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Circle Diffeomorphisms: Quasi-reducibility and Commuting Diffeomorphisms

Mostapha Benhenda∗

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Abstract

In this article, we show two related results on circle diffeomorphisms. The first result is on quasi-reducibility: for a Baire-dense set of α, for any diffeomorphism f of rotation number α, it is possible to accumulate Rα with a sequence h−1 fh, h being a diffeomorphism. The second result is: for a Baire-dense set of α, given two commuting diffeomorphisms f and g, such that f has α for rotation number, it is possible to approach each of them by commuting diffeomorphisms fn and gn that are differentiably conjugated to rotations.

In particular, it implies that if α is in this Baire-dense set, and if β is an irrational number such that (α, β) are not simultaneously Diophantine, then the set of commuting diffeomorphisms (f, g) with singular conjugacy, and with rotation numbers (α, β) respectively, is C∞-dense in the set of commuting diffeomorphisms with rotation numbers (α, β).

1 Introduction

It is well-known that there are circle diffeomorphisms with Liouville rotation numbers (i.e. non-Diophantine) that are not smoothly conjugated to rotations [1, 7, 8, 9]. A natural question arises, namely, the problem of smooth quasi-reducibility: given a smooth diffeomorphism f of rotation number α, is it possible to accumulate Rα in the C∞-norm, with a sequence h−1 fh, h being a smooth diffeomorphism? In this case, we say that f is smoothly quasi-reducible to Rα. Quasi-reducibility is a question that has been studied by Herman [7, pp.93-99], who showed that for any C2-diffeomorphism f of irrational rotation number α, it is possible to accumulate Rα in the C1+bv-norm, with a sequence h−1 fh, h being a C2-diffeomorphism (i.e. h−1 fh → Rα in the C1-norm, and the total variation of D(h−1 fh − Rα) converges towards zero). Quasi-reducibility is also related to a problem solved by Yoccoz [10], who showed that it is possible to accumulate a smooth diffeomorphism f in the C∞-norm with a sequence hRαh−1, h being a smooth diffeomorphism. However, these two problems are not the same, and the method used by Yoccoz does not directly yield our result. In our case, we determine a Baire-dense set of rotation numbers α such that for any smooth diffeomorphism f of rotation number α, f is smoothly quasi-reducible.

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Connected to the problem of quasi-reducibility is the following question, raised by Mather: given two commuting $C^\infty$-diffeomorphisms $f$ and $g$, is it possible to approach each of them in the $C^\infty$-norm by commuting smooth diffeomorphisms that are smoothly conjugated to rotations? In this paper, we determine a Baire-dense set of rotation numbers $\alpha$ such that if $f$ and $g$ are commuting $C^\infty$-diffeomorphisms, with $f$ of rotation number $\alpha$, then $f$ and $g$ are accumulated in the $C^\infty$ norm by commuting $C^\infty$-diffeomorphisms that are $C^\infty$-conjugated to a rotation. This result is related to a theorem of Fayad and Khanin [6]. They showed that if $(\alpha, \alpha')$ is a pair of rotation numbers of Diophantine (i.e. there is $C_d > 0$, $\beta \geq 0$ such that for any $p, p' \in \mathbb{Z}$, any $q \geq 1$, $\max(|\alpha - p/q|, |\alpha' - p'/q|) \geq C_d/q^{2+\beta}$). This set includes some pairs $(\alpha, \alpha')$ with $\alpha$ and $\alpha'$ Liouvillean, and if $f$ and $g$ are commuting $C^\infty$-diffeomorphisms, with $f$ and $g$ of rotation numbers $\alpha$ and $\alpha'$ respectively, then $f$ and $g$ are smoothly linearizable. Fayad and Khanin’s result implies our result of quasi-reducibility in the particular case when the rotation numbers of $f$ and $g$ are simultaneously Diophantine. However, in general, our result is not implied by theirs. Indeed, our result holds for a set $(\alpha, \alpha')$ that is Baire-dense in $\mathbb{R}^2$ (because $\alpha$ belongs to a Baire-dense set of $\mathbb{R}$ and $\alpha'$ is arbitrary), whereas the set of simultaneously Diophantine numbers is not Baire-dense.\footnote{The complementary in $\mathbb{R}^2$ of simultaneously Diophantine numbers (noted $SD'$) is Baire-dense. Indeed, we have:}

\[ SD' = \cap_{n \in \mathbb{N}} \cap_{\sigma \in \mathbb{N}^*} \cup_{q \in \mathbb{Z}} (A_{q,k} \times A_{q,k}) \]

with:

\[ A_{q,k} = \{ \alpha \in \mathbb{R} / \text{there is an integer } p \in \mathbb{Z}, |\alpha - \frac{p}{q}| < \frac{1}{q^2} \}. \]

$A_{q,k}$ is open (and so is $A_{q,k} \times A_{q,k}$), and for any integer $n$, $\cup_{q \in \mathbb{Z}} (A_{q,k} \times A_{q,k})$ is dense, because it contains all pairs of rational numbers if $(\alpha = p_1/q_1, \alpha' = p_2/q_2)$, then $(\alpha, \alpha') \in (A_{p_1,q_1,k} \times A_{p_2,q_2,k})$ for any $j, k \in \mathbb{N}^*$. Therefore, $SD'$ is Baire-dense.
Remark 1.3. The proof of theorem 1.1 also gives that $h_{\alpha}R_{\alpha}h_{\alpha}^{-1} \to f$ in the $C^{\infty}$-topology if $\alpha \in A_1$.

Remark 1.4. Combined with [6, p. 965], theorem 1.2 implies that if $\alpha \in A_2$, and $(\alpha, \beta)$ are not simultaneously Diophantine, then $S_{\alpha, \beta}$, the set of couples $(f, g)$ of smooth commuting circle diffeomorphisms with singular conjugacies to $R_{\alpha}$ and $R_{\beta}$ respectively, is $C^{\infty}$-dense in $F_{\alpha, \beta}$, the set of couples $(f, g)$ of smooth commuting circle diffeomorphisms with rotation numbers $\alpha$ and $\beta$ respectively.

Indeed, our result shows that $O_{\alpha, \beta}$, the set of couples $(f, g)$ of smooth commuting circle diffeomorphisms with smooth conjugacies to rotations $R_{\alpha}$ and $R_{\beta}$ respectively, is $C^{\infty}$-dense in $F_{\alpha, \beta}$. Moreover, in [6, p.965], for $(\alpha, \beta)$ not simultaneously Diophantine, Fayad and Khanin described the construction of a couple $(f, g)$ of smooth commuting circle diffeomorphisms with singular conjugacies to $R_{\alpha}$ and $R_{\beta}$ respectively. This construction relies on the method of successive conjugacies, which can be made $C^{\infty}$-dense in $O_{\alpha, \beta}$ [5].

Moreover, by slightly modifying [7, p.160, p.167], this implies that $(O_{\alpha, \beta})^\circ$, the set of couples $(f, g)$ of smooth commuting circle diffeomorphisms with non-$C^1$ conjugacies to rotations $R_{\alpha}$ and $R_{\beta}$, is $C^{\infty}$-generic in $F_{\alpha, \beta}$. See appendix A for a short proof.

