Conjugate Priors for Exponential Families Having Cubic Variance Functions

Maroua Hamza, Abdelhamid Hassairi

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Marwa Hamza and Abdelhamid Hassairi *

Laboratory of Probability and Statistics. Sfax Faculty of Sciences, B.P. 1171, Tunisia.

Abstract In this paper, we give three equivalent properties of the class of multivariate simple cubic natural exponential families (NEF's). The first property says that the cumulant function of any basis of the family is a solution of some Monge-Ampère equation, the second is that the variance function satisfies a differential equation, and the third is characterized by the equality between two families of prior distributions related to the NEF. These properties represent the extensions to this class of the properties stated in [1] and satisfied by the Wishart and the simple quadratic NEF's. We also show that in the real case, each of these properties provides a new characterization of the Letac-Mora class of real cubic NEF's.

Keywords: Natural exponential family, variance function, cumulant function, Monge-Ampère equation, prior distribution.

1 Introduction and preliminaries

For the convenance of the reader, we first introduce some notations and recall some facts concerning the natural exponential families and their variance functions, our notations are the ones used in [10]. Let E be a linear vector space with finite dimension n, denote by E^* its dual, and let $E^* \times E \to \mathbb{R} : (\theta, x) \mapsto \langle \theta, x \rangle$ be the duality bracket.
If μ is a positive radon measure on E, then

$$L_\mu(\theta) = \int_E \exp(\langle \theta, x \rangle)\mu(dx) \tag{1.1}$$

denotes its Laplace transform. We also denote by $\mathcal{M}(E)$ the set of measures μ such that the set

$$\Theta(\mu) = \text{interior}\{\theta \in E^*;\ L_\mu(\theta) < +\infty\} \tag{1.2}$$

*Corresponding author. *E-mail address: Abdelhamid.Hassairi@fss.rnu.tn*

is non empty and μ is not concentrated on an affine hyperplane of E. The cumulant function of an element μ of $\mathcal{M}(E)$ is the function defined for θ in $\Theta(\mu)$ by

$$k_\mu(\theta) = \log L_\mu(\theta).$$

To each μ in $\mathcal{M}(E)$ and θ in $\Theta(\mu)$, we associate the probability distribution on E

$$P(\theta, \mu)(dx) = exp\left(\langle\theta, x\rangle - k_\mu(\theta)\right)\mu(dx).$$

The set

$$F = F(\mu) = \{P(\theta, \mu);\ \theta \in \Theta(\mu)\}$$

is called the natural exponential family (NEF) generated by μ. We also say that μ is a basis of F.

The function k_μ is strictly convex and real analytic. Its first derivative k'_μ defines a diffeomorphism between $\Theta(\mu)$ and its image M_F. Since $k'_\mu(\theta) = \displaystyle\int_E xP(\theta, \mu)(dx)$, M_F is called the domain of the means of F. The inverse function of k'_μ is denoted by ψ_μ and setting

$$P(m, F) = P(\psi_\mu(m), \mu)$$

the probability of F with mean m, we have

$$F = \{P(m, F);\ m \in M_F\},$$

which is the parametrization of F by the mean.

Now the covariance operator of $P(m, F)$ is denoted by $V_F(m)$ and the map

$$M_F \longrightarrow L_s(E^*, E);\ m \longmapsto V_F(m) = k''_\mu(\psi_\mu(m))$$

is called the variance function of the NEF F. It is easy proved that for all $m \in M_F$,

$$V_F(m) = (\psi'_\mu(m))^{-1},$$

and an important feature of V_F is that it characterizes F in the following sense: If F and F' are two NEFs such that $V_F(m)$ and $V_{F'}(m)$ coincide on a nonempty open set of $M_F \cap M_{F'}$, then $F = F'$.

