin{aligned} &\partial_t A + \partial_x Q = 0 \\ &\partial_t Q + \partial_x (Q^2 / A + gI_1) = gA(S_o - S_f) \end{aligned} \tag{1}$$

where g = gravitational acceleration, S_o = bottom slope, (gI_1) = integral of hydrostatic pressure over the cross-section, i.e. $I_1=A^2/(2B)$ for a rectangular channel of width B, friction slope $S_f = Q|Q| / De^2$, where $De = KAR_h^{2/3}$ with K = Strickler coefficient and R_h = hydraulic radius. Equation (1) corresponds to the conservative form of the equations that is desirable when dealing with transcritical flows. The vector-form for a conserved variable $\boldsymbol{u}=(A \;\; Q)^T$ of flux $\boldsymbol{f}$ with source terms $\boldsymbol{s}$ is

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{s}(\boldsymbol{u}) \tag{2}$$

- *Jacobean matrix and characteristic form of Saint-Venant's equations*

Written in conservative form, the system of Saint-Venant's equations is of hyperbolic nature. This means that it can be diagonalised. Using the Jacobean matrix of the system

$$J = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}\boldsymbol{u}} = \begin{pmatrix} 0 & 1 \\ c^2 - v^2 & 2v \end{pmatrix} = VDV^{-1} \tag{3}$$

$$D = \mathrm{diag}\left\{ a^{(1)}; a^{(2)} \right\} \quad : \quad a^{(1)} = v - c \quad ; \quad a^{(2)} = v + c \tag{4}$$

where $a^{(i)}$ (i=1,2) = Eigenvalues of J, V = matrix of Eigenvectors, $v=Q/A$ = mean flow velocity and c = celerity of pressure waves with $c^2 = g\, \mathrm{d}I_1/\mathrm{d}A = gA/B$. Defining

$$\mathrm{d}w = V^{-1}\mathrm{d}u \tag{5}$$

and $\boldsymbol{s}' = V^{-1}\boldsymbol{s}$, Eq. (2) has the characteristic form

$$\partial_t w + D\partial_x w = s' \tag{6}$$

Each equation of system (6) can be re-written using total derivatives in the (x,t) plane as

$$\frac{\mathrm{d}w^{(i)}}{\mathrm{d}t} = s'^{(i)} \;\; \text{along} \;\; \frac{\mathrm{d}x}{\mathrm{d}t} = a^{(i)} \quad i = 1,2 \tag{7}$$

introducing the so-called characteristic curves along which perturbations propagate.

The nature of the flow regime is associated with these waves propagating in the domain, and similarly with the value of the Froude number $\mathsf{F}=v/c$ representing the relative importance of inertial and pressure governing forces. In a subcritical flow ($\mathsf{F}<1$), pressure forces dominate and the wave velocities ($a^{(1)}$ and $a^{(2)}$) are of opposite signs, while a supercritical flow is velocity driven and waves can only travel in the downstream direction.

If there is a transcritical transition, the sign of the celerity of the slowest wave changes (for a positive discharge). A convergence of the associated characteristic lines would imply an over-determination (couples of characteristics issued from different zones giving different values at one point). The solution is then divided into two domains separated by a discontinuity. The Rankine-Hugoniot shock relations (Godlewski and Ravaiart 1991) connect the values on both sides of the discontinuity, defining the shock speed. On the contrary, if shifting from a subcritical to a supercritical flow, the transition is continuous; the direction of propagation of the characteristic associated with the critical point changes continuously from negative to positive, the fan of characteristics is centred on the critical point. There, $v=c$ and the characteristic is steady, namely

$$\frac{\mathrm{d}w^{(1)}}{\mathrm{d}t} = s'^{(1)} \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{8}$$

and for the critical point

$$\mathrm{d}w^{(1)} = \mathrm{d}A - \mathrm{d}Q/2c \tag{9}$$

2.2 *Classical Preissmann's method for Saint-Venant's equations*

- *Discrete system for Saint-Venant's equations*

Consider a temporal evolution between discrete times ($t_n=n\mathrm{d}t$) over a spatial domain represented by N+1 discrete locations (nodes) $x_0<x_1< \ldots <x_N$. Preissmann's method applies over a cell $[x_j;x_{j+1}]\mathrm{x}[t_n;t_{n+1}]$. This finite difference scheme can be interpreted as an approximation of the integral form of the conservation law (2) over the cell

$$\int_{x_j}^{x_{j+1}}[u(x,t_{n+1})-u(x,t_n)]\mathrm{d}x + \int_{n}^{n+1}[f(u(x_{j+1},t))-f(u(x_j,t))]\mathrm{d}t = \int_{x_j}^{x_{j+1}}\int_{n}^{n+1} s(x,t)\ \mathrm{d}t\mathrm{d}x \tag{10}$$

where the spatial and temporal integrals are given by a linear interpolation using the vertexes of the spatio-temporal domain, with a weighting coefficient θ for values at t_{n+1} and ψ=1/2 for values at x_{j+1}. This relation with the integral form says that the method is conservative, which is essential to obtain the correct shock speeds. Rankine-Hugoniot relations are obtained as a limit of the integral conservation law.

The equations given in Eq. (1) discretized with Preissmann's method are formally written as

$$R_{j+1/2} = 0 \tag{11}$$

where $\boldsymbol{R}=(R^A\ \ R^Q)^T$ is called "cell residual". These vector relations correspond to the discretized form of the conservation equations for mass ("A") and momentum ("Q")

$$\begin{aligned} R^A_{j+1/2} &= \frac{1}{2}\left(\Delta A_j + \Delta A_{j+1}\right) + \lambda\left[\theta\left(\Delta Q_{j+1} - \Delta Q_j\right) + Q^n_{j+1} - Q^n_j\right] \\ R^Q_{j+1/2} &= \frac{1}{2}\left(\Delta Q_j + \Delta Q_{j+1}\right) + \lambda\left[\theta\left(\Delta f_{j+1} - \Delta f_j\right) + f^n_{j+1} - f^n_j\right] - \Delta t\left[\theta\left(\Delta s_{j+1} + \Delta s_j\right) + s^n_{j+1} + s^n_j\right] \end{aligned} \tag{12}$$

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where $\Delta u = u^{n+1} - u^n$ for the scheme at the known time step n, $\lambda = \Delta t/\Delta x$, the flux and source terms of the momentum equation are $f = Q^2/A + gI_1$ and $s = gA(S_o - S_f)$.