2 Preliminaries

2.1 Basic properties

When the rotation number $\alpha$ of $f$ is irrational, and if $f$ is of class $C^2$, Denjoy showed that $f$ is topologically conjugated to $R_{\alpha}$. However, this conjugacy is not always differentiable. It depends on the Diophantine properties of the rotation number $\alpha$.

Let $\alpha = a_0 + 1/(a_1 + 1/(a_2 + ...))$ be the development of $\alpha \in \mathbb{R}$ in continued fraction (see [4]). It is denoted $\alpha = [a_0, a_1, a_2, ...]$. Let $p_{-2} = q_{-2} = 0$, $p_{-1} = q_{-1} = 1$. For $n \geq 0$, we define integers $p_n$ and $q_n$ by:

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2}.$$  

We have $q_0 = 1$, $q_n \geq 1$ for $n \geq 1$. The rationals $p_n/q_n$ are called the convergents of $\alpha$. Remember that $q_{n+2} \geq 2q_n$, for $n \geq -1$.

For any real number $\beta \geq 0$, $\alpha \in \mathbb{R} - \mathbb{Q}$ is Diophantine of order $\beta$ and constant $C_d$ (a set denoted $DC(C_d, \beta)$) if there is a constant $C_d > 0$ such that for any $p/q \in \mathbb{Q}$, we have:

$$\left| \alpha - \frac{p}{q} \right| > \frac{C_d}{q^{2\beta}}.$$  

Each of the following relations characterizes $DC(C_d, \beta)$ (see e.g. [11, pp.50-51]):

1. $|\alpha - p_n/q_n| > C_d/q_n^{2+\beta}$ for any $n \geq 0$
2. $a_{n+1} < 2C_d q_n^\beta$ for any $n \geq 0$
3. $q_{n+1} < C_d q_n^{1+\beta}$ for any $n \geq 0$
4. \( \alpha_{n+1} > C_d \alpha_n^{1+\rho} \) for any \( n \geq 0 \).

\( DC(C_d, 0) \) is the set of irrational numbers of constant type \( C_d \). The first derivative of \( f \in D^1(T^1) \) is denoted \( Df \).

### 2.2 Some useful lemmas

For any integer, let \( \alpha_n = [a_0, \ldots, a_n, 1, \ldots] \).

Let \( V_\alpha : \mathbb{N} \to \mathbb{R} \) defined by: \( V_\alpha(n) = \max_{0 \leq i \leq n} a_i \). Observe that \( \alpha_n \in DC(1/V_\alpha(n), 0) \).

We will need the lemma:

**Lemma 2.1.** Let \( \alpha \) be an irrational number, \( q_n \) its convergents and \( \alpha_n = [a_0, \ldots, a_n, 1, \ldots] \).

We have:

\[ |\alpha_n - \alpha| \leq \frac{2}{q_n^2} \leq \frac{4}{2^n}. \]

**Proof.** Let \( \tilde{\alpha}_n = [a_0, \ldots, a_n, 0, \ldots] \). By induction, we can show that \( \tilde{\alpha}_n = p_n/q_n \). Moreover, \( \tilde{\alpha}_n \) is also the \( n \)th convergent of \( \alpha_n \). Therefore, by the best rational approximation theorem, \( |\alpha - p_n/q_n| \leq 1/q_n^2 \) and \( |\alpha_n - p_n/q_n| \leq 1/q_n^2 \). Moreover, since \( q_{n+2} \geq q_n \), then \( q_n \geq (\sqrt{2})^{n-1} \).

\[ \blacksquare \]

We need the lemma:

**Lemma 2.2.** Let \( \phi : \mathbb{N} \to \mathbb{R}_+ \) be such that \( \phi(n) \to_{n \to +\infty} +\infty \). Let

\[ A = \{ \alpha \in \mathbb{R} / V_\alpha(n) < \phi(n) \text{ for an infinity of } n \} \]

Then \( A \) is Baire-dense.

**Proof.** First, we show that for any positive integers \( n \) and \( i \),

\[ A_{i,n} = \{ \alpha \text{ such that } a_i < \phi(n) \} \] is open. Let \( u(x) = [x] \), \( v(x) = \frac{1}{x} \) and \( w(x) = v(x) - u(v(x)) \). We have: \( a_{i+1} = v(w^k(x)) - w^{k+1}(x) \). Since \( v \) is continuous and \( u \) is upper semi-continuous and non-negative, then \( w \) is lower semi-continuous. Moreover, \( w \) is non-negative. Therefore, \( w^k \) and \( w^{k+1} \) are also lower semi-continuous and non-negative. Since \( v \) is decreasing, then \( v \circ w^k - w^{k+1} \) is upper semi-continuous. We conclude that \( A_{i,n} \) is open.

Moreover, for any \( p \geq 0 \),

\[ \bigcup_{i \geq p} \cap_{n \geq i} A_{i,n} \]

is dense. Indeed, since \( \phi(n) \to +\infty \), then it contains all numbers of constant type, which are dense. This set is also open and therefore,

\[ A = \cap_{i \geq 0} \cup_{i \geq p} \cap_{n \geq i} A_{i,n} \]

is Baire-dense.

\[ \blacksquare \]
2.3 Notations

- For any real numbers $a$ and $b$, $a \lor b$ denotes $\max(a, b)$.
- For $\phi$ a real $\mathbb{Z}$-periodic $C^r$ function, $0 \leq r < +\infty$, we define:

$$\|\phi\|_r = \max_{0 \leq j \leq r} \max_{x \in \mathbb{R}} |D^j \phi(x)|.$$ 

Note that for $f, g \in D^r(\mathbb{T}^1)$, $f - g$ is $\mathbb{Z}$-periodic, and for $1 \leq j \leq r$, $D^j f$ is $\mathbb{Z}$-periodic. For $f \in D^r(\mathbb{T}^1)$, we also define:

$$\|f\|_r = \max \left( \|f - id\|_0, \max_{1 \leq j \leq r} \|D^j f\|_0 \right).$$

Note that the notation $\|f\|_r$ is not a norm when $f \in D^r(\mathbb{T}^1)$, since $D^r(\mathbb{T}^1)$ is not a vector space.

- In all the paper, $C$ denotes a constant depending on $u$. $W(f)$ denotes the total variation of $\log Df$, and $S f$ denotes the Schwartzian derivative of $f$.

2.4 Estimates of the conjugacy

The following theorem gives an estimate of the linearization of a diffeomorphism having a rotation numbers of Diophantine constant type. This estimate, obtained in [2], is necessary to derive our results.