Now, let us examine the influence of an affine transformation and a power convolution on a NEF $F = F(\mu)$. If $\varphi(x) = a(x) + b$, where $a \in GL(E)$ and $b \in E$, is an affine transformation of E, then $\varphi(F(\mu)) = F(\varphi(\mu))$, $M_{\varphi(F)} = \varphi(M_F)$, and

$$V_{\varphi(F)}(m) = a\ V_F(\varphi^{-1}(m))\ a^*,$$

where a^* is the transpose of a. On the other hand the set

$$\Lambda(\mu) = \{\lambda > 0;\ \exists\ \mu_\lambda \in \mathcal{M}(E) \text{ such that } L_{\mu_\lambda}(\theta) = (L_\mu(\theta))^\lambda \text{ for all } \theta \in \Theta(\mu)\}$$

is called the Jorgensen set of μ and the measure μ_λ is the $\lambda-$power of convolution of μ. For λ in $\Lambda(\mu)$, we have that

$$M_{F(\mu_\lambda)} = \lambda M_F, \text{ and } V_{F_\lambda}(m) = \lambda\ V_F(\frac{m}{\lambda}).$$

A very interesting fact is that the most common real and multivariate probability distributions belong to the natural exponential families such that the variance function is a polynomial of degree less then or equal to three in the mean m. For instance, up to affine transformations and power of convolution (up to the type), the Gaussian, Poisson, gamma, binomial, negative binomial and hyperbolic cosine distributions form the class of all real NEF's whose variance function is a polynomial of degree less than or equal to 2 characterized by Morris [14]. Letac and Mora [11] have added six others types of distributions, namely, the inverse Gaussian, Ressel, Abel, Tackàs, strict arcsine and large arcsine, to get the class of real cubic NEF's, that is the class of NEF's such that variance function is a polynomial of degree less than or equal to three. The classification of NEF's with polynomial variance function have been extended to the multivariate NEF's. The multivariate version of the Morris class, called the class of simple quadratic NEF's, has been completely described by Casalis [1], it contains $2n+4$ types. Hassairi [6] has defined and characterized the so-called class of multivariate simple cubic NEF's which is the natural extension of the class of real cubic NEF's. It is worth mentioning here that the simple quadratic NEF's are not the only families which have quadratic variance functions, the Wishart families on symmetric matrices have also quadratic variance functions. The classifications of NEF's by the form of the variance function provide an important tool in the study of distributions. In fact, in many important cases, the variance function is very simple and is easier to use than the distribution itself or the Laplace transform. Moreover, the fact that the variance function is quadratic or cubic, is not only a question of form, but the form corresponds to some very interesting analytical characteristic properties. In this respect, let us mention that for the Morris class of real quadratic NEF's, we have the Meixner characterization based on some families of orthogonal polynomials which generate exactly the Morris class (see[12]). Another characterization due to Feinsilver[4] states that a certain class of polynomials naturally associated to a NEF is orthogonal if and only if the family is in the Morris class. This characterization has been extended to the Casalis class of simple quadratic NEF's by Labeye-Voisin, and Pommeret [9]. Concerning the cubic NEF's, Hassairi and Zarai [7] introduced a notion of 2-orthogonality for a sequence of polynomials to give an extended version of the Meixner and Feinsilver characterization which subsume the Letac-Mora class of real cubic NEF's. Hassairi and Zarai [8] have also introduced a notion of trans-orthogonality for a sequence of multivariate polynomials to extend their characterization result to the class of multivariate simple cubic NEF's. Besides these characterizations based essentially on different notions of orthogonality of polynomials, it is stated in Casalis[1] that the simple quadratic NEF satisfies a property based on two conjugates families of prior distributions related to the NEF. For a NEF $F = F(\mu)$, consider the family of prior distributions Π introduced by Diaconis and Ylvisaker [3] and defined by

$$\Pi = \{\pi_{t,m_0}(\mathrm{d}\theta) = C_{t,m_0}\ \exp t(\langle m_0, \theta\rangle - k_\mu(\theta))\ \mathbf{1}_{\Theta(\mu)}(\theta)d\theta, t > 0, m_0 \in M_F\} \tag{1.3}$$

where C_{t,m_0} is a normalizing constant. Consider also the family Π^* introduced by Consonni *et al* [2], see also [5] and defined by

$$\Pi^* = \{\pi^*_{t,m_0}, t > 0, m_0 \in M_F\} \tag{1.4}$$

where

$$\pi^*_{t,m_0}(dm) = C^*_{t,m_0}\ \exp t(\ \langle m_0, \psi_\mu(m)\rangle - k_\mu(\psi_\mu(m))\ )\ \mathbf{1}_{M_F}(m)dm,$$