- *Stability and accuracy*

While the scheme is only based on two spatial positions, it is second-order accurate in both time and space if $\theta = 1/2$, but is then marginally stable. Practically $\theta > 1/2$ is thus used, leading to an accuracy of first order in time and second order in space. A von Neumann stability analysis also indicates that the linearized Preissmann's scheme, for the Saint-Venant's equations in the frictionless horizontal bed case, is marginally stable if $\mathsf{F} = 1$ (Meselhe and Holly 1997). This particular configuration occurs only locally for transcritical flows.

- *Boundary conditions*

The set of cell residual's equations ($\boldsymbol{R}_{j+1/2} = 0$, $j=0,\ldots, N-1$) has to be completed with exactly two Boundary Conditions (BC) to close the system. If the flow is supercritical both are imposed upstream, whereas for subcritical flow one is imposed upstream and one downstream. This corresponds to giving exactly the BC physically required, namely one at each end where a characteristic line is entering the domain. Other choices for the location of the given BC would not respect the physics of the travelling waves and then the numerical method would not be stable (MacDonald 1996).

Note that the scheme uses an indiscriminate space-difference approximation in subcritical and supercritical flow regions, so nothing changes in the scheme whether information is coming from upstream or downstream. But each unknown is implied in two systems of equations *i.e.* in two boxes. Hence it is appropriate both for fully subcritical and fully supercritical flows. Giving the adapted BCs corresponds to forcing the directions of propagation of the information in the numerical model.

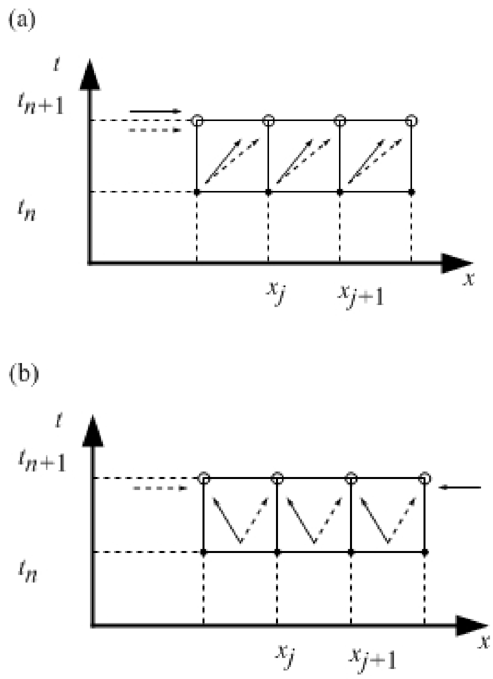
- *Solution algorithm*

Owing to the two-spatial-point formulation of Preissmann's discretisation, the matrix of the implicit system for Saint-Venant's equations has a double-bidiagonal structure allowing for a simple solution algorithmic procedure, depending on the imposed boundary conditions. Figure 1 shows a schematic representation, from where it can be seen that each node, for the new time step, needs two equations or two "arrows" to be solved.

In the supercritical case, a simple sweep is sufficient. The two boundary conditions give the solution at the upstream boundary. One discrete position after the other, Preissmann's scheme gives the solution down to the downstream boundary node.

In the subcritical case, a double-sweep is necessary. The upstream boundary condition permits simplifying Preissmann's scheme for the first cell. One equation is still linking the two nodes (three discrete values) and the other giving a relation for the values of the second node. This last step is used as the subcritical upstream boundary condition for the second cell. An upper-diagonal matrix is obtained at the end of this first sweep. The downstream boundary condition then allows obtaining the solution of the complete system at the last downstream cell. A second sweep calculates all other discrete unknown values, one after the other, from downstream to upstream using, for each cell, the equation linking the nodes. Each method reflects the properties of the corresponding flows: either only upstream information is responsible for the evolution, or both upstream and downstream information have an effect.

The in-cell arrows represent the direction of propagation of information associated with the characteristics as well as the two-cell residual's equations for the Saint-Venant system and the "way of using them" once boundary conditions are given down- and/or upstream.



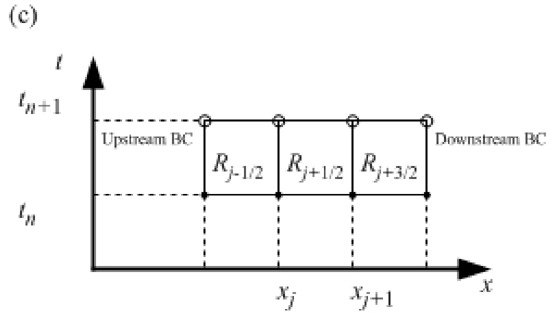


Figure 1 Schematic representation of simple- and double-sweep solution, propagation of information in (a) supercritical state, (b) subcritical state, (c) on how necessary information is given by equations

2.3 *Classical methods for transcritical flows*

Even if many methods and algorithms are available for computing numerical solutions of the shallow water equations, including transcritical flows, few have been applied to real life problems (Alcrudo 2002). No exhaustive list of all the numerical methods solving transcritical flows is presented below, yet the terms of references are described, namely ability to include hydraulic structures, to model networks, to use relatively large time steps, conservativity and robustness.

A practical method to allow for the modelling of transcritical flows uses a model built for subcritical flows but to work with an approximate system for the physical equations (Kutija 1994). The standard (implicit) method is used but gradually reducing the inertia term in the momentum equation as $\mathsf{F} \rightarrow 1$. Then, the characteristic lines of the associated reduced system keep opposite directions. This allows keeping the algorithmic structure of the initial model, imposing boundary conditions inherent to subcritical flow. This advantageous method is used in several commercial packages combined, for example, with Preissmann's or Abbott-Ionescu's scheme. Unfortunately, while maintaining robust solution procedures, these approximations may imply accuracy problems and numerical oscillations. A detailed study of these drawbacks in conjunction with the application to hydraulic networks was conducted by Djordjevic *et al.* (2004).

To solve the complete hyperbolic system for discontinuous solutions, the first idea was to use shock fitting methods, which explicitly calculates the evolution of each discontinuity following its position. This is known to be a difficult problem if real cases are considered (Liu *et al.* 2004). Shock capturing methods have then been advanced. They treat each flux as if it was associated with a discontinuity. Either this involves an (approximate) solution of a Riemann problem as in Godunov-type methods (Guinot 2003, van Leer 1994, León *et al.* 2009), or this calculus is avoided by taking advantage of the finite domain of influence of the discontinuity as in centred methods (Jiang and Tadmor 1998, Toro and Billett 2000). These are appropriate for the precise computation of extreme flows in dam break problems (Paquier 1996, Zoppou and Roberts 2003, Burguete and Garcia-Navarro 2004). One of their drawbacks is that they require small time steps and are highly time-consuming. Toro and Garcia-Navarro (2007) reviewed Godunov-type methods for free-surface flows that are now routinely used by the engineering community to compute dam break problems.