**Theorem 2.3.** Let $l \geq 3$ be an integer and $\eta > 0$. Let $f \in D^l(\mathbb{T}^1)$ be of rotation number $\alpha$, such that $\alpha$ is of constant type $C_d$. There exists a diffeomorphism $h \in D^{l-1-\eta}(\mathbb{T}^1)$ conjugating $f$ to $R_\alpha$, and a function $B$ of $C_d, l, \eta, W(f), \|S f\|_{l-3}$, which satisfy the estimate:

$$\max \left( \min_{\mathbb{T}^1} D^1 h, \|h\|_{l-1-\eta}, \|D^l h\|_{l-3} \right) \leq B(C_d, l, \eta, W(f), \|S f\|_{l-3}).$$

(1)

In particular, we remark that if $f_n$ is a sequence of diffeomorphisms of rotation number $\alpha_n$, if the sequences $W(f_n)$ and $\|S f\|_{l-3}$ are bounded (this will hold in our case, because we will take $f_n = \lambda_n + f$ for a properly chosen $\lambda_n \in \mathbb{R}$, if $V_\alpha(n) \to +\infty$ and if $h_n$ is the conjugacy to a rotation associated with $f_n$, then there is a real function $E(V_\alpha(n))$ such that, for $n$ sufficiently large, we have:

$$\max \left( \frac{1}{\min_{\mathbb{T}^1} D^1 h_n}, \|h_n\|_{l-1-\eta}, \|D^l h_n\|_{l-3} \right) \leq E(V_\alpha(n)).$$

3 Quasi-Reducibility

**Theorem 3.1.** Let $l \geq 3$ be an integer, $f \in D^l(\mathbb{T}^1)$ be of rotation number $\alpha \in \mathbb{T}^1$. Let $\eta > 0$ be a real number. There exists a numerical sequence $F(n)$, going to $+\infty$ as $n \to +\infty$, such that, if

$$\lim inf \frac{V_\alpha(n)}{F(n)} = 0$$
then there is a sequence \( h_n \) of class \( C^{\ell-1-\eta} \) such that \( h_n^{-1} fh_n \to R_\alpha \) in the \( C^{\ell-2-\eta}\)-topology.

By applying lemma 2.2, we obtain the corollary:

**Corollary 3.2.** There is a Baire-dense set \( A_1 \subset \mathbb{R} \) such that if \( \ell \geq 3 \) is an integer, \( f \in D^\ell(T^1) \) of rotation number \( \alpha \in A_1 \) and \( \eta > 0 \), then \( f \) is \( C^{\ell-2-\eta}\)-quasi-reducible: there is a sequence \( h_n \in D^{\ell-1-\eta}(T^1) \) such that \( h_n^{-1} fh_n \to R_\alpha \) in the \( C^{\ell-2-\eta}\)-topology.

The idea of the proof of theorem 3.1 is the following. We observe that for any sequence \( \phi(n) \to +\infty \), the set of numbers \( \alpha \) such that for an infinity of \( n \),

\[
\sup_{k \leq n} a_k \leq \phi(n),
\]

is Baire-dense (lemma 2.2).

The truncated sequence of constant type numbers \( \alpha_n = [a_0, ..., a_n, 1, ...] \) converges towards \( \alpha \) at a controlled speed: \( |\alpha - \alpha_n| \leq 4/2^n \) (lemma 2.1).

Following an idea of Herman [7], we perturbate \( f \) to \( R_\lambda f = f + \lambda_n \) of rotation number \( \alpha_n \), which is linearizable by a conjugacy \( h_n \) (lemma 3.3). By writing:

\[
h_n^{-1} fh_n - R_\alpha = h_n^{-1} fh_n - h_n^{-1} R_\lambda fh_n + R_\alpha - R_\lambda
\]

and by applying the Faa-di-Bruno formula, we obtain a control of the norm of \( h_n^{-1} fh_n - R_\alpha \), which is quasi-reducible.

Thus, if we choose the speed of growth of the sequence \( \sup_{k \leq n} a_k \) sufficiently small with respect to the speed of convergence of \( \alpha_n \) towards \( \alpha \), then \( h_n^{-1} fh_n \) converges towards \( R_\alpha \), and \( f \) is quasi-reducible.

**Proof of theorem 1.1.** We let \( \eta = 3 \) in corollary 3.2. Since \( f \) is smooth, then there is a sequence \( (h_n)_{n \geq 0} \in D^\infty(T^1) \) such that, for any integer \( \ell \geq 3 \) fixed,

\[
\|h_n^{-1} fh_n - R_\alpha\|_{2(\frac{1}{\ell}-1)} \to_{n \to +\infty} 0.
\]

In particular, there is \( n(l) \) such that:

\[
\|h_n^{-1} fh_n - R_\alpha\|_{2(\frac{1}{\ell}-1)} \leq \frac{1}{7}.
\]

Let \( h_l = h_{n(l)l} \). Let \( \epsilon > 0 \), and let \( k > 0 \) be an integer. There is \( l_0 \geq 0 \) such that for any \( l \geq l_0 \), we have: \( \epsilon \geq 1/l, k \leq 2 \left( \frac{1}{\ell} - 1 \right) \) and:

\[
\|h_l^{-1} fh_l - R_\alpha\|_k \leq \|h_l^{-1} fh_l - R_\alpha\|_{2(\frac{1}{\ell}-1)} \leq \frac{1}{7} \leq \epsilon.
\]

Therefore, \( h_l^{-1} fh_l \to_{l \to +\infty} R_\alpha \) in the \( C^k \)-topology, for any \( k \), and therefore, this convergence holds in the \( C^\infty \)-topology.

\[\square\]

### 3.1 The one-parameter family \( R_\lambda f \)

To prove theorem 3.1, we need to consider the one-parameter family \( R_\lambda f = f + \lambda \) (see [7, p.31]). We have the lemma:
Lemma 3.3. Let \( l \geq 3 \) be an integer, \( f \in D^l(T^1) \), \( 0 < \eta \leq l - 3 \), \( \alpha = \rho(f) \). Let \( \tilde{\alpha} \) be an irrational number of constant type. There exists \( \lambda_0 \in \mathbb{R} \) and a \( C^{l-\eta} \)-diffeomorphism \( h \) such that \( h^{-1}R_{\lambda_0} f h = R_{\lambda_0} \). Moreover,

\[
\frac{|\lambda_0|}{\min Dh} \geq |\tilde{\alpha} - \alpha| \geq \frac{|\lambda_0|}{\|Dh\|_0}.
\]

Proof. Let \( \mu(f) = \rho(R_{\lambda_0} f) \). \( \mu \) is continuous, non-decreasing and \( \mu(\mathbb{R}) = \mathbb{R} \) (see [7, p. 31]). Therefore, there exists \( \lambda_0 \in \mathbb{R} \) such that \( \tilde{\alpha} = \rho(R_{\lambda_0} f) \). Since \( \tilde{\alpha} \) is of constant type, there exists a \( C^{l-\eta} \)-diffeomorphism \( h \) such that \( h^{-1}R_{\lambda_0} f h = R_{\lambda_0} \) and that satisfies estimate (1) of theorem 2.3. By the mean value theorem, for any \( x \), there is \( c(x) \) such that:

\[
\tilde{\alpha} + x - h^{-1} f h(x) = R_{\lambda_0}(x) - h^{-1} f h(x) = h^{-1} R_{\lambda_0} f h(x) - h^{-1} f h(x) = D(h^{-1})(c(x)) \lambda_0.
\]

By integrating this equation on an invariant measure of \( h^{-1} f h \), we get lemma 3.3. Note that since \( h \in D^l(T^1) \), then \( Dh(x) > 0 \) for any \( x \), and \( \min Dh > 0 \).

