Volume entropy rigidity of non-positively curved symmetric spaces

Francois Ledrappier

VOLUME ENTROPY RIGIDITY OF NON-POSITIVELY CURVED SYMMETRIC SPACES

FRANÇOIS LEDRAPPIER

To Werner Ballmann for his 60th birthday

ABSTRACT. We characterize symmetric spaces of non-positive curvature by the equality case of general inequalities between geometric quantities.

1. Introduction

Let (M,g) be a closed connected Riemannian manifold, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M,g)$ its universal cover endowed with the lifted Riemannian metric. We denote $p(t,x,y), t \in \mathbb{R}_+, x, y \in \widetilde{M}$ the heat kernel on $\widetilde{M}$, the fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \operatorname{Div} \nabla u$ on $\widetilde{M}$. Since we have a compact quotient, all the following limits exist as $t \to \infty$ and are independent of $x \in \widetilde{M}$:

$$\begin{aligned}
\lambda_0 &= \inf_{f \in C_c^2(\widetilde{M})} \frac{\int |\nabla f|^2}{\int |f|^2} = \lim_t -\frac{1}{t} \ln p(t,x,x) \\
\ell &= \lim_t \frac{1}{t} \int d(x,y) p(t,x,y) d\mathrm{Vol}(y) \\
h &= \lim_t -\frac{1}{t} \int p(t,x,y) \ln p(t,x,y) d\mathrm{Vol}(y) \\
v &= \lim_t \frac{1}{t} \ln \mathrm{Vol} B_{\widetilde{M}}(x,t),
\end{aligned}$$

where $B_{\widetilde{M}}(x,t)$ is the ball of radius t centered at x in $\widetilde{M}$ and Vol is the Riemannian volume on $\widetilde{M}$.

All these numbers are nonnegative. Recall λ_0 is the Rayleigh quotient of $\widetilde{M}$, ℓ the linear drift, h the stochastic entropy and v the volume entropy. There is the following relation:

$$4\lambda_0 \overset{(a)}{\leq} h \overset{(b)}{\leq} \ell v \overset{(c)}{\leq} v^2. \tag{1}$$

See [**L1**] for (a), [**Gu**] for (b). Inequality (c) is shown in [**L3**] as a corollary of (b) and (2):

$$\ell^2 \leq h \tag{2}$$

2000 *Mathematics Subject Classification.* 53C24, 53C20, 58J65.

Key words and phrases. volume entropy, rank one manifolds.

If $(\widetilde{M}, g)$ is a locally symmetric space of nonpositive curvature, all five numbers $4\lambda_0, \ell^2, h, \ell v$ and v^2 coincide and are positive unless $(\widetilde{M}, g)$ is $(\mathbb{R}^n, \text{Eucl.})$. Our result is a partial converse:

Theorem 1.1. *Assume (M, g) has nonpositive curvature. With the above notation, any of the equalities*

$$\ell = v, \quad h = v^2 \quad \text{and} \quad 4\lambda_0 = v^2$$

hold if, and only if, $(\widetilde{M}, \widetilde{g})$ is a symmetric space.

As recalled in [**L3**], Theorem 1.1 is known in negative curvature and follows from [**K**], [**BFL**], [**FL**], [**BCG**] and [**L1**]. The other possible converses are delicate: even for negatively curved manifolds, in dimension greater than two, it is not known that $h = \ell v$ holds only for locally symmetric spaces. This is equivalent to a conjecture of Sullivan (see [**L2**] for a discussion). Sullivan conjecture holds for surfaces of negative curvature ([**L1**], [**Ka**]). It is not known either whether $4\lambda_0 = h$ holds only for locally symmetric spaces. This would follow from the hypothetical $4\lambda_0 \overset{(d)}{\leq} \ell^2$ by the arguments of this note.