Most classical shock capturing methods are explicit. Some implicit methods have also been proposed. But, according to the Total Variation Diminishing (TVD) criterion (Harten 1983) to avoid numerical oscillations, even if they are implicit, they are constrained by a limited time step (except, formally, for a fully implicit method). Furthermore, the implicit systems need to be linearized eventually leading to accuracy problems if large time steps are used (Burguete and Garcia-Navarro 2004). To our knowledge, these methods have not yet been used in practice.

The attractive features of these methods are mostly related to the ability to minimise diffusion in presence of strong gradients. Amongst shortcomings are the computation burden and the difficulty to model inline structures and complex looped network (Mignot *et al.* 2008). None of these methods is really adapted to our requirements.

For an irrigation canal, the simulation can typically last from one week to several months. The main objective is not to model accurately sharp bores, but rather to model correctly the low and medium frequencies dynamics, and the water distribution along the canals using the lowest computational cost. The scheme must also preserve water conservation. The Preissmann scheme meets the present requirements in non-transcritical cases, leading to an adaptation for all flow conditions.

2.4 *Meselhe and Holly's restrictions on Preissmann's scheme*

Meselhe and Holly (1997) have demonstrated that Preissmann's method cannot be used directly to solve the complete Saint-Venant's equations for transcritical flows. The three main points of their instability arguments are discussed hereafter.

- *Problems related to usage for transcritical flows*

The first problem if applying Preissmann's scheme for transcritical flow directly comes from its formulation that imposes exactly two BC for the system. If the flow presents one transition from super- to subcritical (shock/bore) the physics requires two BC upstream and one BC downstream, which numerically leads to one excess equation. If the transition is from subcritical to supercritical (critical point) only one upstream BC is required, which leads to one missing equation. Thus, if one transcritical transition does exist, then imposing exactly the external physical BCs results in an ill-posed numerical system.

Another problem is related to the fact that Preissmann's scheme applied to Saint-Venant's equations is marginally stable if $\mathsf{F} = 1$. Hence, this method cannot solve the case of an even number of transcritical transitions. Even for an apparently well-posed discrete system, the method is likely to diverge.

- *Flexibility according to boundary conditions*

Saying that Preissmann's scheme is not sufficiently flexible to admit any kind of BC, Meselhe and Holly (1997) compared it with Euler's implicit scheme. They argued that the discrete system is closed for the interior nodes and is flexible according to BC as it will accept imposing any number from one to four BC. But this supposes that 'the remaining four unknowns at the boundary nodes can be evaluated by either imposition or extrapolation of BC'. Thus the advantage of flexibility is converted into the problem of defining these extrapolations.

The treatment of boundary conditions is a key point in the design of numerical methods – either external BC or internal BC around hydraulic structures. For a conservative scheme, agreement with physics depends on what enters and leaves the domain, depending on the BCs. Moreover, the error growth factor in the domain also depends on the way BCs are treated. The best ones to be imposed do correspond to the physical conditions.

- *Marginal stability*

Concerning the problem of marginal stability, it must be noted that almost all schemes dealing with transcritical flow add an amount of diffusion in transcritical zones (adding an explicit diffusion term or through limitations of the variations). This necessity can be linked to the invalidity of the initial Saint-Venant approximation in these domains, as they are deduced under the assumptions of hydrostatic pressure and uniform distribution of the velocity along the vertical axis. Accounting for turbulence or non-hydrostatic pressure distribution in models is computationally expensive. Adding numerical diffusion permits to damp the infinite gradients that appear within the classical hypothesis framework. Neither procedure is exact for real cases or susceptible to correct numerical treatment.

3 Johnson's adaptation of Preissmann's scheme

3.1 *Bases of Johnson's scheme*

Johnson *et al*. (2002) proposed a method based on Preissmann's scheme, adapted to the treatment of transcritical flows and using physical boundary conditions. It combines, for each section, the two formulations obtained with Preissmann's scheme for the right and left neighbouring cells. The combination is obtained through a local decomposition (residual distribution concept) along the wave propagation directions (splitting with characteristic decomposition), in the idea of "flux vector splitting" methods (Roe 1981).

3.2 *Residual distribution concept*

The method of residual distribution was proposed by Morton *et al.* (1994) to introduce some up-winding using cell-vertex methods. The problem is approached considering the lack of agreement between the number of discrete unknowns and equations. The idea is to combine, for each section, the cell residuals of the right and left neighbouring cells (Fig. 2 a), thus obtaining one "nodal residual" associated with each section

$$R_j = \rho^+_{j-1/2} R_{j-1/2} + \rho^-_{j+1/2} R_{j+1/2} + B_j \tag{13}$$

where $\rho^{\pm}_{.}$ = transfer matrices and the vector $\boldsymbol{B}$ accounts for external and internal boundary conditions. For $\rho^+_{-1/2}$=0 and $\rho^-_{N+1/2}$=0, the complete system is given by

$$R_j = 0 \quad : \quad j = 0 \ldots N \tag{14}$$

To preserve the conservative property, transfer matrices must satisfy

$$\rho^+_{j+1/2} + \rho^-_{j+1/2} = \Im \tag{15}$$

where $\Im$ = identity matrix.

3.3 *Splitting with characteristic decomposition*

To obtain the distribution according to wave propagating in each cell, a local Jacobian matrix $J_{j+1/2}$ is used as

$$J_0 = \begin{pmatrix} 0 & 1 \\ \widetilde{c}_0^2 - \widetilde{v}_0^2 & 2\widetilde{v}_0 \end{pmatrix} \tag{16}$$

where the particular mean-values $\widetilde{v}$ and $\widetilde{c}$ are given by Roe's (1981) approximation

$$\tilde{c}_{j+1/2} = \sqrt{(c_j^2 + c_{j+1}^2)/2} \qquad \tilde{v}_{j+1/2} = \frac{v_j c_j + v_{j+1} c_{j+1}}{c_j + c_{j+1}} \tag{17}$$