and the constant C^*_{t,m_0} is a normalizing constant. Then, when $F(\mu)$ is a Wishart or a simple quadratic NEF, we have that $k'_\mu(\Pi) = \Pi^*$. It is also shown that this property is equivalent to two other properties expressed in terms of some differential equations satisfied by the cumulant function k_μ. In the real case, the property characterizes the Morris class of real quadratic NEF's, that is $k'_\mu(\Pi) = \Pi^*$ if and only if the NEF F is in the Morris class. In the present paper, we extend these results to the class of multivariate simple cubic NEF's. We construct two families of prior distributions related to a multivariate NEF, and we show that these families coincide when the NEF is simple cubic. We then show that this property is equivalent to the fact that the cumulant function is a solution of some Monge-Ampère equation and also equivalent to the fact that the variance function satisfies a differential equation. As a corollary, we obtain three new characterizations of the Letac-Mora class of real cubic NEF's.

2 Some equivalent properties

Throughout this section, we suppose that $F = F(\mu)$ is a NEF on a linear vector space E with dimension n. Besides the family Π of prior distributions defined in (1.3), we introduce another family $\widetilde{\Pi}$ of prior distributions. Let β be in E^* such that the set

$$\widetilde{\Theta} = \{\theta \in \Theta(\mu); 1 + \langle \beta, k'_\mu(\theta)\rangle > 0\}$$

is nonempty, and denote $\widetilde{M} = k'_\mu(\widetilde{\Theta})$. Consider the family of prior distributions

$$\widetilde{\Pi} = \{\ \widetilde{\pi}_{t,m_0}\ ;\ t \in \mathbb{R}^*_+\ ,\ m_0 \in M_F\}, \tag{2.5}$$

where

$$\widetilde{\pi}_{t,m_0}(dm) = \widetilde{C}_{t,m_0}\ (1 + \langle \beta, m\rangle)^{-n-2}\ \exp t\{\langle m_0, \psi_\mu(m)\rangle - k_\mu(\psi_\mu(m))\}\ \mathbf{1}_{\widetilde{M}}(m)dm.$$

With these notations, we next state and prove our first main result.

Theorem 2.1 *The three following properties are equivalent*
(1) There exists $(a, b, c) \in E \times \mathbb{R}^2$ such that for all m in M_F,

$$\det V_F(m) = \ (1 + \langle \beta, m\rangle)^{n+2}\ \exp\{\langle a, \psi_\mu(m)\rangle + b k_\mu(\psi_\mu(m)) + c\}.$$

(2) There exists $(a, b) \in E \times \mathbb{R}$ such that for all m in M_F and any basis, $(e_i)_{i=1}^n$ of E, with dual basis $(e_i^*)_{i=1}^n$, we have

$$\sum_{i=1}^n [V'_F(m)(e_i)]e_i^* = \frac{n+2}{1 + \langle \beta, m\rangle} V_F(m)(\beta) + a + bm. \tag{2.6}$$

(3) There exists an open subset Ω of $\mathbb{R}^*_+ \times M_F$ such that

$$k'_\mu(\Pi) = \widetilde{\Pi}_\Omega = \{\widetilde{\pi}_{t,m_0}; (t, m_0) \in \Omega\}.$$

Note that (1) may be stated in terms of the cumulant function as there exists $(a, b, c) \in E \times \mathbb{R}^2$ such that for all θ in $\Theta(\mu)$,

$$\det k''_\mu(\theta) = \ (1 + \langle \beta, k'_\mu(\theta) \rangle)^{n+2} \ \exp\{\langle a, \theta \rangle + b k_\mu(\theta) + c\},$$

that is the cumulant function is solution of some Monge-Ampère equation (see [15]).
Proof We will show that (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (2) Suppose that $V_F(m)$ satisfies (1), then we have

$$\log \det V_F(m) = (n + 2) \log(1 + \langle \beta, m \rangle) + \{\langle \psi_\mu(m), a \rangle + b \ k_\mu(\psi_\mu(m)) + c\}.$$

Taking the derivative, we get

$$trace(V_F^{-1}(m) V'_F(m)(.)) = \frac{(n + 2)\langle \beta, . \rangle}{1 + \langle \beta, m \rangle} + \langle \psi'_\mu(m)(.), a \rangle + b \langle m, \psi'_\mu(m)(.) \rangle,$$

which is equivalent to

$$\sum_{i=1}^{n} [V'_F(m)(.) V_F^{-1}(m)(e_i)](e_i^*) = \frac{(n + 2)\langle \beta, . \rangle}{1 + \langle \beta, m \rangle} + \langle \psi'_\mu(m)(.), a \rangle + b \langle m, \psi'_\mu(m)(.) \rangle.$$