We assume henceforth that (M, g) has nonpositive sectional curvature. Given a geodesic γ in M, Jacobi fields along γ are vector fields $t \mapsto J(t) \in T_{\gamma(t)}M$ which describe infinitesimal variation of geodesics around γ. By nonpositive curvature, the function $t \mapsto \|J(t)\|$ is convex. Jacobi fields along γ form a vector space of dimension 2 Dim M. The rank of the geodesic γ is the dimension of the space of Jacobi fields such that $t \mapsto \|J(t)\|$ is a constant function on $\mathbb{R}$. The rank of a geodesic γ is at least one because of the trivial $t \mapsto \dot{\gamma}(t)$ which describes the variation by sliding the geodesic along itself. The rank of the manifold M is the smallest rank of geodesics in M. Using rank rigidity theorem ([**B1**], [**BS**]), we reduce in section 2 the proof of Theorem 1.1 to proving that if (M, g) is rank one, equality in (2) implies that $(\widetilde{M}, \widetilde{g})$ is a symmetric space. For this, we show in section 3 that equality in (2) implies that $(\widetilde{M}, \widetilde{g})$ is asymptotically harmonic (see the definition below). This uses the Dirichlet property at infinity (Ballmann [**B2**]). Finally, it was recently observed by A. Zimmer ([**Z**]) that asymptotically harmonic universal covers of rank one manifolds are indeed symmetric spaces.

2. Generalities and reduction of Theorem 1.1

We recall the notations and results from Ballmann's monograph [**B3**] about the Hadamard manifold $(\widetilde{M}, \widetilde{g})$ that we use. The space $\widetilde{M}$ is homeomorphic to a ball. The covering group $G := \pi_1(M)$ satisfies the duality condition ([**B3**] page 45).

2.1. **Boundary at infinity.** Two geodesic rays γ, γ' in $\widetilde{M}$ are said to be asymptotic if $\sup_{t \geq 0} d(\gamma(t), \gamma'(t)) < \infty$. The set of classes of asymptotic unit speed geodesic rays is called the boundary at infinity $\widetilde{M}(\infty)$. $\widetilde{M} \cup \widetilde{M}(\infty)$ is endowed with the topology of a compact space where $\widetilde{M}(\infty)$ is a sphere and where, for each unit speed geodesic ray γ, $\gamma(t) \to [\gamma]$

as $t \to \infty$. The action of the group G on $\widetilde{M}(\infty)$ is the continuous extension of its action on $\widetilde{M}$. For any $x, \xi \in \widetilde{M} \times \widetilde{M}(\infty)$, there is a unique unit speed geodesic $\gamma_{x,\xi}$ such that $\gamma_{x,\xi}(0) = x$ and $[\gamma_{x,\xi}] = \xi$. The mapping $\xi \mapsto \dot{\gamma}_{x,\xi}(0)$ is a homeomorphism π_x^{-1} between $\widetilde{M}(\infty)$ and the unit sphere $S_x\widetilde{M}$ in the tangent space at x to $\widetilde{M}$. We will identify $S\widetilde{M}$ with $\widetilde{M} \times \widetilde{M}(\infty)$ by $(x, v) \mapsto (x, \pi_x v)$. Then the quotient SM is identified with the quotient of $\widetilde{M} \times \widetilde{M}(\infty)$ under the diagonal action of G.

Fix $x_0 \in \widetilde{M}$ and $\xi \in \widetilde{M}(\infty)$. The *Busemann function* b_ξ is the function on $\widetilde{M}$ given by:

$$b_\xi(x) \;=\; \lim_{y \to \xi} d(y, x) - d(y, x_0).$$

Clearly, $b_{g\xi}(gx) = b_\xi(x) + b_{g\xi}(gx_0)$. Moreover, the function $x \mapsto b_\xi(x)$ is of class C^2 ([**HI**]). It follows that the fonction $\Delta_x b_\xi$ satisfies $\Delta_{gx} b_{g\xi} = \Delta_x b_\xi$ and therefore defines a function B on $G \setminus (\widetilde{M} \times \widetilde{M}(\infty)) = SM$. It follows from the argument of [**HI**] that the function B is continuous on SM (see [**B3**], Proposition 2.8, page 69).