This choice ensures that bores propagate with the correct velocity. Hence, the transfer matrices used by Johnson to define the nodal residuals (13) correspond to the application of up-winding on the characteristic variables defined in the Eigenspace of the system of Eqs. (3) to (5) as

$$\rho^{\pm}_{j+1/2} \;=\; V_{j+1/2}\, \Im^{\pm}_{j+1/2}\, V^{-1}_{j+1/2} \tag{18}$$

where $V_{j+1/2}$ = matrix of Eigenvectors for $J_{j+1/2}$ and the matrices $\Im^{\pm}$ $\Im^{\pm}_{j+1/2}$ determine the up-winding according to the sign of the Eigenvalues

$$\Im_0^{\pm} = \frac{1}{2}\begin{pmatrix} 1 \pm \text{sign}(\tilde{v}_0 - \tilde{c}_0) & 0 \\ 0 & 1 \pm \text{sign}(\tilde{v}_0 + \tilde{c}_0) \end{pmatrix} \tag{19}$$

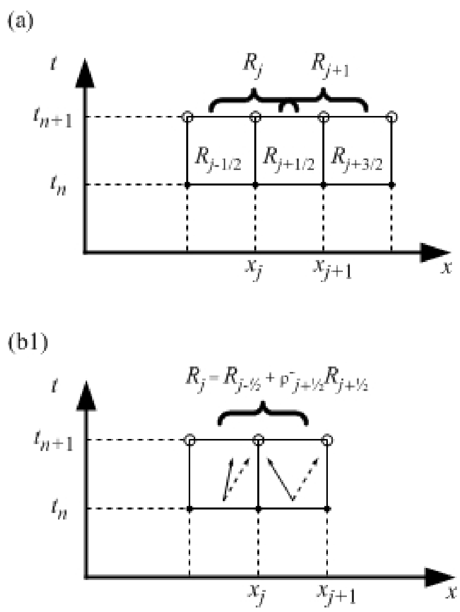
Notably, for a fully supercritical flow (positive speed)

$$\Im_0^{+} = \rho^{+} = \Im \quad \text{and} \quad \Im_0^{-} = \rho^{-} = 0 \tag{20}$$

Hence, if cells do not have the same regime (Fig. 2 b), the nodal residual is

$$j \text{ subcritical}, j{+}1 \text{ supercritical:} \quad R_j = \rho^{+}_{j-1/2}\, R_{j\text{-}1/2} + 0 \cdot R_{j+1/2} + B_j$$

$$j \text{ supercritical}, j{+}1 \text{ subcritical:} \quad R_j = 1 \cdot\, R_{j\text{-}1/2} + \rho^{-}_{j+1/2} R_{j+1/2} + B_j$$



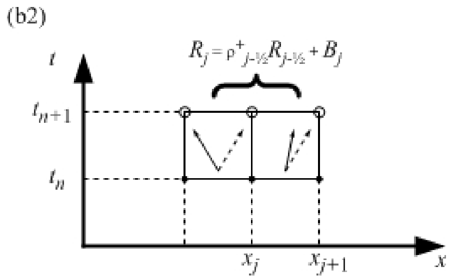


Figure 2 Nodal residual of (a) general from, Johnson's nodal residual for (b1) supercritical-subcritical transition, (b2) subcritical-supercritical transition

3.4 *Resulting scheme*

The use of nodal residuals solves the over-determination at shocks. Nonetheless, this does not ensure that the new problem is well-posed since the equations still have to be independent. In particular, if cell $j-1/2$ is subcritical and $j+1/2$ supercritical, the transfer matrices defining R_j satisfy

$$\text{rank}\ (\rho^+_{j-1/2} + \rho^-_{j+1/2}) < 2 \tag{21}$$

which means that the resulting system is ill-posed. To provide an extra linearly independent equation, Johnson then adds an internal boundary condition

$$\boldsymbol{B}_j = (\Im - \rho^+_{j-1/2} - \rho^-_{j+1/2})\Delta \boldsymbol{u}_j \tag{22}$$

characterizing a critical point located exactly at x_j.

The non-linear implicit system is solved by an iterative method. The approximate Jacobian matrix must be given at the new time step, which means calculating it at each iteration. This is necessary to allow the critical interface to cross more than one computational cell within one time step.

3.5 *Limitation*

The general drawback of Johnson’s method is that it requires, for each cell, calculating mean-value propagation velocities using Eq. (17) and testing the dimension of transfer matrices as given in Eq. (21). This is necessary to obtain an automatic procedure, adapted to any regime. But then, the method leading to Eq. (13) uses three points, even in the subcritical regions where the initial Preissmann’s scheme would be appropriate. The following focuses the necessity of this three points approach.

4 Preissmann's scheme with internal boundary conditions

4.1 *Direction of solution and internal boundary conditions*

To achieve stability, a solution along directions of propagation must be respected. With Preissmann's scheme, this means that the direction of solution has to change if the regime changes. Hence an equivalent/internal boundary condition has to be prescribed at transcritical transitions to permit a change in the solution direction. These relations should describe the particular physical properties associated with transcritical transitions.

Actually, the ill-posedness of the classical implementation of Preissmann's method for both possible transitions can be related to the physical properties of the transitions. In both cases the problem is completely similar to solving the Saint-Venant system with the method of characteristics:

- For a bore, while convergence of characteristics leads to an over-determination; Rankine-Hugoniot jump conditions prove that influences from each side must be taken into account to determine the flow evolution (to obtain right and left states and shock speed). The "over-determination" obtained with Preissmann's classical system corresponds to this fact saying this complete piece of information actually does have to be used to describe the flow at this point.
- At a critical point, the steady characteristic has a proper definition. The "missing equation" observed with Preissmann's classical system corresponds to this fact asking for supplementary information in the description of this specific point.

The idea is thus to translate the physical properties of transcritical transitions to link the different regime sub-domains, within which Preissmann's scheme is adapted. Johnson's method shows that it is possible to treat the transcritical transitions with a through method. The mathematical description of Johnson's combination shows the two systems are exactly equivalent for entirely subcritical or supercritical domains.

4.2 *Treatment of transcritical flows*

Preissmann's scheme is used in each fully subcritical or fully supercritical domain. This means determining the regime at each point (what is generally implicitly done in classical methods for transcritical flows). The flow regime is tested via the local Froude number $\mathsf{F} = |v|/c$ at each point. The characterisation of "fully" subcritical or supercritical means, for the solution at n+1 in cell $\{j,j+1\}$, that values at both points j and j+1 must have the same

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regime, at time steps *n*+1 (with Newton solution, the last approximation is used, hence for the first solution, the regime at time step *n* is used).

Formally, the system of Preissmann's equations over a non-transcritical sub-domain can be reduced to a set of equations at the boundaries (in the idea of simple- and double-sweep methods). The attention is thus restricted to cells where the regime changes. The numerical problem is approached as reduced to giving boundary conditions for each subcritical or supercritical region, consistent with Preissmann's scheme (Fig. 3).