\( \square \)

3.2 The speed of approximation of \( R_{\alpha} \)

The proof of theorem 3.1 is also based on the lemma:

Lemma 3.4. Let \( l \geq 3 \) be an integer, \( f \in D^l(T^1) \), \( 0 < \eta \leq l - 3 \), \( \alpha = \rho(f) \). Let \( \tilde{\alpha} \) be an irrational number of constant type, and let \( \lambda_0 \in \mathbb{R} \) and \( h \) the \( C^{l-\eta} \)-diffeomorphism be given by lemma 3.3. Recall that \( C \) denotes a constant that only depends on \( u, \lambda_0 \) and that satisfies:

\[
C = C(l, \eta, \lambda_0).
\]

We have the estimate:

\[
\|h^{-1} f h - R_{\lambda_0}\|_u \leq C\|f\|_C \|Dh\|^{C-1}_u \left(\frac{1}{\min Dh}\right)^{l} \mid \tilde{\alpha} - \alpha \mid.
\]

Before proving lemma 3.4, we show how theorem 3.1 is derived from it.

proof of theorem 3.1. If \( \alpha \) is of constant type, then \( f \) is reducible and there is nothing to prove. Therefore, we can suppose that \( V_{\alpha}(n) \to_{n \to +\infty} +\infty \). By applying theorem 2.3, there exists a real function \( \tilde{F} \) strictly increasing with \( V_{\alpha}(n) \), such that for \( \alpha_n \), and for its associated diffeomorphism \( h_n \) given by lemma 3.3, we have, for \( n \) sufficiently large:

\[
\|h_n^{-1} f h_n - R_{\alpha}\|_{l-2-\eta} \leq \exp \left( \tilde{F}(V_{\alpha}(n)) \right) |\alpha_n - \alpha|.
\]

Let \( F(n) = \tilde{F}^{-1}(n^{1/2}) \). By extracting, we can suppose that \( \lim V_{\alpha}(n) = 0 \). Therefore, \( V_{\alpha}(n) \leq F(n) \) for \( n \) sufficiently large and therefore,

\[
\tilde{F}(V_{\alpha}(n)) \leq n^{1/2}.
\]

We get, for \( n \) sufficiently large,

\[
\|h_n^{-1} f h_n - R_{\alpha}\|_{l-2-\eta} \leq e^{-\frac{1}{\log n}} \to_{n \to +\infty} 0.
\]

Hence theorem 3.1.

\( \square \)

Now, we show lemma 3.4:
proof of lemma 3.4. We need the Faa-di-Bruno formula (see e.g. [3]):

**Lemma 3.5.** For every integer \( u \geq 0 \) and functions \( \phi \) and \( \psi \) of class \( C^u \), we have:

\[
D^u \left[ \phi(\psi(x)) \right] = \sum_{j=0}^{u} D^j \phi(\psi(x)) B_{u,j} \left( D\phi(x), D^2 \phi(x), \ldots, D^{(u-j+1)} \phi(x) \right).
\]

The \( B_{u,j} \) are the Bell polynomials, defined by \( B_{u,0} = 1 \) and, for \( j \geq 1 \):

\[
B_{u,j}(x_1, x_2, \ldots, x_{u-j+1}) = \sum \frac{u!}{l_1! l_2! \cdots l_{u-j+1}!} \left( \frac{x_1^{l_1}}{l_1!} \right) \left( \frac{x_2^{l_2}}{2!} \right) \cdots \left( \frac{x_{u-j+1}^{l_{u-j+1}}}{(u-j+1)!} \right).
\]

The sum extends over all sequences \( l_1, l_2, l_3, \ldots, l_{u-j+1} \) of non-negative integers such that: \( l_1 + l_2 + \ldots = j \) and \( l_1 + 2l_2 + 3l_3 + \ldots = u \).

Therefore, for any \( x \), we have the estimate:

\[
\left| B_{u,j} \left( D\phi(x), D^2 \phi(x), \ldots, D^{(u-j+1)} \phi(x) \right) \right| \leq C \left( 1 + \| \phi \|_u \right).
\]  

Combining this estimate with lemma 3.5, we obtain the corollary:

**Corollary 3.6.** For every integer \( u \geq 0 \) and functions \( \phi \) and \( \psi \) of class \( C^u \), we have:

\[
\| \phi \circ \psi \|_u \leq C \max_{0 \leq \ell \leq u} \| D^\ell \phi \circ \psi \|_0 \left( 1 + \| \psi \|_u \right).
\]

We apply this corollary to estimate \( \| h^{-1} \|_u \). We let \( \phi(x) = 1/x \) and \( \psi = Dh \circ h^{-1} \). We observe that \( D(h^{-1}) = \frac{1}{Dh} = \phi \circ \psi \). Since there is \( x_0 \) such that \( Dh(x_0) = 1 \), then \( \| Dh \|_0 \geq 1 \) (and we also have \( 1 \geq \min Dh > 0 \)). Therefore, we get:

\[
\| D(h^{-1}) \|_u \leq C \max_{0 \leq \ell \leq u} \frac{1}{\| (Dh \circ h^{-1})^{\ell+1} \|_0} \| Dh \circ h^{-1} \|_0^\ell.
\]

By corollary 3.6, we also have:

\[
\| Dh \circ h^{-1} \|_u \leq C \| Dh \|_u \| h^{-1} \|_u^C.
\]

By combining these two estimates, we get:

\[
\| D(h^{-1}) \|_u \leq C \frac{1}{\min Dh^C} \| Dh \|_u^C \| h^{-1} \|_u^C.
\]

We iterate this estimate to estimate \( \| h^{-1} \|_u \), for \( u \geq 1 \). We get:

\[
\| h^{-1} \|_{u+1} \leq C \frac{1}{\min Dh^C} \| h^{-1} \|_u^C \| h^{-1} \|_u^C.
\]  

Now, we estimate the \( C^\alpha \)-distance of \( h^{-1} fh \) to \( R_\alpha \). Let \( \tilde{\alpha}, \lambda_0 \) be as in lemma 3.3. We have:

\[
h^{-1} fh - R_\alpha = h^{-1} fh - h^{-1} R_{\lambda_0} fh + R_{\lambda_0} - R_\alpha.
\]

Therefore,

\[
\| h^{-1} fh - R_\alpha \|_u \leq \| h^{-1} fh - h^{-1} R_{\lambda_0} fh \|_u + |\tilde{\alpha} - \alpha|.
\]

8
On the other hand, by the Faa-di-Bruno formula, we have:

$$D^l \left[ h^{-1}fh - h^{-1}R_{h_{k_0}}fh \right](x) = \sum_{j=0}^{l} B_{n,j} \left( D(fh)(x), ..., D^{l-1+j}(fh)(x) \right)$$

$$\left[ D^{l}(h^{-1})(fh(x)) - D^{l}(h^{-1})(fh(x) + \lambda_0) \right].$$

Since $|D^j(h^{-1})(fh(x)) - D^j(h^{-1})(fh(x) + \lambda_0)| \leq \|D^{j+1}(h^{-1})\|_0 \lambda_0$, then by applying estimate (2), we get:

$$\|h^{-1}fh - h^{-1}R_{h_{k_0}}fh\|_u \leq C\|h\|_0 \|h^{-1}\|_{u+1} \lambda_0.$$  