2.2. **Jacobi fields.** Let (x, v) be a point in $T\widetilde{M}$. Tangent vectors in $T_{x,v}T\widetilde{M}$ correspond to variations of geodesics and can be represented by Jacobi fields along the unique geodesic $\gamma_{x,v}$ with initial value $\gamma(0) = x, \dot{\gamma}(0) = v$. A Jacobi field $J(t), t \in \mathbb{R}$ along $\gamma_{x,v}$ is uniquely determined by the values of $J(0)$ and $J'(0)$. We describe tangent vectors in $T_{x,v}T\widetilde{M}$ by the associated pair $(J(0), J'(0))$ of vectors in $T_x\widetilde{M}$. The metric on $T_{x,v}T\widetilde{M}$ is given by $\|(J_0, J_0')\|^2 = \|J_0\|^2 + \|J_0'\|^2$. Assume $(x, v) \in SM$. A vertical vector in $T_{x,v}S\widetilde{M}$ is a vector tangent to $S_x\widetilde{M}$. It corresponds to a pair $(0, J'(0))$, with $J'(0)$ orthogonal to v. Horizontal vectors correspond to pairs $(J(0), 0)$. In particular, let X be the vector field on $S\widetilde{M}$ such that the integral flow of X is the geodesic flow. The geodesic spray $X_{x,v}$ is the horizontal vector associated to $(v, 0)$. The orthogonal space to X is preserved by the differential Dg_t of the geodesic flow. More generally, the Jacobi fields representation of $TT\widetilde{M}$ satisfies $D_{x,v}g_t(J(0), J'(0)) = (J(t), J'(t))$.

For any vector $Y \in T_x\widetilde{M}$, there is a unique vector $Z = S_{x,v}Y$ such that the Jacobi field J with $J(0) = Y, J'(0) = Z$ satisfies $\|J(t)\| \leq C$ for $t \geq 0$ ([**B3**] Proposition 2.8 (i)). The mapping $S_{x,v} : T_x\widetilde{M} \to T_x\widetilde{M}$ is linear and selfadjoint. The vectors (Y, SY) describe variations of asymptotic geodesics and the subspace $E^s_{x,v} \subset T_{x,v}T\widetilde{M}$ they generate corresponds to $TW^s_{x,v}$, where $W^s_{x,v}$, the set of initial vectors of geodesics asymptotic to $\gamma_{x,v}$, is identified with $\widetilde{M} \times \pi_x(v)$ in $\widetilde{M} \times \widetilde{M}(\infty)$. Observe that $S_{x,v}\dot{\gamma}_{x,v}(0) = 0$ and that the operator $S_{x,v}$ preserves $(\dot{\gamma}_{x,v}(0))^\perp$. Recall from [**B3**], Proposition 3.2 page 71, that, for $Y \in (\dot{\gamma}_{x,v}(0))^\perp$, with $\pi_x v = \xi$,

$$D_Y(\nabla b_\xi) \;=\; -S_{x,v}Y,$$

and therefore $\Delta_x b_\xi = -\operatorname{Tr} S_{x,v}$ with $\pi_x(v) = \xi$.

Similarly, there is a selfadjoint linear operator $U_{x,v} : T_x\widetilde{M} \to T_x\widetilde{M}$ such that the Jacobi field J with $J(0) = Y, J'(0) = UY$ satisfies $\|J(t)\| \leq C$ for $t \leq 0$. The subspace $E^u_{x,v} \subset$

$T_{x,v}T\widetilde{M}$ they generate corresponds to $TW^u_{x,v}$, where $W^u_{x,v}$ is the set of opposite vectors to vectors in $W^s_{x,-v}$. By definition, $S_{\dot\gamma_{x,(0)}} = -U_{\dot\gamma_{x,-v}(0)}$, so that we also have:

$$B(x,v) \;:=\; -\mathrm{Tr}\; S_{x,v} \;=\; \mathrm{Tr}\; U_{x,-v}.$$

We have Ker $S =$ Ker U and $Y \in$ Ker S if, and only if, the Jacobi field $J(t)$ with $J(0) = Y, J'(0) = 0$ is bounded for all $t \in \mathbb{R}$. The rank of the geodesic $\gamma_{x,v}$ therefore is $\kappa =$ Dim Ker S and the geodesic $\gamma_{x,v}$ is of rank one only if $\mathrm{Det}((U-S)|_{(\dot\gamma_{x,v}(0))^\perp}) = 0$.

Recall that SM is identified with the quotient of $\widetilde{M}\times\widetilde{M}(\infty)$ under the diagonal action of G. Clearly, for $g \in G$, $g(W^s_{x,v}) = W^s_{Dg(x,v)}$ so that the W^s define a foliation $\mathcal{W}^s$ on SM. The leaves of the foliation $\mathcal{W}^s$ are quotient of $\widetilde{M}$, they are naturally endowed with the Riemannian metric induced from $\widetilde{g}$.