Replacing (.) by $V_F(m)(.)$, and using the condition of symmetry

$$V'(m)(V(m)(\alpha))(\beta) = V'(m)(V(m)(\beta))(\alpha) \ \ \forall \ \alpha, \beta \ \in \ E^* \ \ (see \ [10], page \ 103),$$

we obtain

$$\sum_{i=1}^{n} [V'_F(m)(e_i)(.)](e_i^*) = \frac{(n + 2)\langle \beta, V_F(m)(.) \rangle}{1 + \langle \beta, m \rangle} + \langle a, (.) \rangle + b \langle m, (.) \rangle. \tag{2.7}$$

As $V_F(m)$ is symmetric, we get

$$\sum_{i=1}^{n} [V'_F(m)(e_i)](e_i^*) = \frac{(n + 2)}{1 + \langle \beta, m \rangle} V_F(m)(\beta) + a + bm.$$

(2) $\Rightarrow$ (1) Suppose that (2) holds. Then, we easily get (2.7).
Replacing, in (2.7), (.) by $V_F^{-1}(m)(.)$, one obtains

$$\sum_{i=1}^{n} V'_F(m)(e_i) V_F^{-1}(m)(.)(e_i^*) = \frac{(n + 2) \ \langle \beta, (.) \rangle}{1 + \langle \beta, m \rangle} + \langle a, \psi'_\mu(m)(.) \rangle + b \langle m, \psi'_\mu(m)(.) \rangle.$$

This is equivalent to

$$trace(V_F^{-1}(m) V'(m)(.)) = \frac{(n + 2) \ \langle \beta, (.) \rangle}{1 + \langle \beta, m \rangle} + \langle a, \psi'_\mu(m)(.) \rangle + b \langle m, \psi'_\mu(m)(.) \rangle.$$

Integrating, we deduce that there exists c in $\mathbb{R}$ such that

$$\log \det(V_F(m)) = (n + 2) \log(1 + \langle \beta, m \rangle) + \{\langle \psi_\mu(m), a \rangle + b \ k_\mu(\psi_\mu(m)) + c\},$$

and the result follows.

(1) $\Rightarrow$ (3) Suppose that (1) holds, and define

$$\Omega = \{(t, m_0) \in \mathbb{R}_+^* \times M_F \ ; t > b \ and \ m_0 \in (1 - \frac{b}{t})M_F - \frac{a}{t}\}.$$

Take (t, m_0) in Ω and denote ν the image of $\widetilde{\pi}_{t,m_0}$ by ψ_μ. Then it is easy to verify that

$$\nu(d\theta) = \widetilde{C}_{t,m_0} \ e^c \ \exp\{\langle tm_0 + a, \theta\rangle - (t-b)k_\mu(\theta)\} \ \mathbf{1}_{\widetilde{\mathbf{\Theta}}}(\theta)d\theta.$$

Since (t, m_0) is in Ω, we have that $t - b > 0$ *and* $\dfrac{tm_0 + a}{t - b} \in M_F$.
Thus taking $t_1 = t - b$ and $m_1 = \dfrac{tm_0 + a}{t - b}$, we obtain that

$$\nu(d\theta) = C_{t_1,m_1} \ \exp\{t(\langle m_1, \theta\rangle - k_\mu(\theta))\} \ \mathbf{1}_{\widetilde{\mathbf{\Theta}}}(\theta)d\theta.$$

Hence $\psi_\mu(\widetilde{\Pi}_\Omega) \subset \Pi$, and it follows that $\widetilde{\Pi}_\Omega \subset k'_\mu(\Pi)$.
Conversely, if π_{t,m_0} is an element of Π, then its image σ by k'_μ is given by

$$\sigma(dm) = C_{t,m_0} \ e^{-c} \ (1+\langle\beta, m\rangle)^{-n-2} \ \exp\{\langle tm_0 - a, \psi_\mu(m)\rangle - (t+b)k_\mu(\psi_\mu(m))\} \ \mathbf{1}_{\widetilde{\mathbf{M}}}(m)dm.$$

Taking $t_1 = t + b$ and $m_1 = \dfrac{tm_0 - a}{t + b}$. Then (t_1, m_1) is in Ω, and we have

$$\sigma(dm) = \widetilde{C}_{t_1,m_1} \ (1 + \langle\beta, m\rangle)^{-n-2} \ \exp t_1\{\langle m_1, \psi_\mu(m)\rangle - k_\mu(\psi_\mu(m))\}\mathbf{1}_{\widetilde{\mathbf{M}}}(m)dm \ ,$$

which is an element of $\widetilde{\Pi}_\Omega$.