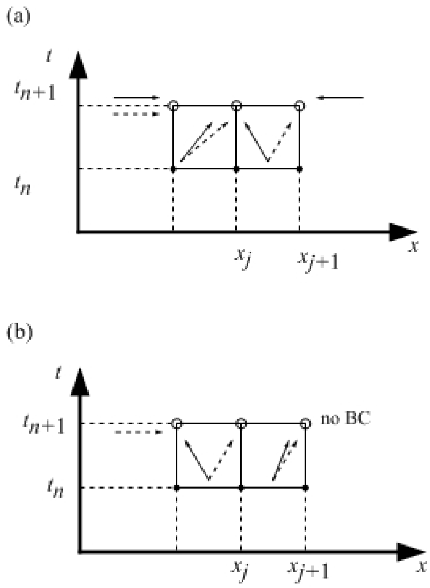

Figure 3 Reduction of solution to neighbour of transcritical transitions for (a) supercritical-subcritical transition, (b) subcritical-supercritical transition

- *Critical point*

The rarefaction fan issued from the critical point shows that, from this point, information is transferred right and left. Hence, the equation defining the critical point is able to furnish the internal boundary condition giving, at the same time, the BC "closing" the upstream subcritical flow and the second BC necessary to define the evolution of the supercritical flow downstream.

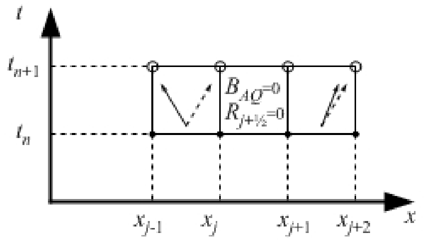

Figure 4 Critical point in transcritical cell

The existence of the critical point does not compromise the conservation property over the cell. Preissmann's system is then retained, but the equation defining the critical point (8) is added to characterize the particular evolution due to the critical point.

If $\mathsf{F}_j < 1 \leq \mathsf{F}_{j+1}$ a critical point exists in this cell, at point $x_c=x_j + \alpha$ dx ($0<\alpha\leq 1$ to be defined), such that $v_c=c_c$. The system can be simply written then

$$\begin{cases} R_{j+1/2} = 0 \\ B_{AQ} = 0 \end{cases} \tag{23}$$

where $B_{AQ} = 0$ is the discrete equation for the steady characteristic written in terms of the conserved variables A and Q, that must be defined at point x_c.

As the solution is continuous across the critical point, the position can be interpolated linearly between the two neighbouring mesh points. This is consistent with Preissmann's discretisation over the cell (with the space weighting ψ=1/2). The point is characterised by $a^{(1)}$=0, resulting in

$$\alpha = \frac{-a_j^{(1)}}{a_{j+1}^{(1)} - a_j^{(1)}} \tag{24}$$

Using Eq. (9) gives

$$B_{AQ} = \alpha\left(\Delta A_{j+1} - \Delta Q_{j+1}/2c_{j+1}\right) + (1-\alpha)\left(\Delta A_j - \Delta Q_j/2c_j\right) \tag{25}$$

where the values characterising the wave are given at the approximate new state.

If this modification is considered in splitting methods, the supplementary equation corresponds to the correction required as the simple Roe-type splitting is applied. This does not always satisfy the entropy condition without introducing appropriate modifications. This is generally achieved either by using another approximate solver (Delis 2003, Ying and Wang 2008) or by adding artificial viscosity (Burguete and Garcia-Navarro 2004).

As compared to classical corrections including Johnson's method, where a supplementary term is added to an equation of the initial system, the above formulation expresses the "physical need" for diffusion terms in classical schemes based on Saint-

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Venant's equations. It is similar to the supplementary "compatibility condition" proposed by Djordjevic *et al.* (2004).

- *Shock*

If a supercritical-subcritical transition is encountered, the direction of shock propagation must be tested to define which sections are influenced by the shock. Roe's definition is appropriate there, since the Eigenvalue of his approximate Jacobian is the shock speed if the values on both sides are linked by a jump.

If $\mathsf{F}_j \geq 1 > \mathsf{F}_{j+1}$, determine $\tilde{a}^{(1)}_{j+1/2}$

- if $\tilde{a}^{(1)}_{j+1/2} \geq 0$, the shock is travelling downstream (Fig. 5a), the solution at point j can be given by Preissmann's scheme in cell j–1/2, while the solution at point j+1 accounts for influences from both upstream and downstream,
- if $\tilde{a}^{(1)}_{j+1/2} < 0$, the shock is travelling upstream (Fig. 5b), the solution at point j must account for influences from both upstream and downstream.

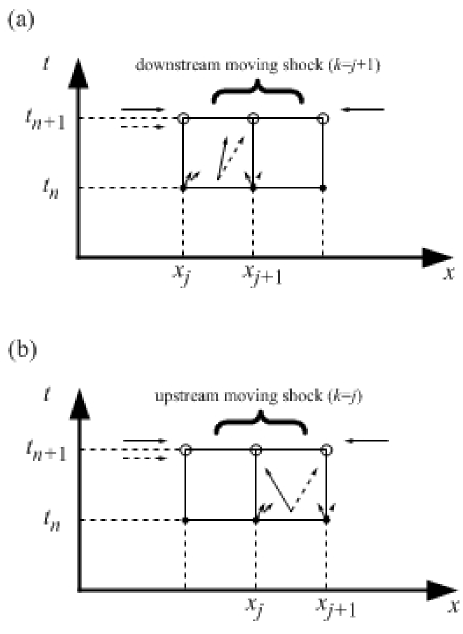


Figure 5 Methods for shock travelling (a) downstream, (b) upstream. The sign of Roe's first Eigenvalue gives direction of shock propagation

To account for upstream and downstream influences at point k (=j, or j+1) if the solution at point k–1 is known, the solution there must remain indeterminate as long as the subcritical region is not solved. At the same time, information must be transmitted downstream to the subcritical region. It is also essential that conservation be respected. To combine these requirements, and following the idea of Johnson's method, Preissmann's solutions for the two cells k–1/2 and k+1/2 are added and the equation for the downstream travelling characteristic issued from cell k+1/2 gives an additional equation in the same unknowns (linked through the definition of d$w^{(2)}$) as

$$\begin{aligned} &R_{k-1/2} + R_{k+1/2} = 0 \\ &\left[\partial_t w^{(2)} + a^{(2)} \partial_x w^{(2)} - s'^{(2)}\right]_{k+1/2} = 0 \end{aligned} \tag{26}$$

The conservation property inherent to Preissmann's method is globally preserved and not compromised by the addition of the second equation giving additional information to be transmitted to the subcritical domain. Furthermore, this summation cancels the flux at the intermediary boundary, which is that of the near-critical node, potentially responsible for instability.