2.3. **Proof of Theorem 1.1.** We continue assuming that $(\widetilde{M},\widetilde{g})$ has nonpositive curvature. By the Rank Rigidity Theorem (see [**B3**]), $(\widetilde{M},\widetilde{g})$ is of the form

$$(\widetilde{M}_0 \times \widetilde{M}_1 \times \cdots \times \widetilde{M}_j \times \widetilde{M}_{j+1} \times \cdots \times \widetilde{M}_k, \widetilde{g})^1,$$

where $\widetilde{g}$ is the product metric $\widetilde{g}^2 = (\widetilde{g}_0)^2 + (\widetilde{g}_1)^2 + \cdots + (\widetilde{g}_j)^2 + (\widetilde{g}_{j+1})^2 + \cdots + (\widetilde{g}_k)^2$, $(\widetilde{M}_0,\widetilde{g}_0)$ is Euclidean, $(\widetilde{M}_i,\widetilde{g}_i)$ is an irreducible symmetric space of rank at least two for $i = 1,\cdots,j$ and a rank-one manifold for $i = j+1,\cdots,k$. If the $(\widetilde{M}_i,\widetilde{g}_i), i = j+1,\cdots k$, are all symmetric spaces of rank one, then $(\widetilde{M},\widetilde{g})$ is a symmetric space. Moreover in that case, all inequalities in (1) are equalities: this is the case for irreducible symmetric spaces (all numbers are 0 for Euclidean space; for the other spaces, $4\lambda_0$ and v^2 are classically known to coincide ([**O**]) and we have:

$$4\lambda_0(\widetilde{M}) \;=\; \sum_i 4\lambda_0(\widetilde{M}_i), \quad v^2(\widetilde{M}) \;=\; \sum_i v^2(\widetilde{M}_i).$$

To prove Theorem 1.1, it suffices to prove that if $\ell^2 = h$, all $\widetilde{M}_i$ in the decomposition are symmetric spaces. This is already true for $i = 0,1,\cdots j$. It remains to show that $(\widetilde{M}_i,\widetilde{g}_i)$ are symmetric spaces for $i = j+1,\cdots k$. Eberlein showed that each one of the spaces $(\widetilde{M}_i,\widetilde{g}_i)$ admits a cocompact discrete group of isometries (see [**Kn**], Theorem 3.3). This shows that the linear drifts ℓ_i and the stochastic entropies h_i exist for each one of the spaces $(\widetilde{M}_i,\widetilde{g}_i)$. Moreover, we clearly have

$$\ell^2 \;=\; \sum_i \ell_i^2, \quad h \;=\; \sum h_i.$$

Therefore Theorem 1.1 follows from

Theorem 2.1. *Assume (M,g) is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\widetilde{M},\widetilde{g})$ is a symmetric space.*

[1]With a clear convention for the cases when Dim $\widetilde{M}_0 = 0$, $j = 0$ or $k = j$.

A Hadamard manifold $\widetilde{M}$ is called asymptotically harmonic if the function $B(=\Delta_x b)$ is constant on $S\widetilde{M}$. Theorem 2.1 directly follows from two propositions:

Proposition 2.2. *Assume (M,g) is a closed connected rank one manifold of nonpositive curvature and that $\ell^2 = h$. Then $(\widetilde{M},\widetilde{g})$ is asymptotically harmonic.*

Proposition 2.3. *[[**Z**], Theorem 1.1] Assume (M,g) is a closed connected rank one manifold of nonpositive curvature such that $(\widetilde{M},\widetilde{g})$ is asymptotically harmonic. Then, $(\widetilde{M},\widetilde{g})$ is a symmetric space.*

3. Proof of Proposition 2.2

We consider the foliation $\mathcal{W}$ of subsection 2.2. Recall that the leaves are endowed with a natural Riemannian metric. We write $\Delta^{\mathcal{W}}$ for the associated Laplace operator on functions which are of class C^2 along the leaves of $\mathcal{W}$. A probability measure m on SM is called harmonic if it satisfies, for any C^2 function f, we have:

$$\int_{SM} \Delta^{\mathcal{W}} f dm \;=\; 0.$$

Let M be a closed connected manifold such that $\ell^2 = h$. In [**L3**] it is shown that then, there exists a harmonic probability measure m on SM such that, at m-a.e. (x,v), $B(x,v) = \ell$. Since B is a continuous function, Proposition 2.2 follows from