(3) $\Rightarrow$ (1) Suppose that $k'_\mu(\Pi) = \widetilde{\Pi}_\Omega$. Then, for an element π_{t,m_0} of Π, we have on the one hand,

$$k'_\mu(\pi_{t,m_0})(dm) = (\det V_F(m))^{-1} \ C_{t,m_0} \ \exp t\{\langle m_0, \psi_\mu(m)\rangle - k_\mu(\psi_\mu(m))\} \ \mathbf{1}_{\widetilde{\mathbf{M}}}(m)dm.$$

On the other hand, since $k'_\mu(\pi_{t,m_0})$ is in $\widetilde{\Pi}_\Omega$, there exists (t_1, m_1) in Ω such that

$$k'_\mu(\pi_{t,m_0})(dm) = \widetilde{C}_{t_1,m_1} \ (1 + \langle\beta, m\rangle)^{-n-2} \ \exp t_1\{\langle m_1, \psi_\mu(m)\rangle - k_\mu(\psi_\mu(m))\} \ \mathbf{1}_{\widetilde{\mathbf{M}}}(m)dm.$$

Comparing these two expressions of $k'_\mu(\pi_{t,m_0})$ gives

$$\det V(m) = (1 + \langle\beta, m\rangle)^{n+2} \ \exp\{\langle a, \psi_\mu(m)\rangle + bk_\mu(\psi_\mu(m)) + c\},$$

where $a = tm_0 - t_1m_1$, $b = t_1 - t$, and $c = \log(\dfrac{c_{t,m_0}}{\widetilde{c}_{t_1,m_1}})$.

$\square$

3 Characterizations of the Letac-Mora class of real cubic NEFs

In this section, we prove that a multivariate simple cubic NEF satisfies the properties in Theorem (2.1), and that the real versions of these properties characterize the real cubic

NEFs. Recall that a simple cubic NEF is obtained form a simple quadratic NEF by the so-called action of the linear group $GL(\mathbb{R} \times E)$ on the NEFs of E. For more details, we refer the reader to [6], where a complete description of this class is given. This action is in fact an extension of the way in which the Letac-Mora class of real cubic NEFs is obtained from the Morris class of real quadratic NEF's. For our purposes here, we need only to mention that, up to affine transformations and power of convolution, a simple cubic variance function is of the form

$$V(m) = (1 + \langle\beta, m\rangle)\ (I + m \otimes \beta)\ V_1(\frac{m}{1 + \langle\beta, m\rangle})\ (I + \beta \otimes m), \tag{3.8}$$

where V_1 is the variance function of a simple quadratic NEF F_1, and m is in $(M_{F_1})_\beta$, where

$$(M_{F_1})_\beta = \{m \in M_{F_1}; 1 + \langle\beta, m\rangle > 0 \ and\ \frac{m}{1 + \langle\beta, m\rangle} \in M_{F_1}\}.$$

The relation (3.8) is invertible and conversely, we have

$$V_1(M) = (1 - \langle\beta, M\rangle)\ (I - M \otimes \beta)\ V(\frac{M}{1 - \langle\beta, M\rangle})\ (I - \beta \otimes M), \tag{3.9}$$

where M is in $(M_F)_{-\beta}$.
We also mention that the relation between a simple cubic NEF $F(\mu)$ and a simple quadratic NEF $F(\nu)$ may also be expressed in terms of the cumulant functions by

$$\begin{cases} k_\mu(\lambda) = k_\nu(\theta) - k_0 \\ \\ \lambda = -\beta k_\nu(\theta) + \theta - \lambda_0 \end{cases} \tag{3.10}$$

or equivalently by

$$\begin{cases} k_\nu(\theta) = k_\mu(\lambda) - k_1 \\ \\ \theta = \beta k_\mu(\lambda) + \lambda - \theta_1 \end{cases} \tag{3.11}$$

where (k_0, λ_0) and (k_1, θ_1) are constants in $\mathbb{R} \times E$. Note that if $\beta = 0$ in (3.8) we obtain the simple quadratic class. Then for more accuracy we exclude this case and we keep only β in $E^* \setminus \{0\}$.
We now prove that the multivariate simple cubic NEF's satisfy the properties in Theorem 2.1.