According to the definition of d$w^{(2)}$, the supplementary equation can be written with the physical increments dA and dQ. Using the matrix of Eigenvectors, a combination of the equations of conservation of mass and momentum is found as

$$\left[a^{(1)} R^A - R^Q\right]_{k+1/2} = 0 \tag{27}$$

which gives the third equation for the two cells implicated in the shock.

4.3 *Resulting scheme*

Preissmann's scheme is directly used in each fully subcritical or fully supercritical domain and specific equations are added at transcritical transitions. The external boundary conditions are then physical.

The procedure begins with computing the Froude number for each node. The system of equations is then developed by determining the equations one cell after the other, according to the regimes of the nodes, tested at n+1 as:

1) Impose the boundary conditions corresponding to the inflow regime.

2) For $j = 0\ldots N-1$

- If $(\mathsf{F}_j-1)(\mathsf{F}_{j+1}-1)>0$, use Preissmann scheme for cell j+1/2
- If $\mathsf{F}_j \geq 1 > \mathsf{F}_{j+1}$, compute $\tilde{a}^{(1)}_{j+1/2}$
  - if $\tilde{a}^{(1)}_{j+1/2} \geq 0$, then use Eq. (28) with k=j+1

    - verify $\mathsf{F}_{j+2} < 1$, otherwise add Eq. (29) for l=j+1 (in cell j+3/2)

    - j:=j+1
  - if $\tilde{a}^{(1)}_{j+1/2} < 0$, previous cell ($j$–1/2) is not solved with Preissmann's scheme

- use Eq. (28) with $k=j$

$$\begin{aligned} R_{k-1/2} + R_{k+1/2} &= 0 \\ \left\lfloor a^{(1)} R^A - R^Q \right\rfloor_{k+1/2} &= 0 \end{aligned} \tag{28}$$

- If $\mathsf{F}_j < 1 \leq \mathsf{F}_{j+1}$
- compute α from Eq. (24), add Eq. (29) for $l=j$

$$B_{AQ} = \alpha\left(\Delta A_{l+1} - \Delta Q_{l+1}/2c_{l+1}\right) + (1-\alpha)\left(\Delta A_l - \Delta Q_l/2c_l\right) \tag{29}$$

- use Preissmann system for cell j+1/2

3) Impose the boundary conditions corresponding to the outflow regime.

Test on external boundary conditions

In the described procedure, it is supposed that the numerical regime corresponds to the one imposed by the user-prescribed boundary conditions. Tests must be added at the boundaries to verify this consistency. The user has to specify if the proposed boundary conditions are to prevail or not.

4.4 *Solution of the system*

This non-linear system is solved by Newton's iterative method; transcritical positions and their characteristics are computed for each iteration. The solution of the linearized system, for (half-) implicit schemes, requires inverting a matrix. One of the advantages of Preissmann's scheme is that simple algorithmic solutions can be used. Using the present transcritical system, these methods can be adopted by just localising the transcritical transitions.

Bore: Combination of sections $j-1, j, j+1$

The sweep in the supercritical regime gives the solution from the inflow to cell $j-1$.

The system (28) can then be simplified giving the equivalent of an inflow BC at j and a system of two equations for the cell j+1/2. The subcritical region is then classically solved with a double sweep method.

Critical point: Addition of one equation for sections $j, j+1$

The first sweep in the subcritical domain, from the inflow boundary condition to cell j–1/2, leads to one equation in (Q_j, A_j). This statement simplifies the equations of cell j+1/2, giving three equations with three unknowns. It correspond to obtaining the solution at j+1, being able to perform the second subcritical sweep, and giving the required upstream boundary conditions for the supercritical zone which can be solved by a classical simple sweep-method.

5 Numerical results

The method is applied on a transient problem involving appearance and disappearance of a supercritical region between a critical point and a bore. Starting from a steady subcritical state, in a canal of mild slope (0.01%) followed by a steeper (1%) slope, the outflow boundary condition is varied as dh/dt=5 mm/s. First, the depth is reduced toward a steady transcritical state, and then raised back to the original depth. The inflow discharge is constant at 50 m^3/s. The channel is rectangular of width 5 m and has a Strickler roughness coefficient of K=50 $m^{1/3}$/s. The solution involves space and time steps of dt=10 s and dx=10 m, and Preissmann scheme is used with θ=0.55.

For this problem, along with a classical implementation using Preissmann's scheme, divergence appears as soon as the transcritical zone appears (Djordjevic *et al.* 2004). The result of the proposed method is generally the same as with Johnson's method, except for the upstream moving bore, where a divergence can appear with Johnson's method.

Results are presented for different specific states of the solution. The transcritical zone appears at t=480 s in Fig. 6a. The entire subcritical domain is then separated at the location of the change of bed slope by a critical point immediately followed by a shock. As the downstream water level is continuously reduced, a shock moves downstream (Fig. 6b). Then, the boundary conditions are kept constant and a steady transcritical state is established (Fig. 6c). As the downstream flow depth is raised back to its initial value, the shock moves upstream. Figure 6d shows the stronger variation appearing at the shock location. As the shock attains the critical point, the supercritical zone disappears and the initial state is found back.

Some details are given in Fig. 7 to show both transcritical transitions at the intermediate steady state and the evolution of the solution when the transcritical domain appears and disappears.

At the intermediate steady state the flow characteristics may be described with an inflow velocity v=2.5 ms^{-1}, $c\approx$6.26 ms^{-1}; in the steep slope region, the normal depth is $h\approx$1.90 m, corresponding to $v\approx$5.25 ms^{-1}, $c\approx$4.32 ms^{-1}. The Courant number is thus around 9.

The solution is also shown for a high Courant number in Fig. 6c. The same problem is treated but taking dt=5dx=50.