Theorem 3.1. *Let (M,g) be a closed connected rank one manifold of nonpositive curvature, $\mathcal{W}$ the stable foliation on SM endowed with the natural metric as above. Then, there is only one harmonic probability measure m and the support of m is the whole space SM.*

Proof. Let m be a $\mathcal{W}$ harmonic probability measure on SM. Then, there is a unique G-invariant measure $\widetilde{m}$ on $S\widetilde{M}$ which coincide with m locally. Seen as a measure on $\widetilde{M}\times\widetilde{M}(\infty)$, we claim that $\widetilde{m}$ is given, for any f continuous with compact support, by:

$$\int f(x,\xi) d\widetilde{m}(x,\xi) \;=\; \frac{1}{\mathrm{Vol} M}\int_{\widetilde{M}}\left(\int_{\widetilde{M}(\infty)} f(x,\xi) d\nu_x(\xi)\right) dx, \tag{3}$$

where the family $x\mapsto \nu_x$ is a family of probability measures on $\widetilde{M}(\infty)$ such that, for all φ continuous on $\widetilde{M}(\infty)$, $x\mapsto \int \varphi(\xi) d\nu_x(\xi)$ is a harmonic function on $\widetilde{M}$ and the measure dx is the Riemannian volume on $\widetilde{M}$. The claim follows from [**Ga**]. For convenience, let us reprove it: on the one hand, the measure $\widetilde{m}$ projects on $\widetilde{M}$ as a G-invariant measure satisfying $\int \Delta f dm \,=\, 0$. The projection of $\widetilde{m}$ on $\widetilde{M}$ is proportional to Volume, gives measure 1 to fundamental domains and formula (3) is the desintegration formula. On the other hand, if one projects $\widetilde{m}$ first on $\widetilde{M}(\infty)$, there is a probability measure ν on $\widetilde{M}(\infty)$ such that

$$\int f(x,\xi) d\widetilde{m}(x,\xi) \;= \int_{\widetilde{M}(\infty)}\left(\int_{\widetilde{M}} f(x,\xi) dm_\xi(dx)\right) d\nu(\xi).$$

For ν-a.e. ξ, the measure m_ξ is a harmonic measure on $\widetilde{M}$; therefore, for ν-a.e. ξ, there is a positive harmonic function $k_\xi(x)$ such that $m_\xi = k_\xi(x)\mathrm{Vol}$. Comparing the two expressions for $\int f d\widetilde{m}$, we see that the measure ν_x is given by

$$\nu_x \;=\; k_\xi(x)\nu$$

and $x \mapsto \int_{\widetilde{M}(\infty)} \varphi(\xi) d\nu_x(\xi)$ is indeed a harmonic function.

The G-invariance of $\widetilde{m}$ implies that, for all $g \in G$, $g_*\nu_x = \nu_{gx}$. In particular, the support of ν is G-invariant. By [**E**] (see [**B3**], page 48), the support of ν is the whole $\widetilde{M}(\infty)$ and therefore the support of m is the whole SM. This result would be sufficient for proving Proposition 2.2, but using discretization, we are going to identify the measure ν_x on $\widetilde{M}(\infty)$ as the hitting measure of the Brownian motion on $\widetilde{M}$ starting from x. This shows Theorem 3.1.

Fix $x_0 \in \widetilde{M}$. The discretization procedure of Lyons and Sullivan ([**LS**]) associates to the Brownian motion on $\widetilde{M}$ a probability measure μ on G such that $\mu(g) > 0$ for all g and that any bounded harmonic function F on $\widetilde{M}$ satisfies

$$F(x_0) \;=\; \sum_{g \in G} F(gx_0)\mu(g).$$

Recall that for all φ continuous on $\widetilde{M}(\infty)$, $x \mapsto \nu_x(\varphi)$ is a harmonic function and that $\nu_{gx} = g_*\nu_x$. It follows that the measure ν_{x_0} is stationary for μ, i.e. it satisfies:

$$\nu_{x_0} \;=\; \sum_{g \in G} g_*\nu_{x_0}\mu(g).$$

Since the support of μ generates G as a semigroup (actually, it is already the whole G), there is only one stationary probability measure on $\widetilde{M}(\infty)$ (see [**B3**], Theorem 4.11 page 58). We know one already: the hitting measure m_{x_0} of the Brownian motion on $\widetilde{M}$ starting from x_0. This shows that $\nu_{x_0} = m_{x_0}$. Since x_0 was arbitrary in the above reasoning, we have $\nu_x = m_x$ for all $x \in \widetilde{M}$ and the measure $\widetilde{m}$ is given by:

$$\int f(x,\xi) d\widetilde{m}(x,\xi) \;=\; \frac{1}{\mathrm{Vol}M} \int_{\widetilde{M}} \left(\int_{\widetilde{M}(\infty)} f(x,\xi) dm_x(\xi) \right) dx.$$

□

Acknowledgements I am very grateful to Gerhard Knieper for his interest and his comments, in particular for having attracted my attention to [**Z**]. I also acknowledge partial support of NSF grant DMS-0811127.

References

[**B1**] W. Ballmann, Nonpositively curved manifolds of higher rank, *Ann. Math*, **122** (1985), 597–609.

[**B2**] W. Ballmann, On the Dirichlet problem at infinity for manifolds of nonpositive curvature, *Forum Mathematicum*, **1** (1989), 201–213.

[**B3**] W. Ballmann, Lectures on spaces of nonpositive curvature, *DMV Seminar*, **25** (1995).

[**BCG**] G. Besson, G. Courtois and S. Gallot, Entropies et rigidités des espaces localement symétriques de courbure strictement négative, *Geom. Func. Anal.* **5** (1995), 731–799.

[**BFL**] Y. Benoist, P. Foulon and F. Labourie, Flots d'Anosov à distributions stables et instables différentiables, *J. Amer. Math. Soc.* **5** (1992), 33–74.

[**BS**] K. Burns and R. Spatzier, Manifolds of nonpositive curvature and their buildings, *Publications math. IHES*, **65** (1987), 35–59.

[**E**] P. Eberlein, Geodesic flows on negatively curved manifolds, II, *Transactions Amer. math. Soc.*, **178** (1973), 57–82.

[**FL**] P. Foulon and F. Labourie, Sur les variétés compactes asymptotiquement harmoniques, *Invent. Math.* **109** (1992), 97–111.

[**Ga**] L. Garnett, Foliations, the ergodic theorem and Brownian motion, *J. Funct. Anal.* **51** (1983), 285–311.

[**Gu**] Y. Guivarc'h, Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire, *Astérisque,* **74** (1980) 47–98.

[**HI**] E. Heintze and H.-C. Im Hof, Geometry of horospheres, *J. Diff. Geom.* **12** (1977), 481–491.

[**K**] V. A. Kaimanovich, Brownian motion and harmonic functions on covering manifolds. An entropic approach, *Soviet Math. Dokl.* **33** (1986) 812–816.

[**Ka**] A. Katok, Four applications of conformal equivalence to geometry and dynamics, *Ergod. Th. & Dynam. Sys.*, 8* (1988), 139–152.

[**Kn**] G. Knieper, On the asymptotic geometry of non-positively curved manifolds, *GAFA*, **7** (1997), 755–782.

[**L1**] F. Ledrappier, Harmonic measures and Bowen-Margulis measures, *Israel J. Math.* **71** (1990), 275–287.

[**L2**] F. Ledrappier, Applications of dynamics to compact manifolds of negative curvature, *in Proceedings of the ICM Zürich 1994*, Birkhäuser (1995), 1195-1202.

[**L3**] F. Ledrappier, Linear drift and entropy for regular covers, *GAFA*, **20** (2010), 710–725.

[**LS**] T. Lyons and D. Sullivan, Function theory, random paths and covering spaces, *J. Differential Geometry*, **19** (1984), 299–323.

[**O**] M.A. Olshanetsky, Martin boundary for the Laplace-Beltrami operator on a Riemannian symmetric space of non-positive curvature, *Uspehi Mat. Nauk.,* **24:6** (1969), 189-190.

[**Z**] A. M. Zimmer, Asymptotically harmonic manifolds without focal points, *preprint*, (http://arxiv.org/abs/1109.2481),

LPMA, UMR CNRS 7599, Université Paris 6, Boîte Courrier 188, 4, Place Jussieu, 75252 PARIS cedex 05, France

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: `fledrapp@nd.edu`