By applying corollary 3.6, we get:

$$\|h^{-1}fh - h^{-1}R_{h_{k_0}}fh\|_u \leq C\|h\|_0 \|h^{-1}\|_{u+1} \lambda_0.$$  

By applying (3), we obtain:

$$\|h^{-1}fh - h^{-1}R_{h_{k_0}}fh\|_u \leq \|f\|_C \|h\|_0 \|h^{-1}\|_{u+1} \frac{|\tilde{\alpha} - \alpha|}{(\min Dh)^2}.$$  

By estimate (4), we obtain:

$$\|h^{-1}fh - R_n\|_u \leq C\|f\|_C \|h\|_0 \|h^{-1}\|_{u+1} \frac{|\tilde{\alpha} - \alpha|}{(\min Dh)^2}. \quad (5)$$

Hence lemma 3.4.

\[ \square \]

4 Application to commuting diffeomorphisms

**Theorem 4.1.** There exists a numerical sequence $G(n)$, going to $+\infty$ as $n \rightarrow +\infty$, such that, for any $l \geq 3$ an integer, $f \in D^l(T^1)$ of rotation number $\alpha \in \mathbb{R}$, $\eta > 0$ and $g$ of class $C^l$ such that $fg = gf$, if

$$\lim \inf \frac{V_n(n)}{G(n)} = 0$$

then there exists two sequences of diffeomorphisms $f_n$ and $g_n$ that are $C^{l-\eta}$-conjugated to rotations, such that $f_ng_n = g_nf_n$, and with $f_n$ and $g_n$ converging respectively towards $f$ and $g$ in the $C^{l-\eta}$-norm.

**Corollary 4.2.** There is a Baire-dense set $A_2 \subset \mathbb{R}$ such that if $l \geq 3$ is an integer, $f \in D^l(T^1)$ has a rotation number $\alpha \in A_2$, $g$ is of class $C^l$ such that $fg = gf$ and $\eta \in \mathbb{R}^+$, then there exists two sequences of diffeomorphisms $f_n$ and $g_n$ that are $C^{l-\eta}$-conjugated to rotations, such that $f_ng_n = g_nf_n$ and with $f_n$ and $g_n$ converging respectively towards $f$ and $g$ in the $C^{l-2-\eta}$-norm.

We derive theorem 1.2 from corollary 4.2 by following the same argument as in the proof of theorem 1.1.
4.1 The speed of approximation of \( g \) by a linearizable and commuting diffeomorphism

To prove theorem 4.1, we consider \((h_n)_{n \geq 0}\), the sequence of conjugating diffeomorphisms constructed in the proof of theorem 3.1. \((\lambda_n)_{n \geq 0}\) the associated sequence of real numbers such that \(f_n = R_{\lambda_n} f = h_n R_{\lambda_n} h_n^{-1}\). We also consider \(g_n = h_n R_{\tilde{g}_n(0)} h_n^{-1}\). The diffeomorphisms \(f_n\) and \(g_n\) commute, and \(f_n \to f\) in the \(C^{t-2-\eta}\)-norm. To prove theorem 4.1, it suffices to show that \(g_n \to g\) in the \(C^{t-2-\eta}\)-norm. This convergence is based on the lemma:

**Lemma 4.3.** Let \( l \geq 3 \) be an integer, \( f \in D^l(T^1) \) of rotation number \( \alpha \in \mathbb{R} \), \( \eta > 0 \), \( 0 \leq u \leq l - 2 - \eta \), and \( g \in D^l(T^1) \) be such that \( fg = gf \). Let \((\varphi_n)_{n \geq 0}\) be the sequence of denominators of the convergents of \( \alpha \), and let \( r \geq 0 \) be an integer. Let \( \tilde{\alpha} \) be an irrational number of constant type, \( \lambda_0 \in \mathbb{R} \) the associated number and \( h \) the associated \( C^{t-\eta} \) diffeomorphism given by lemma 3.3. Let \( f' = h^{-1} fh \) and \( g' = h^{-1} gh \). We have the estimate:

\[
\|g - h R_{\tilde{g}'(0)} h^{-1}\|_u \leq C \|h\|_{C^{n+1}} \|f\|_{C^{n+1}} \|g\|_{C^{n+1}} \left( \frac{1}{q_r} + |\tilde{\alpha} - \alpha| \right) \left( \frac{(C\|h\|_{C^{n+1}} \|f\|_{C^{n+1}} \|g\|_{C^{n+1}})}{\min Dh}\right).
\]

To show this lemma, the basic idea is the following: we approach modulo 1 points \( x \in \mathbb{R} \) by \( p(x) \mod 1 \), where \( p(x) \leq q_r \) is an integer, and where the integer \( r \) will be fixed later. We have a control of \( |x - p(x)\alpha| \mod 1 \) in function of \( q_r \). Then, by using the assumption of commutation \( g' f^p = f^p g' \), we can write:

\[
g'(x) - R_{\tilde{g}'(0)}(x) = g'(x) - g'(pa) + g'(pa) - g f^p(0) + f^p g'(0) - R_{pa}(g'(0)) + R_{\tilde{g}'(0)}(pa) - R_{\tilde{g}'(0)}(x).
\]

We use the distance of \( f^p \) to \( R_{pa} \), which depends on \( q_r \) and the norm of \( f' - R_{\alpha} \). This distance has been estimated in the proof of the result of quasi-reductibility. We also use \( C^k \) analogues, \( k \geq 2 \), of the mean value theorem, obtained with the Faa-di-Bruno formula. This allows to estimate the norm of \( g - h R_{\tilde{g}'(0)} h^{-1} \) in function of the norm of \( g' - R_{\tilde{g}'(0)} \).

To obtain theorem 4.1 from lemma 4.3, we take \( \tilde{\alpha} = \alpha_n \) and we consider the associated sequences \( f_n, g_n, f'_n, g'_n, h_n \). The integer \( q_r \) must be chosen sufficiently large with respect to the conjugacy \( h_n \), so that \( |x - pa| \mod 1 \) is sufficiently small. However, this integer \( q_r \) must not be too large, to keep the norm of \( f_n^p = R_{pa} \) sufficiently small. This integer \( q_r \) is controlled with \( \sup_{k \leq r} a_k \), which itself controls the norm of \( h_n \). Thus, it suffices to properly choose the integer \( r \) in function of \( n \), in order to obtain the convergence of \( g_n \) towards \( g \).