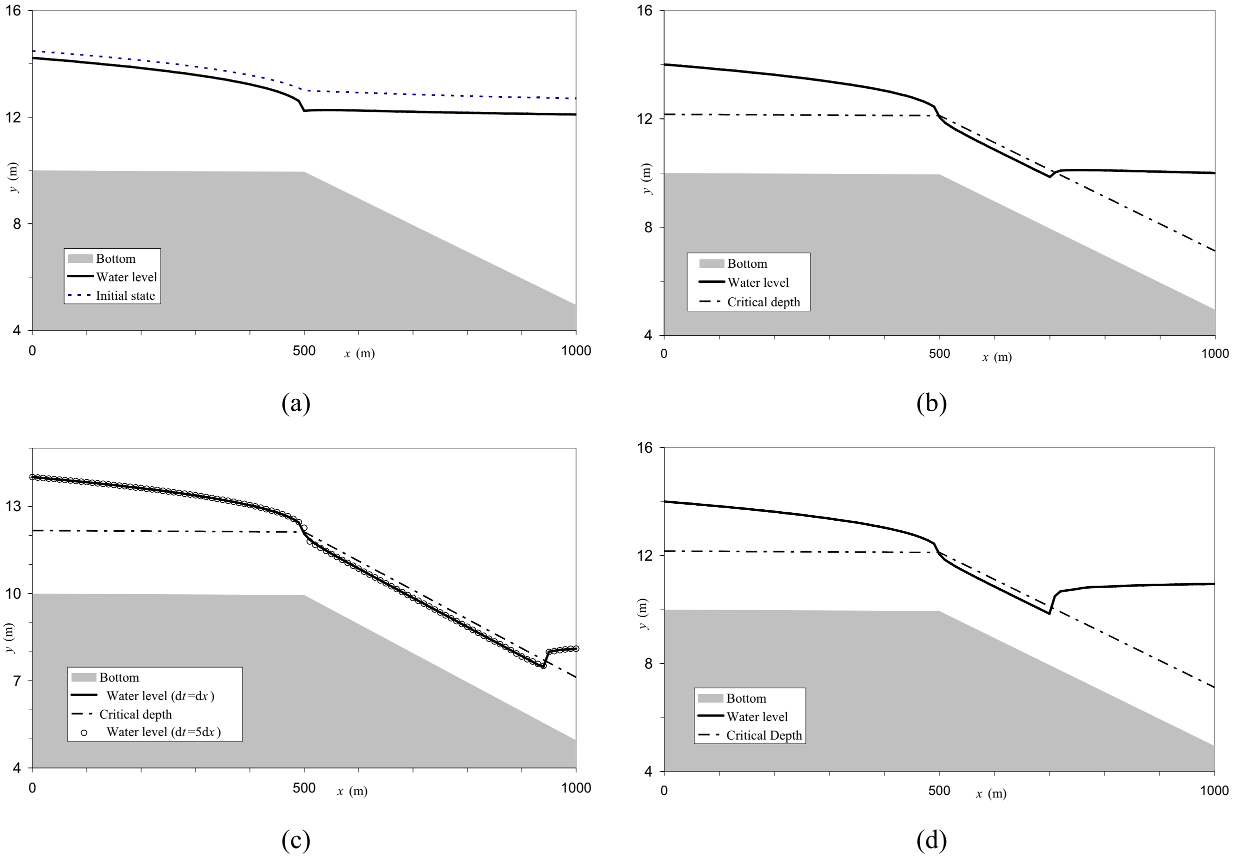
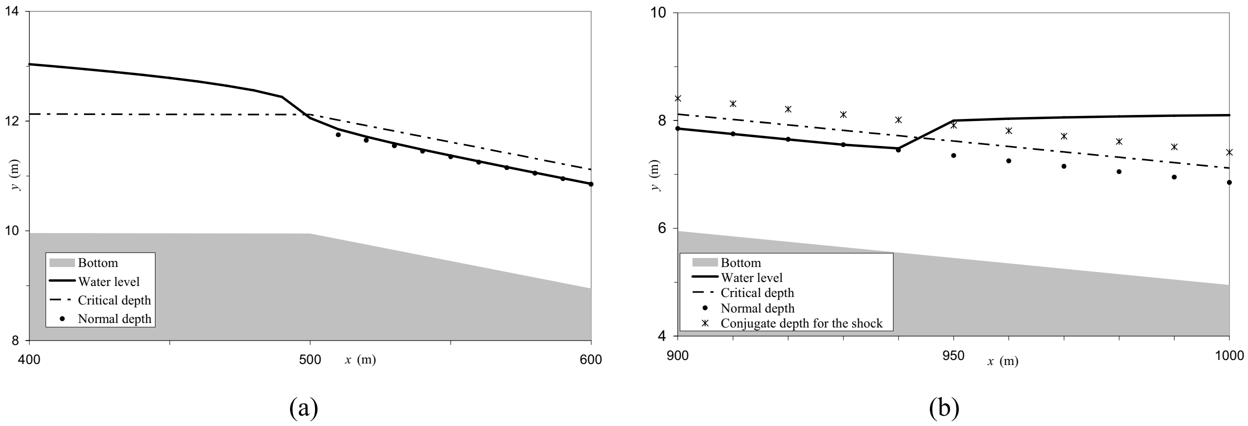
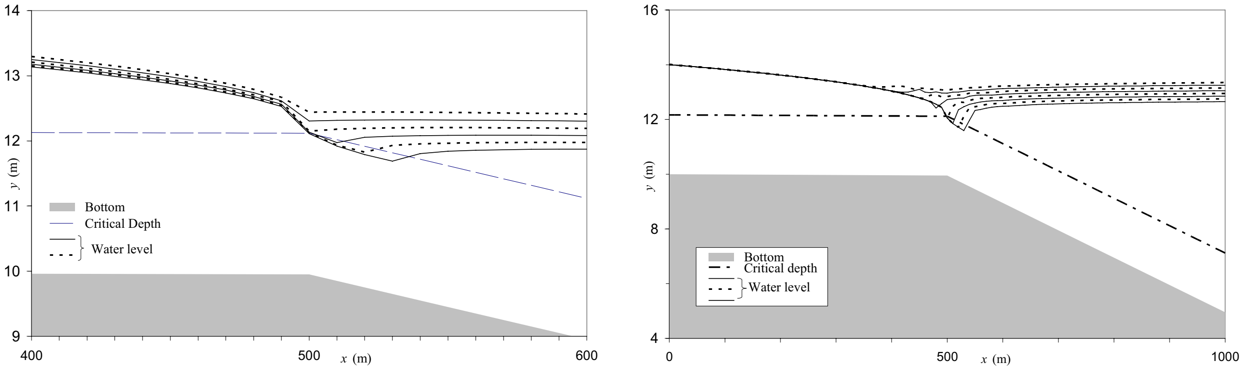


Figure 6 Solution evolution with (a) Appearance of the transcritical zone, (b) Downstream moving shock, (c) Intermediate steady state with dt=dx and dt=5dx, (d) Upstream moving shock





(c) (d)

Figure 7 Details of solution (a) around critical point, (b) around stationary shock, (c) evolution of appearance of transcritical zone (solution shown for each two time steps), (d) evolution of disappearance of transcritical zone (solution shown for each two time steps)

A small local disturbance in discharge is observed at the location of the steady bore (Fig. 8). This remains a problem essentially in the steady case. In the transient case, the disturbance is embedded in the strong modifications of the flow around this point.