Proposition 3.1 *Let $F = F(\mu)$ be a simple cubic NEF on E, then there exists (a, b, c) in $E^* \times \mathbb{R}^2$ such that*

$$\det(V_F(m)) = (1 + \langle\beta, m\rangle)^{n+2}\ \exp\{\langle\psi_\mu(m), a\rangle + b\ k_\mu(\psi_\mu(m)) + c\}.$$

Proof Given that the family F is simple cubic, then there exist β in E^* and $F_1 = F(\nu)$ a simple quadratic NEF such that

$$V_F(m) = (1 + \langle\beta, m\rangle)\ (I + m \otimes \beta)\ V_{F_1}(\frac{m}{1 + \langle\beta, m\rangle})\ (I + \beta \otimes m),$$

see (3.8). As $\det(I + m \otimes \beta) = 1 + \langle \beta, m \rangle$, we obtain

$$\det(V_F(m)) = (1 + \langle \beta, m \rangle)^{n+2} \ \det(V_{F_1}(\frac{m}{1 + \langle \beta, m \rangle})).$$

We now use the fact for a simple quadratic NEF F_1 (see [1]), there exist a' in E^* and b', c' in $\mathbb{R}$ such that, for all M in M_{F_1},

$$\det(V_{F_1}(M)) = \exp\{\langle a', \psi_\nu(M) \rangle + b' k_\nu(\psi_\nu(M)) + c'\}.$$

It follows that

$$\det(V_F(m)) = (1 + \langle \beta, m \rangle)^{n+2} \ \exp\{\langle \psi_\nu(\frac{m}{1 + \langle \beta, m \rangle}), a' \rangle + b' \ k_\nu(\psi_\nu(\frac{m}{1 + \langle \beta, m \rangle})) + c'\}.$$

From (3.10), putting $\lambda = \psi_\mu(m)$ and $\theta = \psi_\nu(\dfrac{m}{1 + \langle \beta, m \rangle})$, we get

$$k_\nu(\psi_\nu(\frac{m}{1 + \langle \beta, m \rangle})) = k_\mu(\psi_\mu(m)) + k_0,$$

$$\psi_\nu(\frac{m}{1 + \langle \beta, m \rangle}) = \psi_\mu(m) + \beta k_\mu(\psi_\mu(m)) + \beta k_0 + \lambda_0.$$

Then

$$\begin{aligned} \det V_F(m) \quad &= \quad (1 + \langle \beta, m \rangle)^{n+2} \exp\{\langle \psi_\mu(m), a' \rangle + (b' + \langle a', \beta \rangle) k_\mu(\psi_\mu(m)) \\ &\qquad + (b' k_0 + \langle a', \beta k_0 + \lambda_0 \rangle + c')\}. \end{aligned}$$

Setting $a = a'$, $b = b' + \langle a', \beta \rangle$ and $c = b' k_0 + \langle a', \beta k_0 + \lambda_0 \rangle + c'$, we obtain the desired result. $\square$

As the Letac-Mora class of real cubic NEFs is nothing but the simple cubic class, when the dimension n is equal to 1, this class satisfies the real version of the properties in Theorem 2.1. We will show that, in this case, these properties are characteristic.

Theorem 3.2 *Let $F = F(\mu)$ be a NEF on the real line, then F is cubic if and only if $k'_\mu(\Pi) = \widetilde{\Pi}$.*

Proof Suppose that $k'_\mu(\Pi) = \widetilde{\Pi}$. Then according to Theorem(2.1), the variance function $V_F(m)$ satisfies the differential equation

$$(1 + \beta m) V'_F(m) - 3\beta \ V_F(m) = (a + bm)(1 + \beta m).$$

Solving this differential equation by standard methods gives

$$V_F(m) = \lambda \ (1 + \beta m)^3 - \frac{b}{\beta^2} \ (1 + \beta m)^2 + \frac{b - \beta a}{2\beta^2} \ (1 + \beta m),$$

which is a polynomial of degree less then or equal to 3. $\square$

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