**Proof of theorem 4.1.** Assuming lemma 4.3, we show theorem 4.1.

Let \( \alpha_n \) and \( h_n \) be the associated diffeomorphism given by lemma 3.3. Since \( V_n(a) \to +\infty \), by applying the estimate for the conjugacy \( h_n \), there exists \( G(x) \) strictly increasing with \( x \) such that, for \( n \) sufficiently large:

\[
\|g - h_n R_{\tilde{g}_n(0)} h_n^{-1}\|_{l-2-\eta} \leq e^{CG(V_n(a))} \left( \frac{1}{q_r} + \frac{e^{CG(V_n(a))q_r}}{2^n} \right).
\]

Moreover, since \( q_{a_n} = \alpha_n q_{a_n-1} + q_{a_n-2} \), and \( q_{a_n-2} \leq q_{a_n-1} \), then

\[
(\sqrt{2})^{a_n-1} \leq q_n \leq \prod_{k=1}^{n} (a_k + 1).
\]
Therefore, we get:

\[ \| g - h_n R_{\delta}(0) h_n^{-1} \|_{-2-\eta} \leq e^{CG(V_a(n)) - \frac{1}{4}(r-1)\log 2} + e^{CG(V_a(n)) + CG(V_a(r))} \leq e^{\log n^{1/2}}. \]  

(7)

Let \( G(n) = \tilde{G}^{-1}(\log n)^{1/2} \). By extracting in the sequence \( V_a(n)/G(n) \), we can suppose that:

\[ \frac{V_a(n)}{G(n)} \to 0. \]

Therefore, for \( n \) sufficiently large, we have:

\[ \tilde{G}(V_a(n)) \leq (\log n)^{1/2}. \]

Moreover, for \( n \) sufficiently large, we can take an integer \( r_n \) such that:

\[ (\log n)^{3/4} \leq r_n \leq (\log n)^{7/8}. \]

We get:

\[ (V_a(r_n) + 1)^{r_n} = e^{r_n \log(V_a(r_n) + 1)} \leq e^{(\log n)^{15/16}}. \]

The first term in estimate (7) tends towards 0. Moreover, since, for \( n \) sufficiently large,

\[ (\log n)^{1/2} e^{(\log n)^{15/16}} \leq \frac{n}{2} \log 2 \]

then the second term also tends towards 0. Hence theorem 4.1.

\[ \square \]

### 4.2 Higher-order analogous of the mean value theorem

**Proof of lemma 4.3.** We need two higher-order analogous of the mean value theorem. The first one is:

**Lemma 4.4.** Let \( u \geq 0 \), \( s, t \in D^u(T^1) \). Let \( \delta \in \mathbb{R} \). We have:

\[ \| st - R_{\delta} t \|_u \leq C\| s \|_{u+1} \| s - R_{\delta} s \|_u \|
\]

Observe the presence of the term \( \| s \|_{u+1} \), which is absent in the mean value formula. This is because of the estimate (2) on the Bell polynomial, in the Faa-di-Bruno formula.

**Proof.** If \( u = 0 \), the estimate is trivial. We suppose \( u \geq 1 \). For any \( x \in \mathbb{R} \), the Faa-di-Bruno formula gives:

\[ D^u(st)(x) - D^u(R_{\delta} t)(x) = \sum_{j=0}^{u} \left( (D^j s)(t(x)) - (D^j R_{\delta} s)(t(x)) \right) B_{u,j}(Dt(x), ..., D^{u-j+1} t(x)). \]

Therefore, by estimate (2), and since \( \| t \|_u \geq 1 \),

\[ |D^u(st)(x) - D^u(R_{\delta} t)(x)| \leq C\| s \|_{u+1} \| s - R_{\delta} s \|_u \|
\]

Hence lemma 4.4.  

\[ \square \]
The second higher-order analogous of the mean value theorem is:

**Lemma 4.5.** Let \( u \geq 0, s \in D^{u+1}(T^1), t \in D^u(T^1), \delta \in \mathbb{R} \). We have:

\[
\|st - sR_\delta\|_u \leq C\|s\|_{u+1}\|t\|^u_0\|t - R_\delta\|_u.
\]

Observe the presence of the term \( \|t\|_u \), which is absent in the mean value formula. As in lemma 4.4, this is because of an estimate on the Bell polynomial, in the Faa-di-Bruno formula.

**Proof.** If \( u = 0 \), the estimate holds. We suppose \( u \geq 1 \). We use the following lemma:

**Lemma 4.6.** Let \( u \geq 1 \), \( j \leq u \) be integers and \( a_1, ..., a_{u-j+1}, x_1, ..., x_{u-j+1} \geq 0 \). Let \( x \geq \max\{x_i | \forall 1 \leq k \leq u - j + 1 \} \) and let \( a \geq \max\{|a_i| | 1 \leq k \leq u - j + 1 \} \). Let \( B_{u,j} \) be a Bell polynomial. We have:

\[
|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq Ca(x + a)^u.
\]

**Proof.** Let \( p \geq 1 \) and \( l_1, ..., l_p \) be integers. Then we have:

\[
(x_1 + a_1)^{l_1}...(x_p + a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_i-1}(x_1 + a_1)^{l_i}...(x_p + a_p)^{l_p} - x_1^{l_1}...x_{i-1}^{l_i-1}(x_1 + a_1)^{l_i}...(x_p + a_p)^{l_p}
\]

\[
(x_1 + a_1)^{l_1}...(x_p + a_p)^{l_p} - x_1^{l_1}...x_p^{l_p} = \sum_{i=1}^{p} x_1^{l_1}...x_{i-1}^{l_i-1}(x_1 + a_1)^{l_i}...(x_p + a_p)^{l_p} \left[(x_1 + a_1)^{l_i} - x_1^{l_i}\right]
\]

(with the conventions \( x_1^{l_1}...x_0^{l_0} = 1 \) and \( x_{p+1}^{l_{p+1}}...x_p^{l_p} = 1 \)).

Since \( (x_1 + a_1)^{l_i} - x_1^{l_i} \leq l_i|a_1|(|x_1| + |a_1|)^{l_i-1} \leq l_i a(|x_1| + a)^{l_i-1}, 1 \leq l_i \leq u \) and \( x + a \geq 1 \) (because \( x \geq 1 \)), we obtain:

\[
|B_{u,j}(x_1 + a_1, ..., x_{u-j+1} + a_{u-j+1}) - B_{u,j}(x_1, ..., x_{u-j+1})| \leq a(u - j + 1)uB_{u,j}(x + a, ..., x + a).
\]

By the formula giving the Bell polynomials, we have:

\[
B_{u,j}(x + a, ..., x + a) \leq C(x + a)^u.
\]

\(\square\)

To show lemma 4.5, For any \( 0 \leq v \leq u \), we write:

\[
D^v(st)(x) - D^v(sR_\delta)(x) = \sum_{j=0}^{v} D^j s(t(x)) \left[ B_{v,j}(Dt(x), ..., D^{v-j+1}t(x)) - B_{v,j}(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)) \right] +
\]

\[
\left[D^j s(t(x)) - D^j s(R_\delta(x))\right] B_{v,j}(DR_\delta(x), ..., D^{v-j+1}R_\delta(x)).
\]

We apply lemma 4.6 with \( a = \|t - R_\delta\|_u \) and \( x = \|R_\delta\|_u \geq 1 \). Since \( t \in D^u(T^1) \), then \( \|t\|_u \geq 1 \). We get:

\[
\left|B_{v,j}(Dt(x), ..., D^{v-j+1}t(x)) - B_{v,j}(DR_\delta(x), ..., D^{v-j+1}R_\delta(x))\right| \leq C\|t - R_\delta\|_u(1 + \|t - R_\delta\|_u)^u
\]

\[
\left|B_{v,j}(Dt(x), ..., D^{v-j+1}t(x)) - B_{v,j}(DR_\delta(x), ..., D^{v-j+1}R_\delta(x))\right| \leq C\|t - R_\delta\|_u(2 + \|t\|_u)^u \leq C\|t - R_\|_u\|t\|_u^u.
\]

\(\square\)
4.3 Successive estimates

To prove lemma 4.3, we also need these successive estimates:

**Lemma 4.7.** Let \( l \geq 3 \) be an integer, \( f \in D^l(\mathbb{T}^1) \) of rotation number \( \alpha \in \mathbb{R} \), \( \eta > 0 \), \( 0 \leq u \leq l - 2 = \eta \), and \( g \in D^l(\mathbb{T}^1) \) be such that \( fg = gf \). Let \( (q_i)_{i \geq 0} \) be the sequence of denominators of the convergents of \( \alpha \). Let \( \tilde{\alpha} \) be an irrational number of constant type, \( \lambda_0 \in \mathbb{R} \) the associated number and \( h \) the associated \( C^{r-1+\eta} \) diffeomorphism given by lemma 3.3. Let \( f' = h^{-1} f h \) and \( g' = h^{-1} g h \). We have the estimates:

\[
A_{1,u} = \|h^{-1}\|_u \leq C\|h\|_u^C \frac{1}{(\min Dh)^C} \tag{8}
\]

\[
A_{2,u} = \|f'\|_u \leq CA_{1,u}\|f\|_u^C \|h\|_u^C \tag{9}
\]

\[
A_{3,u}(m) = \|f'^m\|_u \leq C^m A_{2,u}^m \tag{10}
\]

\[
A_{4,u} = \|f' - R_u\|_u \leq C\|h\|_{u+1}^C \|f\|_u^C \|h\|_u^C \frac{1}{(\min Dh)^C} |\tilde{\alpha} - \alpha| \tag{11}
\]

\[
A_{5,u}(m) = \|f'^m - R_u\|_{u+1} \leq mCA_{4,u}A_{2,u} A_{2,u} \max_{k \leq m} A_{3,u+1}(k) \tag{12}
\]

\[
A_{6,u} = \|g'\|_u \leq CA_{1,u}\|g\|_u^C \|h\|_u^C \tag{13}
\]

and for any integer \( r \geq 0 \), we have:

\[
A_{7,u} = \|g' - R_{g'}(0)\|_u \leq A_{6,u+1} + \frac{1}{q_r} \max_{m \leq 2q_r} \left(A_{6,u+1}A_{5,u}(m)A_{5,u+1}(m) + A_{6,u}A_{3,u+1}(m)A_{5,u}(m)\right) \tag{14}
\]

\[
A_{8,u} = \|g'h^{-1} - R_u\|_u \leq CA_{6,u+1}A_{7,u} A_{1,u}^C \tag{15}
\]

\[
A_{9,u} = \|hg'h^{-1} - hR_{g'}(0)\|_u \leq C\|g\|_u^C A_{8,u} A_{1,u}^C \|h\|_{u+1} \tag{16}
\]

The crucial estimate is (14), which is obtained by approaching modulo 1 each \( x \in \mathbb{R} \) by a \( m(x)\alpha \), with \( m(x) \leq q_r \). If \( q_r \) increases, \( x - m(x)\alpha \) is smaller modulo 1, but the bound on \( A_{3,u}(m(x)) \) and \( A_{5,u}(m(x)) \) increases. In the proof of theorem 4.1, we make a proper choice of \( r \) (and \( q_r \)).

Estimate (11) corresponds to estimate (5) of the proof of the result of quasi-reducibility.

The other estimates, namely, estimates (8),(9),(10), (12),(13), (15) and (16) are derived from applications of the Faa-di-Bruno formula: either corollary 3.6, lemma 4.4 or lemma 4.5.

**Proof of lemma 4.7.** For \( A_{1,u} \), by estimate (3), we have:

\[
\|h^{-1}\|_u \leq C\|h\|_u^C \frac{1}{(\min Dh)^C}.
\]

Hence estimate (8).

For \( A_{2,u} \), by applying corollary 3.6 twice, we have,
\[
\|f^\prime\|_u \leq CA_{1,0}\|f\|_u^C\|h\|_u^C.
\]

Hence estimate (9).

For \(A_{3,0}\), by applying corollary 3.6 again, we have, for any \(m\),
\[
\|f^{m+1}\|_u \leq C\|f^{m}\|_u\|f^\prime\|_u^C
\]
and therefore, by iteration, we get:
\[
\|f^m\|_u \leq C^m\|f^\prime\|_u^mC.
\]

Hence (10).

Estimate (11) is a direct application of estimate (5).

For estimate (12), we observe that for any \(0 \leq v \leq u\):
\[
D^v f^{m} - D^v R_{m} = D^v \sum_{k=0}^{m-1} f^{m-k} R_{k0} - f^{m-k-1} R_{(k+1)0}
\]
\[
D^v f^{m} - D^v R_{m} = \sum_{k=0}^{m-1} D^v (f^{m-k-1} f^\prime) R_{k0} - D^v (f^{m-k-1} R_{a}) R_{k0}.
\]

By applying lemma 4.5, and by noting that for any \(k\), \(\|f^{m-k-1}\|_{u+1} \leq \max_{0 \leq k \leq m-1} \|f^k\|_{u+1}\), we get:
\[
\|f^m - R_{m0}\|_u \leq mC\|f\|_u^C \max_{0 \leq k \leq m-1} \|f^k\|_{u+1}\|f^\prime - R_{a}\|_u.
\]

Hence (12).

For \(A_{6,0}\), estimate (13) is the same as (9):
\[
\|g^\prime\|_u \leq C\|h^{-1}\|_u\|g\|_u^C\|h\|_u^C.
\]

Hence (13).

For \(A_{7,0}\), let \(m \geq 0\) and \(u \geq v \geq 1\). For any \(x\),
\[
D^v R_{a}(x) = \int_0^1 D^y g^\prime(y) dy.
\]

Therefore,
\[
|D^v g^\prime(x) - D^v R_{a}(x)| = \left| D^v g^\prime(x) - \int_0^1 D^y g^\prime(y) dy \right| = \left| \int_0^1 (D^v g^\prime(x) - D^y g^\prime(y)) dy \right| \leq \max_{x,y\in[0,1]} |D^v g^\prime(x) - D^y g^\prime(y)|.
\]

On the other hand, we have:
\[ D' g'(x) - D' g'(y) = D' g'(x) - D' g'(y + ma) + D' g'(R_{ma}(y)) - D' (g' f''(y)) + D' (f'' m g'(y)) - D' g'(y). \]

Moreover, we have:

\[ |D' g'(x) - D' g'(y + ma)| \leq |D^{m+1} g'_{|x-y- ma}. \]

By lemma 4.5, we also have:

\[ |D' g'(R_{ma}(y)) - D' (g' f''(y))| \leq C \|g'\|_{L^{i+1}} \|f''\|_u^C \|f'' - R_{ma}\|_u. \]

Finally, by lemma 4.4, we have:

\[ |D' (f'' m g'(y)) - D' (R_{ma} g'(y))| \leq C \|f'' m\|_{L^{i+1}} \|f'' - R_{ma}\|_u. \]

Since \( R_{ma} g'(y) = g'(y) + ma \), and \( v \geq 1 \), then \( D' (R_{ma} g'(y)) = D' (R_{ma} g'(y)) \). Therefore, the same estimate holds for \( |D' (f'' m g'(y)) - D' (g'(y))| \).