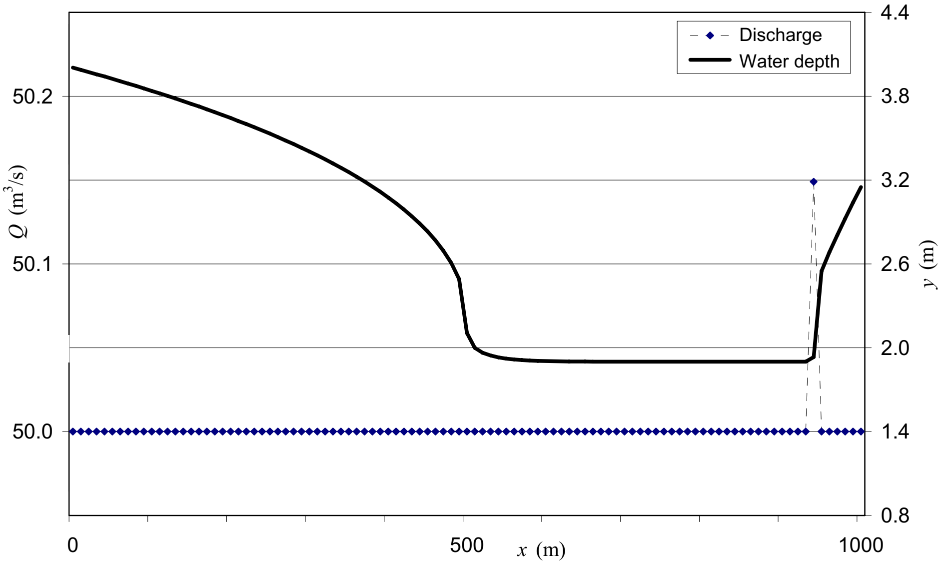


Figure 8 Profile of $Q(x)$ at intermediate steady state

6 Conclusions

Using Preissmann's scheme, the inherent directional property of signal propagation of Saint-Venant's equations can automatically be reproduced once the appropriate boundary conditions are given. To treat transcritical flows, it is physically meaningful to introduce formulations to capture the flow behaviour at transcritical transitions, while it corresponds to a numerical necessity.

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The method obtained for the solution of transcritical flows directly uses Preissmann's scheme. The classical use of this two-point scheme is preserved in the major flow part. Even though there is an explicit determination of the cells experiencing the transcritical changes, it is a shock capturing method. It can be directly related to the three-point method of Johnson whose formulation is more systematic. The regime test is not explicitly done in the definition of the algorithm but implicitly calculated in the scheme. Johnson's method is elegant but finally requires more calculation.

The problem of transcritical transitions is solved locally, either by associating the cells involved in a bore and adding an equation to characterize the information transferred in the subcritical domain, or by the addition of an internal boundary condition to characterize the expansion fan at the critical point.

The internal boundary condition is on the one hand an alternative to the introduction of entropy corrections to solve the classical problem of rarefaction-shocks linked to the use of so-called Roe-type splitting. On the other hand, the solution of shocks is primarily based on Preissmann's scheme written for non-transcritical sections, and considering the solution in this domain, the supplementary discretisation of the characteristic equation adds some diffusion to the Preissmann's system.

The required test on the discrete Froude number is not expensive as compared to other particular treatments that can be proposed for transcritical flows. This is a simple way to conserve the two-point discretisation with its robustness and adapted form for the inclusion of hydraulic structures. Besides, the double-sweep algorithm can be used. Moreover, this "regime-detector" is also a useful tool to organize external boundary conditions. It is intended to make the scheme able to support any flow evolution or any change imposed by hydraulic structures.

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Notations

$a^{(i)}$ = Eigenvalues of Jacobian matrix (i = 1 or 2)

A = Wetted cross-sectional area

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B	=	Channel width
$\boldsymbol{B}_j$	=	Boundary condition at node j
B_{AQ}	=	Scalar internal boundary condition
c	=	$(gA/B)^{1/2}$ = Celerity of infinitesimal wave
$\tilde{c}$	=	Roe's approximation for infinitesimal wave celerity
d$\boldsymbol{w}$	=	V^{-1} d$\boldsymbol{u}$ = variation of Eigenvariable
D	=	diag$\{a^{(1)};a^{(2)}\}$ = Diagonal matrix formed with Eigenvalues $a^{(1)}$ and $a^{(2)}$
f	=	Flux of momentum equation
$\boldsymbol{f}$	=	General vector flux function
F	=	Froude number
g	=	Gravitational acceleration
I_1	=	Hydrostatic-pressure term
J	=	Jacobian matrix of Saint-Venant system
K	=	Strickler coefficient
Q	=	Discharge
$\boldsymbol{R}_{j+1/2}$	=	$(R^A_{j+1/2} \quad R^Q_{j+1/2})^T$ = Preissmann's cell residual
$\boldsymbol{R}_j$	=	Nodal residual
s	=	Source term of momentum equation
$\boldsymbol{s}$	=	General vector source term
$\boldsymbol{s}'$	=	V^{-1} $\boldsymbol{s}$ = Vector source term in Eigen-system
S_o	=	Bottom slope
S_f	=	Friction slope
t	=	Time
$\boldsymbol{u}$	=	$(A \quad Q)^T$ = General vector conserved variable
v	=	Q/A = Mean celerity of flow
$\tilde{v}$	=	Roe's approximation for mean celerity of flow
V	=	Matrix of eigenvectors of J
$\boldsymbol{w}$	=	Vector-conserved variables in Eigensystem
x	=	Space coordinate

X_p = Any variable X taken at position p, either node j or cell j+1/2

Δt = Time step

Δx = Space step

Δy = $y^{n+1} - y^n$ for known time step n (for any variable y)

$\Im^{\pm}_{j+1/2}$ = Matrices defining up-winding according to Eigenvalue sign

Greek Symbols

α = Relative location of critical point in a cell

ψ = Spatial weighting in cell (=1/2 for Preissmann's scheme)

λ = $\Delta t/\Delta x$

$\rho^{\pm}_{j+1/2}$ = Transfer matrices

θ = Temporal weighting in semi-implicit discretisation

Superscripts

(i) = Sign of i^{th} scalar value of a vector (i = 1 or 2)

T = Sign of vector transposition

Subscripts

c = critical point

j = Mesh point location

j+1/2 = Cell location

n = Time step number

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