By combining these estimates, we obtain:

\[ |D' g'(x) - D' g'(y)| \leq \|g'\|_{L^{i+1}} |x-y- ma| + C \|g'\|_{L^{i+1}} \|f''\|_u^C \|f'' - R_{ma}\|_u + C \|f'' m\|_{L^{i+1}} \|f'' - R_{ma}\|_u \|g'\|_u^C. \]

Moreover, for any \( r \geq 0 \), any \( x, y \in \mathbb{R} \), there is an integer \( m(x, y) \leq 2q_r \), there are real numbers \( x', y' \) such that \( x'- x \in \mathbb{Z} \), \( y'- y \in \mathbb{Z} \) and such that \( |x'- y' - m(x, y)n| \leq 1/q_r \). Since \( v \geq 1 \), then \( |D' g'(x) - D' g'(y)| = |D' g'(x') - D' g'(y')| \). We apply the former estimate with \( x' \) and \( y' \) and we get:

\[ \max_{1 \leq m \leq u} \|D' g' - D' R_{g'}(0)\|_u \leq \frac{A_{6u+1}}{q_r} + \max_{m \geq 2q_r} \left( A_{6u+1} A_{5u}^C (m) A_{5u}(m) + A_{6u+1} A_{5u}(m) \right). \]

If \( v = 0 \), we note that for any \( r \geq 0 \), any \( x \in \mathbb{R} \), there is an integer \( m(x) \leq q_r \), and a real number \( x' \) is \( \mathbb{R} \) such that \( x' - x \in \mathbb{Z} \), and such that \( |x' - m(x)n| \leq 1/q_r \). Moreover, we have: \( g'(x) - R_{g'}(0)(x) = g'(x') - R_{g'}(0)(x') \), and

\[ g'(x') - R_{g'}(0)(x') = g'(x') - g'(ma) + g'(ma) - g' f''(0) + f'' m g'(0) - R_{ma} (g'(0)) + R_{g'}(0)(ma) g'(x') - R_{g'}(0)(x'). \]

Hence estimate (14).

For \( A_{8u} \), estimate (15) follows immediately from lemma 4.4.

For \( A_{9u} \), let \( x \in \mathbb{R} \). Let \( 0 \leq v \leq u \). By the Faa-di-Bruno formula:

\[ D' \left( h g'h^{-1} \right)(x) - D' \left( h R_{g'}(0)h^{-1} \right)(x) = \sum_{j=0}^v D^j h (g'h^{-1})(x) B_{e,j} \left( D \left( g'h^{-1} \right)(x), ..., D^{j-1} \left( g'h^{-1} \right)(x) \right) - D^j h(g'h^{-1})(x) B_{e,j} \left( D \left( R_{g'}(0)h^{-1} \right)(x), ..., D^{j-1} \left( R_{g'}(0)h^{-1} \right)(x) \right) \]
\[= \sum_{j=0}^{\infty} D^j h(g' h^{-1})(x) \]

\[\left| B_{x,j} \left( D \left( g' h^{-1}\right)(x), ..., D^{\alpha-1} (g' h^{-1})(x) \right) - B_{x,j} \left( D \left( R_{\alpha}(0) h^{-1}\right)(x), ..., D^{\alpha-1} (R_{\alpha}(0) h^{-1})(x) \right) \right| - \]

\[\left| D^j h(R_{\alpha}(0) h^{-1})(x) \right| - D^j h(g' h^{-1})(x) \right] B_{x,j} \left( D \left( R_{\alpha}(0) h^{-1}\right)(x), ..., D^{\alpha-1} (R_{\alpha}(0) h^{-1})(x) \right). \]

Since \( \|h^{-1}\|_u \geq 1 \), then lemma 4.6 gives,

\[\left| B_{x,j} \left( D \left( g' h^{-1}\right)(x), ..., D^{\alpha-1} (g' h^{-1})(x) \right) - B_{x,j} \left( D \left( R_{\alpha}(0) h^{-1}\right)(x), ..., D^{\alpha-1} (R_{\alpha}(0) h^{-1})(x) \right) \right| \leq \]

\[C\|g' h^{-1}\|^C\|g' h^{-1} - R_{\alpha}(0) h^{-1}\|_u. \]

Since \( g' h^{-1} = h^{-1} g \) and \( \|h^{-1} g\|_u \leq C\|h^{-1}\|_u g_C \), we get,

\[\left| D^\alpha \left( h g' h^{-1}\right) (x) - D^\alpha \left( h R_{\alpha}(0) h^{-1}\right) (x) \right| \leq \]

\[C\|g\|^C\|h^{-1}\|_u g_C^C - R_{\alpha}(0) h^{-1}\|_u + C\|h^{-1}\|_u g' h^{-1} - R_{\alpha}(0) h^{-1}\|_u. \]

Hence estimate (16). This completes the proof of lemma 4.7.

\[\Box\]

By combining these estimates, we obtain:

\[A_{\alpha, u} \leq C A_{1,u+1}^C \|h\|_u\|f\|_u^C g_C^C \left( 1 + \max_{m \geq 2u} \left( A_{3,u+1}^C(m) A_{5,u}(m) \right) \right) \]

\[A_{\alpha, g} \leq C \|h\|_u^C \|f\|_u g_C^C \left( 1 + \alpha^C \left( 1 + \frac{\|f\|_u}{(\min D h)^C} \right) \right)) \]

Hence lemma 4.3. Notice the loss of one derivative for \( h \).

\[\Box\]

**A Appendix: proof of the \( C^{\infty}\)-genericity of \( (O_{\alpha, \beta}^1)^{\infty} \) in \( F_{\alpha, \beta} \)**

To show that \( (O_{\alpha, \beta}^1)^{\infty} \) is \( C^{\infty}\)-generic in \( F_{\alpha, \beta} \), we slightly modify [7, p.160, p.167]. Let \( H : F_{\alpha, \beta} \to \mathbb{R}_+ \cup \{+\infty\} \) be defined by \( H(f, g) = \sup_{p \geq 1} (\|D f^n\|_b, \|D g^n\|_b) \).

The map \( H \) is lower semi-continuous, because it is an upper bound of a family of continuous maps. Therefore, \( (f, g) \in F_{\alpha, \beta} \|H(f, g) > n\) is open, and

\[H^{-1}(+\infty) = \bigcap_{n \geq 1} (f, g) \in F_{\alpha, \beta} \|H(f, g) > n\] is a \( G_\delta \)-set (i.e. a countable intersection of open sets).

By [7, p.52], \( (O_{\alpha, \beta}^1)^{\infty} = H^{-1}(+\infty) \) (if and only if \( H(f, g) = +\infty \)). By the first part of remark 1.4, \( S_{\alpha, \beta} \subset (O_{\alpha, \beta}^1)^{\infty} \) is \( C^{\infty}\)-dense.

Since \( C^1 \)-open sets are \( C^{\infty}\)-open (if \( \phi_\alpha \) does not converge to \( \phi \) in the \( C^1 \) norm, then \( \phi_\alpha \) does not converge to \( \phi \) in the \( C^{\infty} \) norm), we conclude that \( (O_{\alpha, \beta}^1)^{\infty} \) is \( C^{\infty}\)-generic in \( F_{\alpha, \beta} \).
References


